Partielle Differenzialgleichungen

Delio Mugnolo

(delio.mugnolo@uni-ulm.de)

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KAPITEL 1

Partial differential equations and what they are good for

A **partial differential equation** (PDE) is an equation whose unkwnown is a function (say, u) of two or more variables (say, $x_1, x_2, \ldots, x_N$) and in which two or more of whose partial derivatives (of first or higher order) appear. (PDEs involving real functions of real variables are most commonly considered, although PDEs involving functions acting on other differential structures and/or taking values in other general fields also exist.)

Generally speaking, a PDE **of m-th order** takes the form

$$F\left(x_1, \ldots, x_N, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}, \ldots, \frac{\partial^m u}{\partial x_1^m}, \ldots, \frac{\partial^m u}{\partial x_N^m}\right) = 0,$$

for a given function F. Such a PDE is called **linear** if it can be written as

$$f_0(x_1, \ldots, x_n)u + \sum_{k=1}^{N} f_{1k}(x_1, \ldots, x_N)\frac{\partial u}{\partial x_k} + \sum_{j,k=1}^{N} f_{2jk}(x_1, \ldots, x_N)\frac{\partial^2 u}{\partial x_k \partial x_j} + \ldots = 0$$

for suitable functions $f_0, f_{11}, \ldots, f_{1N}, \ldots, f_{m1\ldots1}, \ldots, f_{mN\ldots N}$.

For example, the *transport equation*

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial u}{\partial x}(t,x),$$

the *wave equation*

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x),$$

or the *Korteweg–de Vries equation*

$$\frac{\partial u}{\partial t}(t,x) + \frac{\partial^3 u}{\partial x^3}(t,x) = u(t,x)\frac{\partial u}{\partial x}(t,x),$$

are partial differential equations of first, second, and third order, respectively. The transport and wave equations are linear, while de Korteweg–de Vries equation is nonlinear.

Differential equations involving only a function of one variable (or, which is the same, only the partial derivatives with respect to one specific variable of a function of several variables) are well-known *ordinary differential equations* and will not be treated in this course.

Just like ordinary differential equations, partial ones arise in many different contexts as one tries to describe the behaviour of a system ruled by some law. Typically, this has to do with some physical process, like heat diffusion in a conductor material, vibrations of a bridge, circulation of fluids in thin channels, distribution of electric charge on the surface of a metallic sphere... However, in the last century modelling by means of partial differential equations has proved successful also in non-physical disciplines, like in the case of the *Hodgkin–Huxley equation*

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) - \frac{1}{2}u(t,x)\Big(u(t,x) - 1\Big)\Big(u(t,x) - \alpha\Big), \tag{1.1}$$

and of the *Black–Scholes equation*[1]

$$\frac{\partial u}{\partial t}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}(t,x) + rx\frac{\partial u}{\partial x}(t,x) - ru(t,x) = 0. \tag{1.2}$$

The Hodgkin–Huxley equation models the firing patterns of a neuron while the Black–Scholes describes the price of an European stock-option. Both were eventually celebrated with a Nobel Prize: in 1961 (Medicine) and 1997 (Economics), respectively.

Modelling a well-understood process by a mathematical law is often a demanding task, called **derivation** of an equation. Just like in the ordinary case, the derivation of a partial differential equation is often a rough work. Usually, many approximations and simplifications have to be made. In some cases, the resulting PDE gives little insight in the corresponding motivating problem: the PDE can nevertheless be highly interesting at a purely mathematical level.

[1] Here, as in many cases, α, σ, r are constant that have to be determined taking into account the properties of the model specifically considered.

KAPITEL 2

Hyperbolic equations and the method of characteristics

We want a model a transport process (of some incompressible fluid and neglecting turbulences: for instance neutrons in a reactor) inside a (thin) tube of constant section A (such that the transport is forced to take place along the axis of the tube only). We consider a coordinate system such that the x-axis is parallel to the tube. We denote by $u(t,x)$ the density of transported matter at point x and time x. The quantity of matter contained in the tube between the points x and $x+\Delta x$ at time t is

$$\int_x^{x+\Delta x} u(t,\xi) A d\xi.$$

The difference between the quantity of matter leaving this piece of tube at time $t+\Delta t$ and that engering it at time t is given by

$$\int_x^{x+\Delta x} u(t+\Delta t,\xi) A d\xi - \int_x^{x+\Delta x} u(t,\xi) A d\xi = \int_x^{x+\Delta x} \left(u(t+\Delta t,\xi) - u(t,\xi)\right) A d\xi.$$

The **flow** of matter (i.e., the quantity of matter crossing a certain section of the tube in unitary time) is described by a function ψ. Between times t and $t+\Delta t$ and at the section corresponding to the point x, it is measured by

$$\int_t^{t+\Delta t} A\psi(\tau,x) d\tau.$$

Assuming that there are neither sources nor sinks in the tube, the matter has to be conserved: the difference between the matter contained in the piece of tube between times $t+\Delta t$ and t agrees with the difference between the matter that has been flowing through the tube at times t to $t+\Delta t$ between the sections at x and $x+\Delta x$; in other words,

$$\int_x^{x+\Delta x} A\left(u(t+\Delta t,\xi) - u(t,\xi)\right) d\xi = \int_t^{t+\Delta t} A\left(\psi(\tau,x) - \psi(\tau,x+\Delta x)\right) d\xi.$$

It is a reasonable physical assumption that the density and flow functions u and ψ are continuosly differentiable, so that dividing by Δt and passing to the limit $\Delta t \to 0$ we obtain

$$\begin{aligned} \int_x^{x+\Delta x} A\frac{\partial u}{\partial t}(t,\xi) d\xi &= \lim_{\Delta t\to 0} \frac{1}{\Delta t}\int_t^{t+\Delta t} A\left(\psi(\tau,x) - \psi(\tau,x+\Delta x)\right) d\xi \\ A\left(\psi(t,x) - \psi(t,x+\Delta x)\right) d\xi. \end{aligned}$$

Now, dividing again (this time by Δx) and passing to the limit

$$\begin{aligned} \lim_{\Delta x\to 0} \frac{1}{\Delta x}\int_x^{x+\Delta x} A\frac{\partial u}{\partial t}(t,\xi) d\xi &= A\frac{\partial u}{\partial t}(t,x) \\ &= -A\frac{\partial \psi}{\partial x}(t,x). \end{aligned}$$

It is possible to see that if a source or sink modeled by a function $f : \mathbb{R}_+ \times (a, b) \to \mathbb{R}$ are present[1], then the above differential equation can be generalized to

$$A\frac{\partial u}{\partial t}(t,x) = -A\frac{\partial \psi}{\partial x}(t,x) + f(t,x).$$

In the special case of transport of matter with constant velocity and small density, flow and density can be assumed to be proportional, say, $\psi(t,x) = cu(t,x)$, and we finally obtain the partial differential equation

$$\frac{\partial u}{\partial t}(t,x) = -c\frac{\partial u}{\partial x}(t,x) + f(t,x), \qquad t \geq 0, \tag{2.1}$$

where x denotes a coordinate along the transport axis. This is the (inhomogeneous) **transport equation**, obtained under the assumptions that

- the tube is thin enough that the only transport processes take place in the longitudinal direction,
- the density and flow functions are continuously differentiable,
- the velocity of the matter inside the tube is constant, and
- flow and density are proportional by the above velocity constant.

This model can be generalized to the higher-dimensional case: the tranport of a matter inside a d-dimensional container Ω in the direction given by a vector $b \in \mathbb{R}^d$ with space-dependent velocity c is given by

$$\frac{\partial u}{\partial t}(t,x) = -c(x)b \cdot \nabla u(t,x) + f(t,x), \qquad t \geq 0,\ x \in \Omega, \tag{2.2}$$

where $\nabla u(t,x)$ denotes the gradient of u at point x and time t.

Generally speaking, this equation does not determine a unique solution: e.g., it is clear that any function that is constant (that is, constant in both time and space) solves (2.2) if $f \equiv 0$. Since the partial differential equation is first order in the time variable, it is still necessary to impose an initial condition

$$u(0,x) = u_0(x), \qquad x \in \Omega, \tag{2.3}$$

for some $u_0 : \Omega \to \mathbb{R}$. Since the equation is also first order in the space variable, one can expect that a boundary condition is also necessary, say

$$u(t,z) = \phi(z), \qquad t \geq 0,\ z \in \Gamma, \tag{2.4}$$

for some $\phi : \Gamma \to \mathbb{R}$, where Γ is a subset of the topological boundary $\partial\Omega$ of Ω[2].

Definition 2.1. *A* **solution** *of the initial-boundary value problem associated with the transport equation is a function $u : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that*

- *u is continuously differentiable in both variables (one often writes $u \in C^{1,1}(\mathbb{R}_+ \times \Omega)$),*
- *u satisfies* (2.2) *for all $t \geq 0$ and all $x \in \Omega$,*
- *u satisfies* (2.3) *for all $x \in \Omega$,*
- *u satisfies* (2.4) *for all $t \geq 0$ and all $z \in \Gamma$.*

In order to solve the transport equation, a common tool is the so-called *method of characteristics*. For the sake of simplicity we restrict to the 1-dimensional case. Let first $f = 0$ and consider a curve Γ parametrised by a continuously differentiable function $\gamma : \mathbb{R}_+ \to \mathbb{R}$, i.e.,

$$\Gamma := \{(s, \gamma(s)) \in \mathbb{R}_+ \times \mathbb{R}\}$$

[1] We parametrise the tube as an interval (a, b), with $-\infty \leq a < b \leq +\infty$.

[2] This is issue is delicate: in fact, one can see that a boundary condition has to be imposed in the region of the boundary where the transport process begins, and only there. No boundary condition is necessary if the spacial region is unbounded in the direction the transport comes from, cf. Exercise 2.9.

inside the space-time region where the transport equation is considered – e.g., $\mathbb{R}_+ \times \mathbb{R}$ for a (thin) tube of infinite length or $\mathbb{R}_+ \times [0, \ell]$ for a tube of length ℓ. Such a curve Γ is called a **characteristic** of (2.2) if each solution to (2.2) is constant along Γ, i.e., whenever $u(s, \gamma(s)) \equiv const$ for all $s \geq 0$.

Restricting ourselves for the sake of simplicity to the case of $f \equiv 0$, this means that Γ is a characteristic if and only if

$$0 = \frac{du}{ds}(s, \gamma(s)) = \frac{\partial u}{\partial t}(s, \gamma(s)) + \gamma'(s)\frac{\partial u}{\partial x}(s, \gamma(s)), \tag{2.5}$$

i.e., taking into account (2.2), if

$$\gamma'(s) \equiv c(\gamma(s)), \qquad s \geq 0. \tag{2.6}$$

This is the ordinary differential equation defining the characteristic of (2.2). We assume that a solution u satisfies $u(t, x) \equiv u(t, \gamma(t))$, i.e., that k is chosen in such a way that the curve Γ given by $\gamma(s) = cs + k$ crosses (t, x). More explicitly, we are looking for the solution to (2.6) with initial data

$$\gamma(t) = x.$$

In the case of constant c, this yields

$$\gamma(s) = cs + k, \qquad s \in \mathbb{R},$$

for some $k \in \mathbb{R}$. That is, in the 1-dimensional case each characteristic of (2.2) is a line of slope c. The initial condition is then satisfied if and only if

$$\gamma(s) = x + c(s - t), \qquad s \geq 0, \tag{2.7}$$

(and hence for $k = x - ct$). Since u is constant along characteristics, we deduce that

$$u(t, x) = u(0, \gamma(0)) = u_0(\gamma(0)).$$

In the above discussed special case, this means that a solution u is given by

$$u(t, x) = u_0(\gamma(0)) = u_0(x - ct),$$

whenever this expression makes sense.

Similarly, if we consider a d-dimensional transport process defined by (2.2) one can see that for $f \equiv 0$ a solution is given by

$$u(t, x) = u_0(\gamma(0)) = u_0(x - ct\, b), \tag{2.8}$$

since the d-dimensional characteristic is given by

$$\gamma(s) = x + c(s - t)b. \tag{2.9}$$

The above formula (2.8) is quite explicit. Examining it shows that the given solution is a wave that propagates to the right (if $c > 0$) or to the left (if $c < 0$) with speed c and whose profile is constantly that of u_0: one sometimes refers to it as a **shift** of u_0.

We finally consider the case of inhomogeneous transport equations, i.e., of (2.1) with $f \neq 0$. A general principle allows to obtain a solution, in a way that resembles the variation of constant formula of ordinary differential equations. Namely, the solution is given by

$$u(t, x) = u_0(\gamma(0)) + \int_0^t f(\tau, \gamma(\tau))d\tau,$$

where $\Gamma = \{(t, \gamma(t))\}$ is a characteristic of the associated homogeneous problem. This is due to the fact that

$$\begin{aligned}\frac{du}{ds}(s,\gamma(s)) &= \frac{\partial u}{\partial t}(s,\gamma(s)) + \frac{\partial u}{\partial x}(s,\gamma(s))\gamma'(s)\\ &= \frac{\partial u}{\partial t}(s,\gamma(s)) + c(\gamma(s))\frac{\partial u}{\partial x}(s,\gamma(s))\\ &= f(s,\gamma(s)).\end{aligned}$$

Accordingly, by the fundamental theorem of calculus

$$u(t,x) = u(t,\gamma(t)) = u_0(\gamma(0)) + \int_0^t \frac{du}{ds}(s,\gamma(s))ds = u_0(\gamma(0)) + \int_0^t f(s,\gamma(s))ds.$$

Summing up, we have proved the following.

Theorem 2.2. *Let $u_0 \in C^1(\mathbb{R}^d)$ and $f \in C^{0,1}(\mathbb{R}_+ \times \Omega)$, i.e., such that f is continuous with respect to the first variable and continuously differentiable with respect to the second one. If the velocity function c is constant, then the initial value problem (2.2)–(2.3) has a solution given by*

$$u(t,x) = u_0(x - ct) + \int_0^t f(\tau, x + c(\tau - t)b)d\tau, \tag{2.10}$$

for all $t \geq 0$ and $x \in \Omega$ such that this expression makes sense. If $u_0 \in C^k(\mathbb{R}^d)$ and $f \in C^{h,k}(\mathbb{R}_+ \times \Omega)$, then this solution is in fact of class $C^{h+1,k}(\mathbb{R}_+ \times \Omega)$. If $f \equiv 0$, then $u_0(x) \geq 0$ for all $x \in \Omega$ if and only if $u(t,x) \geq 0$ for all $t \geq 0$ and all $x \in \Omega$. If $f(t,x)$ for all $t \geq 0$ and all $x \in \not\leq$ and $u_0(x) \geq 0$ for all $x \in \Omega$, then $u(t,x) \geq 0$ for all $t \geq 0$ and all $x \in \Omega$.

Positivity of the solution is an important property. Physically speaking, positivity of the density function in dependence on positivity of the initial density is a feature that heuristically confirms the modelling qualities of the transport equation.

Remark 2.3. *Let us check that the formula provided in (2.10) is actually a solution. First of all, $u \in C^{1,1}$ due to the regularity properties of u_0, f. Moreover,*

$$-\frac{1}{c}\frac{\partial u}{\partial t}(t,x) = -\frac{1}{c}\frac{\partial u_0}{\partial t}(x - ct) = u_0'(x - ct) = \frac{\partial u_0}{\partial x}(x - ct) = \frac{\partial u}{\partial x}(x - ct).$$

By construction,

$$u(0,x) = u_0(x - c \cdot 0) = u_0(x), \qquad t \in \mathbb{R}.$$

Corollary 2.4. *Let $u_0 \in C^1(\mathbb{R}^d)$ and $f \in C^{0,1}(\mathbb{R}_+ \times \Omega)$. Then the initial value problem (2.2)–(2.3) has at most one solution.*

Beweis. If the initial value problem admits a solution u, then by definition u is regular enough that the computation in (2.5) can be repeated along a characteristic curve parametrised by γ. Thus, γ is necessarily a solution to the ordinary differential equation (2.6) with initial data $\gamma(t) = x$. Since such a Cauchy problem has at most one solution (under suitable regularity assumptions on γ, say, γ is Lipschitz continuous) on some maximal solvability interval J, the solution u is uniquely determined by the formula (2.10). □

Remark 2.5. *This shows that Theorem 2.2 can be easily generalised and yields solutions to transport equations with arbitrary coefficients, under the sole assumption that (2.6) with initial condition $\gamma(t) = x$ admits a solution. In other words, we have transformed a partial differential equation into an ordinary differential equation.*

Exercise 2.6. *Repeat the arguments leading to the g to (2.2) in the general d-dimensional case.*

Exercise 2.7. *The method of characteristics is also suitable for dealing with transport equations with non-constant coefficients. Consider the transport equation* (2.2) *on* $\Omega = \mathbb{R}$ *with* $f \equiv 0$ *and velocity function* c *defined by* $c(x) := x$.

(1) Solve the differential equation defining the characteristic γ *and determine* $\gamma(0)$.
(2) Find the solution to the initial value problem associated with (2.2).

May it happen that for some special c *the initial value problem has no solution?*

Remark 2.8. *In fact, one sees that the formula in Theorem 2.2 for the solution to* (2.2) *makes perfectly sense also if we extend it to a function defined on* $\mathbb{R} \times \mathbb{R}$, *provided that also* u_0 *and* f *are extended accordingly. This means that the transport equation on* $\mathbb{R}$ *is uniquely solvable backward as well as forward in time. This is in sharp contrast to the case of a transport equation on a bounded domain – say, on* $(0, 1)$.

On a bounded domain, boundary conditions necessarily have to be imposed: intuitively, since (for $f \equiv 0$*) the solution is a wave shifting the profile of* u_0 *to the right (if* $c > 0$*) or to the left (if* $c < 0$*), then we should impose a rule on completing this waveform after it has left the left or right boundary, respectively. Possible boundary conditions include*

- *the periodic one –* $u(1) = u(0)$ *– which effectively provides an extension of the solution by prescribing that what is flowing out has to flow in again (it is just like considering the transport equation on a circle, or a torus);*
- *the Dirichlet one –* $u(0) = 0$ *(if* $c > 0$*) or* $u(1) = 0$ *(if* $c < 0$*) – which prescribes that the wave has to be completed by 0.*

Observe that the solution is continuous in space if and only if the initial data satisfies the boundary condition.

Observe finally that although the boundary conditions are a delicate issue, the formula for the solution to the transport problem shows the profile of a function (the initial data) moving (towards the boundary) with finite speed c. *Therefore, if the initial data are given by a function concentrated in a ball far from the boundary, this formula keeps its complete validity as long as the travelling initial density function does not reach the boundary.*

Exercise 2.9. *1) Derive a formula for the transport equation on* $[0, 1]$ *with periodic boundary condition.*

2) Let $c > 0$. *Derive a formula for the transport equation on* $[0, 1]$ *with Dirichlet boundary condition in* $x = 0$.

3) Is it possible to impose a Dirichlet boundary condition in $x = 0$ and also *in* $x = 1$*? Explain your answer.*

Exercise 2.10. *Find a solution to* (2.2) *on* $\mathbb{R}$ *for* $c \equiv 2$ *and* $f(t, x) := tx^2$, $t \geq 0$, $x \in \mathbb{R}$.

2.1. The 1-dimensional wave equation

Another important application of the method of characteristics is the discussion of the following important class of partial differential equations.

Consider a string of linear density ρ. Assume the string to have horizontally constant tension T, i.e., $\frac{\partial T}{\partial x}(t, x) \equiv 0$ (but T can depend on time). We want to model the vertical oscillations of the string, assuming that they are small enough that the horizontal extension can be neglected. We denote by $u(t, x)$ the vertical extension at each time t and each point x of the string, with respect to a reference level which can be set at 0 without loss of generality. At each point x and time t the row undergoes an acceleration

$$\frac{\partial^2 u}{\partial t^2}(t, x).$$

By Newton's second law ($F = m \cdot a$) the force applied to each "infinitesimally small" piece of string of length Δx (which has mass $\rho \Delta x$, by assumption) is

$$\rho(x)\Delta x \frac{\partial^2 u}{\partial t^2}(t,x).$$

Since the tension is assumed to be constant in the horizontal direction, the force acting on the considered piece is given by the difference of the vertical component of the tension at $x+\Delta x$ and x, i.e.,

$$T(t)\sin\theta(t,x+\Delta x) - T(t)\sin\theta(t,x),$$

where $\theta(t,x)$ denotes the angle of the string at each point x at time t. We also ought to add external forces (like the gravity) acting vertically, whose net magnitude we denote by $\Phi(t,x)\Delta x$. All in all the balance equation is

$$\rho(x)\Delta x \frac{\partial^2 u}{\partial t^2}(t,x) = T(t)\sin\theta(t,x) - T(t)\sin\theta(t,x) + \Phi(t,x)\Delta x,$$

i.e.,

$$\rho(x)\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{T(t)\sin\theta(t,x+\Delta x) - T(t)\sin\theta(t,x)}{\Delta x} + \Phi(t,x).$$

Passing to the limit for $\Delta x \to 0$ we obtain

$$\begin{aligned}
\rho(x)\frac{\partial^2 u}{\partial t^2}(t,x) &= \frac{\partial}{\partial x}\left(T(t)\sin\theta(t,x)\right) + \Phi(t,x) \\
&= T(t)\frac{\partial}{\partial x}\sin\theta(t,x) + \Phi(t,x) \\
&= T(t)\cos\theta(t,x)\frac{\partial\theta}{\partial x}(t,x) + \Phi(t,x).
\end{aligned}$$

Moreover, the vertical extensions are supposed to be small, and accordingly $\cos\theta(t,x) \cong 1$ and $\theta(t,x) \cong \sin\theta(t,x) \cong \frac{\partial u}{\partial x}(t,x)$, whence

$$\frac{\partial\theta}{\partial x}(t,x) \cong \frac{\partial^2 u}{\partial x^2}(t,x)$$

for all t, x. Neglecting the gravity we obtain

$$\rho(x)\frac{\partial^2 u}{\partial t^2}(t,x) = T(t)\frac{\partial^2 u}{\partial x^2}(t,x),$$

which can be (approximately) written as

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{T}{\rho(x)}\frac{\partial^2 u}{\partial x^2}(t,x) + F(t,x),$$

provided T does not depend on time. Here

$$F(t,x) := \frac{1}{\rho(x)}\Phi(t,x), \qquad t \geq 0,\ x \in I.$$

One usually calls the constant

$$c(x) := \sqrt{\frac{T}{\rho(x)}}$$

the **wavespeed**. We are thus led to

$$\frac{\partial^2 u}{\partial t^2}(t,x) = c^2(x)\frac{\partial^2 u}{\partial x^2}(t,x) + F(t,x), \qquad t \geq 0,\ x \in I, \tag{2.11}$$

which is the most common form of the 1-dimensional **wave equation** on a (possibly unbounded) open interval $I \subset \mathbb{R}$: we recall that it has been obtained under the assumptions that

- the horizontal tension T does not depend on space and on time,
- the gravity is ignored and
- the vertical extensions are small and the horizontal ones negligible.

