Variationsrechnung Blatt 9

(a) $\int_\Omega v\,(x\cdot\nabla v)\,dx$

$= \int_\Omega x \cdot v\cdot(\nabla v)\,dx$

$= \int_\Omega x \;\nabla\left(\frac{v^2}{2}\right)\,dx$

$= \sum_{i=1}^{n} \int_\Omega x_i \frac{\partial}{\partial x_i} \frac{v^2}{2}\,dx$

$= \sum_{i=1}^{n} \left(\int_{\partial\Omega} x_i \frac{v^2}{2} \nu_i \, dS(x) - \int_\Omega \underbrace{\frac{\partial}{\partial x_i} x_i}_{=1} \frac{v^2}{2}\,dx \right)$

$= \frac{1}{2} \int_{\partial\Omega} v^2 \langle x, \nu \rangle \, dS(x) - \frac{n}{2} \int_\Omega v^2\,dx$

(b)

(i) Sei $p \in \partial\Omega$, $\nu(p)$ äußere Normale an p.

Ergänze $\{\nu(p)\}$ zu einer ONB von $\mathbb{R}^n$.

$\leadsto$ $\{\nu(p), e_1(p), \ldots e_{n-1}(p)\}$ bilde eine ONB von $\mathbb{R}^n$.

da $e_i(p) \perp \nu(p)$ $\forall i = 1 \ldots n-1$ gilt:

$e_i(p)$ ist tangential zu $\partial\Omega$ in p, d.h

$\forall i = 1, \ldots n-1$ $\exists$ $c^{(i)} \in C^1((-\varepsilon, \varepsilon); \mathbb{R}^n)$:

$c^{(i)}(t) \in \partial\Omega$ $\forall$ $t \in (-\varepsilon, \varepsilon)$, $c^{(i)}(0) = p$, $\dot{c}^{(i)}(0) = e_i$

Nun gilt

$$\nabla u(p) \cdot e_i(p) = \nabla u(c^{(i)}(0)) \, \dot{c}^{(i)}(0)$$

$$= \frac{d}{dt}\Big|_{t=0} \underbrace{u(c(t))}_{\equiv 0 \text{ da } c(t) \in \partial\Omega} = \frac{d}{dt}\Big|_{t=0} 0 = 0$$

$\Rightarrow$ $\langle \nabla u(p), e_i(p) \rangle = 0$ $\forall i = 1, \ldots n-1$

Da $\{\nu(p), e_1(p), \ldots, e_{n-1}(p)\}$ eine ONB

$$\nabla u(p) = \langle \nabla u(p), \nu(p) \rangle \nu(p) + \sum_{i=1}^{n} \langle \nabla u(p), e_i(p) \rangle e_i(p)$$

$$= \langle \nabla u(p), \nu(p) \rangle \nu(p) = \frac{\partial u}{\partial \nu}(p) \cdot \nu(p) \qquad \text{qed}$$

(ii) Hu bezeichne die Hessematrix von u

$$\Delta u(x) = \operatorname{div}(Du(x))$$

$$= \operatorname{Spur}(Hu(x))$$

$$= \sum_{i=1}^{n-1} e_i(x)^T Hu(x) e_i(x) + \nu(x)^T Hu(x) \nu(p)$$

Spur ist Basiswechselinvariant.

Beh $\forall i = 1, \dots, n-1$ $\quad e_i(x)^T Hu(x) e_i(x) = 0$

Bew Sei $c = c^{(i)}$ für irgendein i

$$0 = \frac{d^2}{dt^2} u(c(t))$$

$$= \frac{d}{dt} Du(c(t)) \cdot c'(t)$$

$$= Du(c(t)) c''(t) + c'(t)^T Hu(c(t)) c'(t)$$

$$\underset{c(t) \in \partial\Omega}{=} \nu(c(t)) \frac{\partial u}{\partial \nu}(c(t)) c''(t)$$

$$+ c'(t)^T Hu(c(t)) c'(t)$$

$$= c'(t) Hu(c(t)) c'(t)$$

$t = 0 \quad c = c^{(i)} \implies 0 = e_i(x)^T Hu(x) e_i(x)$

$$\Rightarrow \Delta u(x) = \gamma^T H u \, \gamma$$

$$\xrightarrow[\text{wie oben}]{\text{Rechnung}} \left.\frac{d^2}{dt^2}\right|_{t=0} u(x+t\gamma) = \frac{\partial^2 u}{\partial \gamma^2}$$

