Variationsrechnung Blatt 4

A 12

a) Sei $u \in W^{1,2}(B) \cap C^1(\bar{B})$

$$\int_{\partial B} u^2 \, dS(x) = \int_{\partial B} \langle u^2 x, \nu(x) \rangle \, dS(x)$$

$$\overset{\text{Gauß}}{=} \int_B \operatorname{div}(u^2(x) x) \, dx$$

$$= \int_B 2u(x) \langle Du(x) \cdot x \rangle \, dx + \int_B u^2(x) \operatorname{div}(x) \, dx$$

$$= \int_B 2u(x) \langle Du(x), x \rangle \, dx + n \int_B u^2(x) \, dx$$

$$\leq \int_B 2|u(x)| \, |Du(x)| \, dx + n \int u^2(x) \, dx$$

$$\leq \int_B |u(x)|^2 + |Du(x)|^2 + n \int |u(x)|^2 \, dx$$

$$\leq (n+1) \left(\int_B |u(x)|^2 + \int_B |Du(x)|^2 \, dx \right)$$

$$\Longrightarrow \quad \|u\|_{L^2(\partial B)} \leq \sqrt{(n+1)} \, \|u\|_{W^{1,2}(B)}$$

b) B C^1-glatt berandet

$\Longrightarrow \forall u \in W^{1,2}(B) \quad \exists (u_n) \subset W^{1,2}(B) \cap C^1(\bar{B})$:

$u_n \longrightarrow u$ in $W^{1,2}(B)$.

Nun (u_n) ist Cauchy in $L^2(\partial B)$ denn

$$\|u_n - u_m\|_{L^2(\partial B)} \leq C \|u_n - u_m\|_{W^{1,2}(B)}$$

$$\Rightarrow \forall \varepsilon > 0 \; \exists N \in \mathbb{N} : n, m \geq N \Rightarrow \|u_n - u_m\|_{L^2(\partial B)} < \varepsilon$$

Nun $L^2(\partial B)$ ist Banachraum (?? $L^2(\Omega, \Sigma, \mu)$ ist Banachraum $\Omega = \partial B$, Σ = Borelmengen $\cap\, \partial B$ $\mu = S$?? Maß?)

$\hookrightarrow$ Hausdorff-Maße

$\hookrightarrow$ Riemannsche Maß.

$$\Rightarrow \exists w \in L^2(\partial B) : \quad u_n \longrightarrow w \text{ in } L^2(\partial B)$$

$tr(u) := w$

<u>Man kann zeigen:</u>

- Die Definition hängt nicht von der gewählten Folge ab.
- tr ist linear,
- tr ist stetig.
- $tr\big|_{W^{1,2}(B) \cap C^1(\bar{B})} = (\cdot)\big|_{\partial\Omega}$

(c)

Sei $u \in W^{1,2}(B) \cap C^0(\bar{B})$

Nun $u_n := i(u) * \varphi_{\frac{1}{n}} \in W^{1,2}(B) \cap C^1(\mathbb{R}^n)$.

$u_n := i(u) * \varphi_{\frac{1}{n}}\big|_B \longrightarrow u$ in $W^{1,2}(B)$ und

$u_n := \cancel{i(u)}\, u * \varphi_{\frac{1}{n}}\big|_{\bar{B}_\varepsilon} \longrightarrow i(u)$ in $C^0(\bar{B})$ da $i(u)$, u stetig auf $\bar{B}$

c) $u \in W^{1,2}(B) \cap C^0(\bar{B})$

$$\Rightarrow u_r := u(r\cdot) \in W^{1,2}\left(\frac{B}{r}\right) \cap C^0\left(\frac{B}{r}\right) \quad r<1$$

$$\int |u(rx) - u(x)|^2 \, dx \to 0, \quad \int |r D_j u(rx) - D_j u(x)|^2 \, dx \to 0$$

$$\Rightarrow u_r \to u \text{ in } W^{1,2}(B)$$

$$u_{r,n} := i(u_r) * \varphi_{\frac{1}{n}} \Big|_{\bar{B}} \to u_r \text{ in } W^{1,2}(B)$$

$$\text{und } u_{r,n}\Big|_{\bar{B}} \xrightarrow{\text{glm, also in } C^0(\bar{B})} i(u_r)\Big|_{\bar{B}} = u_r$$

$$\text{denn } i(u_r) \in C^0\left(\frac{B}{r}\right)$$

$\Longrightarrow$

$$tr(u) = \overset{L^2(\partial B)}{\lim_{r \to 1}} tr(u_r)$$

$$= \overset{L^2(\partial B)}{\lim_{r \to 1}} \overset{L^2(\partial B)}{\lim_{n \to \infty}} tr(u_{r,n})$$

$$= \overset{L^2(\partial B)}{\lim_{r \to 1}} \underbrace{\overset{L^2(\partial B)}{\lim_{n \to \infty}} \left(u_{r,n}\Big|_{\partial B} \right)}_{= u_r|_{\partial B} \text{ da glm konv} \Rightarrow L^2(\partial B)\text{-konv}}$$

$$= \overset{L^2(\partial B)}{\lim_{r \to 1}} u_r\Big|_{\partial B} = \overset{\text{glm}}{\lim_{r \to 1}} u(r\cdot)\Big|_{\partial B} = u(\cdot)\Big|_{\partial B}.$$

(d)

$u \in W^{1,2}_c(B)$

$\Longrightarrow \exists\, (\varphi_n) \subset C_0^\infty(B) \;:\; \varphi_n \longrightarrow u \quad \text{in } W^{1,2}(B)$

$$tr(u) = \lim_{n\to\infty}^{L^2(\partial B)} tr(\varphi_n)$$

$$= \lim_{n\to\infty}^{L^2(\partial B)} tr(0) = \lim_{n\to\infty}^{L^2(\partial B)} 0 = 0.$$

Für $u \in C^0(\bar{B}) \cap W^{1,2}(B)$: klar

(e) $(u_n) \subset W^{1,2}(B) \cap C^1(\bar{B})$:

$u_n \longrightarrow u \quad \text{in } W^{1,2} \Longrightarrow |u_n| \longrightarrow |u| \quad \text{in } W^{1,2}$

$$tr(|u|) = \lim_{n\to\infty}^{L^2(\partial B)} tr(|u_n|)$$

$$= \lim_{n\to\infty}^{L^2(\partial B)} |u_n|\big|_{\partial B} = \left| \lim_{n\to\infty}^{L^2(\partial B)} u_n\big|_{\partial B} \right|$$

$$= \cancel{\lim_{n\to\infty}} \; |tr(u)|$$

(f) Seien $(u_n) \in W^{1,2}(B) \cap C^1(\bar{B}) : u_n \to u \text{ in } W^{1,2}(B)$

$$\int_B \langle Du \cdot \varphi \rangle \, dx$$

$$= \lim_{n\to\infty} \int_B \langle Du_n \cdot \varphi \rangle \, dx$$

$$= \lim_{n\to\infty} \int_{\partial B} u_n \langle \varphi, \nu \rangle \, dx - \int_B u_n \operatorname{div} \varphi \, dx$$

$$= \int_{\partial B} tr(u) \langle \varphi, \nu \rangle \, d\sigma(x) - \int_B u \operatorname{div} \varphi \, dx$$