If the above argument is repeated in a second or a third (or d^{th}) spacial dimension, we formally arrive to the 2- or 3- (or d-) dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2}(t,x) = c^2(x)\Delta u(t,x), \qquad t \geq 0,\ x \in \Omega, \tag{2.12}$$

in domains $\Omega \subset \mathbb{R}^d$ representing membranes or eleastic bodies. Here we have introduced the common notation

$$\Delta u(t,x) := \sum_k \frac{\partial^2 u}{\partial x_k^2}(t,x),$$

for the **Laplacian** Δu of the function u, where the sum is taken over the spacial dimensions.

Since the differential equation is of second order, in analogy with the case of ODEs of second order we are led to introduce two boundary conditions

$$u(0,x) = u_0(x), \qquad \frac{\partial u}{\partial t}u(0,x) = u_1(x), \qquad x \in \Omega. \tag{2.13}$$

We will see that these two boundary conditions are indeed necessary in order to solve the problem. Moreover, if Ω is bounded, also a boundary condition is necessary. In the 1-dimensional case, common boundary conditions include

$$u(t,a) = u(t,b) \equiv 0, \qquad t \geq 0, \qquad \text{(Dirichlet b.c.)},$$

$$\frac{\partial u}{\partial x}(t,a) = \frac{\partial u}{\partial x}(t,b) = 0, \qquad t \geq 0, \qquad \text{(Neumann b.c.)},$$

$$\frac{\partial^2 u}{\partial x^2}(t,a) = \frac{\partial^2 u}{\partial x^2}(t,b) = 0, \qquad t \geq 0. \qquad \text{(Wentzell b.c.)}.$$

Definition 2.11. *A* **solution** *of the initial-boundary value problem associated with the wave equation is a function $u : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that*

- $u \in C^{2,2}(\mathbb{R}_+ \times \Omega))$,
- *u satisfies* (2.11) *for all $t \geq 0$ and all $x \in \Omega$,*
- *u satisfies* (2.13) *for all $x \in \Omega$,*
- *u satisfies the imposed boundary condition for all $t \geq 0$ and al $z \in \partial\Omega$.*

A possible and common approach to solve the 1-dimensional wave equations is based on the method of characteristics. The basic idea is the formal factorisation

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u$$

of the wave equation (observe that this solely holds if c is a constant), which suggests to introduce the unknown v defined by

$$v(t,x) := \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u(t,x), \qquad t \geq 0,\ x \in \mathbb{R},$$

and to study the PDE

$$\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\right)v(t,x)=0, \qquad t\geq 0,\ x\in\mathbb{R},$$

in the unknown v. The latter equation is just a linear transport equation. The solution to this equation has been obtained in Chapter 2: it is given by

$$v(t,x)=\tilde{\gamma}(x-ct), \qquad t\geq 0\ x\in\mathbb{R},$$

where $\tilde{\gamma}(\cdot)$ is the (not explicitly known) function defining the initial data of v, i.e., $v(0,\cdot)$. In particular, by definition of v we get

$$v(t,x)=\tilde{\gamma}(x-ct)=\frac{\partial u}{\partial t}-c\frac{\partial u}{\partial x}, \qquad t\geq 0,\ x\in\mathbb{R},$$

which is an inhomogeneous transport equation in the unknown u with inhomogeneous term $\tilde{\gamma}$. By Theorem 2.2 we obtain (due to $b=-1$)

$$\begin{aligned}
u(t,x) &= u_0(x+ct)+\int_0^t v(\tau,x+c(t-\tau))d\tau\\
&= u_0(x+ct)+\int_0^t \tilde{\gamma}(x+c(t-\tau)-c\tau)d\tau\\
&= u_0(x+ct)+\int_0^t \tilde{\gamma}(x+ct-2c\tau)d\tau\\
&= u_0(x+ct)+\frac{1}{2c}\int_{x-ct}^{x+ct}\tilde{\gamma}(y)dy.
\end{aligned}$$

We still have to determine $\tilde{\gamma}$: the second boundary condition and the definition of v imply

$$\tilde{\gamma}(x)=v(0,x)=\frac{\partial u}{\partial t}(0,x)-c\frac{\partial u}{\partial x}(0,x)=u_1(x)-cu_0'(x),$$

whence

$$u(t,x)=u_0(x+ct)+\frac{1}{2}\int_{x-ct}^{x+ct}\big(u_1(y)-u_0'(y)\big)\,dy.$$

Accordingly, we have proved the following.

Theorem 2.12. *Let $u_0\in C^2(\mathbb{R})$ and $u_1\in C^1(\mathbb{R})$. If the wavespeed c is constant, then the initial value problem for the 1-dimensional wave equation on $\mathbb{R}$ has a solution: this is given by*

$$u(t,x)=\frac{1}{2}\left(u_0(x+ct)+u_0(x-ct)\right)+\frac{1}{2c}\int_{x-ct}^{x+ct}u_1(y)dy, \qquad t\geq 0,\ x\in\mathbb{R}.$$

We will see that the possibility of writing the solution as the sum of two terms, each depending only on one of the initial conditions, is a typical feature of the wave equation. These are often referred to as "cosine" and "sine" term, respectively. This is known as **D'Alembert formula**. Sometimes one writes it as

$$u(t,x):=\big(C(t)u_0\big)(x)+\big(S(t)u_1\big)(x),$$

where (in this 1-dimensional case) the transformations $C(t)$ and $S(t)$, $t\geq 0$, are function-valued functions defined by

$$\big(C(t)u_0\big)(x):=\frac{1}{2}\left(u_0(x+ct)+u_0(x-ct)\right)$$

and

$$\big(S(t)u_1\big)(x) := \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(y)dy.$$

In other words, we can represent all solutions as the sum of solutions to the wave equation with initial data $u_0, u_1 \equiv 0$ (given by $u(t,x) = \big(C(t)u_0\big)(x)$) and $u_0 \equiv 0, u_1$ (given by $u(t,x) = \big(S(t)u_1\big)(x)$), respectively. We will see that this is a general feature of the wave equation.

Exercise 2.13. *Show that $C(0)f \equiv f$ and moreover $2C(t)\left(C(s)f\right) \equiv (C(t+s)f) + (C(t-s)f)$ as well as $(S(t+s)f) \equiv C(s)(S(t)f) + (S(s)C(t)f)$ for all $t, s \geq 0$ and all functions $f \in C^1$. Why are C and S called "cosine" and "sine", respectively?*

Exercise 2.14. *Let $A, k, \omega \in \mathbb{R}$. Show that both functions u, v defined by*

$$u(t,x) := A\sin(kx - \omega t) \qquad \text{and} \qquad v(t,x) := A\sin(kx + \omega t), \qquad t, x \in \mathbb{R}$$

are solutions to the 1-dimensional wave equation on $\mathbb{R}$. They are called **travelling waves**. *Due to linearity, their sum is also a solution to the wave equation. Show that, however $u + v$ is not a travelling wave, and that in fact it can be written as a product $\eta \cdot \xi$ of two functions, where η only depends on time and ξ only depends on space. Such a solution is called a* **stationary wave**.

Exercise 2.15. *One may try to apply the method of characteristics directly. In this case, one should introduce a change of variables setting $\lambda := x - ct$ and $\mu := x + ct$ and consider those curves along which the solutions to the wave equation are not constant (like in the case of the transport equation), but rather satisfy*

$$\frac{\partial^2 u}{\partial \lambda \partial \mu} \equiv 0.$$

Work out the details.

Exercise 2.16. *Discuss the wave-type partial differential equation*

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) - x\frac{\partial u}{\partial x}(t,x), \qquad t \geq 0,\ x \in \mathbb{R},$$

with wavespeed given by $c(x) = x$.

(Hint: try to perform a factorisation similar to that crucial in order to solve the standard wave equation.)

Remark 2.17. *Observe that even if we were only looking for a* **forward solution** *of the wave equation, the solution we have derived is also a* **backward** *one, i.e., it is a function that solves the wave equation also for negative times: knowing the state of a wave at a certain time t_0, it is possible to reconstruct the state of the wave at any given previous time $t < t_0$.*

We have derived an explicit solution. Would it be possible to have more solutions, at least for some special initial data? No, it would not. This is the first application of the so-called *variational method*, which we will more extensively discuss later on.

Theorem 2.18. *Let I be a (possibly unbounded) open interval of $\mathbb{R}$. Let $u_0 \in C^2(\overline{I})$ and $u_1 \in C^1(\overline{I})$. Then the initial value problem for the 1-dimensional wave equation on I has at most one solution in the class of those functions that are such that are unifomly bounded on any compact domain, provided that*

- *either $I = \mathbb{R}$*
- *or the boundary condition is of Dirichlet or Neumann type (possibly inhomogeneous), if $I \neq \mathbb{R}$ has a boundary.*

Beweis. Assume u, v to be two solutions to the equation. Due to linearity, also $w := u - v$ solves the 1-dimensional wave equation with initial conditions

$$w(0,x) = 0, \qquad \frac{\partial w}{\partial t}(0,x) = 0, \qquad x \in I.$$

Moreover, since u and v agree at the boundary, their difference (or the difference of their normal derivatives) vanishes identically at the boundary (if there is a boundary), *constantly in time*. We introduce a function $E : \mathbb{R}_+ \to \mathbb{R}$ by

$$E(t) := \frac{1}{2}\int_0^1 \left|\frac{\partial w}{\partial t}(t,x)\right|^2 + \left|c\frac{\partial w}{\partial x}(t,x)\right|^2 dx.$$

We differentiate this expression and obtain (integrating by parts)

$$\begin{aligned} E'(t) &= \int_0^1 \frac{\partial w}{\partial t}(t,x)\frac{\partial^2 w}{\partial t^2}(t,x)dx + \int_0^1 c^2\frac{\partial w}{\partial x}(t,x)\frac{\partial^2 w}{\partial x \partial t}(t,x)dx \\ &= \int_0^1 \frac{\partial w}{\partial t}(t,x)\frac{\partial^2 w}{\partial t^2}(t,x) + c^2\frac{\partial w}{\partial x}(t,x)\frac{\partial w}{\partial t}(t,x)\Big|_{x=0}^{x=1} - \int_0^1 c^2\frac{\partial^2 w}{\partial x^2}(t,x)\frac{\partial w}{\partial t}(t,x)dx \\ &= \int_0^1 \frac{\partial w}{\partial t}(t,x)\left(\frac{\partial^2 w}{\partial t^2}(t,x) - c^2\frac{\partial^2 w}{\partial x^2}(t,x)\right) + c^2\frac{\partial w}{\partial x}(t,x)\frac{\partial w}{\partial t}(t,x)\Big|_{x=0}^{x=1} \\ &= 0. \end{aligned}$$

The last identity holds because, as observed above, the boundary term vanishes identically at the boundary. We conclude that E is a constant function, i.e., $E(t) = E(0) = 0$, and therefore

$$\left|\frac{\partial w}{\partial t}(t,x)\right| = \left|\frac{\partial w}{\partial x}(t,x)\right| = 0$$

for all $t \geq 0$ and all $x \in I$, i.e., w is constant in time and in space. Because w vanishes constantly at the boundary and also everywhere for $t = 0$, we deduce that $w \equiv 0$ everywhere and at any time. This concludes the proof. □

The above proof motivates the introduction of the following.

Definition 2.19. *The* **total energy** *of a wave on a domain $\Omega \subset \mathbb{R}^d$ is given at any time t by the sum*

$$E := E_p + E_k$$

of its **potential energy**

$$E_p(t) := \int_\Omega |\nabla u(t,x)|^2 dx$$

and its **kinetic energy**

$$E_k(t) := \int_\Omega \left|\frac{\partial u}{\partial t}(t,x)\right|^2 dx,$$

where u denotes the solution to the wave equation on Ω.

We conclude this section by discussing an extension of the D'Alembert formula to the case of the semi-bounded domain $(0, \infty)$. This explains the method of reflections that is typical for the wave equation. It will be used in the next chapter. For the sake of simplicity we consider only the case of unitary wavespeed, the general one being analogous.

Theorem 2.20. *Let $c = 1$. For $u_0 \in C^2([0,\infty))$ and $u_1 \in C^1([0,\infty))$ the solution to the wave equation for $(t,r) \in \mathbb{R}_+ \times (0,\infty)$ with Dirichlet boundary condition*

$$u(t,0) = 0, \qquad t \geq 0,$$

is given by

$$u(t,x) := \begin{cases} \frac{1}{2}\left(u_0(x+t) + u_0(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)dy, & x \geq t \geq 0, \\ \frac{1}{2}\left(u_0(x+t) - u_0(t-x)\right) + \frac{1}{2}\int_{t-x}^{x+t} u_1(y)dy, & t \geq x \geq 0. \end{cases}$$

Observe that the above formula yields a function u that is twice continuously differentiable everywhere apart from $t = x$. However, if one is interested in the classical notion of solution, continuous differentiability in $t = x$ becomes an issue and we have to impose the Wentzell boundary condition on u_0 at 0. In particular, the second derivative (both with respect to space and to time) of u has a jump equal to $2u_0''(0)$.

Beweis. Since we have a solution formula at hand for the wave equation on $\mathbb{R}$, it is natural to try to extend the wave equation on $(0,\infty)$ to the whole line. This is simply done by introducing the odd extension of u

$$\tilde{u}(t,x) := \begin{cases} u(t,x), & t \geq 0,\ x \geq 0, \\ -u(t,-x), & t \geq 0,\ x \leq 0, \end{cases}$$

i.e., for all $t \geq 0$ $\tilde{u}(t,\cdot)$ is the unique odd function that agrees with u on $\mathbb{R}$. Similarly, we introduce the odd functions

$$\tilde{u}_0(x) := \begin{cases} u_0(x), & x \geq 0, \\ -u_0(-x), & x \leq 0, \end{cases}$$

and

$$\tilde{u}_1(x) := \begin{cases} u_1(x), & x \geq 0, \\ -u_1(-x), & x \leq 0. \end{cases}$$

Let us prove that if u is a solution to the 1-dimensional wave equation on $(0,\infty)$ with Dirichlet boundary condition, then $\tilde{u}$ solves the 1-dimensional wave equation on $\mathbb{R}$ (clearly, *without* boundary conditions). This is due to the fact that $\tilde{u}$ satisfies the differential equation on $(0,\infty)$, where it agrees with u, and that moreover for all $\tilde{x} := -x$, $x \geq 0$, there holds

$$\begin{aligned} \frac{\partial^2 \tilde{u}}{\partial t^2}(t,\tilde{x}) &= -\frac{\partial^2 u}{\partial t^2}(t,x) \\ &= -\frac{\partial^2 u}{\partial x^2}(t,x) \\ &= \frac{\partial^2 u}{\partial x^2}(t,\tilde{x})\frac{\partial \tilde{x}}{\partial x} \\ &= \frac{\partial^2 u}{\partial \tilde{x}^2}(t,\tilde{x}), \end{aligned}$$

i.e., $\tilde{u}$ solves the wave equation on the whole line. Moreover

$$u(0,\tilde{x}) = -u(0,x) = -u_0(x) = -\tilde{u_0}(\tilde{x})$$

and likewise

$$\frac{\partial u}{\partial t}(0,\tilde{x}) = -u(0,x) = -u_1(x) = -\tilde{u_1}(\tilde{x}).$$

Thus, $\tilde{u}$ is given by the D'Alembert formula, from where the claimed identities hold. $\square$

Remark 2.21. *A relevant consequence of the above D'Alembert formula is that even for positive-valued* u_0 *and* $u_1 \equiv 0$ *the solution needs not be positive for all* $t \geq 0$. *This is in sharp contrast to the behaviour of the transport equation – and, as we will see, of the heat equation, too.*

Exercise 2.22. *A similar strategy can also be applied to find the solution to the wave equation on a bounded interval, say, on* $(0, 1)$, *with Dirichlet boundary condition*

$$u(t, 0) = u(t, 1) = 0, \qquad t \geq 0.$$

The idea is that the initial conditions have to be reflected and *extended by periodicity. Work out the details.*

Exercise 2.23. *While odd extensions are useful to solve the 1-dimensional wave equation with the Dirichlet boundary condition, even extensions come into play when the Neumann boundary condition*

$$u'(t, 0) = u'(t, 1) = 0, \qquad t \geq 0,$$

is considered. Deduce an explicit formula for the solution to this wave equation. Is the solution positive-valued, provided the initial data are?

KAPITEL 3

The wave equation in higher dimensions and the method of descent

In this chapter we discuss the d-dimensional generalization of the wave equation, which appears in several problems of theoretical acoustics or electromagnetism. We follow the classical approach as presented, e.g., in [**5**, §2.4]. In order to obtain an explicit formula for the solutions to such wave equations, our strategy will be the following:

- We show that certain spherical means w of any solution u to the d-dimensional (d odd!) wave equation solve a modified 1-dimensional wave equation – the so-called Euler–Poisson–Darboux equation (Lemma 3.5.
- While w only solves the Euler–Poisson–Darboux equation, one can rescale it and introduce a function U that solves the (classical) 1-dimensional wave equation (Lemma 3.8).
- Because the explicit solution to the 1-dimensional wave equation is known given (locally) by the D'Alembert formula, we deduce a formula for U that, in turn, yields a formula for u in terms of spherical means of the initial data (Theorem 3.9).
- The case of d even is solved by the method of descent: any solution u to the $(d-1)$-dimensional wave equation (d even) is embedded in a solution $\tilde{u}$ to the d-dimensional wave equation in a suitable way, and then the above mentioned results yield a formula for $\tilde{u}$ and thus, in turn, for u. The details are carried out for the 2-dimensional case only (Theorem 3.11).

In the above strategy there is a hidden assumption, namely that to each initial value problem for the heat equation there is at most one solution (hence exactly one in the special cases where an explicit formula is known): in the 1-dimensional case this assumption is justified by Theorem 2.18, but an analogous assertion also holds in the general d-dimensional case.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^d$ be a (possibly unbounded) domain such that the Gauß–Green formulae hold. Let $u_0 \in C^2(\overline{\Omega})$ and $u_1 \in C^1(\overline{\Omega})$. Then the initial value problem for the d-dimensional wave equation on $\mathbb{R}^d$ has at most one solution (in the class of those solutions that are uniformly bounded on compact domains) provided that*

- *either $\Omega = \mathbb{R}^d$*
- *or the boundary condition is of Dirichlet or Neumann type (possibly inhomogeneous), if Ω has a boundary.*

Exercise 3.2. *Prove Theorem 3.1.*

In order to solve the wave equation in $\mathbb{R}^d$ for $d \geq 2$ we first discuss a modification of the 1-dimensional wave equation. Here and in the following we denote by $B_r(x)$ the d-dimensional ball of radius r centered at x, by $\partial B_r(x)$ its surface (i.e., its topological boundary), and by $|B_r| := |B_r(x)|$ and $|\partial B_r| := |\partial B_r(x)|$ their respective measures, i.e.,

$$|B_r| = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)} r^d \qquad \text{and} \qquad |\partial B_r| = \frac{d\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)} r^{d-1},$$

where Γ denotes the gamma-function. Here and in the following, $d\sigma$ denotes the $(d-1)$-dimensional Hausdorff (surface) measure. An important feature of the above formulae is that

$$\frac{d}{dr}|B_r| = |\partial B_r| \qquad \text{and} \qquad \frac{d}{dr}\frac{1}{|\partial B_r|} = \frac{1-d}{d}\frac{1}{|B_r|}, \qquad r > 0. \tag{3.1}$$

Lemma 3.3. *Let $u \in C^2(\mathbb{R}^d)$. Then for all $x \in \mathbb{R}^d$ the function ϕ defined by*

$$\phi(r) := \frac{1}{|\partial B_r|}\int_{\partial B_r(x)} u(y)d\sigma(y)$$

is continuously differentiable and

$$\frac{\partial \phi}{\partial r}(r) = \frac{r}{d}\frac{1}{|B_r|}\int_{B_r(x)} \Delta u(y)dy.$$

Beweis. Substituting $y \mapsto \tilde{y} := x + rz$ yields

$$\phi(r) = \frac{1}{|\partial B_1|}\int_{\partial B_1(0)} u(x+rz)d\sigma(z).$$

Now, the chain rule yields

$$\begin{aligned} \phi'(r) &= \frac{1}{|\partial B_1|}\int_{\partial B_1(0)} \nabla u(x+rz)\cdot z d\sigma(z) \\ &= \frac{1}{|\partial B_r|}\int_{\partial B_r(x)} \nabla u(y)\cdot \frac{y-x}{r} d\sigma(y) \\ &= \frac{1}{|\partial B_r|}\int_{\partial B_r(x)} \frac{\partial u}{\partial n}(y)d\sigma(y) \\ &= \frac{r}{d}\frac{1}{|B_r|}\int_{B_r(x)} \Delta u(y)dy, \end{aligned}$$

where the last identity follows from Green's formula. This completes the proof. □

Exercise 3.4. *Let $u \in C^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Determine*

- $\frac{\partial}{\partial r}\left(\frac{1}{|B_r|}\int_{\partial B_r(x)} u(y)d\sigma(y)\right)$,
- $\frac{\partial}{\partial r}\left(\frac{1}{|\partial B_r|}\int_{B_r(x)} u(y)dy\right)$, *and*
- $\frac{\partial}{\partial r}\left(\frac{1}{|B_r|}\int_{B_r(x)} u(y)dy\right)$,

for $r > 0$. Are you able to find a general formula for

$$\frac{\partial^k}{\partial r^k}\phi(r)?$$

Lemma 3.5. *Let $u_0 \in C^2(\mathbb{R})$ and $u_1 \in C^1(\mathbb{R})$. Let $u \in C^m([0,\infty)\times\mathbb{R}^d)$ be a solution to the d-dimensional wave equation (2.12) with initial conditions*

$$u(0,x) = u_0(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(0,x) = u_1(x), \qquad x \in \mathbb{R}.$$

Then, for any $x \in \mathbb{R}^d$

$$w(t,r,x) := \frac{1}{|\partial B_r|}\int_{\partial B_r(x)} u(t,y)d\sigma(y), \qquad t \geq 0,\ x \in \mathbb{R},$$

defines a solution to the so-called **Euler–Poisson–Darboux equation**

$$\frac{\partial^2 w}{\partial t^2}(t,r,x) = \frac{\partial^2 w}{\partial r^2}(t,r,x) + \frac{d-1}{r}\frac{\partial w}{\partial r}(t,r,x), \qquad t \geq 0,\ r \geq 0,$$

with initial data defined for all $x \in \mathbb{R}$ *by*

$$w(0,r,x) = w_0(r,x) := \frac{1}{|\partial B_r|}\int_{\partial B_r(x)} u_0(y)d\sigma(y)$$

and

$$\frac{\partial w}{\partial t}(0,r,x) = w_1(r,x) := \frac{1}{|\partial B_r|}\int_{\partial B_r(x)} u_1(y)d\sigma(y).$$

Moreover, $w(\cdot,\cdot,x) \in C^m([0,\infty)\times[0,\infty))$ *for all* $x \in \mathbb{R}^d$.

