Cheat Sheet

Die Fouriertransformation

$$S(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) \mid \forall (k_1 \dots k_n) \in \mathbb{N}_0^n$$
$$\forall (m_1 \dots m_n) \in \mathbb{N}_0^n$$
$$x_1^{k_1} \dots x_n^{k_n} \partial_{x_1}^{m_1} \dots \partial_{x_n}^{m_n} f(x) \longrightarrow 0$$
$$\text{für } |x| \longrightarrow \infty \}$$

<u>Def</u> $f \in S(\mathbb{R}^n)$

Setze

$$\mathcal{F}(f)(\lambda) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i x \cdot \lambda} f(x)\, dx$$

<u>Eigenschaften</u>

- $\mathcal{F}(f) \in S(\mathbb{R}^n) \quad \forall f \in S(\mathbb{R}^n)$
- $(i\lambda_1)^{m_1} \dots (i\lambda_n)^{m_n} \partial_{\lambda_1}^{k_1} \dots \partial_{\lambda_n}^{k_n} \mathcal{F}(f)$
$$= \mathcal{F}\left(\partial_{x_1}^{m_1} \dots \partial_{x_n}^{m_n} (-ix_1)^{k_1} \dots (-ix_n)^{k_n} f \right)$$
- $\mathcal{F}(f(\cdot - \alpha e_k)) = e^{-i\alpha\lambda_k} \mathcal{F}(f)$

- $\mathcal{F}(f(\alpha\cdot))(\lambda) = \frac{1}{|\alpha|^n} \mathcal{F}(f)\left(\frac{\lambda}{\alpha}\right)$
- $f \in S(\mathbb{R}^n), g \in S(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \overline{f(x)}\, g(x)\, dx = \int_{\mathbb{R}^n} \overline{\mathcal{F}(f)(\lambda)}\, \mathcal{F}(g)(\lambda)\, d\lambda$$

- $\mathcal{F}\left(e^{-a|x|^2/2}\right) = \frac{1}{\sqrt{a}} e^{-\frac{|\lambda|^2}{2a}}$
- $f \in S(\mathbb{R}^n) \Longrightarrow f * g \in S(\mathbb{R}^n)$

und $\mathcal{F}(fg) = \frac{1}{(2\pi)^{n/2}} \mathcal{F}(f) * \mathcal{F}(g)$

$$\mathcal{F}(f * g) = (2\pi)^{n/2} \mathcal{F}(f)\mathcal{F}(g)$$

- Umkehrformel $f \in S(\mathbb{R}^n) \Rightarrow \mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\lambda x} f(\lambda)\, d\lambda$

<u>Satz von Plancherel</u>

$\exists!\ \mathcal{F} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ unitär

derart dass

$$\mathcal{F}\big|_{S(\mathbb{R}^n)}(f) = \int_{\mathbb{R}^n} f(x)\, e^{-itx}\, dx$$

Riemann-Lebesgue-Lemma

$f \in L^1(\mathbb{R}^n)$. Dann

$$\mathcal{F}(f)(\lambda) := \int f(x)\, e^{-i\lambda x}\, dx \,\frac{1}{(2\pi)^{n/2}}$$

$$\in C_0(\mathbb{R}^n)$$

$$:= \left\{ f \in C(\mathbb{R}^n) : \lim_{|x|\to\infty} f(x) = 0 \right\}$$

Dazu $\|\mathcal{F}(f)\|_\infty \leq \frac{1}{(2\pi)^{n/2}} \|f\|_{L^1}$

Fouriertransformation und $W^{1,2}(\mathbb{R}^d)$

$f \in W^{1,2}(\mathbb{R}^d)$

$$\Longleftrightarrow \int_{\mathbb{R}^d} |\mathcal{F}(f)|^2 (1+|x|^2)\, dx < \infty$$

$g \in L^2(\mathbb{R}^d)$: $\int_{\mathbb{R}^d} |g|^2 (1+|x|^2)\, dx < \infty$

$$\Longrightarrow \exists f \in W^{1,2}(\mathbb{R}^d) : g = \mathcal{F}(f)$$

- Es gilt noch $\mathcal{F}(D_{x_j} f) = i x_j \mathcal{F}(f)$

Analog

$$\hat{H}^k(\mathbb{R}^d) := \left\{ f : \mathbb{R}^d \to \mathbb{C} \text{ messbar } \int_{\mathbb{R}^d} |f(x)|^2 (1+|x|^2)^k \, dx < \infty \right\}$$

Dann

$$\mathcal{F}(W^{k,2}(\mathbb{R}^d)) = \hat{H}^k(\mathbb{R}^d)$$

Sobolev'scher Einbettungssatz und d Fouriertransformation

$$k > \frac{d}{2} \Rightarrow W^{k,2}(\mathbb{R}^d) \subset C_0(\mathbb{R}^d)$$

Bew Sei $f \in W^{k,2}(\mathbb{R}^d)$

$$\Rightarrow \int_{\mathbb{R}^d} |\mathcal{F}(f)|^2 (1+|x|^2)^k \, dx < \infty$$

Daher

$$\int |\mathcal{F}(f)| \, dx \leq \int_{\mathbb{R}^d} |\mathcal{F}(f)|^2 (1+|x|^2)^k \, dx \cdot \int \frac{1}{(1+|x|^2)^k} \, dx < \infty \text{ falls } k > \frac{n}{2}$$

$$\Longrightarrow \mathcal{F}(f) \in L^1(\mathbb{R}^d)$$

$$\Longrightarrow \mathcal{F}(\mathcal{F}(f)) \in C_0(\mathbb{R}^d)$$

Aber $\mathcal{F}(\mathcal{F}(f))(x) = \mathcal{F}^{-1}(\mathcal{F}(f))(-x)$

$$= f(-x)$$

$$\Longrightarrow f(-\,\cdot\,) \in C_0(\mathbb{R}^d)$$

$$\Longrightarrow f \in C_0(\mathbb{R}^d)$$