(g)

$$\eta_K(x) := (1-|x|^2) \max(-k, \min(u,k))$$

$$\eta_k^+(x) = (1-|x|^2) \min(u^+, k)$$

$$\Rightarrow 0 \leq \eta_k^+ \leq K(1-|x|^2) \in W_0^{1,2}(B)$$

denn $f(x) := K(1-|x|^2)$ ist $Lip(B)$ und $\equiv 0$ auf ∂B.

am Rand $\xrightarrow[\text{Aufgabe 11(a)}]{\text{Blatt 3}}$ $f \in W_0^{1,2}(B)$

Analog

$$\eta_k^-(x) = (1-|x|^2) \min(u^-, K)$$

$$\Rightarrow 0 \leq \eta_k^- \leq \underbrace{K(1-|x|^2)}_{\in W_0^{1,2}(B)} \Rightarrow \eta_k \in W_0^{1,2}(B)$$

$$\Rightarrow \eta_k = \eta_k^+ - \eta_k^- \in W_0^{1,2}(B)$$

$$\Rightarrow 0 = \int_B u \eta_k + \int_B Du \, D\eta_k$$

$$= \int_{-k<u<k} (1-|x|^2) u^2 + k \int_{u>k} (1-|x|^2) u + (-k) \int_{u<-k} (1-|x|^2) u$$

$$+ \int_{-k<u<k} \left((1-|x|^2)|Du|^2 - 2xu Du\right) dx$$

Was passiert wenn

$K \to \infty$

$$\Rightarrow \left| K \int_{u>k} (1-|x|^2)\, u \right| \leq \int_{u>k} (1-|x|^2)\, |u|^2 \to 0 \quad (k\to\infty)$$

Analog

$$\left| -k \int_{u<-k} (1-|x|^2)\, u \right| \to 0 \quad (k\to\infty)$$

$$\Rightarrow 0 = \int_B (1-|x|^2)\, u^2 + \int_B (1-|x|^2)\, |Du|^2$$

$$- \int_B 2x\, u\, Du\, dx$$

Nun sei $(u_\ell) \subset C^1(\bar{B}) \cap W^{1,2}(B)$: $u_n \to u$ in $W^{1,2}$

$$\Rightarrow \int_B 2x\, u\, Du\, dx = \lim_{\ell\to\infty} \int_B \underbrace{2x\, u_\ell}_{=:\phi}\, Du\, dx$$

$$\overset{\text{A12f Blatt 4}}{=} \lim_{k\to\infty} \Big(\underbrace{\int_{\partial B} 2\,tr(u_\ell)\, (u_\ell x \cdot x)\, dS(x)}_{=0 \text{ da } tr(u)=0}$$

$$- \int_B (\mathrm{div}(x \phi u_\ell))\, u\, dx \Big)$$

$$= \lim_{\ell\to\infty} - \int_B 2(\mathrm{div}\, x\, u_\ell + x\, Du_\ell)\, u\, dx$$

$$\underset{\text{div}\, x = n}{=} \lim_{l \to \infty} \left(-\int_B 2n\, u_l\, u \, dx - \int_B 2x \cdot \nabla u_l \, u \, dx \right|$$

$$= -\int_B 2n u^2 - \underbrace{\int_B 2x \cdot \nabla u \, u \, dx}_{=:A}$$

$$\Rightarrow A = -\int 2nu^2 - A \Rightarrow 2A = -\int 2nu^2$$

$\Rightarrow$ Phönix:

$$4\int_B x \, \nabla u \, u \, dx = -\int_B 2n u^2 dx$$

$$\Rightarrow 2\int_B x \, \nabla u \, u \, dx = -n \int_B u^2 dx$$

$$\Rightarrow 0 = \int_B (1-|x|^2) u^2 dx + \int_B (1-|x|^2) |\nabla u|^2 dx$$

$$+ n \int_B u^2 dx$$

$\Rightarrow u \equiv 0$ auf B für da alle Integranden positiv auf B.

(h)

Sei nun $u \in \ker(tr)$ Nun $W^{1,2} = W_0^{1,2} \oplus (W_0^{1,2})^\perp$

$\Rightarrow u = u_1 + u_2$ für $u_1 \in W_0^{1,2}(B)$

$u_2 \in (W_0^{1,2}(B))^\perp$

$$\Rightarrow 0 = tr(u) = \underbrace{tr(u_1)}_{\substack{=0 \\ \text{da } u_1 \in W_0^{1,2}}} + tr(u_2) = tr(u_2)$$

$\Rightarrow \operatorname{tr}(u_2) = 0$ und $u_2 \in (W_0^{1,2}(B))^{\perp}$

$\overset{(g)}{\Longrightarrow}$ $u_2 \equiv 0$ auf B

$\Longrightarrow$ $u = u_1 \in W_0^{1,2}(B)$

<u>Aufgabe 13</u>

(a) Zunächst zeigen wir für alle a, φ:

(1) $\phi_{a,\varphi}(B_1(0)) \subset B_1(0)$

denn sei $|z| < 1$

$$\left| e^{i\varphi} \frac{z-a}{1-\bar{a}z} \right|^2 = \frac{|z-a|^2}{|1-\bar{a}z|^2} = \frac{|z|^2 + |a|^2 - 2\operatorname{Re} \bar{a}z}{1 + |a|^2|z|^2 - 2\operatorname{Re}\bar{a}z}$$

<u>Nun</u> $\quad |z|^2 + |a|^2 - 2\operatorname{Re}\bar{a}z \leq 1 + |a|^2|z|^2 - 2\operatorname{Re}\bar{a}z$

$$\Longleftrightarrow \quad 0 \leq 1 + |a|^2|z|^2 - |z|^2 - |a|^2$$

$$= \underbrace{(1-|a|^2)}_{>0} \underbrace{(1-|z|^2)}_{>0}$$

$\Longrightarrow$ Beh

<u>Wir berechnen nun die Umkehrabbildung:</u>

$$w = e^{i\varphi} \frac{z-a}{1-\bar{a}z}$$

$$\Rightarrow (1-\bar{a}z)w = e^{i\varphi}(z-a)$$

$$\Rightarrow z(e^{i\varphi} + \bar{a}w) = w + ae^{i\varphi}$$

$$\Rightarrow \quad z = \frac{w + ae^{i\varphi}}{e^{i\varphi} + \bar{a}w} = e^{-i\varphi} \frac{(w + ae^{i\varphi})}{(1 + \bar{a}e^{-i\varphi}w)}$$

$$= e^{-i\varphi} \frac{w + ae^{i\varphi}}{(1 + \overline{ae^{i\varphi}}\, w)}$$

$\Rightarrow \phi_{a,\varphi}^{-1} = \phi_{ae^{i\varphi},-\varphi}$

$\Rightarrow \phi_{a,\varphi}^{-1}(B_1(0)) \subset B_1(0)$

$\Rightarrow \phi_{a,\varphi}(B_1(0)) = B_1(0)$

and $\phi_{a,\varphi}$ ist bijektiv auf $B_1(0)$.