(c)

$$\int_\Omega \Delta^2 u\,(x\cdot \nabla u)\,dx$$

$$= \int_\Omega \Delta(\Delta u)\,(x\cdot \nabla u)\,dx$$

$$= \sum_{k=1}^n \int_\Omega (\Delta u)_{x_k x_k}\,(x\cdot \nabla u)\,dx$$

$$= \sum_{k=1}^n \underbrace{\int_{\partial\Omega} (\Delta u)_{x_k}\,(x\cdot \nabla u)\,\nu^k\,dS}_{=0 \text{ da auf } \partial\Omega:\ \nabla u = \nu\frac{\partial u}{\partial\nu} = 0} - \int_\Omega (\Delta u)_{x_k}\,(x\cdot \nabla u)_{x_k}\,dx$$

$$= -\sum_{k=1}^n \int_\Omega (\Delta u)_{x_k}\,(x\cdot \nabla u)_{x_k} = -\int_\Omega \nabla(\Delta u)\cdot \nabla(x\cdot \nabla u)\,dx$$

Nun

$$-\int \nabla(\Delta u)\ \nabla(x\cdot \nabla u)\,dx$$

$$= -\sum_{k=1}^n \int (\Delta u)_{x_k}\,(x\cdot \nabla u)_{x_k}\,dx$$

$$= -\sum_{k,i=1}^n \int_\Omega (\Delta u)_{x_k}\,(x_i\,u_{x_i})_{x_k}\,dx$$

$$= -\sum_{k,i=1}^n \int_\Omega (\Delta u)_{x_k}\,(\delta_{ik}\,u_{x_i} + x_i\,u_{x_i x_k})\,dx$$

$$= -\sum_{k,i=1}^{n} \left(\int_\Omega (\Delta u)_{x_k} \delta_{ik} u_{x_i} dx + \int_\Omega x_i (\Delta u)_{x_k} u_{x_i x_k} dx \right)$$

$$= -\sum_{k=1}^{n} \int_\Omega (\Delta u)_{x_k} u_{x_k} dx - \sum_{k,i=1}^{n} \int_\Omega (\Delta u)_{x_k} x_i u_{x_i x_k} dx$$

$$= -\int_\Omega \nabla(\Delta u) \nabla u - \sum_{k,i=1}^{n} \int_\Omega (\Delta u)_{x_k} x_i u_{x_i x_k} dx$$

$$= \int_\Omega (\Delta u)^2 dx - \int_{\partial\Omega} (\Delta u) \frac{\partial u}{\partial \nu} dS$$

$$- \sum_{k,i=1}^{n} \int_\Omega (\Delta u)_{x_k} x_i u_{x_i x_k} dx$$

(d) $\sum_{k,i=1}^{n} \int_{\Omega} x_i (\Delta u)_{x_k} u_{x_i x_k} \, dx$

$$= \sum_{k,i=1}^{n} \int_{\partial\Omega} x_i \, u_{x_i x_k} \nu_k \, \Delta u \, ds - \underbrace{\int_{\Omega} (\Delta u) \frac{\partial}{\partial x_k} (x_i \, u_{x_i x_k}) \, dx}_{(**)}$$

Nun

$$\sum_{k,i=1}^{n} \int_{\partial\Omega} (x_i \, u_{x_i x_k} \nu_k) \, \Delta u \, dS$$

$$= \int_{\partial\Omega} x^T Hu \, \nu \, \Delta u \, dS$$

Nun gilt für festes $x \in \partial\Omega$, e_i wie in (b)

$$x^T Hu \, \nu = \left(\sum_{i=1}^{n-1} \langle x, e_i(x) \rangle e_i(x) + \langle x, \nu \rangle \nu \right)^T Hu \, \nu$$

$$= \sum_{i=1}^{n-1} \langle x, e_i(x) \rangle \quad e_i(x)^T Hu \, \nu$$

$$+ \langle x, \nu \rangle \, \nu^T Hu \, \nu$$

$$= \langle x, \nu \rangle \Delta u + \sum_{i=1}^{n} \langle x, e_i(x) \rangle \; e_i(x)^T Hu \, \nu$$

Nun: Beh

$$e_i^T(x)\, Hu\, \gamma = \frac{d}{dt}\Big|_{t=0} \frac{\partial u}{\partial \gamma}(c(t)) \quad [= 0]$$

