Kinetic maximal $L^p$-regularity

and applications to quasilinear kinetic equations

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Kolmogorov equation

Interested in solutions $u = u(t, x, v): [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of

$$
\begin{cases}
\partial_t u + v \cdot \nabla_x u = -(-\Delta_v)^{\beta/2} u + f \\
u(0) = g.
\end{cases}
$$

(1)

with data $f, g$ and $\beta \in (0, 2]$.

**Key points:**

- Studied first by Kolmogorov in 1934 ($\beta = 2$).
- The transport operator $\partial_t + v \cdot \nabla_x$ is called kinetic term.
- Degenerate as the Laplacian acts in half of the variables.
- Unbounded coefficient in front of the lower order term.
- Prototype for the Boltzmann equation.
Particles at position \( x \) with velocity \( v \). We describe the movement of the particles with the SDE

\[
\begin{aligned}
\frac{dX(t)}{dt} &= V(t) dt \\
\frac{dV(t)}{dt} &= dW(t),
\end{aligned}
\]

where \( (W(t))_{t \geq 0} \) is the Wiener process. \( \sim \) Kolmogorov equation with \( \beta = 2 \).

The Boltzmann equation models the particle collision, i.e. the change of velocity, more precisely.
The Boltzmann equation can be written as

\[ \partial_t u + v \cdot \nabla_x u = Q(u, u) + \text{1.o.t.}, \]

where

\[
Q(u, g) = \text{p.v.} \int_{\mathbb{R}^n} \frac{u(t, x, v + h) - u(t, x, v)}{|h|^{n+\beta}} m(g)(t, x, v, h) \, dh
\]

with

\[
m(g)(t, x, v, h) = \int_{w \perp h} g(t, x, v + w) |w|^{\gamma+\beta+1} \, dw
\]

and \( \beta \in (0, 2), \gamma > -n \) depend on physical assumptions. For fixed \( g \) the operator \( Q(u, g) \) is the fractional Laplacian in velocity with variable density.
Maximal regularity

Let us consider a PDE of the form

\[
\begin{cases}
\partial_t u = Au + f, \; t > 0 \\
\quad u(0) = g,
\end{cases}
\]

where $A$ is an operator on a Banach space $B$ and $u$ a function of time with values in $B$.

General Principle:

Find a function space $Z$ for the solution $u$, a function space $X$ for the inhomogeneity $f$ and a function space $X_\gamma$ for the initial value $g$ such that the equation admits a unique solution $u \in Z$ if and only if $f \in X$ and $g \in X_\gamma$.

Here: Maximal $L^p$-regularity, i.e. $X = L^p(B)$ for some base space $B$. 
Maximal $L^p$-regularity

Example - Heat equation

For all $p \in (1, \infty)$ the heat equation

\[
\begin{aligned}
\partial_t u &= \Delta u + f \\
u(0) &= g
\end{aligned}
\]

admits a unique solution $u \in Z = H^{1,p}(0, \infty) \cap L^p((0, \infty); H^{2,p}(\mathbb{R}^n))$ if and only if

- $f \in X = L^p((0, \infty); L^p(\mathbb{R}^n))$,

- $g \in X_\gamma = B^{2(1-1/p)}_{pp}(\mathbb{R}^n)$ (Besov space).

Moreover, $u \in C([0, \infty); B^{2(1-1/p)}_{pp}(\mathbb{R}^n))$. 
Towards kinetic maximal regularity
Which is the right choice for the solution space $Z$?

For simplicity $\beta = 2$, every result presented here holds true for $\beta \in (0, 2)$.

We choose $X = L^p(\mathbb{R}; L^p(\mathbb{R}^{2n}))$. Singular integral theory on homogeneous groups developed by Folland and Stein in 1974 allows to prove the following. If $f \in L^p(\mathbb{R}; L^p(\mathbb{R}^{2n}))$, then the solution $u$ of the Kolmogorov equation satisfies

$$\| \partial_t u + \nu \cdot \nabla_x u \|_p + \| \Delta \nu u \|_p \lesssim \| f \|_p.$$  

No control of the time-derivative. We prove classical maximal $L^p$-regularity is not applicable.
Our choice of function space for the solution is:

$$Z = \{ u: u, \Delta \nu u, \partial_t u + \nu \cdot \nabla_x u \in L^p((0, T); L^p(\mathbb{R}^{2n})) \}.$$
Singular integrals on homogeneous groups

Three important underlying structures of the Kolmogorov equation, $\mathcal{K}(u) = \partial_t u + \nu \cdot \nabla_x u - \Delta_\nu u$.