$\phi_{a,\varphi}$ ist holomorph da Quotient holomorphen auf $B_1(0)$. Funktionen und der Nenner $1-\bar{a}z$ wird nicht Null, da $|z| < 1 < \frac{1}{|a|}$

Nun $\phi_{a,\varphi}^{-1} = \phi_{ae^{i\varphi},-\varphi}$ ist auch auf $B_1(0)$ holomorph.

(b) Es gilt $\begin{cases} \frac{\partial \mathrm{Re} f}{\partial x} = \frac{\partial \mathrm{Im} f}{\partial y} \\ \frac{\partial \mathrm{Im} f}{\partial x} = -\frac{\partial \mathrm{Re} f}{\partial y} \end{cases}$

$$\det Dg(x,y) = \det \begin{pmatrix} \frac{\partial \mathrm{Re} f}{\partial x} & \frac{\partial \mathrm{Re} f}{\partial y} \\ \frac{\partial \mathrm{Im} f}{\partial x} & \frac{\partial \mathrm{Im} f}{\partial y} \end{pmatrix}$$

$$= \frac{\partial \mathrm{Re} f}{\partial x}\frac{\partial \mathrm{Im} f}{\partial y} - \frac{\partial \mathrm{Re} f}{\partial y}\frac{\partial \mathrm{Im} f}{\partial x}$$

$$= \left(\frac{\partial \mathrm{Re} f}{\partial x}\right)^2 + \left(\frac{\partial \mathrm{Im} f}{\partial x}\right)^2 = \left|\frac{\partial}{\partial x} \underbrace{(\mathrm{Re} f + i \,\mathrm{Im} f)}_{=f}(x+iy)\right|^2$$

$$\overset{\frac{\partial}{\partial x} = \frac{d}{dz} \text{ sofern } f \text{ holomorph}}{=} \left|\frac{df}{dz}(x+iy)\right|^2$$

(c)

$$D(\tilde{F}) = \int_{B} \left(|\partial_{v_1} \tilde{F}|^2 + |\partial_{v_2} \tilde{F}|^2 \right) \, d(v_1, v_2)$$

$$\partial_{v_1} \tilde{F} = \partial_{v_1} (F \circ \phi)$$

$$= \partial_{u_1} F \circ \partial_{v_1} \phi^1 + \partial_{u_2} F \, \partial_{v_1} \phi^2$$

$$= \partial_{u_1} F \, \partial_{v_1} \mathrm{Re}\, \phi_{a,4} + \partial_{u_2} F \, \partial_{v_1} \mathrm{Im}\, \phi_{a,4}$$

$\underline{\text{Analog}}$

$$\partial_{v_2} \tilde{F} = \partial_{u_1} F \, \partial_{v_2} \mathrm{Re}\, \phi_{a,4} + \partial_{u_2} F \, \partial_{v_2} \mathrm{Im}\, \phi_{a,4}$$

$$\Rightarrow D(\tilde{F}) = \int_{B} \left(|\partial_{v_1} \tilde{F}|^2 + |\partial_{v_2} \tilde{F}|^2 \right) d(v_1, v_2)$$

$$= \int \Big(|\partial_{u_1} F \, \partial_{v_1} \mathrm{Re}\, \phi_{a,4} + \partial_{u_2} F \, \partial_{v_1} \mathrm{Im}\, \phi_{a,4}|^2$$

$$+ |\partial_{u_1} F \, \partial_{v_2} \mathrm{Re}\, \phi_{a,4} + \partial_{u_2} F \, \partial_{v_2} \mathrm{Im}\, \phi_{a,4}|^2 \Big) \, d(v_1, v_2)$$

$$= \int \Big\{ |\partial_{u_1} F|^2 (\partial_{v_1} \mathrm{Re}\, \phi_{a,4})^2 + 2 \langle \partial_{u_1} F, \partial_{u_2} F \rangle \, \partial_{v_1} \mathrm{Re}\, \phi_{a,4} \, \partial_{v_1} \mathrm{Im}\, \phi_{a,4}$$

$$+ |\partial_{u_2} F|^2 (\partial_{v_1} \mathrm{Im}\, \phi_{a,4})^2 + |\partial_{u_1} F|^2 (\partial_{v_2} \mathrm{Re}\, \phi_{a,4})^2$$

$$+ 2 \langle \partial_{u_1} F, \partial_{u_2} F \rangle \, \partial_{v_2} \mathrm{Re}\, \phi_{a,4} \, \partial_{v_2} \mathrm{Im}\, \phi_{a,4}$$

$$+ |\partial_{u_2} F|^2 (\partial_{v_2} \mathrm{Im}\, \phi_{a,4})^2 \Big\} \, d(v_1, v_2)$$

$$\int |\partial_{u_1} F|^2 \Big(\underbrace{(\partial_{v_1} \operatorname{Re} \phi_{a,4})^2 + (\partial_{v_2} \operatorname{Re} \phi_{a,4})^2}_{①} \Big) \, d(v_1, v_2)$$

$$+ \int 2 \langle \partial_{u_1} F, \partial_{u_2} F \rangle \Big(\underbrace{\partial_{v_1} \operatorname{Re} \phi_{a,4} \, \partial_{v_1} \operatorname{Im} \phi_{a,4} + \partial_{v_2} \operatorname{Re} \phi_{a,4} \, \partial_{v_2} \operatorname{Im} \phi_{a,4}}_{②} \Big) \, d(v_1, v_2)$$

$$+ \int |\partial_{u_2} F|^2 \Big(\underbrace{(\partial_{v_1} \operatorname{Im} \phi_{a,4})^2 + (\partial_{v_2} \operatorname{Im} \phi_{a,4})^2}_{③} \Big) \, d(v_1, v_2)$$

Now

② $\partial_{v_1} \operatorname{Re} \phi_{a,4} \, \partial_{v_1} \operatorname{Im} \phi_{a,4} + \partial_{v_2} \operatorname{Re} \phi_{a,4} \, \partial_{v_2} \operatorname{Im} \phi_{a,4}$

$$= \partial_{v_2} \operatorname{Im} \phi_{a,4} \left(- \partial_{v_2} \operatorname{Re} \phi_{a,4} \right) + \partial_{v_2} \operatorname{Re} \phi_{a,4} \, \partial_{v_2} \operatorname{Im} \phi_{a,4}$$

$$= 0$$

① $(\partial_{v_1} \operatorname{Re} \phi_{a,4})^2 + (\partial_{v_2} \operatorname{Re} \phi_{a,4})^2$

$$= (\partial_{v_1} \operatorname{Re} \phi_{a,4})^2 + (- \partial_{v_1} \operatorname{Im} \phi_{a,4})^2$$

$$= \left| \frac{\partial}{\partial v_1} \phi_{a,4} (v_1 + i v_2) \right|^2 = \left| \frac{d\phi_{a,4}}{dz} (v_1 + i v_2) \right|^2$$

