Kinetic maximal L^p-regularity

and applications to quasilinear kinetic equations

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Kolmogorov equation

Interested in solutions $u = u(t, x, v)\colon [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of

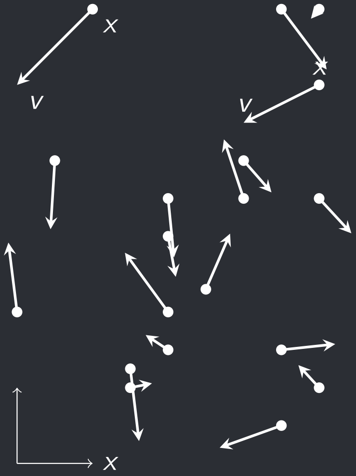
$$\begin{cases} \partial_t u + v \cdot \nabla_x u = -(-\Delta_v)^{\beta/2} u + f \\ u(0) = g. \end{cases} \tag{1}$$

with data f, g and $\beta \in (0, 2]$.

Key points:

- Studied first by Kolmogorov in 1934 ($\beta = 2$).
- The transport operator $\partial_t + v \cdot \nabla_x$ is called kinetic term.
- Degenerate as the Laplacian acts in half of the variables.
- Unbounded coefficient in front of the lower order term.
- Prototype for the Boltzmann equation.

Motivation - Particle Physics



Particles at position x with velocity v. We describe the movement of the particles with the SDE

$$\begin{cases} \mathrm{d}X(t) = V(t)\mathrm{d}t \\ \mathrm{d}V(t) = \mathrm{d}W(t), \end{cases}$$

where $(W(t))_{t\geq 0}$ is the Wiener process. $\rightsquigarrow$ Kolmogorov equation with $\beta = 2$.
The Boltzmann equation models the particle collision, i.e. the change of velocity, more precisely.

Boltzmann equation

The Boltzmann equation can be written as

$$\partial_t u + v \cdot \nabla_x u = Q(u, u) + \mathrm{l.o.t.},$$

where

$$Q(u, g) = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{u(t, x, v + h) - u(t, x, v)}{|h|^{n+\beta}} m(g)(t, x, v, h) \mathrm{d}h$$

with

$$m(g)(t, x, v, h) = \int_{w \perp h} g(t, x, v + w) \, |w|^{\gamma+\beta+1} \, \mathrm{d}w$$

and $\beta \in (0, 2)$, $\gamma > -n$ depend on physical assumptions. For fixed g the operator $Q(u, g)$ is the fractional Laplacian in velocity with variable density.

Maximal regularity

Let us consider a PDE of the form

$$\begin{cases} \partial_t u = Au + f, t > 0 \\ u(0) = g, \end{cases}$$

where A is an operator on a Banach space B and u a function of time with values in B.

General Principle:

Find a function space Z for the solution u, a function space X for the inhomogeneity f and a function space X_γ for the initial value g such that the equation admits a unique solution $u \in Z$ if and only if $f \in X$ and $g \in X_\gamma$.

Here: Maximal L^p-regularity, i.e. $X = L^p(B)$ for some base space B.

Maximal L^p-regularity

Example - Heat equation

For all $p \in (1, \infty)$ the heat equation

$$\begin{cases} \partial_t u = \Delta u + f \\ u(0) = g \end{cases}$$

admits a unique solution
$u \in Z = H^{1,p}((0,\infty); L^p(\mathbb{R}^n)) \cap L^p((0,\infty); H^{2,p}(\mathbb{R}^n))$ if and only if

- $f \in X = L^p((0,\infty); L^p(\mathbb{R}^n))$,
- $g \in X_\gamma = B^{2(1-1/p)}_{pp}(\mathbb{R}^n)$ (Besov space).

Moreover, $u \in C([0,\infty); B^{2(1-1/p)}_{pp}(\mathbb{R}^n))$.

Towards kinetic maximal regularity

Which is the right choice for the solution space Z?

