The problem

We study $L^p$-solutions $u = u(t,x,v) : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ to linear kinetic equations of the type

$$\begin{cases}
\partial_t u + v \cdot \nabla_x u = Au + f \\
u(0) = g,
\end{cases}$$

where $A$ is an operator in a suitable function space $X$ and $f, g$ are given data. Goal: Characterize unique solutions $u$ to equation (1) with $\partial_t u + v \cdot \nabla_x u, Au \in L^p((0,T); X)$ in terms of functions spaces for the data $f$ and $g$. In particular, show that the solutions to equation (1) define a semi-flow in the trace space. If $A$ admits such a characterization we say that $A$ enjoys kinetic maximal $L^p(X)$-regularity.

Divide and conquer

- The case of vanishing initial data, i.e. $g = 0$. Using singular integral theory and the solution representation given by a fundamental solution.
- Complicated operators can often be reduced to simpler cases.
- The homogeneous case, i.e. $f = 0$. Make sense of the temporal trace. How does the kinetic term transfer regularity from $v$ to $x$ on this level?

The most important example

For $\beta \in (0,2)$ consider the (fractional) Kolmogorov equation in $\mathbb{R}^{2n}$,

$$\begin{cases}
\partial_t u + v \cdot \nabla_x u = (-\Delta)_x^\beta u + f \\
u(0) = g,
\end{cases}$$

The equation dictates that for strong $L^p$-solutions the right solution space is

$$E(0,T) = \{ u : u, \partial_t u + v \cdot \nabla_x u, (-\Delta)_x^\beta u \in L^p((0,T); L^p(\mathbb{R}^{2n})) \},$$

i.e. $X = L^p(\mathbb{R}^{2n})$. Using operator theoretic properties of the characteristics $(t,x,v) \mapsto (t,x+tv,v)$ we can prove

$$E(0,T) \hookrightarrow C([0,T]; L^p(\mathbb{R}^{2n})),
$$

whence $X_T$, the trace space of $E(0,T)$, is well-defined and $E(0,T) \hookrightarrow C([0,T]; X_T)$. A theorem of Bouchut (2002) shows that

$$E(0,T) = E(0,T) \cap L^p((0,T); H^{\beta/2,q}((\mathbb{R}^{2n}))).
$$

Hence, the trace space should also have some regularity in $x$. Indeed, using methods from harmonic analysis and the fundamental solution we prove

$$X_T \equiv B^{1-\beta/2}_{p,q} (\mathbb{R}^{2n}) \cap B^{0}_{p,q} (\mathbb{R}^{2n}).$$

In particular, $A = (-\Delta)_x^\beta$ admits kinetic maximal $L^p$-regularity in $L^p(\mathbb{R}^{2n})$.

More examples

The linearization of many nonlinear kinetic models leads to the (fractional) Kolmogorov equation with variable coefficients.

The characterization of strong $L^p$-solutions for the Kolmogorov equation can be extended to

$$\begin{cases}
\partial_t u + v \cdot \nabla_x u = a(t,x,v) \cdot \nabla_x u + b \cdot \nabla_x u + cu + f \\
u(0) = g
\end{cases}$$

provided that $a(t,x,v) \geq \lambda > 0$, $b \in L^\infty$, $c \in L^\infty$ and that

- $(t,x,v) \mapsto a(t,x,v)$ is bounded and uniformly continuous
- $(t,x,v) \mapsto a(t,x+tv,v)$ is bounded and uniformly continuous, see [2].

If $a$ has a special structure we are also able to treat the case of non-uniform ellipticity.

In [3] we study non-local operators with variable coefficients

$$[A_{t,x,v}] (t,x,v) = p.v. \int_{\mathbb{R}^n} (u(t,x,v+h) - u(t,x,v)) \frac{a(t,x,v,h)}{|h|^{n+p}} dh,$$

where $a$ is symmetric in $h$, satisfies a similar continuity property in $(t,x,v)$ and is also Hölder continuous in $v$.

An application

The precise solution theory allows to study quasi-linear kinetic partial differential equations of the type

$$\begin{cases}
\partial_t u + \nabla_x \cdot (\kappa (w) \nabla_x u) = 0 \\
u(0) = u_0
\end{cases}$$

We are interested in $L^p(X)$ solutions, that is functions such that

$$\partial_t u + \nabla_x \cdot (\kappa (w) \nabla_x u) \in L^p(X).$$

Under a local Lipschitz assumption on the operators $A$ and $F$ we are able to prove local in time existence for all $u_0 \in X_T$ such that $(A(u_0))$ admits kinetic maximal $L^p(X)$-regularity.

Two Examples:

- A quasilinear diffusion problem, $A(w)u = \nabla_x \cdot (\kappa (w) \nabla_x u)$ for suitable functions $\kappa$.
- A kinetic toy model of the form $A(w)u = - \int_{\mathbb{R}^n} w dv \cdot (-\Delta_x)^\beta u$ for $\beta \in (0,2]$.

The precise characterization of the trace space is very helpful to control the nonlinearities. For example embeddings such as

$$B^{2-2\beta}_{p,q}(\mathbb{R}^{2n}) \cap B^{1-1-\beta}_{p,q}(\mathbb{R}^{2n}) \hookrightarrow C_0(\mathbb{R}^{2n})$$

are available for $p > 2n + 1$.

Ongoing research

- Establish the kinetic maximal $L^p$-regularity for more operators, such as for example the kinetic Fokker-Planck equation $Au = \Delta_x u + v \cdot \nabla_x u$.
- Weak $L^p$-solutions.
- Study the local existence of solutions for more complicated quasilinear equations.
- Global in time existence for quasilinear equations. Here, one needs to incorporate a priori estimates from the kinetic De Giorgi-Nash-Moser theory.

References