③ $(\partial_{v_1} \operatorname{Im} \phi_{a,4})^2 + (\partial_{v_2} \operatorname{Im} \phi_{a,4})^2$

$$= (- \partial_{v_2} \operatorname{Re} \phi_{a,4})^2 + (\partial_{v_2} \operatorname{Im} \phi_{a,4})^2$$

$$= \left| \frac{\partial}{\partial v_2} \phi_{a,4} (v_1 + i v_2) \right|^2 = \left| \frac{d\phi_{a,4}}{dz} (v_1 + i v_2) \right|^2$$

$$\Longrightarrow D(\tilde{F}) = \int_B \left(|\partial_{u_1} F|^2 + |\partial_{u_2} F|^2 \right) \left| \frac{d\phi_{a,4}}{dz} (v_1 + i v_2) \right|^2$$

$$= \int_B \left(|\partial_{u_1} F|^2 + |\partial_{u_2} F|^2\right) \det D\phi(v_1, v_2) \; d(v_1, v_2)$$

$$= \int_B \left(|\partial_{u_1} F(\phi(v_1,v_2))|^2 + |\partial_{u_2} F(\phi(v_1,v_2))|^2\right) \underbrace{\det D\phi(v_1,v_2)}_{=|\det D\phi(v_1,v_2)|} d(v_1,v_2)$$

$$= \int_{\phi(B)=B} \left(|\partial_{u_1} F|^2 + |\partial_{u_2} F|^2\right) d(u_1,u_2) = D(F).$$

(d) (i) Sei $r<1$. Es gilt dass $\frac{f(z)}{z}$ holomorph ist, da $f(0)=0$.

$$\Rightarrow \max_{B_r(0)} \frac{|f(z)|}{|z|} = \max_{\partial B_r(0)} \frac{|f(z)|}{|z|} = \frac{1}{r} \underbrace{\max_{\partial B_r(0)} |f(z)|}_{\leq 1 \text{ da } f: B_1(0) \to B_1(0)}$$

$$\leq \frac{1}{r}$$

$\underline{r \to 1} \Rightarrow$

$$\sup_{z \in B_1(0)} \frac{|f(z)|}{|z|} \leq 1$$

$$\Rightarrow |f(z)| \leq |z|.$$

Falls

$\frac{f(z)}{z} = 1$ für ein $z \in B_1(0)$

etwa $|z| = r < 1$

~~so gi~~

So gilt

$\max_{|z| \leq \frac{r+1}{2}} \left|\frac{f(z)}{z}\right|$ wird im inneren angenommen

$\Rightarrow f(z)/z = \text{Const.} \Rightarrow f(z) = cz$

(ii) $|1-\bar{a}w|^2 = 1 + |a|^2|w|^2 - 2\,\mathrm{Re}(\bar{a}w)$

$|a-w|^2 = |a|^2 + |w|^2 - 2\,\mathrm{Re}(\bar{a}w)$

~~$\Rightarrow |a-w|^2 - |1-\bar{a}w|^2 = 1+|a|$~~

$$|1-\bar{a}w|^2 - |a-w|^2 = 1 + |a|^2|w|^2 - |a|^2 - |w|^2$$

$$= (1-|a|^2)(1-|w|^2) \geq 0 \quad \forall\, w : |w| \leq 1$$

$$\Longrightarrow |a-w|^2 \leq |1-\bar{a}w|^2 \Longrightarrow |a-w| \leq |1-\bar{a}w|$$

(iii) $f : B_1(0) \longrightarrow B_1(0)$ $\quad f(0) = a$

$$h(z) := \frac{f(z) - a}{1 - \bar{a} f(z)}$$

$h(B_1(0)) \subset B_1(0)$.

denn

$z \in B_1(0) \Longrightarrow f(z) \in B_1(0)$

$$\Longrightarrow |h(z)| = \frac{|f(z) - a|}{|1 - \bar{a} f(z)|} \leq 1 \quad \text{wegen (ii) für } w = f(z)$$

h bijektive denn

$$\frac{f(z) - a}{1 - \bar{a} f(z)} = w$$

$$\Longrightarrow f(z) - a = (1 - \bar{a} f(z))\, w$$

$$\Longrightarrow f(z)(1 + \bar{a}w) = w + a$$

$$\Longrightarrow \quad z = f^{-1}\Big(\underbrace{\frac{w+a}{1+\bar{a}w}}_{\in B_1(0).}\Big) \quad \in 1$$

$$\text{da } \left|\frac{w+a}{1+\bar{a}w}\right| = \left|\frac{w-(-a)}{1-\overline{(-a)}\,w}\right|$$

Nun

$$h^{-1}(w) := f^{-1}\left(\frac{w+a}{1+\bar{a}w}\right)$$ erfüllt

$h^{-1}(B_1(0)) \subset B_1(0)$

$\Longrightarrow h(B_1(0)) = B_1(0)$ und h ist ein Automorphismus der Kreisscheibe.

Nun

$$h(0) = \frac{f(0)-a}{1-\bar{a}f(0)} = \frac{a-a}{1-|a|^2} = 0,$$

$$\Longrightarrow \quad |h(z)| \leq |z| \quad \forall z \in B_1(0)$$

Analog

$$h^{-1}(0) = 0$$

$$\Longrightarrow \quad |h^{-1}(w)| \leq |w| \quad \forall w \in B_1(0)$$

$$\Longrightarrow \quad |w| = |h^{-1}(h(w))| \leq |h(w)| \leq |w| \quad \forall w \in B_1(0)$$

$$\Longrightarrow \quad |h(w)| = |w| \quad \forall w \in B_1(0)$$

$\Longrightarrow \quad h(w) = cw$ für ein $c \in \mathbb{C} : |c| = 1$

$$\Longrightarrow \quad h(w) = e^{i\varphi} w \quad \forall w \in B_1(0)$$

für ein $\varphi \in \mathbb{R}$

$$\Longrightarrow \quad \frac{f(w) - a}{1 - \bar{a} f(w)} = e^{i\varphi} w \quad \forall w \in B_1(0)$$

$$\Longrightarrow \quad f(w) = \frac{e^{i\varphi} w + a}{1 + \bar{a} e^{i\varphi} w}$$

$$= e^{i\varphi} \frac{w + a e^{-i\varphi}}{1 + \overline{\dot{a} e^{-i\varphi}} w}$$

$$= \phi_{-a e^{-i\varphi}, \varphi} (w) \quad \forall w \in B_1(0)$$

qed

Aufgabe 14

(a)

$$\mathrm{Spur}(AB) = \sum_{i=1}^{n_2} (AB)_{ii}$$

$$= \sum_{i=1}^{n_2} \left(\sum_{j=1}^{n_1} a_{ij} b_{ji} \right)$$

$$= \sum_{i=1}^{n_2} \sum_{j=1}^{n_1} a_{ij} b_{ji}$$

$$= \sum_{j=1}^{n_1} \left(\sum_{i=1}^{n_2} b_{ji} a_{ij} \right)$$

$$= \sum_{j=1}^{n_1} (BA)_{jj} = \mathrm{Spur}(BA)$$

(b) 1) $\mathrm{Spur}\, A = \mathrm{Spur}(S^{-1}AS) \quad \forall S \in GL_n(\mathbb{C})$,

da

$$\mathrm{Spur}(S^{-1}AS) = \mathrm{Spur}(S^{-1}(AS))$$

$$\overset{(a)}{=} \mathrm{Spur}((AS)S^{-1})$$

$$= \mathrm{Spur}(A(SS^{-1})) = \mathrm{Spur}\, A.$$

Nun wähle $S \in GL_n(\mathbb{C})$ sodass

$$S^{-1}AS = JNF(A)$$ „Jordan Normalform"