For simplicity $\beta = 2$, every result presented here holds true for $\beta \in (0, 2)$.

We choose $X = L^p(\mathbb{R}; L^p(\mathbb{R}^{2n}))$. Singular integral theory on homogeneous groups developed by Folland and Stein in 1974 allows to prove the following. If $f \in L^p(\mathbb{R}; L^p(\mathbb{R}^{2n}))$, then the solution u of the Kolmogorov equation satisfies

$$\|\partial_t u + v \cdot \nabla_x u\|_p + \|\Delta_v u\|_p \lesssim \|f\|_p \, .$$

No control of the time-derivative. We prove classical maximal L^p-regularity is not applicable.
Our choice of function space for the solution is:

$$Z = \{u \colon u, \; \Delta_v u, \; \partial_t u + v \cdot \nabla_x u \in L^p((0, T); L^p(\mathbb{R}^{2n}))\}.$$

Singular integrals on homogeneous groups

Three important underlying structures of the Kolmogorov equation, $\mathcal{K}(u) = \partial_t u + v \cdot \nabla_x u - \Delta_v u$.

- Scaling: For $\delta_\lambda u = u(\lambda^2 t, \lambda^3 x, \lambda v)$ we have $\mathcal{K}(\delta_\lambda u) = \delta_\lambda \mathcal{K}(u)$.
- Translation For $z_0 = (t_0, x_0, v_0), (t, x, v) \in \mathbb{R}^n$ we define

  $$(t, x, v) \circ (t_0, x_0, v_0) = (t + t_0, x + x_0 + tv_0, v + v_0).$$

  Then, $\mathcal{K}(u((t, x, v) \circ z_0)) = \mathcal{K}(u)((t, x, v) \circ z_0))$.
- Fundamental solution: There exists a $\gamma \in C^\infty(\mathbb{R}^{2n+1} \setminus \{0\})$ such that

  $$u = \int_{\mathbb{R}^{2n+1}} \gamma(z^{-1} \circ x) f(z) \mathrm{d}z$$

  solves $\mathcal{K}u = f$ for suitable f.

The pair $(\mathbb{R}^{2n+1}, \circ)$ defines a homogeneous group. $\rightsquigarrow$ CZO theory

Towards kinetic maximal regularity

Divide and conquer

We can split the characterization of solutions in Z in two separate problems.

Inhomogeneous eq. with zero intial-value X

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = 0 \end{cases}$$

Classical Method: L^p-estimates, singular integrals,...
Done, Folland/Stein.

Homogeneous eq. with non-zero intial-value X_γ

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u \\ u(0) = g \end{cases}$$

Classical Method: Studying the trace space of Z.

TODO!

Towards kinetic maximal regularity

The trace space of Z - 1

Does a function $u \in Z$ admit a trace? Yes!

Sketch of the proof

Define

$$[\Gamma u](t,x,v) = u(t, x+tv, v) \text{ and } [\Gamma(t)w](x,v) = w(x+tv, v)$$

on functions $u\colon [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}$ and $w\colon \mathbb{R}^{2n} \to \mathbb{R}$. Then,

$$\partial_t \Gamma u = \Gamma(\partial_t u + v \cdot \nabla_x u).$$

If $u \in Z$, then $\Gamma u \in H^{1,p}((0,T); L^p(\mathbb{R}^{2n}))$, whence

$$\Gamma u \in C([0,T]; L^p(\mathbb{R}^{2n})).$$

As $(\Gamma(t))_{t\in\mathbb{R}}$ is a C_0-group, it follows

$$u = \Gamma^{-1}(t)\Gamma(t) u \in C([0,T]; L^p(\mathbb{R}^{2n})).$$

Consequently, $\mathrm{Tr}(Z)$ well-defined and $Z \hookrightarrow C([0,T]; \mathrm{Tr}(Z))$.