- **Scaling**: For $\delta_\lambda u = u(\lambda^2 t, \lambda^3 x, \lambda \nu)$ we have
  $\mathcal{K}(\delta_\lambda u) = \delta_\lambda \mathcal{K}(u)$.

- **Translation** For $z_0 = (t_0, x_0, \nu_0), (t, x, \nu) \in \mathbb{R}^n$ we define
  $$(t, x, \nu) \circ (t_0, x_0, \nu_0) = (t + t_0, x + x_0 + t\nu_0, \nu + \nu_0).$$

  Then, $\mathcal{K}(u((t, x, \nu) \circ z_0)) = \mathcal{K}(u((t, x, \nu) \circ z_0))$.

- **Fundamental solution**: There exists a $\gamma \in C^\infty(\mathbb{R}^{2n+1} \setminus \{0\})$ such that
  $$u = \int_{\mathbb{R}^{2n+1}} \gamma(z^{-1} \circ x)f(z)dz$$

  solves $\mathcal{K}u = f$ for suitable $f$.

The pair $(\mathbb{R}^{2n+1}, \circ)$ defines a homogeneous group. ⇝ CZO theory
Towards kinetic maximal regularity

Divide and conquer

We can split the characterization of solutions in $Z$ in two separate problems.

Inhomogeneous eq. with zero initial-value $X$

\[
\begin{aligned}
\partial_t u + v \cdot \nabla_x u &= \Delta_v u + f \\
\quad u(0) &= 0
\end{aligned}
\]

Classical Method: $L^p$-estimates, singular integrals,...
Done, Folland/Stein.

Homogeneous eq. with non-zero initial-value $X_\gamma$

\[
\begin{aligned}
\partial_t u + v \cdot \nabla_x u &= \Delta_v u \\
\quad u(0) &= g
\end{aligned}
\]

Classical Method: Studying the trace space of $Z$.

TODO!
Towards kinetic maximal regularity

The trace space of $Z - 1$

Does a function $u \in Z$ admit a trace? Yes!

Sketch of the proof

Define

$$[\Gamma u](t, x, \nu) = u(t, x + tv, \nu) \quad \text{and} \quad [\Gamma(t)w](x, \nu) = w(x + tv, \nu)$$

on functions $u: [0, T] \times \mathbb{R}^{2n} \to \mathbb{R}$ and $w: \mathbb{R}^{2n} \to \mathbb{R}$. Then,

$$\partial_t \Gamma u = \Gamma (\partial_t u + \nu \cdot \nabla_x u).$$

If $u \in Z$, then $\Gamma u \in H^{1,p}((0, T); L^p(\mathbb{R}^{2n}))$, whence

$$\Gamma u \in C([0, T]; L^p(\mathbb{R}^{2n})).$$

As $(\Gamma(t))_{t \in \mathbb{R}}$ is a $C_0$-group, it follows

$$u = \Gamma^{-1}(t)\Gamma(t)u \in C([0, T]; L^p(\mathbb{R}^{2n})).$$

Consequently, $\text{Tr}(Z)$ well-defined and $Z \hookrightarrow C([0, T]; \text{Tr}(Z))$. 
Towards kinetic maximal regularity

*The trace space of Z - 2*

The trace space of $Z$ cannot be characterized by classical interpolation theory. Recalling the heat equation we expect at least

$$\text{Tr}(Z) \hookrightarrow B^{2(1-1/p)}_{pp,v}(\mathbb{R}^{2n}).$$

Is there any control of regularity in $x$? Yes!
Towards kinetic maximal regularity

The phenomenon of regularity transfer from \( v \) to \( x \).