Wobei $JNF(A) = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_\ell \end{pmatrix}$

$\forall i = 1 \_ \ell \quad A_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$

λ_i sind dann die Eigenwerte von $JNF(A)$

und da A, $JNF(A)$ ähnlich sind

gilt

$$\operatorname{Spur} A = \operatorname{Spur}(JNF(A)) = \sum_{\substack{\lambda \text{ EW von } JNF(A) \\ \text{mit alg VF}}} \lambda$$

$$= \sum_{\substack{\lambda \text{ EW von } A \\ \text{mit alg VF.}}} \lambda$$

3) A symmetrisch positiv definit

$\Rightarrow \exists\, C \in GL_n(\mathbb{R}) : A = C C^T$

((Cholesky Zerlegung)

$$\operatorname{Spur}(AB) = \operatorname{Spur}(C C^T B) = \operatorname{Spur}(C^T B C)$$

$$= \sum_{i=1}^{n} \underbrace{e_i^T C^T B C e_i}_{= (C^T B C)_{ii}} = \sum_{i=1}^{n} \underbrace{(C e_i)^T B (C e_i)}_{> 0 \text{ da } B \text{ spd.}} > 0$$

14.b)

Sei S orientierbar & regulär.

g_p symmetrisch.

Z $g_p(W_p(X), Y) = g_p(X, W_p(Y))$ $\left[= g_p(W_p(Y), X) \right]$

$\forall X, Y \in T_pS$

d.h.

$\mathrm{II}_p(X, Y) = \mathrm{II}_p(Y, X)$

In Bemerkung zu Def 5.17 wird gezeigt

(V, F, U) lokale Parametrisierung $\circ$ $F(\bar{u}) = p$

$\Rightarrow \mathrm{II}_p\left(\partial_{u^i} F(\bar{u}), \partial_{u^j} F(\bar{u})\right) = \left\langle \frac{\partial^2 F}{\partial u_i \partial u_j}(\bar{u}), N(F(\bar{u})) \right\rangle$

$=: h_{ij}$

Nun $h_{ij} = h_{ji}$ da

$h_{ij} = \left\langle \frac{\partial^2 F}{\partial u_i \partial u_j}(\bar{u}), N(F(\bar{u})) \right\rangle \overset{\text{Schwarz}}{=\!=} \left\langle \frac{\partial^2 F}{\partial u_j \partial u_i}, N(F(\bar{u})) \right\rangle$

$= h_{ji}$

Nun ist $\{ \partial_{u^1} F(\bar{u}), \partial_{u^2} F(\bar{u}) \}$ Basis von T_pS.

$\Rightarrow X, Y \in T_pS \Rightarrow \exists x^1, x^2, y^1, y^2 \in \mathbb{R} : \begin{array}{l} X = x^1 \partial_{u^1} F(\bar{u}) + x^2 \partial_{u^2} F(\bar{u}) \\ Y = y^1 \partial_{u^1} F(\bar{u}) + y^2 \partial_{u^2} F(\bar{u}) \end{array}$

$\Rightarrow \mathrm{II}_p(X, Y) \overset{\mathrm{II}_p \text{ bilinear}}{=} \sum_{i=1}^{2} \sum_{j=1}^{2} x^i \, \mathrm{II}_p\left(\partial_{u^i} F(\bar{u}), \partial_{u^j} F(\bar{u})\right) y^j$

$$= \sum_{i=1}^{2} \sum_{j=1}^{2} x^i h_{ij} y^j = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Analog

$$\mathrm{II}_p(y, X) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^T \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Nun da $h_{12} = h_{21}$

folgt

$$\mathrm{II}_p(X, y) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right)^T$$

$$= \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^T \begin{pmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^T \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathrm{II}_p(y, X)$$

Nun

$$\det(g) = \left(1+\left(\frac{\partial f}{\partial u}\right)^2\right)\cdot\left(1+\left(\frac{\partial f}{\partial v}\right)^2\right) - \left(\frac{\partial f}{\partial u}\frac{\partial f}{\partial v}\right)^2$$

$$= 1+\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2 = 1+|Df|^2$$

$$\Longrightarrow (g^{ij})_{ij} = \begin{pmatrix} \frac{1+f_v^2}{1+|Df|^2} & \frac{-f_u f_v}{1+|Df|^2} \\[2ex] \frac{-f_u f_v}{1+|Df|^2} & \frac{1+f_u^2}{1+|Df|^2} \end{pmatrix}$$

<u>Schritt 5</u> Darstellungsmatrix der Weingartenabbildung

$$w_{ij} = \sum_{k=1}^{2} g^{ik} h_{kj}$$

$$w_{11} = g^{11}h_{11} + g^{12}h_{21} = \frac{(1+f_v^2)}{(1+|Df|^2)^{\frac{3}{2}}}\cdot f_{uu} - \frac{f_u f_v}{(1+|Df|^2)^{\frac{3}{2}}} f_{uv}$$

$$w_{22} = g^{21}h_{12} + g^{22}h_{22} = -\frac{f_u f_v}{(1+|Df|^2)^{\frac{3}{2}}} f_{uv} + \frac{(1+f_u^2)}{(1+|Df|^2)^{\frac{3}{2}}} f_{vv}$$

<u>Schritt 6</u> Mittlere Krümmung

$= \mathrm{Spur}(W)$

$$H = \frac{1}{2}(w_{11}+w_{22}) = \frac{1/2}{(1+|Df|^2)^{3/2}}\left(f_{uu} + f_{vv} + f_v^2 f_{uu} + f_u^2 f_{vv} - 2 f_u f_v f_{uv}\right)$$

(c) Lokale Parametrisierung

$(B, F, \mathbb{R}^3)$ $\quad F(u,v) = (u, v, f(u,v))$

Sei $p = F(u,v)$

Schritt 1 Tangentialraum bestimmen

Basis von T_pS $\quad \{\partial_u F, \partial_v F\}$

also $\left\{ \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial u} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial v} \end{pmatrix} \right\}$

Schritt 2 Einheitsnormalenfeld bestimmen

$$\tilde{N}(F(u,v)) := \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial u} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial v} \end{pmatrix} = \begin{pmatrix} -\frac{\partial f}{\partial u} \\ -\frac{\partial f}{\partial v} \\ 1 \end{pmatrix}$$

