Partielle Differenzialgleichungen WS 2010/11

Musterlösung zu Blatt 6

1.

$$\frac{d}{d\,t}E(t) = \frac{\partial}{\partial t}\frac{1}{2}\int_\Omega |u_t(t,x)|^2 + \|\nabla u(t,x)\|^2\,dx = \int_\Omega u_t u_{tt} + \nabla u_t \cdot \nabla u\,dx$$
$$= \int_\Omega u_t(u_{tt} - \Delta u)\,dx + \int_{\partial\Omega} \frac{\partial u}{\partial n}\frac{\partial u}{\partial t}\,d\sigma = \int_{\partial\Omega} \frac{\partial u}{\partial n}\frac{\partial u}{\partial t}\,d\sigma$$

Die Energie $E(t)$ wächst also für $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial t}$ und nimmt ab für $\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial t}$.

2. Mit $E(t) := \frac{1}{2}\int_\Omega \frac{\rho}{c^2}|\phi_t(t,x)|^2 + \rho\|\nabla\phi(t,x)\|^2\,dx + \frac{1}{2}\int_{\partial\Omega} k\delta(t,y)^2 + m\delta_t(t,y)^2\,d\sigma(y)$ gilt

$$\frac{d}{d\,t}T(t) = \int_\Omega \rho\phi_t(\frac{1}{c^2}\phi_{tt} - \Delta\phi)\,dx + \int_{\partial\Omega} \rho\frac{\partial\phi}{\partial n}\frac{\partial\phi}{\partial t} + k\delta\delta_t + m\delta_t\delta_{tt}\,d\sigma$$
$$= \int_{\partial\Omega} \delta_t(\rho\phi_t + k\delta - d\delta_t - k\delta - \rho\phi_t)\,d\sigma = \int_{\partial\Omega} -d\delta_t^2\,d\sigma \le 0.$$

3. Man hat $T_0(t,x,z) = (t,x,z)$ und

$$T_\delta(T_\epsilon(t,x,z)) = T_\delta(t, x+2\epsilon t, e^{-\epsilon x - \epsilon^2 t}z) = (t, x + 2\epsilon t + 2\delta t, e^{-\delta(x+2\epsilon t) - \delta^2 t}e^{-\epsilon x - \epsilon^2 t}z)$$
$$= (t, x + 2(\epsilon+\delta)t, e^{-(\epsilon+\delta)x - (\epsilon+\delta)^2 t}z) = T_{\epsilon+\delta}(t,x,z).$$

Ferner ist $A = (\frac{d}{d\,\epsilon}T_\epsilon)|_{\epsilon=0} = (0, 2t, -xz)$.

4. Setze $\tilde t = t$, $\tilde x = x + 2\epsilon t$ bzw. $x = \tilde x - 2\epsilon\tilde t$, dann ist die Funktion $\tilde u_\epsilon$ gegeben durch

$$\tilde u(\tilde t, \tilde x) = e^{-\epsilon(\tilde x - 2\epsilon\tilde t) - \epsilon^2\tilde t}u(\tilde t, \tilde x - 2\epsilon\tilde t) = e^{-\epsilon\tilde x + \epsilon^2\tilde t}u(\tilde t, \tilde x - 2\epsilon\tilde t)$$

und

$$\tilde u_{\tilde t} = e^{-\epsilon\tilde x + \epsilon^2\tilde t}(\epsilon^2 u(\tilde t, \tilde x - 2\epsilon\tilde t) + u_t(\tilde t, \tilde x - 2\epsilon\tilde t) - 2\epsilon u_x(\tilde t, \tilde x - 2\epsilon\tilde t))$$
$$= e^{-\epsilon\tilde x + \epsilon^2\tilde t}(\epsilon^2 u(\tilde t, \tilde x - 2\epsilon\tilde t) + u_{xx}(\tilde t, \tilde x - 2\epsilon\tilde t) - 2\epsilon u_x(\tilde t, \tilde x - 2\epsilon\tilde t)) = \tilde u_{\tilde x\tilde x}(\tilde t, \tilde x).$$