Beweis. The definition of w does not modify the dependence on t of u, and we conclude that w is m-times continuously differentiable in its first variable, provided that u is.

Now, apply Lemma 3.3 and observe that for all $r > 0$

$$\frac{\partial w}{\partial r}(t,r,x) = \frac{\partial}{\partial r}\frac{1}{|\partial B_r|}\int_{\partial B_r(x)} u(t,y)d\sigma(y) = \frac{r}{d}\frac{1}{|B_r|}\int_{B_r(x)}\Delta u(t,y)dy = \frac{1}{|\partial B_r|}\int_{B_r(x)}\Delta u(t,y)dy. \tag{3.2}$$

Hence, $w(\cdot,\cdot,x) \in C^1([0,\infty)\times(0,\infty))$. Passing to the limit for $r \to 0$, by Lebesgue's differentiation theorem we obtain

$$\lim_{r\to 0}\frac{\partial w}{\partial r}(t,r,x) = 0:$$

thus, $w(\cdot,\cdot,x) \in C^1([0,\infty)\times[0,\infty))$. One can further differentiate and apply (3.2) and (3.1) in order to obtain

$$\begin{aligned}
\frac{\partial^2 w}{\partial r^2}(t,r,x) &= \frac{\partial^2}{\partial r^2}\frac{1}{|\partial B_r|}\int_{\partial B_r(x)} u(t,y)d\sigma(y)\\
&= \frac{\partial}{\partial r}\left(\frac{1}{|\partial B_r|}\int_{B_r(x)}\Delta u(t,y)dy\right)\\
&= \frac{1-d}{d}\frac{1}{|B_r|}\int_{B_r(x)}\Delta u(t,y)dy + \frac{1}{|\partial B_r|}\frac{\partial}{\partial r}\int_0^r\int_{\partial B_s(x)}\Delta u(t,y)d\sigma(y)\\
&= \frac{1-d}{d}\frac{1}{|B_r|}\int_{B_r(x)}\Delta u(t,y)dy + \frac{1}{|\partial B_r|}\int_{\partial B_r(x)}\Delta u(t,y)d\sigma(y).
\end{aligned}$$

Therefore, using (3.2) and the fact that u solves the wave equation we deduce

$$\begin{aligned}
\frac{\partial^2 w}{\partial r^2}(t,r,x) &= -\frac{d-1}{d}\frac{d}{r}\frac{1}{|\partial B_r|}\int_{B_r(x)}\Delta u(t,y)dy + \frac{1}{|\partial B_r|}\int_{\partial B_r(x)}\frac{\partial u}{\partial t^2}(t,y)d\sigma(y)\\
&= -\frac{d-1}{r}\frac{\partial w}{\partial r}(t,r,x) + \frac{\partial^2 w}{\partial t^2}(t,r,x).
\end{aligned}$$

Thus, w is actually a solution to the Euler–Poisson–Darboux equation.

It remains to check that $\frac{\partial^2 w}{\partial r^2}$ is continuous up to the boundary, i.e., up to $r = 0$. Just like before, passing to the limit for $r \to 0$ yields

$$\lim_{r\to 0}\frac{\partial^2 w}{\partial r^2}(t,r,x) = \Delta u(t,x) + \left(\frac{1}{d} - 1\right)\Delta u(t,x) = \frac{1}{d}\Delta u(t,x).$$

One can further differentiate and obtain recursively that in the radial direction w is as regular as u. □

Remark 3.6. *The Euler–Poisson–Darboux is a wave equation whose energy increases in time, cf. Remark 3.16, since the damping term $\frac{d-1}{x}\frac{\partial w}{\partial x}$ has a positive-valued coefficient. It probably gives more insight to observe that the term $\frac{\partial^2 w}{\partial x^2} + \frac{d-1}{x}\frac{\partial w}{\partial x}$ that appears in the Euler–Poisson–Darboux equation agrees with the radial compoonent of Δw, whenever we write it in spherical coordinates.*

In the following we are going to transform the odd- and even-dimensional wave equation into the 1-dimensional wave equation, via the formalism introduced to study the Euler–Poisson–Darboux equation. Since a solution to the 1-dimensional wave equation is known, we will be able to find general solutions in these higher-dimensional cases. Explicit formulae are also known for the general d-dimensional case.

To begin with, we consider the case of d odd, i.e., $d = 2k+1$. In the following, $n!!$ denotes the **double factorial** of an integer n, i.e.,

$$n!! := \begin{cases} n \cdot (n-2) \cdot \ldots 2 & \text{if } n \text{ is even,} \\ n \cdot (n-2) \cdot \ldots 1 & \text{if } n \text{ is odd.} \end{cases}$$

Lemma 3.7. *Let $k \in \mathbb{N}$. Then the following identities hold for all $\phi \in C^{k+1}(\mathbb{R})$.*

(1) $\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right) = (2k-1)!!\, r\phi(r) + \sum_{j=1}^{k-1} \beta_j^k r^{j+1} \frac{d^j\phi}{dr^j}(r)$, *for some constants β_j^k independent on ϕ.*

(2) $\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right) = \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k}\frac{d\phi}{dr}(r)\right)$.

Here

$$\left(\frac{1}{r}\frac{d}{dr}\right)^0 f = f, \quad \left(\frac{1}{r}\frac{d}{dr}\right)^1 f = \frac{1}{r}f', \quad \left(\frac{1}{r}\frac{d}{dr}\right)^2 f = \frac{1}{r}\left(\frac{1}{r}f'\right)' = -\frac{1}{r^3}f' + \frac{1}{r^2}f''$$

and so on.

Beweis. Both identities are proven by induction on k.

1) The assertion is clearly true for $k = 1$, since both left and right hand sides agree with $r\phi(r)$. If the assertion holds for k, then we see that

$$\begin{aligned} \left(\frac{1}{r}\frac{d}{dr}\right)^k \left(r^{2k+1}\phi(r)\right) &= \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(\frac{1}{r}\frac{d}{dr}\right)\left(r^{2k+1}\phi(r)\right) \\ &= \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\left((2k+1)\phi(r) + r\frac{d\phi}{dr}(r)\right)\right) \\ &:= \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\tilde{\phi}(r)\right). \end{aligned}$$

Apply the induction assumption to $\tilde{\phi}(r)$ and obtain

$$\begin{aligned} \left(\frac{1}{r}\frac{d}{dr}\right)^k \left(r^{2k+1}\phi(r)\right) &= (2k-1)!!\, \tilde{\phi}(r) + \sum_{j=1}^{k-1} \beta_j^k r^{j+1} \frac{d^j}{dr^j}\tilde{\phi}(r) \\ &= (2k+1)!!\, r\phi(r) + (2k-1)!! r^2 \frac{d\phi}{dr}(r) + \\ &\quad + \sum_{j=1}^{k-1}(2k+1)\beta_j^k r^{j+1}\frac{d^j\phi}{dr^j}(r) + \sum_{j=1}^{k-1}\beta_j^k r^{j+1}\frac{d^j}{dr^j}\left(r\frac{d\phi}{dr}(r)\right). \end{aligned}$$

Now, it is immediate (induction!) that for any function $f \in C^j$ there holds

$$\frac{d^j}{dr^j}(rf(r)) = r\frac{d^j f}{dr^j}(r) + j\frac{d^{j-1}f}{dr^{j-1}}(r), \qquad j \in \mathbb{N}.$$

Therefore, for $f = \frac{\partial \phi}{\partial r}$ we obtain

$$\begin{aligned}
\left(\frac{1}{r}\frac{d}{dr}\right)^k \left(r^{2k+1}\phi(r)\right) &= (2k+1)!!\, r\phi(r) + (2k-1)!! r^2 \frac{d\phi}{dr}(r) + \\
&\quad + \sum_{j=1}^{k-1}(2k+1)\beta_j^k r^{j+1}\frac{d^j\phi}{dr^j}(r) + \sum_{j=1}^{k-1}\beta_j^k r^{j+1}\left(r\frac{d^{j+1}\phi}{dr^{j+1}}(r) + j\frac{d^j\phi}{dr^j}(r)\right) \\
&= (2k+1)!!\, r\phi(r) + (2k-1)!! r^2 \frac{d\phi}{dr}(r) + \\
&\quad + \sum_{j=1}^{k-1}(2k+1)\beta_j^k r^{j+1}\frac{d^j\phi}{dr^j}(r) + \sum_{j=2}^{k}\beta_{j-1}^k r^{j+1}\frac{d^j\phi}{dr^j}(r) + \sum_{j=1}^{k-1} j\beta_j^k r^{j+1}\frac{d^j\phi}{dr^j}(r) \\
&= (2k+1)!!\, r\phi(r) + \sum_{j=1}^{k}\beta_j^{k+1} r^{j+1}\frac{d^j\phi}{dr^j}(r),
\end{aligned}$$

where the explicit definition of the constants β_j^{k+1} is left as an easy exercise.

2) The assertion is true for k=1, because

$$\frac{d^2}{dr^2}(r\phi(r)) = \frac{d}{dr}\left(\phi(r) + r\frac{d\phi}{dr}(r)\right) = 2\frac{d\phi}{dr}(r) + r\frac{d^2\phi}{dr^2}(r) = \frac{1}{r}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}(r)\right), \qquad r \in \mathbb{R}.$$

Let now assume that the assertion holds for k. Then

$$\begin{aligned}
\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^k \left(r^{2k+1}\phi(r)\right) &= \frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(\frac{1}{r}\frac{d}{dr}\right)\left(r^{2k+1}\phi(r)\right) \\
&= \frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left((2k+1)r^{2k-1}\phi(r) + r^{2k}\frac{d\phi}{dr}(r)\right) \\
&= \ldots \\
&= \left(\frac{1}{r}\frac{d}{dr}\right)^{k+1}\left(r^{2k+2}\frac{d\phi}{dr}(r)\right)
\end{aligned}$$

The missing step is left as an exercise to the reader. $\square$

Lemma 3.8. *If u is a solution to the initial value problem*

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x), & t > 0,\ x \in \mathbb{R}^{2k+1}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^{2k+1}, \\ \frac{\partial u}{\partial x}(0,x) = u_1(x), & x \in \mathbb{R}^{2k+1}, \end{cases} \tag{3.3}$$

then for all $x \in \mathbb{R}^{2k+1}$ the function U defined by $U(t,r,x) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}w(t,r,x)\right)$, i.e.,

$$U(t,r,x) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(\frac{r^{2k-1}}{|\partial B_r|}\int_{\partial B_r(x)} u(t,y)d\sigma(y)\right), \tag{3.4}$$

solves the 1*-dimensional wave equation for* $(t,r) \in (0,\infty) \times (0,\infty)$ *with Dirichlet boundary condition*

$$U(t,0,x) = 0, \qquad t > 0,$$

and with initial data

$$U(0,r,x) = G(r,x), \qquad \frac{\partial U}{\partial t}(0,r,x) = H(r,x), \qquad r \in (0,\infty),$$

where

$$\begin{aligned} G(r,x) &:= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1} w_0(r,x)\right) \\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(\frac{r^{2k-1}}{|\partial B_r|}\int_{\partial B_r(x)} u_0(r,y)d\sigma(y)\right) \end{aligned}$$

and

$$\begin{aligned} H(r,x) &:= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1} w_1(r,x)\right) \\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(\frac{r^{2k-1}}{|\partial B_r|}\int_{\partial B_r(x)} u_1(y)d\sigma(y)\right). \end{aligned}$$

Beweis. For fixed $x \in \mathbb{R}^{2k+1}$, let $r > 0$ and $t \geq 0$. Then it is possible to apply Lemma 3.7.(2) and obtain

$$\begin{aligned} \frac{\partial^2 U}{\partial r^2}(t,r,x) &= \frac{\partial^2}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1} w(t,r,x)\right) \\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k} \left(r^{2k}\frac{\partial w}{\partial r}(t,r,x)\right) \\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(\frac{1}{r}\frac{\partial}{\partial r}\right) \left(r^{2k}\frac{\partial w}{\partial r}(t,r,x)\right) \\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1}\frac{\partial^2 w}{\partial r^2}(t,r,x) + 2kr^{2k-2}\frac{\partial w}{\partial r}(t,r,x)\right) \\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1}\left(\frac{\partial^2 w}{\partial r^2}(t,r,x) + \frac{2k}{r}\frac{\partial w}{\partial r}(t,r,x)\right)\right) \\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1}\frac{\partial^2 w}{\partial t^2}(t,r,x)\right) \\ &= \frac{\partial^2 U}{\partial t^2}(t,r,x), \end{aligned}$$

where of course the last step is due to the fact that w solves the Euler–Poisson–Darboux equation.

Finally, the fact that U satisfies the initial conditions is a direct consequence of the fact that w satisfies the initial conditions of the Euler–Poisson–Darboux equation, which in turn is just an application of the definition of w. □

We are finally in the position to solve the initial value problem associated with the odd-dimensional wave equation, in a suitable sense.

If u is a solution to the $(2k+1)$-dimensional wave equation, then U defined in (3.4) is a solution to the 1-dimensional one. Thus, by the D'Alembert formula (suitably localised in time), we deduce that U is given by

$$U(t,r,x) = \frac{1}{2}\left(G(t+r,x) - G(t-r,x)\right) + \frac{1}{2}\int_{t-r}^{t+r} H(y,x)dy, \qquad t \geq 0,\ r \in [0,t].$$

It remains to re-write this expression in terms of u, u_0 and u_1. To this aim, we want to consider the above formula in the limit $r \to 0+$. Then, by Lebesgue's differentiation theorem, Lemma 3.7.(1) yields

$$\begin{aligned}
u(t,x) &= \lim_{r\to 0+} w(t,r,x) \\
&= \lim_{r\to 0+} w(t,r,x) + \lim_{r\to 0+} \sum_{j=1}^{k-1} \frac{\beta_j^k}{(2k-1)!!} r^j \frac{\partial^j w}{\partial r^j}(t,r,x) \\
&= \lim_{r\to 0+} \frac{1}{(2k-1)!!\, r} \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1} w(t,r,x)\right) \\
&= \lim_{r\to 0+} \frac{U(t,r,x)}{(2k-1)!!\, r} \\
&= \frac{1}{(2k-1)!!} \lim_{r\to 0+} \left(\frac{G(t+r,x) - G(t-r,x)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r} H(y,x)dy\right) \\
&= \frac{1}{(2k-1)!!}\left(G'(t) + H(t)\right).
\end{aligned}$$

Thus, if the $(2k+1)$-dimensional wave equation has a solution, it must have the above form. In order to check that u defined above actually solves the equation, we proceed as follows.

Theorem 3.9. *Let $k \in \mathbb{N}$. For $u_0 \in C^{k+2}(\mathbb{R}^{2k+1})$ and $u_1 \in C^{k+1}(\mathbb{R}^{2k+1})$, define a function u by*

$$\begin{aligned}
u(t,x) \;:=\; & \frac{1}{(2k-1)!!}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(\frac{t^{2k-1}}{|\partial B_t|}\int_{\partial B_t(x)} u_0(z)d\sigma(z)\right) \\
& + \frac{1}{(2k-1)!!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(\frac{t^{2k-1}}{|\partial B_t|}\int_{\partial B_t(x)} u_1(z)d\sigma(z)\right), \qquad t > 0,\ x \in \mathbb{R}^{2k+1}.
\end{aligned}$$

Then $u \in C^{2,2}(\mathbb{R}_+ \times \mathbb{R}^{2k+1})$. Moreover, u is a solution to the $(2k+1)$-dimensional wave equation away from $t = 0$ (where u is not defined). However, $\lim_{t\to 0+} u(t,x_0)$ and $\lim_{t\to 0+} \frac{\partial u}{\partial t}(t,x_0)$ exist for all $x_0 \in \mathbb{R}^{2k+1}$, and in fact

$$\lim_{(t,x)\to(0,x_0)} u(t,x) = u_0(x_0) \qquad \text{and} \qquad \lim_{(t,x)\to(0,x_0)} \frac{\partial u}{\partial t}(t,x) = u_1(x_0)$$

for all $x_0 \in \mathbb{R}^{2k+1}$.

Beweis. We exploit linearity of the wave equation: in fact, we check separately the two cases

- $u_0 \equiv 0$ and u_1 arbitrary,
- u_0 arbitrary and $u_1 \equiv 0$.

In the former case we obtain that

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(t,x) &= \frac{1}{(2k-1)!!}\frac{\partial^2}{\partial t^2}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(\frac{t^{2k-1}}{|\partial B_t|}\int_{\partial B_t(x)} u_1(z)d\sigma(z)\right)\\
&= \frac{1}{(2k-1)!!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k}\left(t^{2k}\frac{\partial}{\partial t}\frac{1}{|\partial B_t|}\int_{\partial B_t(x)} u_1(z)d\sigma(z)\right)\\
&= \frac{1}{(2k-1)!!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k}\left(\frac{t^{2k}}{|\partial B_t|}\int_{B_t(x)} \Delta u_1(y)dy\right)
\end{aligned}$$

where the second-last equality follows from Lemma 3.7.(2) and the last one from Lemma 3.3. Observe that

$$\alpha_k := \frac{t^{2k}}{|\partial B_t|}$$

only depends on k, but not on t, hence

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(t,x) &= \frac{\alpha_k}{(2k-1)!!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k}\left(\alpha_k\int_{B_t(x)} \Delta u_1(y)dy\right)\\
&= \frac{1}{(2k-1)!!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(\frac{\alpha_k}{t}\frac{\partial}{\partial t}\int_{B_t(x)} \Delta u_1(y)dy\right)\\
&= \frac{1}{(2k-1)!!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(\frac{\alpha_k}{t}\frac{\partial}{\partial t}\int_0^t\int_{\partial B_s(x)} \Delta u_1(y)d\sigma(y)ds\right)\\
&= \frac{1}{(2k-1)!!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(\frac{\alpha_k}{t}\int_{\partial B_t(x)} \Delta u_1(y)dy\right)\\
&= \frac{1}{(2k-1)!!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(\frac{\alpha_k}{t}\int_{\partial B_t(0)} \Delta u_1(x+y)dy\right)\\
&= \Delta\left(\frac{1}{(2k-1)!!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(\frac{\alpha_k}{t}\int_{\partial B_t(0)} u_1(x+y)dy\right)\right)\\
&= \Delta\left(\frac{1}{(2k-1)!!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(\frac{t^{2k-1}}{|\partial B_t|}\int_{\partial B_t(x)} u_1(y)dy\right)\right)\\
&= \Delta u(t,x).
\end{aligned}$$

The second case is similar. We left as an exercise to check that the claimed solution also satisfies the initial data. □

Exercise 3.10. *Check that the function u defined in Theorem 3.9 is actually a solution to the $(2k+1)$-dimensional wave equation.*

Let us now consider the case of even dimension. The main idea is that if $d = 2k+1$, then the solution for the wave equation in $\mathbb{R}^{d-1} = \mathbb{R}^{2k}$ can be obtained by extending each function on $\mathbb{R}^{2k}$ to a function on $\mathbb{R}^{2k+1}$ in a trivial way, i.e., simply prescribing no dependence on x_d. We explain the details only for the case of $d = 2$ and refer to [**5**, §2.4.d] for the general even-dimensional case.

Theorem 3.11. *Let $k \in \mathbb{N}$. For $u_0 \in C^{k+2}(\mathbb{R}^{2k})$ and $u_1 \in C^{k+1}(\mathbb{R}^{2k})$, define a function u by*

$$\begin{aligned} u(t,x) \quad &:= \quad \frac{1}{(2k)!!}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(\frac{t^{2k}}{|\partial B_t|}\int_{\partial B_t(x)} \frac{u_0(y)}{\sqrt{t^2-|y-x|^2}}dy\right) \\ &\qquad + \frac{1}{(2k)!!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(\frac{t^{2k}}{|\partial B_t|}\int_{\partial B_t(x)} \frac{u_1(y)}{\sqrt{t^2-|y-x|^2}}dy\right), \qquad t>0,\ x \in \mathbb{R}^{2k}. \end{aligned}$$

Then $u \in C^{2,2}(\mathbb{R}_+ \times \mathbb{R}^{2k+1})$. Moreover, u is a solution to the $(2k)$-dimensional wave equation away from $t=0$ (where u is not defined). However, $\lim_{t\to 0+} u(t,x_0)$ and $\lim_{t\to 0+} \frac{\partial u}{\partial t}(t,x_0)$ exist for all $x_0 \in \mathbb{R}^{2k+1}$, and in fact

$$\lim_{(t,x)\to(0,x_0)} u(t,x) = u_0(x_0) \qquad \text{and} \qquad \lim_{(t,x)\to(0,x_0)} \frac{\partial u}{\partial t}(t,x) = u_1(x_0)$$

for all $x_0 \in \mathbb{R}^{2k}$.