Normieren $\Rightarrow$ $N(F(u,v)) = \dfrac{\left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right)^T}{\sqrt{1 + \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2}}$

$$= \frac{(-\nabla f, 1)}{\sqrt{1 + |\nabla f|^2}}$$

Schritt 3 zweite Fundamentalform

$\rightsquigarrow$ Darstellungsmatrix bezüglich der gewählten Basis

$$\mathrm{II}_p\left(\partial_{u^1}F(\bar u), \partial_{u^2}F(\bar u)\right) = h_{ij} := \left\langle \frac{\partial^2 F}{\partial u_i \partial u_j}(u,v), N(F(u,v)) \right\rangle$$

<u>Konkret</u>

$$h_{11} = \left\langle \begin{pmatrix} 0 \\ 0 \\ \frac{\partial^2 f}{\partial u^2} \end{pmatrix}, \begin{pmatrix} -\frac{\partial f}{\partial u} \\ -\frac{\partial f}{\partial v} \\ 1 \end{pmatrix} \right\rangle \frac{1}{\sqrt{1+|Df|^2}} = \frac{\frac{\partial^2 f}{\partial u^2}}{\sqrt{1+|Df|^2}}$$

$$h_{12} = h_{21} = \left\langle \begin{pmatrix} 0 \\ 0 \\ \frac{\partial^2 f}{\partial u \partial v} \end{pmatrix}, \begin{pmatrix} -\frac{\partial f}{\partial u} \\ -\frac{\partial f}{\partial v} \\ 1 \end{pmatrix} \right\rangle \frac{1}{\sqrt{1+|Df|^2}} = \frac{\frac{\partial^2 f}{\partial u \partial v}}{\sqrt{1+|Df|^2}}$$

$$h_{22} = \left\langle \begin{pmatrix} 0 \\ 0 \\ \frac{\partial^2 f}{\partial v^2} \end{pmatrix}, \begin{pmatrix} -\frac{\partial f}{\partial u} \\ -\frac{\partial f}{\partial v} \\ 1 \end{pmatrix} \right\rangle \frac{1}{\sqrt{1+|Df|^2}}$$

$$= \frac{\frac{\partial^2 f}{\partial v^2}}{\sqrt{1+|Df|^2}}$$

<u>Schritt 4</u> Inverse des Metrischen Tensors (1. Fundamental form) bzgl der gewählten Basis

$$(g_{ij})_{ij} = \begin{pmatrix} |\partial_u F|^2 & \langle \partial_u F, \partial_v F\rangle \\ \langle \partial_u F, \partial_v F\rangle & |\partial_v F|^2 \end{pmatrix} = \begin{pmatrix} 1+\left(\frac{\partial f}{\partial u}\right)^2 & \frac{\partial f}{\partial u}\frac{\partial f}{\partial v} \\ \frac{\partial f}{\partial u}\frac{\partial f}{\partial v} & 1+\left(\frac{\partial f}{\partial v}\right)^2 \end{pmatrix}$$

$$\Longrightarrow (g^{ij})_{ij} = \frac{1}{\det g} \begin{pmatrix} 1+\left(\frac{\partial f}{\partial v}\right)^2 & -\frac{\partial f}{\partial u}\frac{\partial f}{\partial v} \\ -\frac{\partial f}{\partial u}\cdot\frac{\partial f}{\partial v} & 1+\left(\frac{\partial f}{\partial u}\right)^2 \end{pmatrix}$$

$\Longrightarrow$ $H(F(u,v))$

$$= \frac{1/2}{(1+|Df|^2)^{3/2}} \left(\Delta f + f_v^2 f_{uu} + f_u^2 f_{vv} - 2 f_u f_v f_{uv} \right)$$

(d)

$$\operatorname{div}\left(\frac{Df}{\sqrt{1+|Df|^2}}\right) = \operatorname{div}(Df)\,\frac{1}{\sqrt{1+|Df|^2}} + Df^T \cdot \operatorname{grad}\left(\frac{1}{\sqrt{1+|Df|^2}}\right)$$

$$= \operatorname{div}[Df]\,\frac{1}{\sqrt{1+|Df|^2}} + Df^T \cdot \left(\frac{-1}{(1+|Df|^2)^{3/2}} H_f\, Df\right)$$

$$= \frac{\Delta f\,(1+|Df|^2) - Df^T H_f\, Df}{(1+|Df|^2)^{3/2}}$$

$$= \frac{\Delta f + \Delta f\,|Df|^2 - Df^T H_f\, Df}{(1+|Df|^2)^{3/2}}$$

$$= \frac{\Delta f + (f_{uu}+f_{vv})(f_u^2+f_v^2) - f_u^2 f_{uu} - f_v^2 f_{vv} - 2 f_u f_v f_{uv}}{(1+|Df|^2)^{3/2}}$$

$$= \frac{\Delta f + f_u^2 f_{vv} + f_v^2 f_{uu} - 2 f_u f_v f_{uv}}{(1+|Df|^2)^{3/2}}$$

$\Longrightarrow$ Beh (Multipliziere mit $\frac{1}{2}$)

(e)

$$H = \frac{1}{2}\,\mathrm{Spur}\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$$

$$= \frac{1}{2}\,\mathrm{Spur}\left(\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \right)$$

Nun

$$h_{ij} = \left\langle \frac{\partial^2 F}{\partial u^i \partial u^j}, N \right\rangle = \frac{\frac{\partial^2 f}{\partial u_i \partial u_j}}{\sqrt{1+|\nabla f|^2}}$$

$$\Rightarrow H = \frac{1}{2}\,\mathrm{Spur}\left(\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \frac{1}{\sqrt{1+|\nabla f|^2}}\, H_f \right)$$

$$= \frac{1/2}{\sqrt{1+|\nabla f|^2}}\,\mathrm{Spur}\Bigg(\underbrace{\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1}}_{\text{pos. def}} \underbrace{H_f}_{\text{pos. def}} \Bigg) \geq 0$$

da

$$g_{11} = |\partial_u F|^2 > 0$$

$$g_{11} g_{22} - g_{12} g_{21} = |\partial_u F|^2 |\partial_v F|^2 - \langle \partial_u F, \partial_v F \rangle^2$$

(*) Falls H_f negativ definit:

$\Rightarrow -H_f$ positiv definit

$$H = \frac{1/2}{\sqrt{1+|\nabla f|^2}} \operatorname{Spur}\left(\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} H_f \right)$$

$$= \frac{-1/2}{\sqrt{1+|\nabla f|^2}} \underbrace{\operatorname{Spur}\left(\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} (-H_f) \right)}_{\geq 0} \leq 0$$

(f)

$$F(\varphi, \theta) = \begin{pmatrix} \cos\varphi \cos\theta \\ \sin\varphi \cos\theta \\ \sin\theta \end{pmatrix} \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \varphi \leq 2\pi$$