Towards kinetic maximal regularity

The trace space of Z - 2

The trace space of Z cannot be characterized by classical interpolation theory. Recalling the heat equation we expect atleast

$$\mathrm{Tr}(Z) \hookrightarrow B^{2(1-1/p)}_{pp,v}(\mathbb{R}^{2n}).$$

Is there any control of regularity in x? Yes!

Towards kinetic maximal regularity

The phenomenon of regularity transfer from v to x.

Theorem (Bouchut 2002)

Let $u \in L^p((0,T); L^p(\mathbb{R}^{2n}))$ with $\partial_t u + v \cdot \nabla_x u \in L^p((0,T); L^p(\mathbb{R}^{2n}))$ and $u \in L^p((0,T); H_v^{2,p}(\mathbb{R}^{2n}))$, then

$$u \in L^p((0,T); H_x^{2/3,p}(\mathbb{R}^{2n})).$$

In words: If u is the solution of a kinetic equation and u has two derivatives in velocity we obtain 2/3 of a derivative in space, too.
Very useful and powerful result!
It is proven by Fourier analytic methods. For $p = 2$ one can see how the characteristics (i.e. Γ) transfer the regularity.

Towards kinetic maximal regularity

The initial value problem - 1

Consequently:

$$Z = Z \cap L^p((0,T); H_x^{2/3,p}(\mathbb{R}^{2n})).$$

Similar to Bouchut we also get some regularity in x for the trace space.

Theorem (N., Zacher, 2020)

Let $p \in (1, \infty)$, then

$$\mathrm{Tr}(Z) \cong B_{pp,x}^{2/3(1-1/p)}(\mathbb{R}^{2n}) \cap B_{pp,v}^{2(1-1/p)}(\mathbb{R}^{2n})$$

An anisotropic Besov spaces with a kinetic scaling.

Towards kinetic maximal regularity

The initial value problem

Proof - Part 1/3

We prove that the inclusion mapping

$$\iota\colon \mathrm{Tr}\left(\mathbb{E}_\mu(0,T)\right) \to B^{2/3(1-1/p)}_{pp,x}(\mathbb{R}^{2n}) \cap B^{2(1-1/p)}_{pp,v}(\mathbb{R}^{2n})$$

is a well-defined linear, bounded and surjective operator. It follows that ι defines an isomorphism.

Towards kinetic maximal regularity

The initial value problem

Proof - Part 2/3

The norm of the kinetic Besov space can be equivalently characterized by

$$\|\varphi_0 * g\|_p + \left(\int_0^1 \left(\frac{\|\varphi_t * g\|_p}{t^{1-1/p}} \right)^p \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}},$$

where $\varphi_t(x, v) = t^{-2n}\varphi(t^{-3/2}x, t^{-1/2}v)$ for a suitable function φ whose Fourier transform admits compact support in an ellipsoid and is positive in its interior.
We choose φ related to the fundamental solution of the Kolmogorov equation. Let $g \in \mathrm{Tr}(Z)$ then w.l.o.g. we may choose $u \in Z$ is a solution of the Kolmogorov equation with $u(0) = g$. It follows that ι is bounded.

Towards kinetic maximal regularity
The initial value problem

Proof - Part 3/3

Regarding the surjectivity of ι let g be an element of the kinetic Besov space. In Fourier variables (ξ for v and k for x) the solution of the Kolmogorov equation is given by

$$\hat{u}(t,k,\xi) = \hat{g}(k,\xi + tk)\exp\left(-|\xi|^2 t - \xi \cdot kt^2 - |k|^2 \frac{t^3}{3}\right).$$

Using the Littlewood-Paley decomposition of u we can directly show that $\Delta_v u \in L^p((0,T); L^p(\mathbb{R}^{2n}))$, i.e. $u \in Z$, under the assumption that $B^{2/3(1-1/p)}_{pp,x}(\mathbb{R}^{2n}) \cap B^{2(1-1/p)}_{pp,v}(\mathbb{R}^{2n})$.