Beweis. As already announced, we are only going to prove this theorem in the case of $k=1$. Let a function $u : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ solve the 2-dimensional wave equation and define a function $\tilde{u} : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$ by

$$\tilde{u}(t,x_1,x_2,x_3) := u(t,x_1,x_2), \qquad t \ge 0\ , x_1,x_2,x_3 \in \mathbb{R}.$$

Extending similarly u_0 and u_1 to $\tilde{u}_0, \tilde{u}_1 : \mathbb{R}^3 \to \mathbb{R}$ we immediately see that u is a solution to the 3-dimensional wave equation with initial condition given by $\tilde{u}_0$ and $\tilde{u}_1$. Thus, by Theorem 3.1 the function $\tilde{u}$ necessarily agrees with the solution defined in Theorem (3.9), i.e.,

$$\tilde{u}(t,x) = \frac{\partial}{\partial t}\left(\frac{t}{4\pi t^2}\int_{\partial B_t^{(3)}(x)} \tilde{u}_0(z)d\sigma(z)\right) + \frac{t}{4\pi t^2}\int_{\partial B_t^{(3)}(x)} \tilde{u}_1(z)d\sigma(z), \qquad t>0,\ x\in\mathbb{R}^3. \tag{3.5}$$

(Here $B_t^{(3)}(x)$ and $B_t^{(2)}(x)$ denote, for the sake of clarity, the 3- and 2-dimensional balls of radius t centered at x, respectively.) By construction, the same formula also holds for u. We want to make it handlier by simplifying the term containing the spherical mean of u_0 by some elementary tools of vector analysis. Observe that

$$\{(y, \sqrt{t^2-|y-x|^2}) : y \in B_t^{(2)}(x)\}$$

is the (upper) (3-dimensional) halfsphere of radius t constructed over the 2-dimensional ball $B_t^{(2)}(x)$. Hence, we introduce the parametrising function $\gamma : B_t^{(2)}(x) \to \mathbb{R}$ defined by $\gamma(y) := \sqrt{t^2-|y-x|^2}$, so that the integral of $\tilde{u}_0$ over the sphere $\partial B_t^{(3)}(x)$, hence over the two halfspheres, agrees with

$$\frac{2}{4\pi t^2}\int_{B_t^{(2)}(x)} u_0(y)\sqrt{1+|\nabla\gamma(y)|^2}dy = \frac{1}{2\pi t^2}\int_{B_t^{(2)}(x)} \frac{u_0(y)t}{\sqrt{t^2-|y-x|^2}}dy.$$

Accordingly,

$$\frac{t}{4\pi t^2}\int_{\partial B_t^{(3)}(x)} \tilde{u}_i(z)d\sigma(z) = \frac{1}{2|B_t^{(2)}|}\int_{B_t^{(2)}(x)} \frac{u_i(y)t}{\sqrt{t^2-|y-x|^2}}dy, \qquad i=0,1. \tag{3.6}$$

We therefore have

$$u(t,x) = \tilde{u}(t,x) \quad = \quad \frac{\partial}{\partial t}\left(\frac{1}{2|B_t^{(2)}|}\int_{B_t^{(2)}(x)} \frac{u_0(y)t^2}{\sqrt{t^2-|y-x|^2}}dy\right) \tag{3.7}$$

$$+\frac{1}{2|B_t^{(2)}|}\int_{B_t^{(2)}(x)} \frac{u_1(y)t^2}{\sqrt{t^2-|y-x|^2}}dy, \qquad t>0,\ x\in\mathbb{R}^3. \tag{3.8}$$

This concludes the proof. □

In the 2-dimensional case it is possible to further simplify the solution formula. In order to reduce the first term on the RHS we perform a change of variable $y \mapsto x + tz$ and obtain

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{t^2}{2|B_t^{(2)}|}\int_{B_t^{(2)}(x)}\frac{u_0(y)}{\sqrt{t^2-|y-x|^2}}dy\right) &= \frac{\partial}{\partial t}\left(\frac{t}{2|B_1^{(2)}|}\int_{B_1^{(2)}(0)}\frac{u_0(x+tz)}{\sqrt{1-|z|^2}}dz\right)\\ &= \frac{1}{2|B_1^{(2)}|}\int_{B_1^{(2)}(0)}\frac{u_0(x+tz)}{\sqrt{1-|z|^2}}dz\\ &\quad +\frac{t}{2|B_1^{(2)}|}\int_{B_1^{(2)}(0)}\frac{\nabla u_0(x+tz)\cdot z}{\sqrt{1-|z|^2}}dz\end{aligned}$$

Back-substituting finally yields

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{t^2}{2|B_t^{(2)}|}\int_{B_t^{(2)}(x)}\frac{u_0(y)}{\sqrt{t^2-|y-x|^2}}dy\right) &= \frac{t}{2|B_t^{(2)}|}\int_{B_t^{(2)}(x)}\frac{u_0(y)}{\sqrt{t^2-|x-y|^2}}dy\\ &\quad +\frac{t}{2|B_t^{(2)}|}\int_{B_t^{(2)}(x)}\frac{\nabla u_0(y)\cdot(y-x)}{\sqrt{t^2-|y-x|^2}}dy\end{aligned}$$

Combining this identity with (3.7) we finally arrive at the formula

$$u(t,x)=\frac{1}{2\pi t}\int_{B_t^{(2)}(x)}\frac{u_0(y)+\nabla u_0(y)\cdot(y-x)+tu_1(y)}{\sqrt{t^2-|y-x|^2}}dy,\qquad t\geq 0,\ x\in\mathbb{R}^2,$$

which is easily seen to be the special case of the general formula for $d = 2$.

Remark 3.12. *In other words, if we consider two functions u_0 and $\tilde{u_0}$ that only differ inside a bounded set, say $B_1(0)$, then the solutions to the wave equations with initial data u_0 and $\tilde{u_0}$ are seen to differ at points outside the ball only after a certain time. One refers to this behaviour by saying that the wave equations enjoys* **finite speed of propagation***. Equations – like the wave equations – with finite speed of propagation and such that their solutions are not more regular than the initial data are referred to as* **hyperbolic equations***.*

Remark 3.13. *The above formulae for the solution to the wave equation are often interpreted as a heuristic explanation for (or rather, recognition of) the fact that the world we are living in is odd-dimensional (with respect to the space). This is due to the fact that, by the formulae just obtained (and their generalizations to arbitrary natural numbers), in the 3- (and, more generally, odd-)dimensional case any variation of the initial data at, say, $x \in \mathbb{R}^d$ only affect the solution on the* surface *of the* **sound cone**

$$\{y\in\mathbb{R}^d : |x-y|\leq t\},$$

in accordance with of the **Huygens' principle** *of acoustics, whereas in 2- (and, more generally, even-)dimensional case any such variation should also affect the solution in the* interior *of the sound cone, against experimental observations (think of a jet fighter's sonic boom).*

Theorem 3.14. *Let $\Omega \subset \mathbb{R}^d$ be a domain with C^1-boundary (and in particular such that the Gauß–Green formulae hold). Then the total energy $E_p(t) + E_k(t)$ is constant for all times t, i.e.,*

$$E_p(t)+E_k(t)=\int_\Omega\left|\frac{\partial u}{\partial t}(0,x)\right|^2dx+\int_\Omega|\nabla u(0,x)|^2dx,\qquad t\geq 0.$$

where u is any solution to the wave equation on Ω with either Neumann or Dirichlet boundary conditions such that (in the case of Ω unbounded) u is unifomly bounded on any compact domain.

Beweis. We observe that by the Gauß–Green formulae

$$\begin{aligned}
\frac{d}{dt}\left(E_p(t)+E_k(t)\right) &= \frac{1}{2}\frac{d}{dt}\int_\Omega \left|\frac{\partial u}{\partial t}(t,x)\right|^2 dx + \frac{d}{dt}\int_\Omega |\nabla u(t,x)|^2 dx \\
&= \int_\Omega \frac{d}{dt}\left(\frac{\partial u}{\partial t}(t,x)|\frac{\partial u}{\partial t}(t,x)\right) dx + \int_\Omega \frac{d}{dt}\left(\nabla u(t,x)|\nabla u(t,x)\right) dx \\
&= \frac{d}{dt}\int_\Omega \left(\frac{\partial^2 u}{\partial t^2}(t,x)|\frac{\partial u}{\partial t}(t,x)\right) dx + \int_\Omega \left(\nabla u(t,x)|\nabla \frac{\partial u}{\partial t}(t,x)\right) dx \\
&= \frac{d}{dt}\int_\Omega \left(\frac{\partial^2 u}{\partial t^2}(t,x)|\frac{\partial u}{\partial t}(t,x)\right) dx - \int_\Omega \left(\Delta u(t,x)|\frac{\partial u}{\partial t}(t,x)\right) dx + \int_{\partial\Omega} \left(\frac{\partial u}{\partial n}(t,z)|\frac{\partial u}{\partial t}(t,z)\right) d\sigma(z) \\
&= \frac{d}{dt}\int_\Omega \left(\frac{\partial^2 u}{\partial t^2}(t,x) - \Delta u(t,x)|\frac{\partial u}{\partial t}(t,x)\right) dx,
\end{aligned}$$

where the boundary term disappears owing to the imposed boundary conditions. However, since u solves the wave equation on Ω, the identity $\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta\frac{\partial u}{\partial t}(t,x)$ holds pointwise and for any time t. It follows that

$$\frac{d}{dt}\left(E_p(t)+E_k(t)\right) = \frac{d}{dt}\int_\Omega \left(\frac{\partial^2 u}{\partial t^2}(t,x) - \Delta u(t,x)|u(t,x)\right) dx = 0.$$

This concludes the proof. □

Exercise 3.15. *Consider the (undamped) 1-dimensional wave equation and show that if the initial data u_0, u_1 have compact support, then there exists $T > 0$ such that*

$$E_k(t) = E_p(t) \qquad \text{for all } t > T.$$

Remark 3.16. *Of course, this feature is not very realistic. All the wave phenomena we can commonly observe are not eternal: due to internal frictions and possibly to the influence of external forces (like the gravity) the wave fades away and the system eventually comes to quiet. In other words, the system dissipates energy. This would only be possible if the computations in the proof of Theorem 3.14 could be shown to eventually yield an estimate which is less than 0, say, less than $\int_\Omega |\alpha(x)\frac{\partial u}{\partial t}(t,x)|^2 dx$ for some function $\alpha : \Omega \to (-\infty, 0)$ and any t. In other words, we would like to have*

$$\frac{d}{dt}\int_\Omega \left(\frac{\partial^2 u}{\partial t^2}(t,x) - \Delta u(t,x) - \alpha(x)\frac{\partial u}{\partial t}(t,x)|\frac{\partial u}{\partial t}(t,x)\right) dx \leq 0.$$

This is surely the case if u does not solve the original wave equation, but rather

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \alpha(x)\frac{\partial u}{\partial t}(t,x).$$

Since energy is absorbed throughout the process, this is called a **damped wave equation**. *If on the other hand $\alpha(x) > 0$, $x \in \Omega$, then the system's energy increases during time (in absence of external force such a behaviour is of course unrealistic).*

Exercise 3.17. *Another way to force a vibrating model to have decaying energy is to apply a so-called* **closed feedback***. This means that the system is controlled by some gadget that is able to modify the boundary conditions in real time. Mathematically speaking, this amounts to considering a modified boundary condition – e.g.,*

$$\frac{\partial u}{\partial n}(t,z) = \frac{\partial u}{\partial t}(t,z), \qquad t \geq 0,\ z \in \partial\Omega.$$

Check that the wave equation endowed with this boundary condition actually enjoys energy decay.

Exercise 3.18. *Certain investigations have led theoretical physicists (Morse 1968; Beale–Rosencrans 1974) to consider wave equations equipped with* **acoustic boundary conditions***, which can be written in the form*

$$\left\{\begin{array}{rcll} \frac{\partial^2 u}{\partial t^2}(t,x) &=& c^2\Delta u(t,x), & t \geq 0,\ x \in \Omega, \\ m\frac{\partial^2 \delta}{\partial t^2}(t,z) &=& -d\frac{\partial \delta}{\partial t}(t,z) - k\delta(t,z) - \rho\frac{\partial \phi}{\partial t}(t,z), & t \geq 0,\ z \in \partial\Omega, \\ \dot{\delta}(t,z) &=& \frac{\partial u}{\partial n}(t,z), & t \geq 0,\ z \in \partial\Omega. \end{array}\right.$$

Here ϕ is the velocity potential of a fluid filling an open domain $\Omega \subset \mathbb{R}^d$, either bounded or exterior (i.e., the complement of a compact domain), such that the Gauß–Green formulae hold; δ is the normal displacement of the boundary $\partial\Omega$ of Ω; m, d, and k are the mass per unit area, the resistivity, and the spring constant of the boundary, respectively; finally, ρ and c are the unperturbed density of, and the speed of sound in the medium, respectively. Assume m, k, d, ρ to be positive constants.

Introduce a suitable energy function and show that this energy is decaying. (Hint: also the term δ on $\partial\Omega$ contributes to the total energy of the system.)

KAPITEL 4

The heat equation and the method of symmetries

Consider a thin metal rod, which we can think of as 1-dimensional. If at time $t=0$ the rod is heated at some point(s), the heat will diffuse along the rod. If the initial temperature distribution is known at any point x, can we foresee the temperature distribution at any point x at any future time $t>0$?

Such a heat diffusion problem leads to the introduction of the so-called **heat equation**. First of all, we simplify the setting by assuming the rod to be homogeneous, i.e., its linear density $\rho>0$ to be spatially constant. It is physically meaningful to assume the temperature to be proportional to the heat capacity of the body, i.e., to the thermal energy that it has to be given in order to become 1^o warmer. In a way similar to the case of the transport equation, in an isolated system the temperature change over time in each small region has to be balanced by the heat flow ϕ through the region's boundary. Passing to the limit in time and in space we obtain the differential relation

$$\frac{\partial u}{\partial t}(t,x)=\frac{\partial \phi}{\partial x}(t,x),$$

where $u(t,x)$ denotes the temperature at time t at the point x of the rod and $\phi(t,x)$ the quantity of heat flowing per time and section unit, again at time t at the point x. Since however it has been observed by Fourier that the heat flow is proportional to the temperature gradient $-\frac{\partial u}{\partial x}$. The proportionality factor is usually not dependent on time, leading to the partial differential equation

$$\frac{\partial u}{\partial t}(t,x)=\frac{\partial}{\partial x}\left(\alpha\frac{\partial u}{\partial x}\right)(t,x),$$

where α is a function of space modelling the conductivity at each point. This is usually called **diffusion equation**. In the special case of $\alpha\equiv c$ we obtain the common heat equation

$$\frac{\partial u}{\partial t}(t,x)=c\frac{\partial^2 u}{\partial x^2}(t,x), \tag{4.1}$$

but in general the chain rule yields

$$\frac{\partial u}{\partial t}(t,x)=\alpha(x)\frac{\partial^2 u}{\partial x^2}(t,x)+\alpha'(x)\frac{\partial u}{\partial x}(t,x).$$

If we also consider dissipation and drift phenomena, we can also introduce a drift term $\beta-\alpha'$ and a damping term γ (this will be better motivated in Remark 4.1) and study the equation

$$\frac{\partial u}{\partial t}(t,x)=\alpha(x)\frac{\partial^2 u}{\partial x^2}(t,x)+\beta(x)\frac{\partial u}{\partial x}(t,x)+\gamma(x)u(t,x).$$

This is the most general 1-dimensional diffusion equation.

If we are considering a heat diffusion problem in higher dimension, formally repeating the above procedure yields the general d-dimensional diffusion equation

$$\frac{\partial u}{\partial t}(t,x) = \nabla\left(A \cdot \nabla u\right)(t,x) + B(x)\cdot \nabla u(t,x) + \Gamma(x)u(t,x), \tag{4.2}$$

where $A : \mathbb{R}^d \to \mathbb{R}^{d^2}$ is a general matrix-valued function that models the spacial conductivity in any direction, $B : \mathbb{R}^d \to \mathbb{R}^d$ a vector-valued function modelling the drift and $\Gamma : \mathbb{R}^d \to \mathbb{R}$ a scalar-valued function modelling the damping. Evolution equations like 4.2 are commonly referred to as **parabolic equations**.

One may suspect that the method of characteristics also works for the heat equation. In order to solve the heat equation on – say – $\mathbb{R}$, one can assume that in $\mathbb{R}_+ \times \mathbb{R}$ a curve parametrised by γ exists such that each solution is constant along it, i.e., such that

$$u(s,\gamma(s)) \equiv const, \qquad s \geq 0.$$

However, repeating the computations in (2.5) one only obtains that u and γ satisfy the not quite helpful condition

$$\gamma'(s)\frac{\partial u}{\partial x}(s,\gamma(s)) = -\frac{\partial u}{\partial x}\left(\alpha(\gamma(s))\frac{\partial u}{\partial x}(s,\gamma(s))\right), \qquad s \geq 0.$$

Thus, other techniques have to be developed.

Remark 4.1. *Before looking for a solution formula, let us recall the variational method, which we applied in order to show conservation of energy for the wave equation. A similar reasoning can be made for the case of heat equations. In this case, the thermal energy of the system at time t is given by*

$$E(t) := \int_\Omega |\nabla u(t,x)|^2 dx,$$

where u is the solution to the heat equation (4.2) *without drift ($B \equiv 0$) and with negative-valued damping coefficient ($\Gamma(x) \leq 0$ for all $x \in \Omega$) with Dirichlet or Neumann boundary conditions. Then it can be seen that*

$$E(t) \leq E(0), \qquad t \geq 0,$$

where the strict inequality holds whenever the system is endowed with Dirichlet boundary conditions and/or Γ is strictly *negative-valued.*

4.1. Point symmetries of a PDE

The aim of this section is to provide a brief introduction to jet calculus and the theory of symmetries for partial differential equations. This is an old field that goes back to the pioneering work of Sophus Lie and Felix Klein in the 1880s. Although most of the abstract results have been known ever since, a thorough application to different classes of differential equations is more recent, see e.g. [**9**]. A clean and precise formulation of this theory is quite technical, but pursuing this plan is in our opinion worth it. In this section we will mostly follow [**7**].

If a curve on a finite-dimensional space $\mathbb{R}^n$ is given, say by a parametrisation $\{(s,\phi(s)) \in I \times \mathbb{R}^n\}$ for a smooth function $\phi \equiv (\phi_1,\ldots,\phi_n) : I \to \mathbb{R}^n$, I an interval, then at each point $x = \phi(\epsilon)$ the curve has a tangent vector

$$\phi'(\epsilon) = (\phi_1'(\epsilon),\ldots,\phi_n'(\epsilon)) \equiv \sum_{j=1}^n \phi_j'(\epsilon)\frac{\partial}{\partial x_j}.$$

The tangent space at x, $T_x\mathbb{R}^n$, is defined as the n-dimensional space spanned by the tangent vectors to all curves $I \to \mathbb{R}^n$ at x. Hence, $\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\}$ is a basis of $T_x\mathbb{R}^n$.

Example 4.2. *Define* $\phi : \mathbb{R} \to \mathbb{R}^3$ *by*

$$\phi(\epsilon) := (\cos\epsilon, \sin\epsilon, \epsilon)$$

(i.e., a helix). Then

$$\phi'(\epsilon) := (-\sin\epsilon, \cos\epsilon, 1) = (-\phi_2(\epsilon), \phi_1(\epsilon), 1).$$

Interpreting ϕ *as a curve, i.e.,* $(\phi_1, \phi_2, \phi_3) \equiv (x, y, z)$*, this means that*

$$\phi'(\epsilon) \equiv -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Definition 4.3. *A* **vector field on** $\mathbb{R}^n$ *is a mapping defined by assigning to each* $x \in \mathbb{R}^n$ *a vector in its tangent space,*

$$v : \mathbb{R}^n \ni x \mapsto v(x) = \sum_{j=1}^{n} \xi_j(x)\frac{\partial}{\partial x_j} \in T_x\mathbb{R}^n,$$

for some coefficients $\xi_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, n$.

An **integral curve** *of* v *is a smooth curve* $\{(s, \phi(s)) \in I \times \mathbb{R}^n\}$ *whose tangent vector at any point* ϵ *agrees with* $v|_{\phi(\epsilon)}$*, i.e., such that*

$$\frac{d\phi_i}{d\epsilon}(\epsilon) = \xi_i(\phi(\epsilon)) \qquad \text{for all } i = 1, \ldots, n. \tag{4.3}$$

Given $x_0 \in \mathbb{R}^d$*, the maximal integral curve through* $x_0 = \phi(0)$*, which exists provided that* v *is smooth, is called* **orbit of the flow generated** *by* v*. It is commonly denoted by* $\exp(\epsilon v)x_0$, $\epsilon \in I_{x_0}$*, where* $I_{x_0} \subset \mathbb{R}$ *is the maximal interval (which might be empty).*

Remark 4.4. *Observe that the flow generated by a vector field satisfies the group laws*

$$\exp(0v)x = x$$

and

$$\exp((\epsilon_1 + \epsilon_2)v)x = \exp(\epsilon_2 v)(\exp(\epsilon_1 v)x), \qquad \text{if } \epsilon_1, \epsilon_2, \epsilon_1 + \epsilon_2 \in I_x.$$

It may well be that the maximal interval depends on x*: this happens already in the simple case of* $d = 1$ *and* $\xi_1(x) := x^3$*, since then*

$$\exp(\epsilon v)x_0 = \frac{1}{\sqrt{x_0^{-2} - 2\epsilon}},$$

so that

$$I_{x_0} = \left(-\infty, \frac{1}{2x_0^2}\right), \qquad x_0 \in \mathbb{R}.$$

Remark 4.5. *It is important for the following to emphasize two alternative ways of looking at vector fields. First of all, as we have already pointed out, a vector field is uniquely identified by the coefficients* ξ_i*, once a basis of each tangent space is fixed. In other words,*

$$v \equiv (\xi_1, \ldots, \xi_n) : \mathbb{R}^n \to \mathbb{R}^n,$$

A more interesting point of view is the following. Take a function $f : \mathbb{R}^n \to \mathbb{R}$ *and* $x_0 \in \mathbb{R}^n$*. Then,* $f(\exp(\epsilon v)x_0$ *is a well-defined element of* $\mathbb{R}$ *for all* $\epsilon \in I_{x_0}$*. If* I_{x_0} *is an open neighbourhood of* 0*, then it is possible to differentiate the above expression with respect to* ϵ*: in fact the chain rule yields*

$$\frac{\partial}{\partial\epsilon}f(\exp(\epsilon v)x_0) = \sum_{i=1}^{n} \xi_i(\exp(\epsilon v)x_0)\frac{\partial f}{\partial x_i}(\exp(\epsilon v)x_0) = v(f)(\exp(\epsilon v)x_0),$$

by definition. Evaluating in particular this expression at $\epsilon = 0$ shows that

$$v(f)(x_0) = \sum_{i=1}^{n} \xi_i(x_0) \frac{\partial f}{\partial x_i}(x_0), \tag{4.4}$$

or, to put it more mnemonically:

$$v \equiv \xi \Rightarrow v(f)(x_0) \equiv \xi(x) \nabla f(x).$$

Thus, we can also regard v as a first order differential operator acting on functions from $\mathbb{R}^n$ to $\mathbb{R}$.