**Theorem (Bouchut 2002)**

Let \( u \in L^p((0, T); L^p(\mathbb{R}^{2n})) \) with
\[
\partial_t u + v \cdot \nabla_x u \in L^p((0, T); L^p(\mathbb{R}^{2n}))
\]
and \( u \in L^p((0, T); H_v^{2/p}(\mathbb{R}^{2n})) \),
then
\[
u \in L^p((0, T); H_x^{2/3,p}(\mathbb{R}^{2n})).
\]

**In words:** If \( u \) is the solution of a kinetic equation and \( u \) has two derivatives in velocity we obtain \( 2/3 \) of a derivative in space, too.

Very useful and powerful result!

It is proven by Fourier analytic methods. For \( p = 2 \) one can see how the characteristics (i.e. \( \Gamma \)) transfer the regularity.
Towards kinetic maximal regularity

The initial value problem - 1

Consequently:

\[ Z = Z \cap L^p((0, T); H^{2/3,p}_x(\mathbb{R}^{2n})). \]

Similar to Bouchut we also get some regularity in \( x \) for the trace space.

**Theorem (N., Zacher, 2020)**

Let \( p \in (1, \infty) \), then

\[ \text{Tr}(Z) \cong B^{2/3(1-1/p)}_{pp,x}(\mathbb{R}^{2n}) \cap B^{2(1-1/p)}_{pp,v}(\mathbb{R}^{2n}) \]

An anisotropic Besov spaces with a kinetic scaling.
Towards kinetic maximal regularity

The initial value problem

Proof - Part 1/3

We prove that the inclusion mapping

\[ \iota : \text{Tr} \left( E_{\mu} (0, T) \right) \rightarrow B_{pp, x}^{2/3 (1 - 1/p)} (\mathbb{R}^{2n}) \cap B_{pp, v}^{2 (1 - 1/p)} (\mathbb{R}^{2n}) \]

is a well-defined linear, bounded and surjective operator. It follows that \( \iota \) defines an isomorphism.
Towards kinetic maximal regularity

The initial value problem

Proof - Part 2/3

The norm of the kinetic Besov space can be equivalently characterized by

\[ \| \varphi_0 * g \|_p + \left( \int_0^1 \left( \frac{\| \varphi_t * g \|_p}{t^{1-1/p}} \right)^p \frac{dt}{t} \right)^{\frac{1}{p}}, \]

where \( \varphi_t(x, v) = t^{-2n} \varphi(t^{-3/2}x, t^{-1/2}v) \) for a suitable function \( \varphi \) whose Fourier transform admits compact support in an ellipsoid and is positive in its interior.

We choose \( \varphi \) related to the fundamental solution of the Kolmogorov equation. Let \( g \in \text{Tr}(Z) \) then w.l.o.g. we may choose \( u \in Z \) is a solution of the Kolmogorov equation with \( u(0) = g \). It follows that \( u \) is bounded.
Towards kinetic maximal regularity

The initial value problem

Proof - Part 3/3

Regarding the surjectivity of \( \nu \) let \( g \) be an element of the kinetic Besov space. In Fourier variables (\( \xi \) for \( \nu \) and \( k \) for \( x \)) the solution of the Kolmogorov equation is given by

\[
\hat{u}(t, k, \xi) = \hat{g}(k, \xi + tk) \exp \left( -|\xi|^2 t - \xi \cdot kt^2 - |k|^2 \frac{t^3}{3} \right).
\]

Using the Littlewood-Paley decomposition of \( u \) we can directly show that \( \Delta_{\nu} u \in L^p((0, T); L^p(\mathbb{R}^{2n})) \), i.e. \( u \in Z \), under the assumption that \( B_{pp, x}^{2/3(1-1/p)}(\mathbb{R}^{2n}) \cap B_{pp, \nu}^{2(1-1/p)}(\mathbb{R}^{2n}) \).
Kinetic maximal $L^p$-regularity
for the (fractional) Kolmogorov equation

Theorem (N., Zacher, 2020)

Let $T \in (0, \infty)$. For all $p \in (1, \infty)$ the Kolmogorov equation
\[
\begin{align*}
\partial_t u + \nu \cdot \nabla_x u &= \Delta_{\nu} u + f \\
u u(0) &= g
\end{align*}
\]
admits a unique solution $u \in Z$ if and only if

- $f \in X = L^p((0, T); L^p(\mathbb{R}^n))$,
- $g \in X_{\gamma} = B_{pp, x}^{2/3(1-1/p)}(\mathbb{R}^{2n}) \cap B_{pp, \nu}^{2(1-1/p)}(\mathbb{R}^{2n})$.