<u>Bew</u>

$$\frac{d}{dt}\Big|_{t=0} \frac{\partial u}{\partial \gamma}(c(t))$$

$$= \frac{d}{dt}\Big|_{t=0} \left(\langle \gamma(c(t)), Du(c(t)) \rangle \right)$$

$$= \langle \gamma(c(0)), \underbrace{\frac{d}{dt}\Big|_{t=0} Du(c(t))}_{= Hu(x) e_i'(x)} \rangle$$

$$+ \langle \frac{d}{dt} \gamma(c(t)), \underbrace{Du(c(t))}_{= \frac{\partial u}{\partial \gamma} \cdot \gamma(c(t))} \rangle \Big|_{t=0}$$

$$= \gamma^T Hu\, e_i + \underbrace{\frac{\partial u}{\partial \gamma}(c(t))}_{= 0} \underbrace{\langle \frac{d}{dt} \gamma(c(t)), \gamma(c(t)) \rangle}_{= 0 \text{ auch! (aber nicht so wichtig)}} \Big|_{t=0}$$

$$= \gamma^T Hu\, e_i'(x) = e_i^T(x)\, Hu(x)\, \gamma(x)$$

$$\Longrightarrow x^T H u \, \nu = \langle x, \nu \rangle \Delta u$$

$$\overset{(*)}{\Longrightarrow} \sum_{k,i=1}^{n} \int_{\Omega} x_i (\Delta u)_{x_k} u_{x_i x_k} \, dx$$

$$= \int_{\partial\Omega} \langle x, \nu \rangle (\Delta u)^2 \, dS - \sum_{k,i=1}^{n} \int_{\Omega} \Delta u \frac{\partial}{\partial x_k} (x_i \, u_{x_i x_k}) \, dx$$

$$= \int_{\partial\Omega} \langle x, \nu \rangle (\Delta u)^2 \, dS - \sum_{k,i=1}^{n} \int_{\Omega} \Delta u \left\{ \delta_{ik} u_{x_i x_k} + x_i u_{x_i x_k x_k} \right\} dx$$

$$= \int_{\partial\Omega} \langle x, \nu \rangle (\Delta u)^2 \, dS - \sum_{k=1}^{n} \int_{\Omega} \Delta u \; u_{x_k x_k} \, dx$$

$$- \sum_{k,i=1}^{n} \int_{\Omega} \Delta u \, x_i \, u_{x_i x_k x_k} \, dx$$

$$= \int_{\partial\Omega} \langle x, \nu \rangle (\Delta u)^2 \, dS - \int_{\Omega} (\Delta u)^2 \, dx$$

$$- \sum_{i=1}^{n} \int_{\Omega} \Delta u \; x_i \, (\Delta u)_{x_i} \, dx$$

$$= \int_{\partial\Omega} \langle x, \nu \rangle (\Delta u)^2 \, dS - \int_{\Omega} (\Delta u)^2 \, dx - \int \Delta u \cdot (x \cdot \nabla(\Delta u)) \, dx$$

1e)

$$\int_\Omega \Delta u \, (x \, \nabla(\Delta u)) \, dx$$

$$\underset{v=\Delta u}{\overset{(a)\text{ mit}}{=}} \frac{1}{2}\int_{\partial\Omega} \langle x,\nu\rangle (\Delta u)^2 \, dS - \frac{n}{2}\int_\Omega (\Delta u)^2 \, dx$$

$(**)$

$$\Rightarrow \sum_{k,i=1}^{n} \int_\Omega x_i \, (\Delta u)_{x_k} \, u_{x_i x_k} \, dx$$

$$\overset{(d)+(*)}{=} -\int_\Omega (\Delta u)^2 \, dx - \left(\frac{1}{2}\int_{\partial\Omega} \langle x,\nu\rangle (\Delta u)^2 \, dS - \frac{n}{2}\int_\Omega (\Delta u)^2 \, dx\right)$$

$$+ \int_{\partial\Omega} (\Delta u)^2 \langle x,\nu\rangle \, dS$$

$$= \frac{1}{2}\int_{\partial\Omega} (\Delta u)^2 \langle x,\nu\rangle \, dS + \frac{n-2}{2}\int_\Omega (\Delta u)^2 \, dx$$

(f)

