UNIVERSITÄT ULM

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Prof. Dr. Friedmar Schulz
Marius Müller
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Übungen Introduction to Complex Analysis: Sheet 1

In this worksheet, we are going to practice some basic arithmetic for complex numbers. In order to do so, brush up on basic presentations of complex numbers and operations to deal with them. Even though it wasn't explicitly mentioned in lecture, we might need the geometric sum formula and the generalized binomial formula, which can be shown to hold for complex numbers as well as for the reals. Even more, the proofs should be identical. In this sheet, you may use these formulas for complex numbers right away without proof.

3. Let $z = 1 + \sqrt{3}i$. Find real and imaginary part of $\frac{1}{z}$, as well as all complex numbers w such that $w^2 = z$. Additionally, find a simple presentation of z^{10}. (3)

Hint: Instead of just adopting the formula for $\frac{1}{z}$ from the lecture, you might want to use the identity $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$.

4. Compute (1)

$$\left| \frac{3+5i}{5+3i} \right| .$$

5. Let $z \in \mathbb{C}$. Define $w := i\frac{z-i}{z+i}$. Show that $w \in \mathbb{R}$ implies $|z| = 1$. (3)

6. (3)

(a) Show that the following equation holds for arbitrary $x \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\cos(nx) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-1)^k \cos(x)^{n-2k} \sin(x)^{2k},$$

where by $\lfloor t \rfloor$ we mean the smallest integer that does not exceed t. Remark: This formula generalizes $\cos(2x) = \cos^2(x) - \sin^2(x)$.

(b) Show : For any $z \in \mathbb{C}$ we have

$$\sum_{k=0}^{n} \binom{n}{k} |z|^{2k} \overline{z}^{n-2k} \in \mathbb{R}.$$

7. The following problem deals with the complex roots of unity, which we define to be (5+2*)

$$z_j = \cos\left(j\frac{2\pi}{k}\right) + i \sin\left(j\frac{2\pi}{k}\right), \quad j = 0, ..., k-1.$$

, just like we did in lecture. Here is a general hint which can be useful throughout the entire exercise but needs to be justified if used: $z_j = z_1^j$.

(a) Show that $\overline{z_j} = z_{k-j}$ für $j = 1, ..., k-1$.

(b) Show that

$$\sum_{j=0}^{k-1} z_j = 0.$$

(c) For the next part we extrapolate the definition given above and define $z_k = \cos\left(k\frac{2\pi}{k}\right) + i \sin\left(k\frac{2\pi}{k}\right) = 1$. Show

$$\sum_{j=0}^{k} \binom{k}{j} z_j = \pm \left(2 + 2\cos\left(\frac{2\pi}{k}\right)\right)^{\frac{k}{2}} .$$

Bonus question: Show that the sum will indeed always attain the value $-2^k \cos\left(\frac{\pi}{k}\right)^k$ which means that „$-$“is always the case.

Hint: You might want to compute the complex conjugate of the left hand side in order to show that the left hand side is necessarily real.

8. Show that for any $z, w \in \mathbb{C}$ it holds that (2)

$$|z + w| \leq |z| + |w|.$$

9. Let $P(z) = a_0 z^0 + ... + a_n z^n$ be a complex polynomial with real and nonnegative coefficients $a_0, ..., a_n \geq 0$. Show: Given that $a_n > \sum_{k=0}^{n-1} a_k$, any root of $P(z)$ will lie in $D(0, 1)$. Bonus question: Can you think of a similar criterion granting that any root lies in $D(0, r)$ for fixed $r > 0$? (3+1*)

10. For any $w \in \mathbb{C}$ we define (3)

$$z_w := \begin{cases} \sqrt{|w|}\frac{w+|w|}{|w+|w||} & w \notin \mathbb{R}_{<0} \\ i\sqrt{-w} & \text{sonst.} \end{cases}$$

Show that $z_w^2 = w$ and $Re(z_w) \geq 0$. Moreover, show that for $w, w' \in \mathbb{C}$ the desirable identity $z_{ww'} = z_w z_{w'}$ holds if and only if $-\pi < \arg(w) + \arg(w') \leq \pi$.

Hint: Show that $arg(z_w) = \frac{arg(w)}{2}$. You don't have to derive this using just the given formula, the facts that $z_w^2 = w$ and $Re(z_w) \geq 0$ should be (almost) sufficient already.