Universität Ulm

Abgabe: Dienstag, 27.06.2017

Prof. Dr. Friedmar Schulz
Marius Müller
Sommersemester 2017
Punktzahl: 23

Übungen Elemente der Funktionentheorie: Blatt 3

This week, we discoered some unbelievably important concepts: For example series and their convergence behavior, power series and the exponential function as well as the complex logartihm. Another important tool - which we are going to use later - is the Arzéla-Ascoli Theorem.

19. (a) Decide whether or not the given logartihms are well-defined and if so, compute them : $\log i, \log(-1), \log(-1-\sqrt{3}i)$ (2)

(b) Find all complex numbers $i^{i \log i}$. How many are there ? (1)

(c) For the rest of the problem we define: $\sqrt{w} := \exp\left(\frac{1}{2}\log w\right)$ for all $w \in \mathbb{C} \setminus \mathbb{R}_{<0}$. Show that the principal branch or the log satisfies $\sqrt{w} = z_w$, where z_w is the number defined in Problem 10 on Worksheet 1. (2)

(d) Show: $\overline{\sqrt{z}} = \sqrt{\overline{z}}$. (2)

(e) Find a power series expansion of $f : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \to \mathbb{C}, f(z) := \frac{\sin\sqrt{z}}{\sqrt{z}}$ and show that this series converges indeed on the entire domain of definition. Does the expansion make sense for $z \in \mathbb{R}_{\leq 0}$ as well ? (2)

(f) Show that $\sin(x+iy) = \sin(x)\cosh(y) + i\sinh(y)\cos(x)$. Conclude that $\sin(x+iy) = 0$ if and only if $x + iy = k\pi$ for some $k \in \mathbb{Z}$. Find all complex roots of $\sinh : \mathbb{C} \to \mathbb{C}$. (3)

(g) Show that for arbitrary $x \in \mathbb{R}$ and $n \in \mathbb{N}$ (2)

$$\cos^n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos((n-2k)x).$$

20. Let $K \subset \mathbb{C}$ be compact (f_n) a sequence of continuous functions defined on K, that converges uniformly to f. Show that in this case, (f_n) is uniformly bounded and equicontinuous. (4)

21. (a) BONUS: Let $(x_n) \subset \mathbb{C}$ be a sequence. Show that $x_n \to x$ in $(\mathbb{C}, d)$ if and only if every sequence possesses a subsequence, that converges to x. Give an example for a divergent sequence, each subsequence of which possesses a convergent subsequence . (3*)

(b) BONUS: Let (f_n) be a uniformly bounded and equicontinuous sequence, that converges to f pointwise. Show that f_n converges indeed uniformly to f. (3*)

22. Consider the following sequence (f_N) of complex-valued functions (5)

$$f_N(z) = \sum_{n=3}^{N}\frac{(-1)^n}{n+z}.$$

Show that f_N converges uniformly on $D := \{z \in \mathbb{C} | 1 < |z| < 2\}$. Can the Weierstrass-M-test be applied to the series

$$\sum_{n=3}^{\infty}\frac{(-1)^n}{n+z}?$$