UNIVERSITÄT ULM

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Sommersemester 2017
Punktzahl: 24


# Übungen Elemente der Funktionentheorie: Blatt 4

**23.** (a) Show that $f(z) = \overline{z}$ is nowhere complex differentiable. (2)

(b) Determine all the points $z \in \mathbb{C}$ such that $f(z) = z^3 + |z|^2$ is complex differentiable (3)

**24.** Let $D \subset \mathbb{C}$ be an open set and $f : \mathbb{D} \to \mathbb{C}$ holomorphic on $D$. Define for $(x, y)$ satisfying $x + iy \in D$ $u(x, y) = Re(f(x + iy))$ and $v(x, y) = Im(f(x + iy))$.

(a) If $u$ and $v$ are twice continuously partially differentable on $D$, then the identity $u_{xx} + u_{yy} = 0$ is satisfies. Functions that satisfy this identity are called harmonic. (3)

(b) For the sake of simplicity we will assume in this exercise that $D = \mathbb{C}$. Define $g(r, \theta) := u(re^{i\theta})$ and $h(r, \theta) = v(re^{i\theta})$ for $r > 0$ and $\theta \in \mathbb{R}$. Show that the following version of the Cauchy-Riemann-equation holds: (3)

$$g_r = \frac{1}{r} h_\theta, \quad h_r = -\frac{1}{r} g_\theta$$

(c) Let $G \subset D$ be a domain, such that $|f(z)| = const$ for each $z \in G$. Show: $f$ is constant on $G$. (3)

**25.** (a) Compute for $r > 0$ (2)

$$\int_{|z|=r} \overline{z}\, dz, \quad \int_{|z|=r} \frac{1}{z^2} dz,$$

(b) Let $E$ be an ellipse, the semiminor f which has length $a$ and the semimajor of which has length $b$. Let $\gamma_E$ be the curve that runs counterclockwise around $E$. Compute (just using the parametrization) (2)

$$\int_{\gamma_E} \frac{1}{|z|} dz$$

(c) This task justifies the fact that the velocity in which we run through the curve is an invariant of the curve integral. Let $\phi : [0, 1] \to [0, 1]$ be a continuously differentiable bijection fulfilling $\phi'(x) > 0$ for each $x \in (0, 1)$ and $\gamma : [0, 1] \to \mathbb{C}$ a continuously differentiable curve. Define $\widetilde{\gamma}(t) := \gamma(\phi(t))$ for $t \in [0, 1]$. Show that for every function $f : \mathbb{C} \to \mathbb{C}$ it holds that (1)

$$\int_\gamma f dz = \int_{\widetilde{\gamma}} f dz.$$

(d) Show for arbitrary $x_0 \in \mathbb{R}$ , $|x_0| < 1$ (2)

$$\int_{|z|=1} \frac{\overline{z}}{z - x_0} dz = \int_{|z|=1} \frac{1}{1 - x_0 z}$$

(e) Let $f : \mathbb{C} \to \mathbb{C}$ be continuous such that the is a complex differentiable $g$ satisfying $g' = f$. Show: In this case we find that (2)

$$\int_\gamma f dz = 0$$

for each closed curve $\gamma : [a, b] \to \mathbb{C}$ .

(f) For $R > 2$ let $\gamma_R$ be the closed curve, that runs along a straight line from $-R + 0i$ to $R + 0i$, and goes back on a half circle that lies above the $x$- axis. Show that for even $n \in \mathbb{N}$,$n \geq 2$ we have the approximation (3+2*)

$$\int_{-\infty}^{\infty} \frac{e^{-x}}{1 + x^n} dx = \lim_{R \to \infty} \int_{\gamma_R} \frac{e^{-z}}{1 + z^n} dz$$

Extra credit: Find a family of closed curves $\gamma_R$ such that

$$\int_{-\infty}^{\infty} \frac{e^x}{1 + x^n} dx = \lim_{R \to \infty} \int_{\gamma_R} \frac{e^z}{1 + z^n} dz.$$

Übungsblätter sowie aktuelle Informationen unter
https://www.uni-ulm.de/mawi/analysis/lehre/veranstaltungen/sose2017/elemente-der-funktionentheorie/