Example 4.6. *Consider the vector fields on $\mathbb{R}$ defined by*

$$v(x) := \frac{\partial}{\partial x} \qquad \text{and} \qquad w(x) := x\frac{\partial}{\partial x}, \qquad x \in \mathbb{R}.$$

In order to determine the associated flows, observe that $\exp(\epsilon v)x_0$, $\epsilon \in \mathbb{R}$, has to solve the Cauchy problem

$$\frac{dy}{d\epsilon}(\epsilon) = 1, \qquad y(0) = x_0;$$

whereas $\exp(\epsilon w)x$, $\epsilon \in \mathbb{R}$, has to solve the Cauchy problem

$$\frac{dy}{d\epsilon}(\epsilon) = y, \qquad y(0) = x_0.$$

One directly sees that

$$\exp(\epsilon v_1)\, x_0 = x_0 + \epsilon, \qquad \epsilon \in \mathbb{R},\ x_0 \in \mathbb{R},$$

while

$$\exp(\epsilon v_2) x_0 = e^{\epsilon} x_0, \qquad \epsilon \in \mathbb{R},\ x_0 \in \mathbb{R}.$$

Observe that a smooth function $f \in C^K(\mathbb{R}^n)$, has

$$\binom{n+k-1}{k}$$

different partial derivatives of order $1 \leq k \leq K$ (due to the theorem of Schwarz): we denote them generally by

$$f^{(J)} := \frac{\partial^k f}{\partial x_{j_1} \dots \partial x_{j_k}}, \tag{4.5}$$

where $J = (j_1, \dots, j_k)$ is a multi-index of integers between 1 and n.

It is useful to interpret the solutions to a differential equation as a closed subset (actually, a submanifold) of a suitable Euclidean space. To do so, we introduce a geometrical notion as follows.

Definition 4.7. *Let U be an open domain of $\mathbb{R}^d$. Two functions $f, g \in C^K(\overline{U})$ are said to be* **k-jet-equivalent at** *$x_0 \in U$ if their difference $f - g$ vanishes at x_0 along with all its partial derivatives of any order up to k. The* **k^{th}-jet space** *at x_0, denoted by $J^k_{x_0}$, is the quotient space of $C^K(\overline{U})$ with respect to the k-jet-equivalence relation at x_0. The generic element of $J^k_{x_0}$ is therefore of the form $(x_0, j_k f(x_0))$ for some $f \in C^K(\overline{U}^d)$ and is called* **k-jet** *of f at x_0. The resulting function $j_k f : \mathbb{R}^d \to \mathbb{R}^{d_k}$ is called the* **k-jet** *$j_k f$. Finally,*

$$J^k := J^k(U) := \{J^k_{x_0} : x_0 \in U\} \equiv U \times \mathbb{R}^{d_k}$$

is called the **k^{th}-jet bundle***. The same construction can be repeated if we start by considering functions in $C^K(U)$ instead of $C^K(\mathbb{R}^d)$, eventually arriving at $U \times \mathbb{R}^{d_k}$.*

Here

$$d_0 := 1, \qquad d_k := 1 + \sum_{h=1}^{k} \binom{d+h-1}{h}, \quad k \geq 1.$$

For example, a generic element $(x, f(x))$ of J^0 consists of a point $x \in \mathbb{R}^d$ and the value $f(x)$; a generic element $(x, j_1 f(x))$ of J^1 consists of a point of $\mathbb{R}^d$, the value $f(x)$ and the vector[1] $(u_{x_1}(x), \ldots, u_{x_d}(x))$; and more generally a generic element $(x, j_k f) := (x, j_k f(x))$ of J^k consists of the coefficients of the (k^{th}) truncated Taylor polynomial of f at x. The k-th jet of f maps f to the vector consisting of all partial derivatives of f of order between 0 and k.

In other words, for a function $f \in C^K(U)$ the graph of its k-jet satisfies

$$\left\{ \left(x, j_k f(x) \right) : x \in U \right\} \subset J^k.$$

Consider a general (it is not really relevant whether ordinary or partial) differential equation of K^{th} order[2]

$$H \left(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_d}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots \frac{\partial^2 u}{\partial x_d^2}, \ldots, \frac{\partial^K u}{\partial x_d^K} \right) = 0, \tag{4.6}$$

in the unknown u and its partial derivatives, for a function $H : U \times C(U) \times \ldots \times C(U) \to \mathbb{R}$. By definition, a solution to (4.6) is a function $u \in C^K(U)$ such that u (and its derivatives) satisfy (4.6) for all $x \in U$. The subset of $J^K(U)$ consisting of all jets $(x, j_K u)$ such that $H(x, j_K u) = 0$ is called **solution manifold** of the differential equation.

On the other hand, at each point x the equation is fulfilled by u if and only if H vanishes in $(x, j_K u(x))$. Hence, we can regard H as a function from J^K to $\mathbb{R}$ and the differential equation (4.6) as an algebraic equation. Denoting its null set by $\mathcal{N}_H$, u is a solution to (4.6) if and only if $\{(x, j_K u(x)) : x \in \mathbb{R}^d\} \subset \mathcal{N}_H$. We can therefore re-write (4.6) as

$$H(x, j_K u) = 0.$$

Determining the set of C^K-functions that solve (4.6) is therefore equivalent to determining the closed subset of J^K consisting of solutions to (4.6).

Remark 4.8. *In particular, we can introduce the tangent spaces $T_{\mathbf{x}} J^0$, $\mathbf{x} := (x, u(x)) \in J^0$. This is the $(d+1)$-dimensional vector space consisting of tangent vectors at $\mathbf{x}$ to all curves in J^0. Its basis elements are*

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial u}.$$

Similarly, we consider tangent spaces to J^1 with basis elements

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial u}, \frac{\partial}{\partial u_{x_1}}, \ldots, \frac{\partial}{\partial u_{x_d}},$$

and so on for tangent spaces to J^k for any integer k.

[1] Here

$$u_{x_i} := \frac{\partial u}{\partial x_i}, \qquad i = 1, \ldots, d.$$

[2] Clearly, any differential equation of order K is also a differential equation of order $K + m$ for any $m \geq 0$. Therefore, one takes the order of a differential equation to be the least integer K such that the equation is well-defined on $C^K(\mathbb{R}^d)$.

Definition 4.9. *A* **point transformation** *is a mapping from J^0 to itself.*

A **one-parameter point transformation group** *of (4.6) is a family $\mathcal{T} := (T_\epsilon(\mathbf{x}))_{\epsilon\in\mathbb{I}_{\mathbf{x}},\, \mathbf{x}\in J^0}$ of point transformations such that $0 \in I_{\mathbf{x}}$ for all $\mathbf{x} \in J^0$ and such that*

$$T_0(\mathbf{x}) = \mathbf{x}$$

and that

$$T_{\epsilon_1+\epsilon_2}(\mathbf{x}) = T_{\epsilon_2}\left(T_{\epsilon_1}(\mathbf{x})\right), \tag{4.7}$$

for all $\mathbf{x} \in J^0$ and all $\epsilon_1, \epsilon_2 \in \mathbb{R}$ for which the above identity makes sense. In many cases we need the dependence of $\mathcal{T}$ on $\mathbf{x}$ and ϵ to be jointly continuously differentiable, which we therefore assume throughout.

In most applications, $I_{\mathbf{x}}$ is an open interval of $\mathbb{R}$ that contains 0, for any $\mathbf{x} \in J^0$, but in fact $\mathcal{T}$ will generally only be defined on subsets of $\mathbb{R} \times J^0$. For this reasons, such transformation groups are sometimes referred to as *local.*

Remark 4.10. *We emphasize that point transformations do not act on functions, but rather on elements of the functions' graph, i.e., on pairs $(x, f(x))$. The difference is subtle but fundamental, since in particular it lets a point transformation group define a curve (parametrised in ϵ) in the $(d+1)$-dimensional jet space J^0, and hence to apply the elementary vector analysis introduced before.*

Definition 4.11. *The* **infinitesimal generator** *of a one-parameter point transformation group $\mathcal{T}$ is the vector field $A : J^0 \to J^0$ defined as the tangent vector to the curve $(T_\epsilon(\mathbf{x}))_{\epsilon\in I}$ at $\mathbf{x} = T_0\mathbf{x} \in J^0$, i.e.,*

$$A(\mathbf{x}) \equiv \lim_{\epsilon\to 0+} \frac{T_\epsilon(\mathbf{x}) - \mathbf{x}}{\epsilon} =: \big(\xi_1(\mathbf{x}), \ldots, \xi_d(\mathbf{x}), \phi(\mathbf{x})\big).$$

By Remark 4.5 we deduce in particular that for any $F : J^0 \to \mathbb{R}$ and any $(x, u) \in J^0$ we have

$$(A(x, u))(F) = \sum_{i=1}^{d} \xi_i(x, u)\frac{\partial F}{\partial x_i}(x, u) + \phi(x, u)\frac{\partial F}{\partial u}(x, u).$$

We can thus regard $A(\mathbf{x})$ as a differential operator acting on functions from J^0 to $\mathbb{R}$, for all $\mathbf{x} \in J^0$.

Example 4.12. *The mapping*

$$J^0 \ni (x, u) \mapsto (O(x), u) \equiv (Ox, u(Ox)) \in J^0,$$

where O is a $d \times d$ matrix, is a point transformation. If in particular O is a rotation, then this transformation can also be inverted. In the special case of $d = 2$,

$$O = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for some $\theta \in \mathbb{R}$, and one is led to introduce the one-parameter group of transformations

$$T_\theta : (x, y, u) \mapsto (\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}(x, y), u), \qquad \theta \in \mathbb{R}.$$

This represents an action of the special orthogonal group $SO(2)$, corresponding to rotations in the argument of functions u. Its infinitesimal generator is the vector field given by

$$A(x, y, u) = \lim_{\theta\to 0} \frac{(x\cos\theta - y\sin\theta - x, x\sin\theta + y\cos\theta - u, 0)}{\theta} = (x\cos' 0 - y, x + y\cos' 0, 0) = (-y, x, 0).$$

That is,

$$A(x,y,u) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Example 4.13. *Another possible action of $SO(2)$ is given by the one-parameter group of transformations*

$$T_\theta : (x,u) \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} (x,u), \qquad \theta \in \mathbb{R},$$

i.e.,

$$T_\theta(x,u(x)) = \big(x\cos\theta - u(x)\sin\theta, x\sin\theta + u(x)\cos\theta\big) =: (\tilde{x}, \tilde{u}(x)), \qquad \theta \in \mathbb{R}.$$

In order to consistently express $\tilde{u}$ as a function of the new variable $\tilde{x}$, one has to get rid of the old variable x. This corresponds to solving a system of equations, which solvability is typically only granted for some values of θ.

The infinitesimal generator of this transformation group is the vector field given by

$$A(x,u) = \lim_{\theta\to 0} \frac{(x\cos\theta - u\sin\theta - x, x\sin\theta + u\cos\theta - u)}{\theta} = (x\cos' 0 - u, x + u\cos' 0) = (-u, x).$$

That is,

$$A(x,u) = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}.$$

Example 4.14. *Denote by e_i the i^{th} vector of the canonical basis of $\mathbb{R}^d$. For all $i = 1, \ldots, d$ the mappings*

$$J^0 \ni (x,u) \mapsto (x + \epsilon e_i, u) \in J^0, \qquad \epsilon \in \mathbb{R},$$

define a one-parameter point transformation group. Its infinitesimal generator is the vector field with coordinates given by

$$A(x,u) \equiv (e_i, 0), \quad \text{i.e.,} \quad A(x,u) = \frac{\partial}{\partial x_i}.$$

Example 4.15. *The mappings*

$$J^0 \ni (x,u) \mapsto (x, e^\epsilon u) \in J^0, \qquad \epsilon \in \mathbb{R},$$

define a one-parameter point transformation group. Its infinitesimal generator is the vector field with coordinates given by

$$A(x,u) \equiv (0, u), \quad \text{i.e.,} \quad A(x,u) = u\frac{\partial}{\partial u}.$$

Remark 4.16. *Given a vector field, it is possible to obtain the flow generated by it solving the differential equations* (4.3)*; and conversely, given a flow that takes the form of a one-parameter group it is possible to find its infinitesimal generator by differentiating the flow at* 0*.*

The general aim of this section is to investigate symmetries, i.e., transformation groups mapping solutions to differential equations into solutions to differential equations. Naively, we might want to define a symmetry as a one-parameter point transformation group $\mathcal{S}$ such that

$$(x,u) \text{ is a solution to (4.6)} \quad \Rightarrow \quad S_\epsilon(x,u) \text{ is a solution to (4.6) for all } \epsilon \in I_u. \tag{4.8}$$

However, this does not (yet) make much sense, unless H is a function defined on J^0, i.e., unless (4.6) is in fact not a plain algebraic (as opposite to differential) equation.

But we do know how to turn differential equations into algebraic ones: simply pass to the jet space representation! In order to discuss more general cases it is necessary to extend point transformation groups $\mathcal{T}$ acting on J^0 to higher order jet spaces. The most natural way to do so is to define an extended transformation group that for all $\epsilon \in I$ maps (x,u) into $(\tilde{x},\tilde{u}) = T_\epsilon(x,u)$ and higher order Taylor coefficients of u ar x into higher order coefficients of $\tilde{u}$ at $\tilde{x}$.

Definition 4.17. *Let $\mathcal{T}$ be a one-parameter point transformation group with generator A. The* **k-jet of** $\mathcal{T}$ *is the family $j_k\mathcal{T}$ of mappings from J^k to itself defined by*

$$(j_kT_\epsilon)(x, j_kf) := (\tilde{x}, j_k\tilde{f}), \quad \text{whenever } (\tilde{x},\tilde{f}) = T_\epsilon(x,f), \qquad \text{for all } \epsilon \in \tilde{I}_{(x,f)},$$

for some family $(\tilde{I}_{\mathbf{x}})_{\mathbf{x}\in J^0}$ of open intervals, each contained in (and possibly strictly smaller than) $I_{\mathbf{x}}$ and such that $(0,\mathbf{x}) \in \tilde{I}_{\mathbf{x}}$.

The **k-jet** j_kA **of** A *is defined as the (unique) vector field on J^k that is the infinitesimal generator of $j_k\mathcal{T}$, i.e.,*

$$j_kA(x,f) := \lim_{\epsilon\to 0} \frac{j_kT_\epsilon(x,j_kf) - (x,j_kf)}{\epsilon}, \qquad (x,j_kf) \in J^k. \tag{4.9}$$

Remark 4.18. *We strongly emphasize that in* (4.9) *the notation $T_\epsilon(x,j_kf)$ is merely referred to the action of the group onto the jet $(x,j_kf) = (x,j_kf(x))$. In other words, denoting for the sake of simplicity*

$$(x_\epsilon, (j_kf)_\epsilon) := j_kT_\epsilon(x,j_kf),$$

this means that

$$j_kA(x,f) = \left(\lim_{\epsilon\to 0}\frac{x_\epsilon - x}{\epsilon}, \lim_{\epsilon\to 0}\frac{(j_kf)_\epsilon(x) - j_kf(x)}{\epsilon}\right), \qquad (x,j_kf) \in J^k.$$

Definition 4.19. *A* **one-parameter point symmetry group** *of* (4.6) *is a one-parameter point transformation group $\mathcal{T}$ such that its K-jet satisfies*

$$H(x,j_Ku) = 0 \quad \Rightarrow \quad H\big((j_KT_\epsilon)(x,j_Ku)\big) = 0 \qquad \text{for all } \epsilon \in \tilde{I}_{(x,u)}. \tag{4.10}$$

Example 4.20. *Consider again the one-parameter point transformation group introduced in Example 4.13. A direct computation (check!) shows that its first jet acts on general elements $(x,u,u_x) \in J^1$ by*

$$(j_1T_\theta)(x,u,u_x) = \left(x\cos\theta - u\sin\theta, x\sin\theta + u\cos\theta, \frac{\sin\theta + u_x\cos\theta}{\cos\theta - u_x\sin\theta}\right), \qquad \theta \in \mathbb{R}.$$

Differentiating this expression at $\theta = 0$ we obtain

$$(j_1A)(x,u,u_x) = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1+u_x^2)\frac{\partial}{\partial u_x}.$$

It is important to emphasize that the jets of a transformation group (T_ϵ) (and more generally of a flow) might be defined on maximal intervals that are shorter than (T_ϵ). Moreover, it might be that even when the action of a transformation group on elements of a jet bundle are well-defined, the resulting transformed elements cannot be interpreted as functions anymore.

Example 4.21. *Consider the one-parameter point transformation group $(T_\theta)_{\theta\in\mathbb{R}}$ introduced in Example 4.13. Then $(T_\theta)_{\theta\in\mathbb{R}}$ is a one-parameter symmetry group of the 1-dimensional Laplace equation*

$$u''(x) = 0, \qquad x \in \mathbb{R}.$$

In fact, the solutions to this equation are exactly the affine functions, i.e., those functions of the form

$$u(x) = ax + b, \qquad x \in \mathbb{R},$$

for some $a, b \in \mathbb{R}$. A simple computation shows that each T_θ maps affine functions into affine functions. However, this only holds for small θ (in fact, the necessary smallness of θ clearly depends on the parameter a in the above formula), since at some point rotating the graph of u yields a perfectly vertical line – not the graph of a (single-valued) function anymore! Therefore, it is not $\mathcal{S} := (T_\theta \mathbf{x})_{\theta\in\mathbb{R},\, \mathbf{x}\in J^2}$ that satisfies (4.8), but rather its suitable restriction $\mathcal{S} := (T_\theta \mathbf{x})_{\theta\in I_u,\, \mathbf{x}\in J^2}$ defined in such a way that for all $\theta \in I_{(x,j_2u)}$

$$(j_2T_\theta)(x, j_2u) \in \mathbb{R}^4,$$

i.e., in such a way that the slope of a representant $u \in C^2(\mathbb{R}^d)$ of the equivalence class $(j_2T_\theta)(x, j_2u)$ is not infinite for any $\theta \in I_u$.

Example 4.22. *We now consider two one-parameter point transformation groups for the 1-dimensional heat equation. They are defined by*

$$T_\epsilon : (t, x, u) \mapsto (t + \epsilon a, x + \epsilon b, u), \qquad \epsilon \in \mathbb{R},$$

and

$$S_\epsilon : (t, x, u) \mapsto (t, x - 2\epsilon t, e^{-\epsilon x-\epsilon^2 t}u), \qquad \epsilon \in \mathbb{R}.$$

Simply computing the partial derivatives with respect to t and x shows that (the 2-jets of) both groups map solutions to the heat equation into solutions to the heat equations.

While checking that a given transformation group is a symmetry of (4.6) is in many cases feasible, we are interested in performing a systematic search for point symmetry groups.

Theorem 4.23. *Let $\mathcal{T}$ be a one-parameter point transformation group with infinitesimal generator A. Consider a differential equation (4.6) such that at each solution (x_0, j_Ku) of the system the following condition is satisfied:*

$$\begin{aligned} &\text{There exists a local change of coordinates } \mathbf{y} = (y_1, \ldots, y_{d+d_K}),\\ &\qquad \mathbf{y} : (x, j_Ku) \mapsto \mathbf{y}(x, j_Ku),\\ &\text{such that } \mathbf{y}(x_0, j_Ku) \text{ is the new origin and (4.6) can be written as } y_1(x, j_Ku) = 0. \end{aligned} \tag{4.11}$$

Then the following assertions are equivalent.

(a) $\mathcal{T}$ is a one-parameter point symmetry group of (4.6).

(b) A satisfies the implication

$$H(x, j_Ku) = 0 \quad \Rightarrow \quad \big((j_KA)H\big)(x, j_Ku) = 0. \tag{4.12}$$

In the following proog, for the sake of notational simplicity we prefer to write $z_\epsilon := (j_KT_\epsilon)(x, j_Ku)$, $\epsilon \in I_{(x,j_Ku)}$, and in particular $z_0 = (x, j_Ku)$.