Mit (c)

$$\int \Delta u \,(x\cdot Du)\,dx = \int_\Omega (\Delta u)^2$$

$$- \sum_{k,i=1}^{n} \int_\Omega x_i (\Delta u)_{x_k} u_{x_i x_k}\, dx$$

$$= \int_\Omega (\Delta u)^2\, dx - \Big[\frac{1}{2} \int_{\partial\Omega} (\Delta u)^2 \langle x, \nu\rangle \, dS$$

$$+ \frac{n-2}{2} \int_\Omega (\Delta u)^2\, dx \Big]$$

$$= \frac{4-n}{2} \int_\Omega (\Delta u)^2\, dx - \frac{1}{2} \int_{\partial\Omega} (\Delta u)^2 \langle x, \nu\rangle\, dS$$

(h) mit (g)

$$\Delta^2 u = \lambda u + |u|^{q-1} u$$

$$\Rightarrow \int_\Omega \Delta^2 u \, (x \cdot \nabla u) \, dx = \lambda \int_\Omega u \cdot (x \cdot \nabla u) \, dx + \int_\Omega |u|^{q-1} u \, (x \cdot \nabla u) \, dx$$

$$\overset{(g)}{\Longrightarrow} \frac{4-n}{2} \int_\Omega (\Delta u)^2 \, dx - \frac{1}{2} \int_{\partial\Omega} (\Delta u)^2 \langle x, \nu \rangle \, ds$$

$$= -\frac{\lambda n}{2} \int u^2 dx - \frac{n}{q+1} \int |u|^{q+1} dx \quad \text{(I)}$$

Dazu $\Delta^2 u = \lambda u + |u|^{q-1} u$

$$\Rightarrow \Delta^2 u \, u = \lambda u^2 + |u|^{q+1}$$

$$\Rightarrow \int_\Omega \Delta^2 u \, u \, dx = \lambda \int u^2 dx + \int |u|^{q+1} dx$$

Nun

$$\int \Delta^2 u\, u = \sum_{k=1}^{n} \int (\Delta u)_{x_k x_k} u \, dx$$

$$= \sum_{k=1}^{n} \Big(\underbrace{\int_{\partial\Omega} (\Delta u)_{x_k} u \nu^k \, dS}_{=0 \text{ da } u|_{\partial\Omega}=0}$$

$$- \int_\Omega (\Delta u)_{x_k} u_{x_k} \, dx \Big)$$

$$= -\sum_{k=1}^{n} \int_\Omega (\Delta u)_{x_k} u_{x_k} \, dx$$

$$= -\sum_{k=1}^{n} \Big(\int_{\partial\Omega} (\Delta u) u_{x_k} \nu^k \, dS - \int_\Omega \Delta u \, u_{x_k x_k} \, dx \Big)$$

$$= \underbrace{- \int_{\partial\Omega} \Delta u \frac{\partial u}{\partial \nu} \, dS}_{=0} + \int_\Omega (\Delta u)^2 \, d\cancel{x}$$

$$= \int (\Delta u)^2 \, dx$$

$$\Longrightarrow (\mathrm{II}) \quad \int_\Omega (\Delta u)^2 \, dx = \lambda \int_\Omega u^2 \, dx + \int_\Omega |u|^{q+1} \, dx$$

Rechne $(\mathrm{I}) - \frac{4-n}{2} (\mathrm{II})$

$\Longrightarrow$

$$-\frac{1}{2}\int_{\partial\Omega}(\Delta u)^2\langle x,\nu\rangle\,ds$$

$$=\left[-\frac{\lambda n}{2}-\lambda\left(\frac{4-n}{2}\right)\right]\int_\Omega u^2\,dx$$

$$+\left[\frac{-n}{q+1}-\frac{4-n}{2}\right]\int_\Omega |u|^{q+1}\,dx$$

$$\Longrightarrow -\frac{1}{2}\int_{\partial\Omega}(\Delta u)^2\langle x,\nu\rangle\,ds$$

$$=-2\lambda\int_\Omega u^2\,dx$$

$$+\left[-\frac{n}{\frac{n+4}{n-4}+1}-\frac{4-n}{2}\right]\int |u|^{q+1}dx$$

$$=-2\lambda\int u^2\,dx+\underbrace{\left[\frac{-n(n-4)}{2n}-\frac{4-n}{2}\right]\int |u|^{q+1}dx}_{=0}$$

$$\Rightarrow -\frac{1}{2}\int_{\partial\Omega}(\Delta u)^2\langle x,\nu\rangle\,ds=-2\lambda\int u^2\,dx$$

$$\Rightarrow \int_{\partial\Omega}(\Delta u)^2\langle x,\nu\rangle\,ds=4\lambda\int u^2\,dx$$