Moreover, $u \in C([0, T]; X_{\gamma})$.

We say the operator $A = \Delta_{\nu}$ admits kinetic maximal $L^p$-regularity.
Extensions

Change of base space

So far we have only considered the base space $X = L^p(\mathbb{R}^{2n})$.

- We also consider the case $X = L^q(\mathbb{R}^{2n})$ for some $q \in (1, \infty)$ different from $p$ and prove kinetic maximal $L^p(L^q)$-regularity.
 Extensions
From \( L^p(L^p) \) to \( L^p(L^q) \)

The operator \( f \mapsto \Delta_v u \) is bounded in \( L^p((0, T); L^p(\mathbb{R}^{2n})) \). In Fourier variables it can be written as

\[
\hat{f} \mapsto |\xi|^2 \int_0^t \exp \left( -|\xi|^2 s - \xi \cdot ks^2 - |k|^2 \frac{s^3}{3} \right) \hat{f}(t - s, k, \xi + sk) ds
\]

Calderón-Zygmund theory for operator valued problems yields the \( L^p(L^q) \)-boundedness. This idea is inspired by the same result for the maximal regularity of non-autonomous PDE. Note that if \( u \) is a solution of the Kolmogorov equation, then \( w = \Gamma u \) solves the non-autonomous degenerate PDE

\[
\partial_t w = (\nabla_v - t\nabla_x) \cdot (\nabla_v - t\nabla_x) w.
\]
So far we have only considered the base space $X = L^p(\mathbb{R}^{2n})$.

- We also consider the case $X = L^q(\mathbb{R}^{2n})$ for some $q \in (1, \infty)$ different from $p$ and prove kinetic maximal $L^p(L^q)$-regularity.
- For $p \in (1, \infty)$, $q = 2$ we characterize the regularity of weak solutions to the fractional Kolmogorov equation. Here, we again use the solution formula and the availability of the theorem of Plancherel for the $x$ and $v$ variables as $q = 2$. 
Extensions

Temporal weights

Instead of $L^p((0, T); X)$ we consider a Lebesgue space with temporal weight of the form $t^{1-\mu}$ for some $\mu \in (1/p, 1]$ defined as

$$L^p_{\mu}((0, T); X) = \{ u: (0, T) \rightarrow X: \int_0^T t^{p-\mu} \| u(t) \|_X^p \, dt < \infty \}.$$ 

We write $Z_{\mu}$ for $Z$ with temporal weight in the $L^p$-spaces.

Key features:

- Kin. max. $L^p$-reg. $\iff$ Kin. max. $L^p_{\mu}$-reg. for any $\mu \in (1/p, 1]$

- The trace space of $Z_{\mu}$ is given by

  $$\text{Tr}(Z_{\mu}) = X_{\gamma, \mu} = B_{pp, x}^{2/3(\mu-1/p)}(\mathbb{R}^{2n}) \cap B_{pp, v}^{2(\mu-1/p)}(\mathbb{R}^{2n}).$$

- Instantaneous regularization

  $Z_{\mu}(0, T) \hookrightarrow Z(\delta, T) \hookrightarrow C([\delta, T]; X_{\gamma, 1})$ for all $\delta > 0$.
**Extensions**

*Kinetic maximal $L^p_\mu(L^q)$-regularity*

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**Theorem (N., Zacher, 2020)**

Let $T \in (0, \infty)$. For all $p, q \in (1, \infty)$ and any $\mu \in (1/p, 1]$ the Kolmogorov equation

\[
\begin{align*}
\partial_t u + \nu \cdot \nabla_x u &= \Delta_v u + f \\
\partial_t u + \nu \cdot \nabla_x u &= g
\end{align*}
\]

admits a unique solution

\[
u \in Z_\mu = \{u: u, \Delta_v u, \partial_t u + \nu \cdot \nabla_x u \in L^p_\mu((0, T); L^q(\mathbb{R}^{2n}))\}.
\]

if and only if

- $f \in X = L^p_\mu((0, T); L^q(\mathbb{R}^n))$,
- $g \in X_{\gamma, \mu} = B^{2/3(\mu-1/p)}(\mathbb{R}^{2n}) \cap B^{2(\mu-1/p)}_{q \rho, v}(\mathbb{R}^{2n})$.