Beweis. (a)$\Rightarrow$ (b) The assertion follows directly from the definition of point symmetry group, i.e.,

$$H(z_0) = 0 \quad \Rightarrow \quad H\big(z_\epsilon\big) = 0 \qquad \text{for all } \epsilon \in I_{(x,j_Ku)}.$$

Differentiating the second condition in (4.10) at $\epsilon = 0$ we see as in Remark 4.5

$$\frac{d}{d\epsilon}H(z_\epsilon)\Big|_{\epsilon=0} = \sum_{i=1}^{d+d_K} \frac{\partial H}{\partial z_{\epsilon,i}}(z_0) \cdot \frac{\partial z_{\epsilon,i}}{\partial \epsilon}\Big|_{\epsilon=0} = \left(\sum_{i=1}^{d+d_K} \frac{\partial z_{\epsilon,i}}{\partial \epsilon}\Big|_{\epsilon=0} \cdot \frac{\partial}{\partial z_{0,i}}\right) H(z_0),$$

where the operator in parenthesis is a vector field on $T_{(z_0)}J^K = T_{(x,u)}J^K$ and whose coefficients agree the tangent vector to the curve defined by $\big(j_KT_\epsilon)(x, j_Ku)\big)_{\epsilon\in I_{(x,j_Ku)}}$ at $(x, j_Ku) = j_KT_0(x, j_Ku) \in J^K$ – that is, with the infinitesimal generator $j_K A$ of the point transformation group's k-jet. Summing up, we have proved that

$$H(z_0) = 0 \quad \Rightarrow \quad \big((j_K A)H\big)(z_0) = 0.$$

To prove the converse implication, observe first that – by definition – the flow

$$(z_\epsilon)_{\epsilon\in I_{(x,j_K u)}} = (j_K T_\epsilon(x, j_K u))_{\epsilon\in I_{(x,j_K u)}}$$

through $(x, j_K u) = 0$ generated by $j_K A$ satisfies the Cauchy problem

$$\begin{cases} \frac{d\mathbf{y}_\epsilon}{d\epsilon} = (j_K A)(\mathbf{y}_\epsilon), \quad \epsilon \in I_0 \subset I_{(x,j_K u)}, \\ \mathbf{y}_0 = 0, \end{cases}$$

associated with the vector field $j_K A \equiv (\xi_1, \ldots, \xi_{d+d_k}) : J^K \to J^K$. Owing to (4.11), with respect to the new coordinates we can write $j_K A$ as

$$j_K A(\mathbf{y}) = \sum_{i=1}^{d+d_K} \xi_i(\mathbf{y}) \frac{\partial}{\partial \mathbf{y}_i},$$

hence $(z_\epsilon)_{\epsilon\in I_{(x,j_K u)}}$ satisfies in particular

$$\begin{cases} \frac{d\mathbf{y}_{\epsilon,1}}{d\epsilon} = \xi_1(\mathbf{y}_\epsilon), \quad \epsilon \in I_0 \subset I_{(x,j_K u)}, \\ \mathbf{y}_{0,1} = 0, \end{cases}$$

Since in particular (4.6) is equivalent to

$$y_1(x, j_K u) = 0, \tag{4.13}$$

it follows from (4.12) that

$$0 = y_1(x, j_K u) = 0 \quad \Rightarrow \quad ((j_K A)H)(x, j_K u) = \sum_{i=1}^{d+d_K} \xi_i(\mathbf{y}(x, j_K u)) \frac{\partial \mathbf{y}_1}{\partial \mathbf{y}_i}(x, j_K u) = \xi_1(\mathbf{y}(x, j_K u)).$$

Eventually, we have obtained that $(z_\epsilon)_{\epsilon\in I_{(x,j_K u)}}$ solves

$$\frac{d\mathbf{y}_{\epsilon,1}}{d\epsilon} = 0, \qquad \epsilon \in I_0 \subset I_{(x,j_K u)} \qquad \text{and} \qquad \mathbf{y}_0 = 0.$$

Due to the existence and uniqueness theorem, the solution $(z_\epsilon)_{\epsilon\in I_{(x,j_K u)}}$ to the above Cauchy problem has to agree with the trivial one, i.e.,

$$(j_K T_\epsilon)(x, j_K u) = 0, \qquad \epsilon \in I_0 \subset I_{(x,j_K u)}.$$

and in fact even for all $\epsilon \in I_{(x,j_K u)}$, by (4.7)). In particular, $z_{\epsilon,1} = 0$ for all $\epsilon \in I_{(x,j_K u)}$. I.e., taking into account the change of coordinates, it satisfies (4.6).

In other words, taking into account (4.13) we have shown that if $z_0 = (x, j_K u)$ is a solution to (4.6), then so is $z_\epsilon = (j_K T_\epsilon)(x, j_K u)$ for all ϵ small enough. But in fact, due to (4.7) we deduce that $(j_K T_\epsilon)(x, j_K u)$ is a solution for *all* $\epsilon \in I_{(x,j_K u)}$. □

The fundamental class of equations for which condition (4.11) is satisfied consists of those having maximal rank, in the following sense.

Definition 4.24. *The equation* (4.6) *is said to have* **maximal rank** *if at each point* $\mathbf{x} \in J^K$ *such that* $H(\mathbf{x}) = 0$ *the Jacobian of* H *at* $\mathbf{x}$*,*

$$\nabla H(\mathbf{x}) := \left(\tfrac{\partial H}{\partial x_1}, \dots, \tfrac{\partial H}{\partial x_d}, \tfrac{\partial H}{\partial u}, \tfrac{\partial H}{\partial u_{x_1}}, \dots, \tfrac{\partial H}{\partial u_{d\dots d}}\right)(\mathbf{x}) \in \mathbb{R}^{d+d_K},$$

has at least one non-vanishing coordinate.

Exercise 4.25. *Show that each equation of maximal rank satisfies condition* (4.11). (Hint: apply the implicit function theorem.)

Example 4.26. *The inhomogeneous* 1*-dimensional transport equation* (2.1) *is defined by the function* $H : J^1 \to \mathbb{R}$ *given by*

$$H(t, x, j_1 u) = H\left(t, x, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right) := \frac{\partial u}{\partial t}(t, x) + c\frac{\partial u}{\partial x}(t, x) - f(t, x).$$

At each point of $(t, x, j_1 u) \in J^1$*, the Jacobian of* H *has coordinates given by*

$$\frac{\partial H}{\partial t} = -\frac{\partial f}{\partial t}, \quad \frac{\partial H}{\partial x} = -\frac{\partial f}{\partial x}, \quad \frac{\partial H}{\partial u} = 0, \quad \frac{\partial H}{\partial u_t} = 1, \quad \frac{\partial H}{\partial u_x} = c,$$

i.e.,

$$\nabla H\,(t, x, j_1 u) = \left(-\frac{\partial f}{\partial t}(t, x), -\frac{\partial f}{\partial x}(t, x); 0; 1, c\right).$$

Hence, (2.1) *has maximal rank.*

Exercise 4.27. *Assume for simplicity the equation* (4.6) *to be of maximal rank. Show that the conditions (a) and (b) in Theorem 4.23 are equivalent to the following one.*

(c) There exists $Q : J^K \to \mathbb{R}$ *such that*

$$\big((j_K A)H\big)(x, j_K u) = Q(x, j_K u)H(x, j_K u) \qquad \textit{for all } (x, j_K u) \in J^K. \tag{4.14}$$

Observe that the maximal rank condition is sufficient but not necessary, in order to ensure that (4.11) is satisfied.

Example 4.28. *The* 3*-dimensional Laplace equation*

$$\Delta u(x, y, z) = 0, \qquad (x, y, z) \in \mathbb{R}^3,$$

is defined by the function $H : J^2 \to \mathbb{R}$ *given by*

$$H\,(x, y, z, j_2 u) := \frac{\partial^2 u}{\partial x^2}(x, y, z) + \frac{\partial^2 u}{\partial y^2}(x, y, z) + \frac{\partial^2 u}{\partial z^2}(x, y, z).$$

At each point of $(x, y, z, j_2 u) \in J^2$*, the Jacobian of* H *is*

$$\nabla H(x, y, z, j_2 u) = (0, 0, 0; 0; 0, 0, 0; 1, 0, 0, 1, 0, 1).$$

Hence, (2.1) *has maximal rank.*

However, consider the (equivalent) formulation

$$(\Delta u(x, y, z))^2 = 0, \qquad (x, y, z) \in \mathbb{R}^3.$$

Then, at each point $(x, y, z, j_2 u) \in J^2$ *the Jacobian of the associated function* $\tilde{H}$ *is*

$$\nabla H(x, y, z) = (0, 0, 0; 0; 0, 0, 0; 2\Delta u(x, y, z), 0, 0, 2\Delta u(x, y, z), 0, 2\Delta u(x, y, z)),$$

which vanish whenever $\Delta u(x, y, z) = 0$.

Exercise 4.29. *Show that the heat equation (4.1) has maximal rank for all $c \in \mathbb{R}$.*

Example 4.30. *Let $d = 1$ and consider the ordinary differential equation defined by the function $H : J^2 \to \mathbb{R}$ defined by*

$$H(x, y, y', y'') := y'' + y.$$

We want to prove that the point transformation

$$T_\epsilon : (x, y) \mapsto (x, e^\epsilon y), \qquad \epsilon \in \mathbb{R},$$

whose generator is

$$A(x, y) := y \frac{\partial}{\partial y},$$

cf. Example 4.6, is a symmetry group of the above (ordinary) differential equation. In order to check this, consider the 2-jet $j_2 A$, which is defined by

$$j_2 A(x, y) := y \frac{\partial}{\partial y} + y' \frac{\partial}{\partial y'} + y'' \frac{\partial}{\partial y''},$$

and compute $(j_2 A)H : J^2 \to J^2$. This is given by

$$((j_2 A)H)(x, j_2 y) = y \frac{\partial H}{\partial y} + y' \frac{\partial H}{\partial y'} + y'' \frac{\partial H}{\partial y''} = y + y'' \in J^2,$$

which vanishes identically whenever $y + y'' = 0$, i.e., whenever y is a solution to the given ODE.

While Theorem 4.23 is a mighty tool for investigating symmetries, it is usually not easy to find jets $j_K A$ of infinitesimal generators, since also formulae for the generated one-parameter groups (even for those on J^0) are seldom known.

A mechanical (and in fact algorithmically implemented) way to determine such jets is known, but providing its proof is lengthy and goes beyond the scope of this lecture. Instead, we content ourselves in summarizing the results that are most relevant to us and refer to [**7**, §2.3].

Proposition 4.31. *Consider an infinitesimal generator $A : J^0 \to J^0$, defined say by*

$$A(x, u) := \sum_{j=1}^{d} \xi_j(x, u) \frac{\partial}{\partial x_j} + \phi(x, u) \frac{\partial}{\partial u}.$$

The k-jet of A, $k \le K$, is the vector field on J^K given by

$$j_k A(x, j_K u) = A(x, u) + \sum_J \phi^{(J)}(x, j_K u) \frac{\partial}{\partial u^{(J)}}, \tag{4.15}$$

where each $\phi^{(J)}$ maps J^K to $\mathbb{R}$ by

$$\phi^{(J)}(x, j_K u) := D_J \left(\phi - \sum_{i=1}^{d} \xi_i \frac{\partial u}{\partial x_i} \right)(x, j_K u) + \sum_{i=1}^{d} \left(\xi_i \frac{\partial u^{(J)}}{\partial x_i} \right)(x, j_K u).$$

In (4.15) we denote by

$$\sum_J$$

the sum over all multi-indices $J = (j_1, \dots, j_k)$ of integers between 1 and d, with $1 \leq k \leq K$. Here D_J denotes the J^{th} **total derivative** of a vector field,

$$D_J = D_{j_1} \dots D_{j_k},$$

defined for all smooth v and all integers i between 1 and d by[3]

$$D_i v := \frac{\partial v}{\partial x_i} + \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial u} + \sum_J \frac{\partial u^{(J)}}{\partial x_i}\frac{\partial v}{\partial u^{(J)}}.$$

In all the applications (to 1-dimensional partial differential equations) we have in mind $d = 2$ and $K \leq 2$, hence in order to apply our symmetry criterion all we have to do is to find an explicit expression for $j_2 A$, that is, for

$$\phi^{(t)}, \phi^{(x)}, \phi^{(tt)}, \phi^{(tx)}, \phi^{(xx)}.$$

A tedious but straightforward computation yields the following.

Corollary 4.32. *Let $d = 2$. The coefficients $\phi^{(t)}, \phi^{(x)}, \phi^{(tt)}, \phi^{(tx)}, \phi^{(xx)}$ of the 2-jet of an infinitesimal generator*

$$A = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \phi \frac{\partial}{\partial u}$$

are given as follows: the coefficients of the 1$^{\text{st}}$ order terms are

$$\begin{aligned}
\phi^{(t)} &= \frac{\partial \phi}{\partial t} - \frac{\partial \xi}{\partial t}\frac{\partial u}{\partial x} + \left(\frac{\partial \phi}{\partial u} - \frac{\partial \tau}{\partial t}\right)\frac{\partial u}{\partial t} - \frac{\partial \xi}{\partial u}\frac{\partial u}{\partial x}\frac{\partial u}{\partial t} - \frac{\partial \tau}{\partial u}\left(\frac{\partial u}{\partial t}\right)^2, \\
\phi^{(x)} &= \frac{\partial \phi}{\partial x} - \frac{\partial \tau}{\partial x}\frac{\partial u}{\partial t} + \left(\frac{\partial \phi}{\partial u} - \frac{\partial \xi}{\partial x}\right)\frac{\partial u}{\partial x} - \frac{\partial \tau}{\partial u}\frac{\partial u}{\partial t}\frac{\partial u}{\partial x} - \frac{\partial \xi}{\partial u}\left(\frac{\partial u}{\partial x}\right)^2,
\end{aligned}$$

---

[3] Formally $u^{(J)}$ is only defined for any (equivalence class whose representant is a) function in C^k, i.e., for the k-jet of u, but we avoid writing

$$(j_k u)^{(J)} \qquad \text{for} \qquad u^{(J)}.$$

(observe the symmetry with respect to $t \leftrightarrow x$ and $\tau \leftrightarrow \xi$) while the coefficients of the 2nd order terms are

$$
\begin{aligned}
\phi^{(tt)} \;=\;& \frac{\partial^2\phi}{\partial t^2} + \left(2\frac{\partial^2\phi}{\partial t\partial u} - \frac{\partial^2\tau}{\partial t^2}\right)\frac{\partial u}{\partial t} - \frac{\partial^2\xi}{\partial t^2}\frac{\partial u}{\partial x} + \left(\frac{\partial^2\phi}{\partial u^2} - 2\frac{\partial^2\tau}{\partial t\partial u}\right)\left(\frac{\partial u}{\partial t}\right)^2 \\
& -2\frac{\partial^2\xi}{\partial t\partial u}\frac{\partial u}{\partial t}\frac{\partial u}{\partial x} - \frac{\partial^2\tau}{\partial u^2}\left(\frac{\partial u}{\partial t}\right)^3 - \frac{\partial^2\xi}{\partial u^2}\left(\frac{\partial u}{\partial t}\right)^2\frac{\partial u}{\partial x} + \left(\frac{\partial\phi}{\partial u} - 2\frac{\partial\tau}{\partial t}\right)\frac{\partial^2 u}{\partial t^2} \\
& -2\frac{\partial\xi}{\partial t}\frac{\partial^2 u}{\partial x\partial t} - 3\frac{\partial\tau}{\partial u}\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} - \frac{\partial\xi}{\partial u}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t^2} - 2\frac{\partial\xi}{\partial u}\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t\partial x} \\
\phi^{(tx)} \;=\;& \frac{\partial^2\phi}{\partial t\partial x} + \left(\frac{\partial^2\phi}{\partial t\partial u} - \frac{\partial^2\xi}{\partial t\partial x}\right)\frac{\partial u}{\partial x} + \left(\frac{\partial^2\phi}{\partial x\partial u} - \frac{\partial^2\tau}{\partial t\partial x}\right)\frac{\partial u}{\partial t} - \frac{\partial^2\xi}{\partial t\partial u}\left(\frac{\partial u}{\partial x}\right)^2 \\
& + \left(\frac{\partial^2\phi}{\partial u^2} - \frac{\partial^2\xi}{\partial x\partial u} - \frac{\partial^2\tau}{\partial t\partial u}\right)\frac{\partial u}{\partial x}\frac{\partial u}{\partial t} - \frac{\partial^2\tau}{\partial x\partial u}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{\partial^2\xi}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial u}{\partial t} - \frac{\partial^2\tau}{\partial u^2}\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial t}\right)^2 \\
& - \frac{\partial\xi}{\partial t}\frac{\partial^2 u}{\partial x^2} - \frac{\partial\xi}{\partial u}\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial\phi}{\partial u} - \frac{\partial\xi}{\partial x} - \frac{\partial\tau}{\partial t}\right)\frac{\partial^2 u}{\partial t\partial x} - 2\frac{\partial\xi}{\partial u}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t\partial x} \\
& -2\frac{\partial\tau}{\partial u}\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t\partial x} - \frac{\partial\tau}{\partial x}\frac{\partial^2 u}{\partial t^2} - \frac{\partial\tau}{\partial u}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t^2} \\
\phi^{(xx)} \;=\;& \frac{\partial^2\phi}{\partial x^2} + \left(2\frac{\partial^2\phi}{\partial x\partial u} - \frac{\partial^2\xi}{\partial x^2}\right)\frac{\partial u}{\partial x} - \frac{\partial^2\tau}{\partial x^2}\frac{\partial u}{\partial t} + \left(\frac{\partial^2\phi}{\partial u^2} - 2\frac{\partial^2\xi}{\partial x\partial u}\right)\left(\frac{\partial u}{\partial x}\right)^2 \\
& -2\frac{\partial^2\tau}{\partial x\partial u}\frac{\partial u}{\partial x}\frac{\partial u}{\partial t} - \frac{\partial^2\xi}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^3 - \frac{\partial^2\tau}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial u}{\partial t} + \left(\frac{\partial\phi}{\partial u} - 2\frac{\partial\xi}{\partial x}\right)\frac{\partial^2 u}{\partial x^2} \\
& -2\frac{\partial\tau}{\partial x}\frac{\partial^2 u}{\partial x\partial t} - 3\frac{\partial\xi}{\partial u}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - \frac{\partial\tau}{\partial u}\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial\tau}{\partial u}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t\partial x}.
\end{aligned}
$$

Exercise 4.33. *Prove Corollary 4.32.*

4.2. The fundamental solution to the heat equation

In general, determining a point symmetry group is a tiresome task, for a human being. If however this has been performed[4], then it is sometimes possible to extract interesting informations from that. In the following we specialise the results of §4.1 to the case of the one-dimensional heat equation.

Theorem 4.34. *The unique one-parameter point symmetry groups of the heat equation*

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x), \qquad t \geq 0,\ x \in \mathbb{R}, \tag{4.16}$$

[4] This is made easier by using suitable symbolic software, like `MAPLE`: its PDEtools package permits e.g. to determine (via the command *Infinitesimals* and *SymmetryTransformations*, respectively) the infitesimal generators of the one-parameter point symmetry groups of a partial differential equations as well as the generated groups, at least in the more elementary cases.

are those given by

$$\begin{array}{rcll}
T_\epsilon^{(1)}(t,x,u) & := & (t+\epsilon, x, u), & \epsilon \in \mathbb{R},\\
T_\epsilon^{(2)}(t,x,u) & := & (e^{2\epsilon}t, e^{\epsilon}x, u), & \epsilon \in \mathbb{R},\\
T_\epsilon^{(3)}(t,x,u) & := & (\frac{t}{1-4\epsilon t}, \frac{x}{1-4\epsilon t}, \sqrt{1-4\epsilon t}\, e^{\frac{-\epsilon x^2}{1-4\epsilon t}} u), & \epsilon < \frac{1}{4t},\\
T_\epsilon^{(4)}(t,x,u) & := & (t, x+\epsilon, u), & \epsilon \in \mathbb{R},\\
T_\epsilon^{(5)}(t,x,u) & := & (t, x+2\epsilon t, e^{-\epsilon x-\epsilon t^2} u), & \epsilon \in \mathbb{R},\\
T_\epsilon^{(6)}(t,x,u) & := & (t, x, e^{\epsilon} u), & \epsilon \in \mathbb{R}.
\end{array}$$

It will be clear from the proof of the theorem that also

$$S_{\epsilon,w}(t,x,u) := (t,x,u+\epsilon w), \qquad \epsilon \in \mathbb{R},\ w \text{ solution to } (4.16)$$

yields a family of symmetries – in fact a group, due to the vector space structure of the set $\mathcal{S}$ of solutions to (4.16). However, this is not a *one-parameter* group: in fact, its natural generator is

$$B : (t,x,u) \mapsto w(t,x)\frac{\partial}{\partial u},$$

which is however *not* a generator in the sense of our Definition 4.11 since it is a vector field acting on a infinite dimensional vector space $\mathcal{S}$. We omit the detais and refer to **[7]** for a thorough (and more advanced) treatment of this kind of symmetries, which require the introduction of the notion of Lie group.