(g) Angenommen $\exists \lambda < 0$, $u \not\equiv 0$ nichttriviale Lösung

Dann

$$4\lambda \int_\Omega u^2 \, dx < 0$$

aber

$$4\lambda \int_\Omega u^2 \, dx = \int_{\partial\Omega} \langle x, \nu \rangle (\Delta u)^2 \, dx \geq 0 \quad ↯$$

32|

1a) $0 = D\mathcal{E}(u)(u)$

$$= \int |Du|^2 \, dx - \lambda \int u^2 \, dx$$

$$- \int |u|^{p^*+1} \, dx$$

$$\Rightarrow \int |u|^{p^*+1} \, dx = \int \left(|Du|^2 - \lambda u^2\right) dx$$

$$\geq \left(1 - \frac{\lambda}{\lambda_1}\right) \int |Du|^2 \, dx$$

$$\Rightarrow \left(\int |u|^{p^*+1} \, dx\right)^{1 - \frac{2}{p^*+1}} \geq \left(1 - \frac{\lambda}{\lambda_1}\right) \frac{\int |Du|^2 \, dx}{\left(\int |u|^{p^*+1} dx\right)^{2/p^*+1}}$$

$$\geq \left(1 - \frac{\lambda}{\lambda_1}\right) S$$

Beachte $\quad 1 - \frac{2}{p^*+1} = 1 - \frac{2}{\frac{2n}{n-2}} = 1 - \frac{n-2}{n}$

$$= 2/n$$

$$\Longrightarrow \int_\Omega u^{p^*+1}\,dx \geq \left(1 - \frac{\lambda}{\lambda_1}\right)^{\frac{n}{2}} S^{n/2}$$

(b)

Nun

$$\int |\nabla u|^2\,dx \geq S\left(\int u^{p^*+1}\,dx\right)^{\frac{2}{p^*+1}}$$

$$\geq S\left[\left(1 - \frac{\lambda}{\lambda_1}\right)^{\frac{n}{2}} S^{n/2}\right]^{\frac{n-2}{n}}$$

$$= S\left(1 - \frac{\lambda}{\lambda_1}\right)^{\frac{n-2}{2}} S^{\frac{n-2}{2}}$$

$$= \left(1 - \frac{\lambda}{\lambda_1}\right)^{\frac{n-2}{2}} S^{n/2}$$

32(c)

Sei u wie in Lemma 1.4

Dann $\mathcal{E}(u) < \frac{1}{n} S^{n/2}$

$$\Rightarrow \frac{1}{n} S^{n/2} > \mathcal{E}(u) = \frac{1}{2}\int |\nabla u|^2\,dx - \frac{\lambda}{2}\int u^2\,dx$$

$$- \frac{1}{p^*+1}\int |u|^{p^*+1}\,dx$$

$$\geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)\int |\nabla u|^2\,dx$$

$$- \frac{1}{p^*+1}\int |u|^{p^*+1}\,dx$$

$$\geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)\left(1 - \frac{\lambda}{\lambda_1}\right)^{\frac{n-2}{2}} S^{n/2}$$

$$- \frac{1}{p^*+1}\int |u|^{p^*+1}\,dx$$

$$= \frac{1}{2}\left(1 - \lambda/\lambda_1\right)^{n/2} S^{n/2} - \frac{1}{p^*+1}\int |u|^{p^*+1}\,dx$$

$$\Longrightarrow \frac{1}{p^\lambda+1}\int |u|^{p^\lambda+1} \geq \left[\frac{1}{2}\left(1-\frac{\lambda}{\lambda_1}\right)^{n/2} - \frac{1}{n}\right] S^{n/2}$$

$$\Longrightarrow \int |u|^{p^\lambda+1}\,dx \geq \frac{2n}{n-2}\left[\frac{1}{2}\left(1-\frac{\lambda}{\lambda_1}\right)^{n/2} - \frac{1}{n}\right] S^{n/2}$$

$$= \left[\frac{n}{n-2}\left(1-\frac{\lambda}{\lambda_1}\right)^{n/2} - \frac{2}{n-2}\right] S^{n/2}$$

$$= \left[\left(1-\frac{\lambda}{\lambda_1}\right)^{n/2} - \frac{2}{n-2}\left(1+\left(1-\frac{\lambda}{\lambda_1}\right)^{n/2}\right)\right] S^{n/2}$$

Ungleichung ist schlechter geworden!