1) Tangentialraum:

$$\partial_\varphi F = \begin{pmatrix} -\sin\varphi \cos\theta \\ \cos\varphi \cos\theta \\ 0 \end{pmatrix} \qquad \partial_\theta F = \begin{pmatrix} -\cos\varphi \sin\theta \\ -\sin\varphi \sin\theta \\ \cos\theta \end{pmatrix}$$

2) Normalenraum

$$\tilde{N}(\varphi,\theta) = \partial_\varphi F \times \partial_\theta F = \begin{pmatrix} -\sin\varphi \cos\theta \\ \cos\varphi \cos\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} -\cos\varphi \sin\theta \\ -\sin\varphi \sin\theta \\ \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cancel{\cos\theta \sin\theta} \; \cos^2\theta \cos\varphi \\ \cos^2\theta \sin\varphi \\ \cos\theta \sin\theta \end{pmatrix}$$

Einheitsnormale

$$N(\varphi,\theta) = \frac{1}{\sqrt{(\cos^2\theta \cos\varphi)^2 + (\cos^2\theta \sin\varphi)^2 + \cos^2\theta}} \begin{pmatrix} \cos^2\theta \cos\varphi \\ \cos^2\theta \sin\varphi \\ \cos\theta \sin\theta \end{pmatrix}$$

$$= \frac{1}{\cos\theta} \begin{pmatrix} \cos^2\theta \cos\varphi \\ \cos^2\theta \sin\varphi \\ \cos\theta \sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta \cos\varphi \\ \cos\theta \sin\varphi \\ \sin\theta \end{pmatrix}$$

3)

Erste Fundamentalform

$$g_{11} = |\partial_\varphi F|^2 = \cos^2\theta$$

$$g_{12} = \langle \partial_\varphi F, \partial_\theta F \rangle = 0 = g_{21}$$

$$g_{22} = |\partial_\theta F|^2 = 1 \qquad \Rightarrow g = \begin{pmatrix} \cos^2\theta & 0 \\ 0 & 1 \end{pmatrix}$$

4) Zweite Fundamentalform.

$$\partial^2_{\varphi\varphi} F = \begin{pmatrix} -\cos\varphi \cos\theta \\ -\sin\varphi \cos\theta \\ 0 \end{pmatrix} \qquad \partial^2_{\varphi\theta} F = \begin{pmatrix} \sin\varphi \sin\theta \\ -\cos\varphi \sin\theta \\ 0 \end{pmatrix}$$

$$\partial^2_{\theta\theta} F = \begin{pmatrix} -\cos\varphi \cos\theta \\ -\sin\varphi \cos\theta \\ -\sin\theta \end{pmatrix}$$

$$h_{11} = \langle \partial^2_{\varphi\varphi} F, N \rangle = \cos^2\varphi \cos^2\theta + \sin^2\varphi \cos^2\theta$$
$$= \cos^2\theta$$

$$h_{12} = \langle \partial^2_{\varphi\theta} F, N \rangle = \sin\varphi \sin\theta \cos\varphi \cos\theta - \cos\theta \sin\varphi \cos\varphi \sin\theta$$
$$= 0$$

$$h_{22} = \langle \partial^2_{\theta\theta} F, N \rangle = \cos^2\varphi \cos^2\theta + \sin^2\varphi \cos\theta^2$$
$$+ \sin^2\theta = 1$$

$$\Rightarrow H = \frac{1}{2} \operatorname{Spur} g^{-1} H$$

$$= \frac{1}{2} \operatorname{Spur} \left(\begin{pmatrix} \frac{1}{\cos^2\theta} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos^2\theta & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$= \frac{1}{2} \operatorname{Spur} \mathbb{1} = \frac{1}{2} \cdot 2 = 1$$

$$\Longrightarrow H_{S^2} = 1$$

[$H_{S^2} \neq 0 \Rightarrow S^2$ keine Minimalfläche ?
Mit Volumenbeschränkung: Lagrange parameter !]

(g) Sei $t \in \mathbb{R}$

Definiere $\tilde{K} := \{p + t\Phi(p) \mid p \in K\}$

Z $\exists \delta > 0 : |t| < \delta$

$\Rightarrow \forall p \in S_t \; \exists (U_p, F_p, V_p) :$ · $F_p : U_p \longrightarrow \mathbb{R}^3$ glatt

· $F_p(U_p) = V_p \cap S_t$

· $\operatorname{rank} DF_p(u) = 2 \quad \forall u \in U_p$,

· F_p Homöomorphismus auf sein Bild.

Sei (U^*, F^*, V^*) die Parametrisierung so, dass $\operatorname{supp} \Phi \subset\subset V^*$.

Sei $t \in \mathbb{R}$ und betrachte

$$F_t(u) := F^*(u) + t\, \phi(F^*(u)) \qquad u \in U^*$$

1) F_t glatt ✓ $\quad \phi : S \to \mathbb{R}^3$ glatt d.h. $\phi \circ F^*$ glatt

2) $\operatorname{rank} DF_t^*(u) = 2 \quad \forall u \in U_*$ denn

$\operatorname{ranck} DF_t^*(u) = 2 \iff DF_t^*(u)^T DF_t^*(u)$ invertierbar

$$DF_t(u)^T DF_t(u)$$

$$= \Big(DF^*(u) + t\, \underbrace{D(\phi \circ F)^*(u)}_{\in C_0^\infty(U)}\Big)^T \Big(DF^*(u) + t\, \underbrace{D(\phi \circ F)^*(u)}_{\in C_0^\infty(U)}\Big)$$

$$= DF^*(u)^T DF^*(u) + t\, D(\phi \circ F)^*(u)^T DF^*(u) + t\, DF^*(u)^T D(\phi \circ F)^*(u)$$

$$+ t^2 D(\phi \circ F)^*(u)^T D(\phi \circ F)^*(u)$$

Nun

$$\| t\,(D\phi \circ F(u))^T D\mathring{F}(u) + t\,DF^*(u)\,D(\phi\circ F)^*(u)$$

$$+ t^2\, D(\phi\circ F)^*(u)\, D(\phi\circ F)(u)\|$$

$$\leq t\, \|DF^*\|_{\infty,\, \underbrace{\mathrm{supp}(\phi\circ F)}_{\text{Kompakt.}}} + o(t^2)$$

$\Rightarrow \exists \delta>0 : |t|<\delta \Rightarrow DF_t(u)^T DF_t(u)$ invertierbar

$\Rightarrow \mathrm{rank}\, DF_t(u) = 2.\quad \forall u \in U$

für t klein genug

F_t Homöomorphismus auf sein Bild : $\boxed{\text{Technisch}}$

- F_t injektiv

~~$\exists$ inf $\alpha = \min_{(u,v)\in(\mathrm{supp}\,\phi)\times(\mathrm{supp}\,\phi)} \|F(u)-F(v)\|$~~

