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Sommersemester 2017
Punktzahl: 24

Übungen Elemente der Funktionentheorie: Blatt 5

26. Compute the following line integrals. If you use the Cauchy integral theorem, you do not need to show the regularity prerequisite (namely that the partial derivatives of u and v are continuous). The reguarity prerequisite follows from the holomorphy of u and v anyway.

(a) (1)

$$\int_{|z-3|=5} e^{z^5 \sin z} dz$$

(b) (1)

$$\int_{|z|=1} \frac{z^3 e^z}{z - \frac{1}{2}} dz$$

(c) (2)

$$\int_{\gamma} \frac{z^3 e^z}{z - \frac{1}{2}} dz$$

where γ parametrizes an ellipse, that contains $\frac{1}{2}$ as an inner point, in anticlockwise direction.

(d) (2)

$$\int_{|z|=1} \frac{z^3 e^z}{\left(z - \frac{1}{2}\right)^2} dz$$

(e) (2)

$$\int_{|z-1|=3} \frac{\sin(z^4+1)}{(z-7)^{42}}$$

(f) (2)

$$\int_{|z|=\frac{5}{2}} \frac{1}{z^2 - 5z + 6} dz$$

(g) (2)

$$\int_{|z|=4} \frac{1}{z^2 - 5z + 6} dz$$

(h) (2)

$$\int_{|z|=3} \frac{e^z}{(z-1)(z-2)} dz$$

(i) (2)

$$\int_{|z|=1} \frac{e^z}{\frac{1}{2}\overline{z} - 1}$$

(j) (2)

$$\int_{|z|=1} \frac{\overline{z}}{z - \frac{1}{2}} dz$$

(k) (1)

$$\int_{|z|=1} \log(z) dz$$

where the $\log(z)$ means the principal branch of the logarithm. We agree on the following convention for the logarithm: $-\pi \leq \arg(z) \leq \pi$ and for the principal barnch of the logarithm it holds that $\log(z) = \log|z| + i \arg(z)$.

(l) (2)

$$\int_{\gamma_0} \frac{1}{(1+z^2)\exp(z)} dz$$

where γ_0 parametrizes an ellipse satisfying $\frac{x^2}{4} + \frac{y^2}{9} = 1$ which the curve passes through in clockwise (!) direction.

(m) (2)

$$\int_{|z|=1} \exp\left(\frac{1}{z}\right) dz$$

(n) (2)

$$\int_{|z-1-i|=2} \frac{z^7+1}{z^2(z^2+1)} dz$$

27. (a) True of False? (with proof or counterexample) Let Ω be as defined in lecture, such that the Gauss-Green-Theorem is applicable. Further, let $z_0 \in \Omega$ be an inner point of Ω. If $f = u + iv$ is holomorphic on $\Omega \setminus \{z_0\}$ and u and v are continuously differentiable viewed as real functions on $\overline{\Omega}$, we have (2)

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z - z_0} dz.$$

Hint: The statement is true. You can either follow the lines of the proof of the Cauchy integral formula or you can define $g(z) := (z - z_0)f(z)$ and observe that g is holomorphic on Ω.

(b) Berechne (2)

$$\int_{|z|=1} \frac{1}{\exp(z) - 1} dz$$

Hint: Define for each $z \in \overline{D(0,1)}$

$$f(z) := \begin{cases} \frac{z}{\exp(z)-1} & z \neq 0 \\ ??? & z = 0 \end{cases} \tag{1}$$

with ??? appropriately chosen such that f is continuous on $\overline{D(0,1)}$. Now observe that the expression you are supposed to compute coincides with

$$\int_{|z|=1} \frac{f(z)}{z} dz, \tag{2}$$

which can be computed using part a).

Problem 28 shows that complex analysis can be used to compute line integrals. You can either turn it in at the 11.07.2017 or the week afterwards together with the sample exam.

28. (a) For $R > 2$ let γ_R be the curve, that runs through a line from $-R + 0i$ to $R + 0i$, and goes back on a half-circle above the x- axis. Show the for even $n \in \mathbb{N}, n \geq 2$ it holds that (3)

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^n} dx = \lim_{R\to\infty} \int_{\gamma_R} \frac{e^{iz}}{1+z^n} dz$$

(b) Find a family of closed curves $(\gamma_R)_{R>2}$ such that (3)

$$\int_{-\infty}^{\infty} \frac{e^{-ix}}{1+x^n} dx = \lim_{R\to\infty} \int_{\gamma_R} \frac{e^{-iz}}{1+z^n} dz.$$

(c) Compute (2)

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2} dx.$$

(d) Using a similarr technique, find (2)

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx.$$