Beweis. This partial differential equation can be associated with the function

$$H : J^2 \ni (t,x,j_2 u) \mapsto \frac{\partial u}{\partial t}(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x) \in \mathbb{R}.$$

Then, by Theorem 4.23 a vector field A is the infinitesimal generator of a one-parameter symmetry group if and only if

$$((j_2 A)H)(x, j_2 u) = 0 \qquad \text{whenever} \qquad H(x, j_2 u) = 0.$$

Plugging the formulae mentioned in Theorem 4.32 in the above condition, this reads

$$\begin{aligned}
0 = ((j_2 A)H)(x, j_2 u) &= \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x} + \phi\frac{\partial H}{\partial u} + \phi^{(t)}\frac{\partial H}{\partial u_t} + \phi^{(x)}\frac{\partial H}{\partial u_x} + \phi^{(tt)}\frac{\partial H}{\partial u_{tt}} + \phi^{(tx)}\frac{\partial H}{\partial u_{tx}} + \phi^{(xx)}\frac{\partial H}{\partial u_{xx}}\\
&= \phi^{(t)} - \phi^{(xx)} \qquad \text{whenever} \qquad u_t - u_{xx} = 0.
\end{aligned}$$

Now, since

$$\phi^{(t)} = \frac{\partial \phi}{\partial t} - \frac{\partial \xi}{\partial t}\frac{\partial u}{\partial x} + \left(\frac{\partial \phi}{\partial u} - \frac{\partial \tau}{\partial t} - \frac{\partial \xi}{\partial u}\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t} - \frac{\partial \tau}{\partial u}\left(\frac{\partial u}{\partial t}\right)^2,$$

and

$$\begin{aligned}
\phi^{(xx)} = &\frac{\partial^2 \phi}{\partial x^2} + \left(2\frac{\partial^2 \phi}{\partial x \partial u} - \frac{\partial^2 \xi}{\partial x^2}\right)\frac{\partial u}{\partial x} - \frac{\partial^2 \tau}{\partial x^2}\frac{\partial u}{\partial t} + \left(\frac{\partial^2 \phi}{\partial u^2} - 2\frac{\partial^2 \xi}{\partial x \partial u}\right)\left(\frac{\partial u}{\partial x}\right)^2\\
&-2\frac{\partial^2 \tau}{\partial x \partial u}\frac{\partial u}{\partial x}\frac{\partial u}{\partial t} - \frac{\partial^2 \xi}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^3 - \frac{\partial^2 \tau}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial u}{\partial t} + \left(\frac{\partial \phi}{\partial u} - 2\frac{\partial \xi}{\partial x}\right)\frac{\partial^2 u}{\partial x^2}\\
&-2\frac{\partial \tau}{\partial x}\frac{\partial^2 u}{\partial x \partial t} - 3\frac{\partial \xi}{\partial u}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - \frac{\partial \tau}{\partial u}\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial \tau}{\partial u}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t \partial x}
\end{aligned}$$

and since we have to assume that u solves the heat equation, i.e., that $u_t = u_{xx}$, substituting and equating we obtain that

$$\begin{aligned}
&\frac{\partial \phi}{\partial t} - \frac{\partial \xi}{\partial t}\frac{\partial u}{\partial x} + \left(\frac{\partial \phi}{\partial u} - \frac{\partial \tau}{\partial t}\right)\frac{\partial^2 u}{\partial x^2} - \frac{\partial \xi}{\partial u}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - \frac{\partial \tau}{\partial u}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 \\
&= \frac{\partial^2 \phi}{\partial x^2} + \left(2\frac{\partial^2 \phi}{\partial x \partial u} - \frac{\partial^2 \xi}{\partial x^2}\right)\frac{\partial u}{\partial x} - \frac{\partial^2 \tau}{\partial x^2}\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 \phi}{\partial u^2} - 2\frac{\partial^2 \xi}{\partial x \partial u}\right)\left(\frac{\partial u}{\partial x}\right)^2 \\
&\quad -2\frac{\partial^2 \tau}{\partial x \partial u}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 \xi}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^3 - \frac{\partial^2 \tau}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial \phi}{\partial u} - 2\frac{\partial \xi}{\partial x}\right)\frac{\partial^2 u}{\partial x^2} \\
&\quad -2\frac{\partial \tau}{\partial x}\frac{\partial^2 u}{\partial x \partial t} - 3\frac{\partial \xi}{\partial u}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - \frac{\partial \tau}{\partial u}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 - 2\frac{\partial \tau}{\partial u}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}.
\end{aligned}$$

Both on the LHS and the RHS there are polynomials in the variables

$$1, u_x, (u_x)^2, (u_x)^3, u_{xx}, u_x u_{xx}, (u_x)^2 u_{xx}, (u_{xx})^2, u_{tx}, u_x u_{tx} :$$

they are equal if and only if the coefficient of each monomial vanishes.

- Setting to 0 the coefficient of u_{tx} and the coefficient of $u_x u_{tx}$ we obtain that
$$\frac{\partial \tau}{\partial u} = 0 = \frac{\partial \tau}{\partial x},$$
i.e., τ **does not depend either on** x **nor on** u (and in particular also $\tau_{uu} = 0$ and $\tau_{xu} = 0$).
- The condition given by $(u_x)^2$ is empty, since it requires that $\tau_u = \tau_u$.
- The condition on the coefficient of $(u_x)^2 u_{xx}$ shows that $\tau_{uu} = 0$, i.e., τ is independent of u (but this was already known).
- The condition on the coefficient of $(u_x)u_{xx}$ shows that $\xi_u = 2\tau_{xu} + 3\xi_u$, i.e.,
$$\xi_u = 3\xi_u,$$
i.e., ξ **is independent of** u (and in particular $\xi_{xu} = 0$).
- Passing to the coefficient of u_{xx} we see that $\phi_u - \tau_t = -\tau_{xx} + \phi_u - 2\xi_x$, i.e.,
$$\tau_t = 2\xi_x,$$
and integrating with respect to x we obtain
$$\xi(t, x) = \frac{1}{2}\tau_t(t)x + c(t),$$
where the integration constant c (with respect to x) may indeed depend on t. In particular,
$$\xi_t(t, x) = \frac{1}{2}\tau_{tt}(t)x + c_t(t) \qquad \text{and} \qquad \xi_{xx}(t, x) = 0. \tag{4.17}$$
- Since ξ is independent on u, $\xi_{uu} = 0$ and the condition on $(u_x)^3$ is empty.
- Similarly, to the second-last case, the condition on $(u_x)^2$ yields
$$0 = \phi_{uu} - 2\xi_{xu} = \phi_{uu},$$
and integrating with respect to u yields
$$\phi(t, x, u) = b(t, x)u + a(t, x), \tag{4.18}$$

for some integration constants a, b (with respect to u). In particular,

$$\phi_{xu}(t,x,u) = b_x(t,x). \tag{4.19}$$

- The condition on u_x yields
$$-\xi_t = 2\phi_{xu} - \xi_{xx} = 2\phi_{xu}.$$
Plugging (4.17) and (4.19) in the last equation we obtain
$$-\frac{1}{2}\tau_{tt}(t)x + c_t(t) = 2b_x(t,x), \tag{4.20}$$
and integrating with respect to x yields
$$-\frac{1}{8}\tau_{tt}(t)x^2 - \frac{1}{2}c_t(t)x + d(t) = b(t,x), \tag{4.21}$$
for some integration constant d (with respect to x).
- Finally, the coefficient condition in 1 yields
$$\phi_t = \phi_{xx}.$$
Taking into account (4.18) we deduce that
$$b_t u + a_t = b_{xx}u + a_{xx}.$$
This is another polynomial in the variables $1, u$, and equating the coefficients of the terms corresponding to the same monomial yields that
$$b_t = b_{xx} \quad \text{as well as} \quad a_t = a_{xx}.$$
Differentiating (4.21) with respect to t and also (4.20) with respect to x yields
$$-\frac{1}{8}\tau_{ttt}(t)x^2 - \frac{1}{2}c_{tt}(t)x + d_t(t) = b_t(t,x) = b_{xx}(t,x) = -\frac{1}{4}\tau_{tt}(t). \tag{4.22}$$
This is a polynomial in $1, x, x^2$. Equating to 0 all the coefficient of same degree yields
$$\tau_{ttt}(t) = 0, \quad c_{tt}(t) = 0, \quad d_t(t) + \frac{1}{4}\tau_{tt}(t) = 0.$$
Accordingly, τ is at most quadratic and c is at most linear with respect to t, i.e.,
$$\tau(t) = c_1 + 2c_2t + 4c_3t^2$$
and
$$c(t) = c_4 + 2c_5t. \tag{4.23}$$
for some $c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}$.

Integrating both d_t and τ_{tt} with respect to t yields

$$d(t) = -\frac{1}{4}\tau_t(t) + c_7. \tag{4.24}$$

This finally yields

$$\begin{aligned} \tau(t) &= c_1 + 2c_2t + 4c_3t^2, \\ \xi(t,x) &= c_4 + c_2x + 2c_5t + 4c_3tx, \\ \phi(t,x,u) &= (c_6 - c_5x - 2c_3t - c_3x^2)u + a(t,x). \end{aligned}$$

for arbitrary numbers $c_1, c_2, c_3, c_4, c_5, c_6$, and where a is any function that satisfy the heat equation. The equation for ϕ is obtained plugging (4.23)–(4.24) and the equation for τ in (4.21), and then in (4.18).

Setting recursively all but one of these coefficient to 0 we obtain the following six vector fields as generators of one-parameter point symmetry groups:

$$\begin{aligned}
A_1 &= \frac{\partial}{\partial t} \\
A_2 &= 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} \\
A_3 &= 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - (2t+x^2)u\frac{\partial}{\partial u} \\
A_4 &= \frac{\partial}{\partial x} \\
A_5 &= 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u} \\
A_6 &= u\frac{\partial}{\partial u}.
\end{aligned}$$

An easy computation shows that differentiating the group $\mathcal{T}^{(k)}$ at $\epsilon = 0$ one obtains A_k: this yields the claim. For example, let us check that

$$A_3 \equiv \left(4t^2, 4tx, -(x^2+2t)u\right)$$

is actually the infinitesimal generator of

$$T_\epsilon^{(3)}(t,x,u) \equiv \left(\frac{t}{1-4\epsilon t}, \frac{x}{1-4\epsilon t}, \sqrt{1-4\epsilon t}\, e^{-\frac{\epsilon x^2}{1-4\epsilon t}} u\right), \qquad \epsilon < \frac{1}{4t},\ (t,x,u) \in J^0.$$

In fact, one checks directly that

$$\frac{\partial}{\partial \epsilon}\frac{t}{1-4\epsilon t} = -\frac{-4t^2}{(1-4\epsilon t)^2},$$

hence

$$\frac{\partial}{\partial \epsilon}\frac{t}{1-4\epsilon t}\Big|_{\epsilon=0} = 4t^2.$$

Likewise,

$$\frac{\partial}{\partial \epsilon}\frac{x}{1-4\epsilon t} = -\frac{-4xt}{(1-4\epsilon t)^2},$$

hence

$$\frac{\partial}{\partial \epsilon}\frac{x}{1-4\epsilon t}\Big|_{\epsilon=0} = 4tx.$$

Finally,

$$\begin{aligned}
\frac{\partial}{\partial \epsilon}\sqrt{1-4\epsilon t}\, e^{-\frac{\epsilon x^2}{1-4\epsilon t}} u &= \frac{1}{2}\frac{-4t}{\sqrt{1-4\epsilon t}}\, e^{-\frac{\epsilon x^2}{1-4\epsilon t}} u + \sqrt{1-4\epsilon t}\left(-\frac{x^2(1-4\epsilon t)+\epsilon x^2 4t}{(1-4\epsilon t)^2}\right) e^{-\frac{\epsilon x^2}{1-4\epsilon t}} u \\
&= -\left(\frac{2t}{\sqrt{1-4\epsilon t}} + \sqrt{1-4\epsilon t}\left(\frac{x^2}{(1-4\epsilon t)^2}\right)\right) e^{-\frac{\epsilon x^2}{1-4\epsilon t}} u,
\end{aligned}$$

and evaluating at $\epsilon = 0$ we obtain

$$\frac{\partial}{\partial \epsilon}\sqrt{1-4\epsilon t}\, e^{-\frac{\epsilon x^2}{1-4\epsilon t}} u\Big|_{\epsilon=0} = -\left(2t+x^2\right)u,$$

thus completing the proof. $\square$

Exercise 4.35. *Work out the details in order to show that the vector field A_k is actually the infinitesimal generator of the one-parameter point transformation group $\mathcal{T}^{(k)}$, $k = 1, \dots, 6$.*

Exercise 4.36. *Use the above obtained formulae for the 2-jet of vector fields in order to determine the point symmetry groups of the 1-dimensional wave equation*

$$\frac{\partial^2 u}{\partial t^2}(t, x) - \frac{\partial^2 u}{\partial x^2}(t, x) = 0.$$

Exercise 4.37. *Use the above obtained formulae for the 2-jet of vector fields in order to determine the point symmetry groups of the 2-dimensional Laplace equation*

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0.$$

Corollary 4.38. *The function*

$$\Phi : (t, x) \mapsto \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \qquad t > 0,\ x \in \mathbb{R}, \tag{4.25}$$

satisfies the one-dimensional heat equation, i.e., there holds

$$\frac{\partial \Phi}{\partial t}(t, x) = \frac{\partial^2 \Phi}{\partial x^2}(t, x), \qquad t > 0,\ x \in \mathbb{R}.$$

A general problem arising when discussing symmetry groups is the following: when a point transformation T acts on a 0-jet, say (x, u), then we obtain $(\tilde{x}, \tilde{u})$ in dependence of the old variables x, u. In order to express $\tilde{u}$ in dependence of its natural variable $\tilde{x}$[5], it is natural to try to invert the first relation expressing $\tilde{x}$ in function of x and $u(x)$: this is generally possible by an elementary application of the inverse function theorem in a suitably small neighbourhood of $\epsilon = 0$, using the fact that T_0 acts as the identity on J^0. Having done this, it will usually be easy to plug this expression for x (in dependence of $\tilde{x}$ and $u(x)$) into the expression for $\tilde{u}(x)$. Performing this in the concrete case of the transformation groups $\mathcal{T}^{(3)}$ and $\mathcal{T}^{(1)}$ will be the main issue in the proof below.

Beweis. Observe in particular that the fact that $(T_\epsilon^{(3)})_{\epsilon \in \mathbb{R}}$ and $(T_\epsilon^{(1)})_{\epsilon \in \mathbb{R}}$ are symmetry groups implies that if (t, x, f) is a solution to the heat equation, then so are also its transforms

$$(\tilde{t}, \tilde{x}, \tilde{u}) := T_{\epsilon_1}^{(3)}(t, x, u)$$

and

$$(\hat{t}, \hat{x}, \hat{u}) := T_{\epsilon_2}^{(1)}(\tilde{t}, \tilde{x}, \tilde{u}),$$

for any $\epsilon \in \mathbb{R}$ and $\epsilon_2 < \frac{1}{4t}$. We will apply in particular this argument to the trivial solution

$$\gamma(t, x) :\equiv c, \qquad t > 0,\ x \in \mathbb{R},$$

where c is any real number.

To begin with, let us express t and x in function of $\tilde{t}$ and $\tilde{x}$: it follows from

$$\tilde{t}(t) = \frac{t}{1 - 4\epsilon_1 t} \qquad \text{and} \qquad \tilde{x}(t, x) = \frac{x}{1 - 4\epsilon_1 t}$$

that

$$t(\tilde{t}) = \frac{\tilde{t}}{1 + 4\epsilon_1 \tilde{t}} \qquad \text{and} \qquad x(\tilde{t}, \tilde{x}) = \frac{\tilde{x}}{1 + 4\epsilon_1 \tilde{t}}.$$

[5] It is $(\tilde{x}, \tilde{u})$ – and not $(x, \tilde{u})$! – that solves (4.6) if (x, u) does.

Plugging these formulae into

$$\tilde{u}(t,x,u) = \sqrt{1-4\epsilon_1 t}\; e^{-\frac{\epsilon_1 x^2}{1-4\epsilon_1 t}} u(t,x)$$

we obtain the ugly expression

$$\tilde{u}(\tilde{t},\tilde{x},u) = \sqrt{1-\frac{4\epsilon_1\tilde{t}}{1+4\epsilon_1\tilde{t}}}\; e^{-\frac{\epsilon_1\left(\frac{\tilde{x}}{1+4\epsilon_1\tilde{t}}\right)^2}{1-\frac{4\epsilon_1\tilde{t}}{1+4\epsilon_1\tilde{t}}}}\, u\left(\frac{\tilde{t}}{1+4\epsilon_1\tilde{t}},\frac{\tilde{x}}{1+4\epsilon_1\tilde{t}}\right),$$

which can be luckily simplified to

$$\begin{aligned}\tilde{u}(\tilde{t},\tilde{x},u) &= \frac{1}{\sqrt{1+4\epsilon_1\tilde{t}}}\; e^{-\frac{\epsilon_1\left(\frac{\tilde{x}}{1+4\epsilon_1\tilde{t}}\right)^2}{\frac{1}{1+4\epsilon_1\tilde{t}}}}\, u\left(\frac{\tilde{t}}{1+4\epsilon_1\tilde{t}},\frac{\tilde{x}}{1+4\epsilon_1\tilde{t}}\right)\\ &= \frac{1}{\sqrt{1+4\epsilon_1\tilde{t}}}\; e^{-\frac{\epsilon_1\tilde{x}^2}{1+4\epsilon_1\tilde{t}}} u\left(\frac{\tilde{t}}{1+4\epsilon_1\tilde{t}},\frac{\tilde{x}}{1+4\epsilon_1\tilde{t}}\right).\end{aligned}$$

Knowing an explicit expression of u allows in most cases to drop the explicit dependence on u, hence getting a closed formula for $\tilde{u}(\tilde{t},\tilde{x})$. In the special case of the trivial solution γ we obtain

$$\tilde{\gamma}(\tilde{t},\tilde{x}) = \frac{c}{\sqrt{1+4\epsilon_1\tilde{t}}}\; e^{-\frac{\epsilon_1\tilde{x}^2}{1+4\epsilon_1\tilde{t}}}, \tag{4.26}$$

which is therefore a solution for all $c\in\mathbb{R}$ and all $\epsilon_1 > -\frac{1}{4\tilde{t}}$. Let us know compute $(\hat{t},\hat{x},\hat{\gamma}) = T^{(1)}_{\epsilon_2}(\tilde{t},\tilde{x},\tilde{\gamma})$. This is quite easy since all we have to do is to observe that

$$\tilde{t}(\hat{t}) = \hat{t}-\epsilon_2 \qquad \text{since} \quad \hat{t}(\tilde{t}) = \tilde{t}+\epsilon_2.$$

Plugging this into (4.26) yields

$$\hat{\gamma}(\hat{t},\hat{x}) = \frac{c}{\sqrt{1+4\epsilon_1(\hat{t}-\epsilon_2)}}\; e^{-\frac{\epsilon_1\hat{x}^2}{1+4\epsilon_1(\hat{t}-\epsilon_2)}},$$

since $\hat{x}=\tilde{x}$. While such a function $\hat{\gamma}$ is a solution to the heat equation for all small ϵ_1, all $\epsilon_2\in\mathbb{R}$ and all $c\in\mathbb{R}$, the choice of $\epsilon_2 = \frac{1}{4\epsilon_1}$ looks natural. It yields

$$\hat{\gamma}(\hat{t},\hat{x}) = \frac{c}{\sqrt{4\epsilon_1\hat{t}}}\; e^{-\frac{\epsilon_1\hat{x}^2}{4\epsilon_1\hat{t}}} = \frac{c}{\sqrt{4\epsilon_1\hat{t}}}\; e^{-\frac{\hat{x}^2}{4\hat{t}}}.$$

Finally, chosing $c=\sqrt{\epsilon_1\pi^{-1}}$ yields the claim. □

Remark 4.39. *Observe that while the formula* (4.25) *defines a function that is continuous at* $t=0$*, and hence we can consider the continuous extension*

$$\Phi(t,x) := \begin{cases} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, & t>0,\\ 0 & t=0,\end{cases}$$

this function cannot be extended to $t<0$*. In fact, it can be proved by more refined methods (e.g., functional analytical ones) that the heat equation is not uniquely solvable backward in time.*

We are finally in the position to discuss the initial value problem associated with the heat equation on $\mathbb{R}^d$. Even if more solution might be constructed through these symmetry groups, starting from the trivial solution, Φ defined in (4.25) is the most general one (in $\mathbb{R}$), in the sense made precise below.

A natural conjecture is that the solution Φ be extendable to general Euclidean spaces $\mathbb{R}^d$ by considering

$$(t,x) \mapsto \frac{1}{\sqrt{4\pi t}} e^{-\frac{\|x\|^2}{4t}}, \qquad t>0\ x \in \mathbb{R}^d.$$

Unfortunately, this is not a smart choice. This is due to the observation that in the 1-dimensional case one has for all $t>0$

$$\int_{-\infty}^{\infty} \Phi(t,x)dx = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4t}} dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz = 1.$$

This normalisation property will be useful in the following, as we will see, but it does not hold for the naive d-dimensional extension proposed above. However, the above computation suggests a solution: considering instead

$$\Phi : (t,x) \mapsto \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{\|x\|^2}{4t}}, \qquad t>0\ x \in \mathbb{R}^d, \tag{4.27}$$

yields (by Fubini's theorem)

$$\int_{\mathbb{R}^d} \Phi(t,x)dx = \frac{1}{\pi^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-z^2} dz = \frac{1}{\pi^{\frac{d}{2}}} \prod_{i=1}^{d} \int_{-\infty}^{\infty} e^{-z_i^2} dz = 1, \qquad t>0. \tag{4.28}$$

Exercise 4.40. *Check that $\Phi : \mathbb{R}^d \to \mathbb{R}$ defined in (4.27) satisfies the heat equation.*

Definition 4.41. *The function $\Phi : (0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ defined in (4.25) is called the* **Gaussian kernel** *or the* **fundamental solution to the heat equation**.

Theorem 4.42. *Let $u_0 \in C_b(\mathbb{R}^d)$ and define*

$$u(t,x) := \int_{\mathbb{R}^d} \Phi(t,x-y)u_0(y)dy = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{\|x-y\|^2}{4t}} u_0(y)dy, \qquad t>0,\ x \in \mathbb{R}^d.$$

Then the following assertions hold:

(1) $u \in C^\infty((0,\infty) \times \mathbb{R}^d)$,

(2) u satisfies

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x), \qquad t>0,\ x \in \mathbb{R}^d,$$

(3) one has for all $x_0 \in \mathbb{R}^d$

$$\lim_{(t,x)\to(0,x_0)} u(t,x) = u_0(x_0).$$

Beweis. (1) The assertion follows from the fact that the Gaussian kernel is an infinitely differentiable function with bounded derivatives.

Let us first show that $u \in C^1((0,\infty) \times \mathbb{R}^d)$. In fact, for all $i = 1, \ldots, d$ and all $h \in \mathbb{R}$ one can compute the i^{th} incremental quotient, which is given by

$$\frac{u(t,x+he_i) - u(t,x)}{h} = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \frac{\Phi(t,x+he_i-y) - \Phi(t,x-y)}{h} u_0(y)dy.$$

Due to the fact that Φ has bounded derivative we deduce that

$$\lim_{h\to 0}\frac{\Phi(t,x+he_i-y)-\Phi(t,x-y)}{h}=\frac{\partial\Phi}{\partial x_i}(t,x-y)$$

and therefore

$$\lim_{h\to 0}\frac{\Phi(t,x+he_i-y)-\Phi(t,x-y)}{h}u_0(y)=\frac{\partial\Phi}{\partial x_i}(t,x-y)u_0(y)$$

uniformly in $x\in\mathbb{R}^d$, and by Lebesgue's dominated convergence theorem we deduce that also

$$\lim_{h\to 0}\int_{\mathbb{R}^d}\frac{\Phi(t,x+he_i-y)-\Phi(t,x-y)}{h}u_0(y)dx=\int_{\mathbb{R}^d}\frac{\partial\Phi}{\partial x_i}(t,x-y)u_0(y),$$

q.e.d.

A similar claim holds when considering the time derivative, using boundedness on each interval $[\delta,\infty)$ of the Gaussian kernel and of its derivatives with respect to t, for any $\delta>0$. Clearly, the general claim can be proved recursively.