Moreover, $u \in C([0, T]; X_{\gamma, \mu})$. 
Extensions

Different Operators

Question: Do other operators admit kinetic maximal $L^p$-regularity? Yes.

Examples:

- $Au = a(t, x, v) : \nabla_v^2 u + b \cdot \nabla_v u + cu$
- $Au = -(-\Delta_v)^{\beta/2} u$ with $\beta \in (0, 2)$
- non-local integro-differential operators acting in velocity with possibly time, space and velocity dependent density of the form

$$Au = \text{p.v.} \int_{\mathbb{R}^n} \frac{u(t, x, v + h) - u(t, x, v)}{|h|^{n+\beta}} m(t, x, v, h) \, dh$$
Extensions

Different Operators

Theorem (N., Zacher, 2020)

Let $p, q \in (1, \infty)$, $\mu \in (1/p, 1]$, $a \in L^\infty([0, T] \times \mathbb{R}^{2n}; \text{Sym}(n))$, $b \in L^\infty([0, T] \times \mathbb{R}^{2n}; \mathbb{R}^n)$ and $c \in L^\infty([0, T] \times \mathbb{R}^{2n}; \mathbb{R})$. If $a \geq \lambda \text{Id}$ for some $\lambda > 0$ and if the function $(t, x, v) \mapsto a(t, x + tv, v)$ is uniformly continuous, then the family of operators

$$A(t)u = a(t, \cdot) : \nabla^2_\nu u + b(t, \cdot) \cdot \nabla_\nu u + c(t, \cdot)u$$

admits kinetic maximal $L^p_\mu(L^q)$-regularity.
We prove short-time existence of strong $L^p_\mu$-solutions to the following quasilinear kinetic diffusion equation

$$\begin{cases}
\partial_t u + v \cdot \nabla_x u = \nabla_v \cdot (a(u)\nabla_v u) \\
u(0) = g
\end{cases}$$

for $a \in C^2_b(\mathbb{R}; \text{Sym}(n))$ with $a \geq \lambda I_d$ for some $\lambda > 0$, $\mu - 1/p > 2n/p$ and $g \in X_{\gamma, \mu}$.

**Method:** Freeze the equation at the initial value and use kinetic maximal $L^p$-regularity for the frozen equation. Here, we need the kinetic maximal regularity of $A = a(g(x, v)) : \nabla_v^2 u$. 
Another interesting quasilinear Problem is

\[
\begin{aligned}
\partial_t u + \nu \cdot \nabla_x u &= \left( \int_{\mathbb{R}^n} u \mu \, d\nu \right) \Delta_v u \\
u(0) &= g
\end{aligned}
\]

with \( \mu \in L^1(\mathbb{R}^n) \).

The more particles there are at a position \( x \) the more diffusion there is. If \( \mu - 1/p > 2n/p \) and \( g \in X_{\gamma, \mu} \) we can show the existence of a strong \( L^p_\mu \)-solution for a possibly short time.

Models of this type are an important step towards more complicated equations such as the Landau and the Boltzmann equation.
Further research

Possible directions:

– weak $L^p$-solutions

– study quasilinear kinetic problems from physics/economics/biology

– qualitative study of quasilinear problems such as large time behavior

– conditions on the operator $A$ such that it admits kinetic maximal $L^p$-regularity

– different first order terms, for example $\partial_t + \langle x, B\nabla \rangle$ or the relativistic kinetic term $\partial_t + \frac{v}{\sqrt{1+|v|^2}} \nabla_x$
Bibliography