Kinetic maximal L^p-regularity

for the (fractional) Kolmogorov equation

Theorem (N., Zacher, 2020)

Let $T \in (0, \infty)$. For all $p \in (1, \infty)$ the Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

admits a unique solution $u \in Z$ if and only if

- $f \in X = L^p((0, T); L^p(\mathbb{R}^n))$,
- $g \in X_\gamma = B^{2/3(1-1/p)}_{pp,x}(\mathbb{R}^{2n}) \cap B^{2(1-1/p)}_{pp,v}(\mathbb{R}^{2n})$.

Moreover, $u \in C([0, T]; X_\gamma)$.

We say the operator $A = \Delta_v$ admits kinetic maximal L^p-regularity.

Extensions

Change of base space

So far we have only considered the base space $X = L^p(\mathbb{R}^{2n})$.

- We also consider the case $X = L^q(\mathbb{R}^{2n})$ for some $q \in (1, \infty)$ different from p and prove kinetic maximal $L^p(L^q)$-regularity.

Extensions

From $L^p(L^p)$ to $L^p(L^q)$

The operator $f \mapsto \Delta_v u$ is bounded in $L^p((0,T); L^p(\mathbb{R}^{2n}))$. In Fourier variables it can be written as

$$\hat{f} \mapsto |\xi|^2 \int_0^t \exp\left(-|\xi|^2 s - \xi \cdot k s^2 - |k|^2 \frac{s^3}{3}\right) \hat{f}(t-s, k, \xi + sk) \mathrm{d}s$$

Calderón-Zygmund theory for operator valued problems yields the $L^p(L^q)$-boundedness. This idea is inspired by the same result for the maximal regularity of non-autonomous PDE. Note that if u is a solution of the Kolmogorov equation, then $w = \Gamma u$ solves the non-autonomous degenerate PDE

$$\partial_t w = (\nabla_v - t\nabla_x) \cdot (\nabla_v - t\nabla_x) w.$$

Extensions

Change of base space

So far we have only considered the base space $X = L^p(\mathbb{R}^{2n})$.

- We also consider the case $X = L^q(\mathbb{R}^{2n})$ for some $q \in (1, \infty)$ different from p and prove kinetic maximal $L^p(L^q)$-regularity.
- For $p \in (1, \infty)$, $q = 2$ we characterize the regularity of weak solutions to the fractional Kolmogorov equation. Here, we again use the solution formula and the availability of the theorem of Plancherel for the x and v variables as $q = 2$.

Extensions

Temporal weights

Instead of $L^p((0,T);X)$ we consider a Lebesgue space with temporal weight of the form $t^{1-\mu}$ for some $\mu \in (1/p, 1]$ defined as

$$L^p_\mu((0,T);X) = \{u\colon (0,T) \to X\colon \int_0^T t^{p-p\mu} \|u(t)\|_X^p \,\mathrm{d}t < \infty\}.$$

We write Z_μ for Z with temporal weight in the L^p-spaces.

Key features:

- Kin. max. L^p-reg. $\iff$ Kin. max. L^p_μ-reg. for any $\mu \in (1/p, 1]$
- The trace space of Z_μ is given by $\mathrm{Tr}(Z_\mu) = X_{\gamma,\mu} = B^{2/3(\mu-1/p)}_{pp,x}(\mathbb{R}^{2n}) \cap B^{2(\mu-1/p)}_{pp,v}(\mathbb{R}^{2n})$.
- Instantaneous regularization $Z_\mu(0,T) \hookrightarrow Z(\delta,T) \hookrightarrow C([\delta,T];X_{\gamma,1})$ for all $\delta > 0$.

