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Übungen Introduction to Complex Analysis: Sheet 2

For this worksheet, we define

$$S := \left\{ (\xi, \eta, \mu) \in \mathbb{R}^3 : \xi^2 + \eta^2 + \left(\mu - \frac{1}{2}\right)^2 = \frac{1}{4} \right\},$$

and consider the map $\phi : S \to \mathbb{C}_\infty$, that maps a point in S to the associated number in $\mathbb{C}_\infty$ as described in section 1.4.2. Let us also recall the metric spaces (S, χ_1) and $(\mathbb{C}_\infty, \chi_2)$, where both χ_1, χ_2 mean the spherical distance. For this worksheet, we might want to recall two definitions that are relevant for metric spaces.
1) Let (M, d) be a metric space. A sequence $(x_n) \subset M$ is called convergent if there exists an $x \in M$ such that $d(x_n, x) \to 0$ as $n \to \infty$. x is also called the limit of this sequence
2) Let (M, d) be a metric space. A subset $K \subset M$ is called compact if every sequence $(x_n) \subset K$ possesses a convergent subsequence.
3) Let (M, d) be a metric space. A subset $A \subset M$ is called closed, if for every convergent sequence $(x_n) \subset A$ the limit lies in A.
In the metric space $(\mathbb{C}, d)$ we identified the compact sets to be the closed and bounded sets. This is not necessarily true for general metric spaces. This worksheet shall help you understand compactness in the metric spaces we just defined.

11. Sketch the following sets and provide a simple description of them:

- $D(\infty; \frac{1}{2})$ as a subset of $\mathbb{C}_\infty$
- $\{z \in \mathbb{C}_\infty : \chi_2(z, \infty) \leq \frac{1}{2}\} \cup \{\infty\}$ as a subset of $\mathbb{C}_\infty$
- $\phi^{-1}(D(\infty; \frac{1}{2}))$ as a subset of S
- $\phi^{-1}(\mathbb{R}_\infty)$ as a subset of S, where $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$

12. Let $A \subset C_\infty$ be such that $\infty \notin A$. Show: A is closed in the metric space $(\mathbb{C}_\infty, \chi_2)$ if and only if A is compact in the metric space $(\mathbb{C}, d)$, where d denotes the canonical metric on $\mathbb{C}$.

13. Show that $\mathbb{R}_\infty$ is a compact set in $\mathbb{C}_\infty$, this means that for every sequence $(x_n) \subset \mathbb{R}_\infty$ we can find a subsequence (x_{l_n}) and $x \in \mathbb{R}_\infty$ such that $\chi_2(x_{l_n}, x) \to 0$ as $n \to \infty$.

14. Show: For every point $(\xi, \eta, \mu) \in S$ there is a unique $\psi \in (-\pi, \pi]$ such that

$$(\xi, \eta, \mu) = (\sqrt{\mu(1-\mu)} \cos\psi, \sqrt{\mu(1-\mu)} \sin\psi, \mu).$$

The coordinates (μ, ψ) are called spherical polar coordinates. Furthermore, show that $\psi = \arg(\phi(\xi, \eta, \mu))$.

15. (a) Given a matrix $M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$ we define the so-called Möbius transformation $T_M : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by

$$T_M(z) = \begin{cases} \frac{az+b}{cz+d} & cz + d \neq 0 \\ \infty & cz + d = 0 \\ \frac{a}{c} & z = \infty \end{cases},$$

where we agree temporarily on $\frac{a}{c} = \infty$, if $c = 0$. Show: For $M, N \in GL_n(\mathbb{C})$ we have the identity $T_{MN} = T_M \circ T_N$. Moreover, T_M is invertible and $T_M^{-1} = T_{M^{-1}}$.

(b) Show that any Möbius transformation T_M can be written as a composition of maps of the following types:

- Rotation by an angle of ϕ: $D_\phi(z) = e^{i\theta} z$ for $z \in \mathbb{C}$ and $D_\phi(\infty) = \infty$
- Dilation by a factor of $\alpha > 0$: $S_\alpha(z) = \alpha z$. Again $S_\alpha(\infty) := \infty$
- Translation by $c \in \mathbb{C}$: $V_c(z) = z + c$, under the tacit assumption that $V_c(\infty) = \infty$.
- Inversion $I(z) = \begin{cases} \frac{1}{z} & z \neq 0 \\ \infty & z = 0 \\ 0 & z = \infty \end{cases}$.

To which matrices do these maps correspond ?

(c) If T_M is a Möbius transformation, then the map $\phi^{-1} \circ T_M \circ \phi$ maps S to S. Find a geometric description of $\phi^{-1} \circ D_\phi \circ \phi$, $\phi^{-1} \circ S_\alpha \circ \phi$ as well as $\phi^{-1} \circ I \circ \phi$ and provide a mathematical description using either the coordinates (ξ, η, μ) or the spherical polar coordinates, whichever you prefer!

(d) Provide two Möbius transforamtions T_{M_1} and T_{M_2} an, such that $\phi^{-1} \circ T_{M_i} \circ \phi$ maps the great circle given by $\mu = \frac{1}{4}$ to the great circle $\mu = \frac{3}{4}$. Additionally, we require that $T_{M_1} \circ T_{M_2}^{-1}$ may not be a rotation.

16. Let (M, d) be a metric space. Show that a union of finitely many compact sets is again compact.

17. Let $K_1, K_2, ... \subset \mathbb{C}$ be compact and nonempty, such that $K_1 \supset K_2 \supset K_3 \supset ...$. Show that

$$K := \bigcap_{i=1}^{\infty} K_i$$

is compact and nonempty.

The following exercise will be on the next worksheet. If you want, you can work on it already:

18. Consider the following sequence of functions:

$$f_N(z) = \sum_{n=3}^{N} \frac{(-1)^n}{n + z}. \tag{1}$$

Show that it converges uniformly on $D := \{z \in \mathbb{C} | 1 < |z| < 2\}$. Does the Weierstraß-M-Test apply ?