~~Angenommen $F_t(u) = F_t(v)$~~

~~$u,v \in \mathrm{supp}\,\phi$~~

~~$\Rightarrow F(u) + t(\phi\circ F)(u) = F(v) + t(\phi\circ F)(v)$~~

~~$\Rightarrow F(u) - F(v) = t\,((\phi\circ F)(v) - (\phi\circ F)(u))$~~

~~$\Rightarrow \alpha \leq \|F(u)-F(v)\| = |t|\,\|\phi\circ F(u) - \phi\circ F(v)\|$~~

~~$\leq |t|\,\|\phi\|_\infty$~~

~~unmöglich falls $|t| < \frac{\alpha}{\|\phi\|_\infty}$~~

(g)

$$\frac{d}{dt}\Big|_{t=0} A(S_t)$$

$$= \frac{d}{dt}\Big|_{t=0} \int_U \overbrace{\sqrt{|\partial_u F_t|^2 |\partial_v F_t|^2 - (\partial_u F_t, \partial_v F_t)^2}}^{\sqrt{\det g_{ij}}}$$

$$= \int_U \frac{|\partial_v F|^2 (\partial_u F, \partial_u(\phi\circ F)) + |\partial_u F|^2 (\partial_v F, \partial_v(\phi\circ F))}{\sqrt{|\partial_u F|^2|\partial_v F|^2 - (\partial_u F,\partial_v F)^2}}$$

$$- (\partial_u F, \partial_v F)\left(\langle \partial_u F, \partial_v \phi\circ F\rangle + \langle \partial_v F, \partial_u \phi\circ F\rangle\right)$$

$$= \int_U \sqrt{\det g_{ij}} \Big(g^{11} (\partial_u F, \partial_u(\phi\circ F)) + g^{22} (\partial_v F, \partial_v(\phi\circ F))$$

$$+ g^{12} \left(\langle \partial_u F, \partial_v(\phi\circ F)\rangle + \langle \partial_v F, \partial_u(\phi\circ F)\rangle \right) \Big)$$

$$\langle \partial_u F, \partial_u(\phi\circ F)\rangle = \partial_u \underbrace{\langle \partial_u F, \phi\circ F\rangle}_{=0} - \langle \partial_u^2 F, \phi\circ F\rangle$$

$$= -|\phi| \langle \partial_u^2 F, N\circ F\rangle = 0 - h_{11} = -h_{11}|\phi|$$

$$\Longrightarrow \frac{d}{dt}\Big|_{t=0} A(S_t) = -\int_U |\phi| \sqrt{\det g_{ij}} \left(g^{11}h_{11} + g^{22}h_{22} + g^{12}h_{12} + g^{21}h_{22}\right)$$

$$= -2 \int_U |\phi|\, H(F(u,v))\, \cancel{d(u,v)} \sqrt{\det(g)}\, d(u,v)$$

$$= -2 \int_S \langle \mathcal{H}, \phi\rangle\, dA$$

(h) Folgt aus (g):

Wähle endlich viele $V_1, \ldots, V_n$ so dass (U_i, F_i, V_i) lokale Parametrisierung ist und

$K \subset \bigcup_{i=1}^{n} V_i$ und Zerlegung der Eins $(\rho_j)_{j=1}^{n}$:

$\mathrm{supp}(\rho_j) \subset V_j$

Dann
$$\phi = \sum_{j=1}^{N} \underbrace{(\rho_j \phi)}_{\mathrm{Supp} \subset V_j}.$$

(i)

(i) $F_t(u^1,u^2) := F(u_1,u_2) + t\,\phi(u_1,u_2)\,\partial_{u_3}F(u_1,u_2)$
$(u^1,u^2)\in U$

$$\frac{d}{dt}\Big|_{t=0} A(S_t)$$

$$= \frac{d}{dt}\Big|_{t=0} \int_U \sqrt{|\partial_{u_1}F_t|^2|\partial_{u_2}F_t|^2 - (\partial_{u_1}F_t,\partial_{u_2}F_t)^2}$$

$$= \int_U \frac{\langle\partial_{u_1}F,\partial_{u_1}(\phi\,\partial_{u_3}F)\rangle|\partial_{u_2}F|^2 + \langle\partial_{u_2}F,\partial_{u_2}(\phi\,\partial_{u_3}F)\rangle|\partial_{u_1}F|^2}{\sqrt{|\partial_{u_1}F|^2|\partial_{u_2}F|^2-(\partial_{u_1}F,\partial_{u_2}F)^2}}$$

$$- \frac{(\partial_{u_1}F,\partial_{u_2}F)\Big((\partial_{u_1}F,\partial_{u_2}(\phi\,\partial_{u_3}F)) + (\partial_{u_2}F,\partial_{u_1}(\phi\,\partial_{u_3}F))\Big)}{\sqrt{|\partial_{u_1}F||\partial_{u_2}F|-(\partial_{u_1}F,\partial_{u_2}F)^2}}$$

$$\underset{\text{konform}}{=\!=} \int_U \frac{\cancel{|\partial_{u_1}F|^2}\big(\langle\partial_{u_1}F,\partial_{u_1}(\phi\,\partial_{u_3}F)\rangle + \langle\partial_{u_2}F,\partial_{u_2}(\phi\,\partial_{u_3}F)\rangle\big)}{\cancel{|\partial_{u_1}F|^2}}$$

$$= \int_U \langle\partial_{u_1}F,\partial_{u_1}(\phi\,\partial_{u_3}F)\rangle + \langle\partial_{u_2}F,\partial_{u_2}(\phi\,\partial_{u_3}F)\rangle$$

$$= \int_U (\partial_{u_1}F,\partial_{u_3}F\rangle\partial_{u_1}\phi + \langle\partial_{u_1}F,\partial_{u_3}\partial_{u_1}F\rangle\phi$$

$$+ \langle\partial_{u_2}F,\partial_{u_3}F\rangle\,\partial_{u_2}\phi + \langle\partial_{u_2}F,\partial_{u_3}\partial_{u_2}F\rangle\phi$$

Fallunterscheidung

$j=1, j=2$

$$\int_U \epsilon\, |\partial_{u_j} F|^2 \partial_{u_j} \phi + \sum_{k=1}^{2} \langle \partial_{u_k} F, \partial_{u_j} \partial_{u_k} F \rangle \phi$$

$$= \int_U |\partial_{u_j} F|^2 \partial_{u_j} \phi + \sum_{k=1}^{2} \partial_{u_j} |\partial_{u_k} F|^2/2 \; \phi$$

$$\underset{\substack{|\partial_{u_1} F|^2 = |\partial_{u_2} F|^2 \\ = |\partial_{u_j} F|^2}}{=} \int_U |\partial_{u_j} F|^2 \partial_{u_j} \phi + 2\, \partial_{u_j} \frac{|\partial_{u_j} F|^2}{2} \phi$$

$$= \int_U |\partial_{u_j} F|^2 \partial_{u_j} \phi + \partial_{u_j} |\partial_{u_j} F|^2 \phi$$

$$= \int_U \partial_{u_j} \big(\underbrace{\phi\, |\partial_{u_j} F|^2}_{\in C_0^\infty(U),} \big)\, dx = 0$$