Extensions

Kinetic maximal $L^p_\mu(L^q)$-regularity

Theorem (N., Zacher, 2020)

Let $T \in (0,\infty)$. For all $p, q \in (1,\infty)$ and any $\mu \in (1/p, 1]$ the Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \Delta_v u + f \\ u(0) = g \end{cases}$$

admits a unique solution

$$u \in Z_\mu = \{u\colon u,\, \Delta_v u,\, \partial_t u + v \cdot \nabla_x u \in L^p_\mu((0,T); L^q(\mathbb{R}^{2n}))\}.$$

if and only if

- $f \in X = L^p_\mu((0,T); L^q(\mathbb{R}^n))$,
- $g \in X_{\gamma,\mu} = B^{2/3(\mu-1/p)}_{qp,x}(\mathbb{R}^{2n}) \cap B^{2(\mu-1/p)}_{qp,v}(\mathbb{R}^{2n})$.

Moreover, $u \in C([0,T]; X_{\gamma,\mu})$.

Extensions

Different Operators

Question: Do other operators admit kinetic maximal L^p-regularity? Yes.

Examples:

- $Au = a(t,x,v) : \nabla_v^2 u + b \cdot \nabla_v u + cu$
- $Au = -(-\Delta_v)^{\frac{\beta}{2}} u$ with $\beta \in (0,2)$
- non-local integro-differential operators acting in velocity with possibly time, space and velocity dependent density of the form

$$Au = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{u(t,x,v+h) - u(t,x,v)}{|h|^{n+\beta}} m(t,x,v,h) \mathrm{d}h$$

Extensions

Different Operators

Theorem (N., Zacher, 2020)

Let $p, q \in (1, \infty)$, $\mu \in (1/p, 1]$, $a \in L^\infty([0, T] \times \mathbb{R}^{2n}; \mathrm{Sym}(n))$, $b \in L^\infty([0, T] \times \mathbb{R}^{2n}; \mathbb{R}^n)$ and $c \in L^\infty([0, T] \times \mathbb{R}^{2n}; \mathbb{R})$. If $a \geq \lambda \mathrm{Id}$ for some $\lambda > 0$ and if the function $(t, x, v) \mapsto a(t, x + tv, v)$ is uniformly continuous, then then the family of operators

$$A(t)u = a(t, \cdot) \colon \nabla_v^2 u + b(t, \cdot) \cdot \nabla_v u + c(t, \cdot)u$$

admits kinetic maximal $L^p_\mu(L^q)$-regularity.

Quasilinear kinetic diffusion problem

Short-time existence

We prove short-time existence of strong L^p_μ-solutions to the following quasilinear kinetic diffusion equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \nabla_v \cdot (a(u) \nabla_v u) \\ u(0) = g \end{cases}$$

for $a \in C^2_b(\mathbb{R}; \mathrm{Sym}(n))$ with $a \geq \lambda Id$ for some $\lambda > 0$, $\mu - 1/p > 2n/p$ and $g \in X_{\gamma,\mu}$.

Method: Freeze the equation at the initial value and use kinetic maximal L^p-regularity for the frozen equation. Here, we need the kinetic maximal regularity of $A = a(g(x,v)) : \nabla_v^2 u$.

Quasilinear kinetic diffusion problem

Short-time existence

Another interesting quasilinear Problem is

$$\begin{cases} \partial_t u + v \cdot \nabla_x u = \left(\int_{\mathbb{R}^n} u \mu \mathrm{d}v \right) \Delta_v u \\ u(0) = g \end{cases}$$

with $\mu \in L^1(\mathbb{R}^n)$.
The more particles there are at a position x the more diffusion there is. If $\mu - 1/p > 2n/p$ and $g \in X_{\gamma,\mu}$ we can show the existence of a strong L^p_μ-solution for a possibly short time.

Models of this type are an important step towards more complicated equations such as the Landau and the Boltzmann equation.

Further research

Possible directions:

- weak L^p-solutions
- study quasilinear kinetic problems from physics/economics/biology
- qualitative study of quasilinear problems such as large time behavior
- conditions on the operator A such that it admits kinetic maximal L^p-regularity
- different first order terms, for example $\partial_t + \langle x, B\nabla\rangle$ or the relativistic kinetic term $\partial_t + \frac{v}{\sqrt{1+|v|^2}}\nabla_x$

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