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Übungen Elemente der Funktionentheorie: Blatt 3

~~This week, we discovered some unbelievably important concepts: For example series and their convergence~~ behavior, power series and the exponential function as well as the complex logartihm. Another important tool - which we are going to use later - is the Arzéla-Ascoli Theorem.

19. (a) Decide whether or not the given logartihms are well-defined and if so, compute them : $\log i, \log(-1), \log(-1-\sqrt{3}i)$ (2)

(b) Find all complex numbers $i^{i \log i}$. How many are there ? (1)

(c) For the rest of the problem we define: $\sqrt{w} := \exp\left(\frac{1}{2}\log w\right)$ for all $w \in \mathbb{C} \setminus \mathbb{R}_{<0}$. Show that the principal branch or the log satisfies $\sqrt{w} = z_w$, where z_w is the number defined in Problem 10 on Worksheet 1. (2)

Hint: Show $|\sqrt{w}| = |z_w|$ and $\arg\sqrt{w} = \arg z_w$. In order to determine $arg(z_w)$ it might be helpful that $\arg(w + |w|) = \frac{\arg w}{2}$. Showing this identity you might want to use the half-angle formulas $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$ and $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$.

(d) Show: $\overline{\sqrt{z}} = \sqrt{\overline{z}}$. (2)

Hint: Use $\exp(\overline{z}) = \overline{\exp(z)}$ and $\arg(\overline{z}) = -\arg(z)$, but don't leave these sneaky identities unjustified!

(e) Find a power series expansion of $f : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \to \mathbb{C}, f(z) := \frac{\sin\sqrt{z}}{\sqrt{z}}$ and show that this series converges indeed on the entire domain of definition. Does the expansion make sense for $z \in \mathbb{R}_{\leq 0}$ as well ? (2)

Hint: You might want to adopt the power series expansion of the sine function that you discovered in class this week.

(f) Show that $\sin(x+iy) = \sin(x)\cosh(y) + i\sinh(y)\cos(x)$. Conclude that $\sin(x+iy) = 0$ if and only if $x + iy = k\pi$ for some $k \in \mathbb{Z}$. Find all complex roots of $\sinh : \mathbb{C} \to \mathbb{C}$. (3)

(g) Show that for arbitrary $x \in \mathbb{R}$ and $n \in \mathbb{N}$ (2)

$$\cos^n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos((n-2k)x).$$

Hint: First, apply the binomial formula. See what you end up with and simplify the expression using that the left hand side is necessarily real.

20. Let $K \subset \mathbb{C}$ be compact (f_n) a sequence of continuous functions defined on K, that converges uniformly to f. Show that in this case, (f_n) is uniformly bounded and equicontinuous. (4)

21. (a) BONUS: Let $(x_n) \subset \mathbb{C}$ be a sequence. Show that $x_n \to x$ in $(\mathbb{C}, d)$ if and only if every sequence possesses a subsequence, that converges to x. Give an example for a divergent sequence, each subsequence of which possesses a convergent subsequence . (3*)

Hint: The reverse implication is a very nice proof by contradiction: Assuming that (x_n) does not converge to x we can find $\epsilon_0 > 0$ such that for each $N \in \mathbb{N}$ there is an index $n_N \geq N$ such that $|x_{n_N} - x| \geq \epsilon_0$. Actually, the x_{n_N} can be arranged into a subsequence. Can this special subsequence still contain a subsequence that converges to x ?

(b) BONUS: Let (f_n) be a uniformly bounded and equicontinuous sequence, that converges to f pointwise. Show that f_n converges indeed uniformly to f. (3*)

Hint: Again by contradiction. Some of the arguments are very similar to the ones presented above. However, we desperately need to apply the Arzela-Ascoli Theorem.

22. Consider the following sequence (f_N) of complex-valued functions (5)

$$f_N(z) = \sum_{n=3}^{N}\frac{(-1)^n}{n+z}.$$

Show that f_N converges uniformly on $D := \{z \in \mathbb{C} | 1 < |z| < 2\}$. Can the Weierstrass-M-test be applied to the series

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{n+z}?$$

Hinweis: I intend to have you apply the Cauchy-Criterion. In order to help you do that let me show you a computation:

$$\left| \sum_{n=800}^{30001} \frac{(-1)^n}{n+z} \right| = \left| \sum_{n=400}^{15000} \left(\frac{1}{2n+z} - \frac{1}{2n+1+z} \right) \right| = \left| \sum_{n=400}^{15000} \frac{1}{(2n+z)(2n+1+z)} \right| \leq \sum_{n=400}^{15000} \frac{1}{(2n-2)(2n-1)} < \frac{1}{800 -}$$

where I divided the summand up into even indices and odd indices and used the triangle inequality as well as the inverse triangle inequality. Readers with acute minds might have noticed that this computation is at all possible only if the parity of the index where the sum starts differs from the parity of the index where it ends. If this is not the case though, some adjustments have to be made (and can be made).