(2) We also get

$$\frac{\partial u}{\partial t}(t,x)-\Delta u(t,x)=\int_{\mathbb{R}^d}\left(\frac{\partial\Phi}{\partial t}-\Delta\Phi\right)(t,x-y)u_0(y)dy=0,\qquad t>0,\ x\in\mathbb{R}^d,$$

where the latter identity follows from the fact that Φ is itself a solution to the heat equation.

(3) Finally, let us check that u satisfies the initial condition. Let $x_0\in\mathbb{R}^d$ and $\epsilon>0$. By continuity of u_0 there exists $\delta>0$ such that

$$y\in B_\delta(x_0)\quad\Rightarrow\quad g(y)\in B_\delta(g(x_0)).$$

Taking $x\in B_{\frac{\delta}{2}}(x_0)$ we deduce that

$$|u(t,x)-u_0(x_0)|=\left|\int_{\mathbb{R}^d}\Phi(t,x-y)u_0(y)dy-u_0(x_0)\right|=\left|\int_{\mathbb{R}^d}\Phi(t,x-y)\left(u_0(y)-u_0(x_0)\right)dy\right|,$$

where the last equality holds due to (4.28). Splitting the integral into a part in $B_\delta(x_0)$ (where we can use closeness of $g(y)$ to $g(x_0)$) and an exterior part (where we can use the exponential decay of the Gaussian kernel) we obtain

$$|u(t,x)-u_0(x_0)|\leq\int_{\mathbb{B}_\delta(x_0)}\Phi(t,x-y)|(u_0(y)-u_0(x_0)|dy+\int_{\mathbb{R}^d\setminus\mathbb{B}_\delta(x_0)}\Phi(t,x-y)|u_0(y)-u_0(x_0)|dy=:I_1(t)+I_2(t).$$

Then, again by (4.28)

$$I_1(t)\leq\epsilon\int_{\mathbb{R}^d}\Phi(t,x-y)dy=\epsilon\int_{\mathbb{R}^d}\Phi(t,z)dz=\epsilon,$$

and moreover for all $x \in B_{\frac{\delta}{2}}(x_0)$[6]

$$\begin{aligned} I_2(t) &\leq 2\|u_0\|_\infty \int_{\mathbb{R}^d\setminus\mathbb{B}_\delta(x_0)} \Phi(t, x-y)dy \\ &\leq \frac{C}{t^{\frac{n}{2}}} \int_{\mathbb{R}^d\setminus\mathbb{B}_\delta(x_0)} e^{-\frac{|x-y|^2}{4t}} dy \\ &\leq \frac{C}{t^{\frac{n}{2}}} \int_{\mathbb{R}^d\setminus\mathbb{B}_\delta(x_0)} e^{-\frac{|x_0-y|^2}{16t}} dy. \end{aligned}$$

Having obtained an estimate involving the integral of a radial function, it is natural to pass to radial coordinates. We then obtain

$$I_2(t) \leq \frac{C}{t^{\frac{n}{2}}} \int_\delta^\infty e^{-r^2} 16tr^{n-1}dr.$$

Therefore,

$$\lim_{t\to 0+} I_2(t) = 0.$$

This ensures that for t suitably small $I_2(t) \leq \epsilon$. All in all, we conclude that

$$|u(t,x) - u_0(x_0)| \leq I_1(t) + I_2(t) \leq 2\epsilon$$

whenever t is small and $x \in B_{\frac{\delta}{2}}(x_0)$. □

Remark 4.43. *We have just seen that no matter how regular the initial data is, the solution to the heat equation will be of class C^∞ – in fact, by density one can extend such a result to all initial data in $L^p(\mathbb{R}^d)$, for any $p \in [1, \infty]$. Moreover, since the solution $u(t,x)$ is obtained integrating the initial data against a strictly positive* **kernel** *$k_t(x, \cdot) := \Phi(t, x - \cdot)$ over all $\mathbb{R}^d$, it follows that modifications of the initial data inside a bounded subset of $\mathbb{R}^d$ affect the solution everywhere and immediately*[7]*. These properties make a heat equations very different from wave and, more generally, hyperbolic equation. Partial differential equations displaying the above behaviour are called* **parabolic** *.*

Exercise 4.44. *Strictly speaking, what we have just done is not (yet) finding* the *fundamental solution to the heat equation, but rather just* a *solution to it – in particular because picking other symmetry groups might in principle yield different, possibly more general solutions. To feel reassured that this is not possible, introduce a suitable energy function and exploit a method similar to that used in the proof of Theorem* (2.18) *in order to get uniqueness of the solution to the heat equation on any bounded domain (with Dirichlet or Neumann boundary condition), for any given initial condition.*

Exercise 4.45. *Determine the point symmetries of the Hodgkin–Huxley-equation 8.*

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[6] Here we use the fact that

$$|y - x_0| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{1}{2}|y - x_0|,$$

whence

$$\frac{1}{2}|y - x_0| \leq |y - x|,$$

whenever $|x - x_0| \leq \frac{\delta}{2}$ and $|y - x_0| \geq \delta$.

[7] In particular, if the initial data is strictly positive at even only a single point of $\mathbb{R}^d$, the solution is strictly positive everywhere for any $t > 0$ (so-called **strong maximum principle**).

4.3. The Burger equation

The Burgers' equation is one of the easiest nonlinear PDEs. It appears in an elementary study of fluid dynamics and shares some features with more sophisticated models, like the Korteweg–de Vries equation.

Consider a thin pipe containing a viscid fluid, which can only move in one direction (say, due to gravity). Checking the derivation of the linear transport equation in Chapter 2 one sees that the decisive choice was that of the flow function. In the linear case (describing a material with constant velocity), we assumed the flow function to be proportional to the density by $\psi = cu$. While this is a good approximation for very thin pipes and very unviscid fluids, it is not acceptable in several situations (in particular, whenever considering shallow, broad channels). Instead of fomally deriving a fluid function ψ modelling this system, we propose

$$\psi(t,x) := \frac{\partial u^2}{\partial x}(t,x) = u(t,x)(\beta - u(t,x)), \qquad t > 0,\ x \in \mathbb{R},$$

for a suitable parameter $\beta > 0$, and try to convince the reader of its plausibility[8]. If u is small, then $\beta - u \approx \beta$ and $\psi \approx \beta u$ – i.e., we obtain the linear transport equation: a less dense fluid is unlikely to develop vortices (liquids), swirls (gases), congestions (traffic) and so on. If however u grows and gets closer to β, then $\psi \approx 0$ and the flow becomes steadier and steadier. Plugging the above flow function into the general transport equation we obtain

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial}{\partial x}\big(u(t,x)(\beta - u(t,x))\big), \qquad t \geq 0,\ x \in \mathbb{R}, \tag{4.29}$$

and introducing the auxiliary function

$$v(t,x) := \beta - 2u\left(\frac{t}{2}, x\right)$$

we see that it satisfies

$$\begin{aligned} \frac{\partial v}{\partial t}(t,x) + v(t,x)\frac{\partial v}{\partial x}(t,x) &= -\frac{\partial u}{\partial t}\left(\frac{t}{2},x\right) + \beta - 2u\left(\frac{t}{2},x\right)\left(-2\frac{\partial u}{\partial x}\left(\frac{t}{2},x\right)\right) \\ &= -\left(\frac{\partial u}{\partial t} - 2\beta\frac{\partial u}{\partial x} - 4u\frac{\partial u}{\partial x}\right)\left(\frac{t}{2},x\right) \\ &= -\frac{\partial u}{\partial t}\left(\frac{t}{2},x\right) - 2\frac{\partial u}{\partial x}\left(\frac{t}{2},x\right)\left(\beta - u\left(\frac{t}{2},x\right)\right). \end{aligned}$$

Hence, upon time rescaling the nonlinear transport equation (4.29) is equivalent to

$$\frac{\partial v}{\partial t}(t,x) = -2v(t,x)\frac{\partial v}{\partial x}(t,x), \qquad t \geq 0,\ x \in \mathbb{R},$$

i.e., to

$$\frac{\partial v}{\partial t}(t,x) = -\frac{\partial v^2}{\partial x}(t,x), \qquad t > 0,\ x \in \mathbb{R},$$

the so-called **inviscid Burgers' equation**.

While a solution to the original Burgers' equation can be found in very special cases by the method of characteristics, see e.g. [**3**, § 2.1.3], this approach is not practicable in generality. A usual workaround is to *artificially*

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[8] We remark the similarity to the function appearing in the *logistic equation* appearing in the theory of ODEs, which actually is designed to present a similar behaviour, in some sense. Also observe that introducing a flow function $\psi(t,x) := -u(t,x)(\beta - u(t,x))$ simply models a transport in the opposite direction.

add a so-called *viscosity term*[9] and consider instead the family of heat-like partial differential equations

$$\frac{\partial v}{\partial t}(t,x) = \epsilon \frac{\partial^2 v}{\partial x^2}(t,x) - \frac{\partial v^2}{\partial x}(t,x), \qquad t \geq 0,\ x \in \mathbb{R}, \tag{4.30}$$

where $\epsilon > 0$. This is the so-called **viscid Burgers' equation**. A strategy in numerical analysis is then to derive a solution $u = u_\epsilon$ to (4.30). Letting $\epsilon \to 0$ will hopefully yield a solution (and a unique one!) to the original Burgers' equation. This can be made more precise but we do not go into details.

Instead, we would like to discuss (4.30) itself, for $\epsilon = 1$. First of all, we turn instead to the related equation

$$\frac{\partial w}{\partial t}(t,x) = \frac{\partial^2 w}{\partial x^2}(t,x) - \left(\frac{\partial w}{\partial x}\right)^2 (t,x), \qquad t > 0,\ x \in \mathbb{R} : \tag{4.31}$$

observe that if v is a solution to (4.31), then

$$v(t,x) := \frac{\partial w}{\partial x}(t,x), \qquad t > 0,\ x \in \mathbb{R},$$

solves (4.30). To get to grips with (4.31) we will follow [**7**, Exa. 2.42] and apply the method of symmetries developed in Chapter 4: to turn it into an algebraic equation one has to consider $H : J^2 \to \mathbb{R}$ defined by

$$H(t,x,j_2 u) := \frac{\partial u}{\partial t}(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x) + \frac{\partial u}{\partial x}(t,x)\frac{\partial u}{\partial x}(t,x) \in \mathbb{R}.$$

Let A be an arbitrary vector field on J^0,

$$A(t,x,u) := \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \phi(t,x,u)\frac{\partial}{\partial u}.$$

Then, considering the 2-jet of A, i.e.,

$$((j_2 A)H)(t,x,j_2 w) = \tau\frac{\partial H}{\partial t} + \xi\frac{\partial H}{\partial x} + \phi\frac{\partial H}{\partial u} + \phi^{(t)}\frac{\partial H}{\partial w_t} + \phi^{(x)}\frac{\partial H}{\partial w_x} + \phi^{(tt)}\frac{\partial H}{\partial w_{tt}} + \phi^{(tx)}\frac{\partial H}{\partial w_{tx}} + \phi^{(xx)}\frac{\partial H}{\partial w_{xx}}$$

and applying Theorem 4.23 one sees that A is the infinitesimal generator of a one-parameter point symmetry group if and only if

$$\phi^{(t)} - 2\phi^{(x)}\frac{\partial w}{\partial x} + \phi^{(xx)} = 0.$$

[9] This is mainly motivated by the analysis to be performed soon, but can be physically justified by saying it models diffusion (along convection), a phenomenon which in large pipes can never be completely neglected.

Proceeding as in the proof of Theorem 4.42 we deduce that there exist six (linearly independent) vector fields that generate a one-parameter point-symmetry group: these are

$$\begin{aligned} C_1 &= \frac{\partial}{\partial t} \\ C_2 &= 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} \\ C_3 &= 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} + (2t+x^2)\frac{\partial}{\partial w} \\ C_4 &= \frac{\partial}{\partial x} \\ C_5 &= 2t\frac{\partial}{\partial x} + x\frac{\partial}{\partial w} \\ C_6 &= -\frac{\partial}{\partial w} \end{aligned}$$

(along with

$$D = z(t,x)e^w\frac{\partial}{\partial w}$$

for any solution z of the *heat* equation).

Exercise 4.46. *Work out the details leading to determination of the coefficients τ, ξ, ϕ in the above discussion.*

The above vector fields look familiar: in fact, one sees that they correspond to the six generators of one-parameter point symmetry groups of the heat equation, upon formally replacing $\frac{\partial}{\partial w}$ by $-u\frac{\partial}{\partial u}$. This suggests to perform a substitution $w \mapsto u = \Psi(w)$ that yields a PDE whose symmetry generators are $A_1, \ldots, A_6$ instead of $C_1, \ldots, C_6$. While this is only a heuristical argument, introducing

$$u(t,x) = e^{-w(t,x)}, \qquad t > 0,\ x \in \mathbb{R},$$

does transform the Burgers' equation into the heat equation. This is known as **Hopf–Cole transformation**. In fact,

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x) &= \frac{\partial e^{-w}}{\partial t}(t,x) - \frac{\partial^2 e^{-w}}{\partial x^2}(t,x) \\ &= -\frac{\partial w}{\partial t}(t,x)e^{-w(t,x)} + \frac{\partial^2 w}{\partial x^2}(t,x)e^{-w(t,x)} - \left(\frac{\partial w}{\partial x}\right)^2(t,x)e^{-w(t,x)}, \end{aligned}$$

i.e., u satisfies the heat equation if and only if w satisfies the Burgers' equation (4.31). We hence get a fundamental solution

$$(t,x) \mapsto \log\left(\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}\right) = -\log\sqrt{4\pi t} - \frac{x^2}{4t}, \qquad t > 0,\ x \in \mathbb{R},$$

for (4.31).

KAPITEL 5

Hilbert spaces and the method of separation of variables

In this chapter we present some elementary results from the theories of Hilbert spaces and Fourier series. This introduction is not meant to be complete, but only to yield some technical tools that are useful for the study of PDEs. In view of some applications it is useful to develop this theory also in the complex case.

Definition 5.1. *Let H be a vector space over a field $\mathbb{K}$. An **inner product** $(\cdot|\cdot)$ on H is a* **norm**, *i.e., a mapping $H \times H \to \mathbb{K}$ such that for all $x, y \in H$ and all $\lambda \in \mathbb{K}$*
(1) $(x|x) \geq 0$ and $(x|x) = 0 \Leftrightarrow x = 0$,
(2) $(\lambda x|y) = \lambda(x|y)$,
(3) $(x|y + z) = (x|y) + (x|z)$,
(4) $(x|y) = \overline{(y|x)}$.
*Then, H is called a **pre-Hilbert space**.*

Lemma 5.2 (Cauchy–Schwarz inequality)**.** *Let H be a pre-Hilbert space. Then for all $x, y \in H$ one has*

$$|(x|y)|^2 \leq (x|x)(y|y).$$

Beweis. Let $x, y \in H$. If $y = 0$, the assertion is clear. Otherwise, for all $\lambda \in \mathbb{K} = \mathbb{C}$ we have

$$0 \leq (x - \lambda y|x - \lambda y)_H = (x|x)_H + |\lambda|^2(y|y)_H - 2\mathrm{Re}(\overline{\lambda}(y|x)_H).$$

Setting $\lambda = \frac{(x|y)_H}{(y|y)_H}$ we obtain

$$0 \leq (x|x)_H + \frac{|(x|y)_H|^2}{(y|y)_H} - 2\frac{|(x|y)_H|^2}{(y|y)_H} = (x|x)_H - \frac{|(x|y)_H|^2}{(y|y)_H}.$$

Accordingly,

$$(x|x)_H \geq \frac{|(x|y)_H|^2}{(y|y)_H},$$

whence the claim follows. The case of $\mathbb{K} = \mathbb{R}$ can be discussed similarly. □

Remark 5.3. *Let H be a pre-Hilbert space and define a mapping $\|\cdot\|$ by*

$$\|x\| := \sqrt{(x|x)}, \qquad x \in H. \tag{5.1}$$

Then $\|\cdot\|$ is a **norm** *of H, i.e., a mapping $H \to \mathbb{R}_+$ such that for all $x, y \in H$ and all $\lambda \in \mathbb{K}$*
(1) $\|x\| = 0 \Leftrightarrow x = 0$,
(2) $\|\lambda x\| = |\lambda|\|x\|$, and
(3) $\|x + y\| \leq \|x\| + \|y\|$.

Corollary 5.4 (Triangle inequality)**.** *Let H be a pre-Hilbert space. Then for all $x, y \in H$ one has*

$$\|x + y\| \leq \|x\| + \|y\|.$$

Beweis. There holds

$$\begin{aligned} \|x+y\|^2 &= (x+y|x+y) \\ &= \|x\|^2 + 2\mathrm{Re}(x|y) + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2. \end{aligned}$$

This concludes the proof. □

Corollary 5.5 (Young inequality)**.** *Let H be a pre-Hilbert space. Then for all $x, y \in H$ one has*

$$|(x|y)_H| \leq \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2.$$

Beweis. Apply the Cauchy–Schwarz inequality and the fact that $(a-b)^2 \geq 0$ for all $a, b \in \mathbb{R}$, hence in particular for $\|x\|, \|y\|$. □

Exercise 5.6. *Prove the following extensions of the Young inequality.*
(1) $(x|y)_H \leq \epsilon\|x\|^2 + \frac{1}{4\epsilon}\|y\|^2$ *for all* $\epsilon > 0$ *and all* $x, y \in H$*.*
(2) $(x|y)_H \leq \frac{1}{p}\|x\|^p + \frac{1}{q}\|y\|^q$ *for all* $p, q \in (1, \infty)$ *such that* $p^{-1} + q^{-1} = 1$ *and all* $x, y \in H$*.*
(3) $(x|y)_H \leq \epsilon\|x\|^p + \frac{1}{q(\epsilon p)^{\frac{q}{p}}}\|y\|^q$ *for all* $p, q \in (1, \infty)$ *such that* $p^{-1} + q^{-1} = 1$*, all* $\epsilon > 0$ *and all* $x, y \in H$*.*
(Hint for (2): use convexity of the exponential function.)

Remark 5.7. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open domain and consider the mapping*

$$(f|g) := \int_\Omega f(x)\overline{g(x)}dx, \qquad f, g \in C(\overline{\Omega}; \mathbb{C}).$$

Although one can prove that $C(\overline{\Omega}; \mathbb{C})$ is not complete with respect to the associated norm (why?), one can actually easily see that this $(\cdot|\cdot)$ does define an inner product, i.e., $C(\overline{\Omega})$ is a pre-Hilbert space with respect to it. In particular, the Cauchy–Schwarz and Young inequalities

$$\int_\Omega f(x)\overline{g(x)}dx \leq \left(\int_\Omega |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |g(x)|^2 dx\right)^{\frac{1}{2}} \leq \frac{1}{2}\int_\Omega |f(x)|^2 dx + \frac{1}{2}\int_\Omega |g(x)|^2 dx$$

hold along with all the other Young-type inequalities.

Definition 5.8. *Let H be a pre-Hilbert space. If H is* **complete**[1]*, then H is called a* ***Hilbert space****.*

Example 5.9. *The main examples of Hilbert spaces are the Euclidean spaces $\mathbb{R}^d$ or $\mathbb{C}^d$ with respect to the usual inner product defined by*

$$(x|y)_{\mathbb{C}^d} := \sum_{k=1}^d x_k \overline{y_k},$$

and their infinite dimensional counterparts, the sequence space ℓ^2 and the Lebesgue spaces $L^2(\Omega)$ with respect to the inner products

$$(x|y)_{\ell^2} := \sum_{n\in\mathbb{N}} x_n \overline{y_n}$$

[1] I.e., if $(x_n)_{n\in\mathbb{N}} \subset H$ is a **Cauchy sequence** – that is, a sequence such that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ with $\|x_n, x_m\| < \varepsilon$ for all $n, m > N$ –, then it converges – that is, there exists $x \in H$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$, and in this case x is called **limit**.

and

$$(f|g)_{L^2} := \int_\Omega f(x)\overline{g(x)}dx,$$

respectively, for any open domain $\Omega \subset \mathbb{R}^d$. In particular, it is clear that $(\cdot|\cdot)_{L^2}$ satisfies conditions (2)–(3)–(4) of the definition of inner product. In order to check condition (1), observe that if

$$\int_\Omega |f|^2 dx = 0,$$

then $|f(x)|^2$ has to vanish for a.e. $x \in \Omega$. It is well-known that ℓ^2 as well as $L^2(\Omega)$ are complete: see e.g. [**4**, Thm. 4.8].

Definition 5.10. *Let H be a pre-Hilbert space. Two vectors $x, y \in H$ are said to be* **orthogonal to each other** *if $(x|y)_H = 0$, and we denote $x \perp y$.*

If two subsets A, B of H satisfy $(x|y)_H = 0$ for all $x \in A$ and all $y \in B$, also A, B are said to be **orthogonal to each other**. *Moreover, the set of all vectors of H that are orthogonal to each vector in A is called* **orthogonal complement** *of A and is denoted by $A^\perp$.*

Definition 5.11. *Let $(H, (\cdot|\cdot)_H)$ be a pre-Hilbert space. Then a family $\{e_n \in H \setminus \{0\} : n \in J\}$, $J \subset \mathbb{N}$, is called* **orthogonal** *if $(e_n|e_m)_H = 0$ for all $n \neq m$, and* **orthonormal** *if $(e_n|e_m)_H = \delta_{mn}$ for all $m, n \in J$, where δ_{mn} denotes the Kronecker delta.*

Moreover, $\{e_n \in H : n \in J\}$ is called **total** *if its linear span (i.e., the set of all* finite *linear combinations of elements of the family) is dense in H*[2]

An orthonormal and total family is called a Hilbert space basis of H, or simply a **basis**.

Exercise 5.12. *Let H be a pre-Hilbert space. Prove the following assertions.*

(1) If x, y are orthogonal to each other, then $\|x\|_H^2 + \|y\|_H^2 = \|x+y\|_H^2$. (This is nothing but the **theorem of Pythagoras** *if $H = \mathbb{R}^2$).*
(2) More generally, $2\|x\|_H^2 + 2\|y\|_H^2 = \|x+y\|_H^2 + \|x-y\|_H^2$ for all $x, y \in H$.
(3) Also, for all $x, y \in H$ one has

$$\begin{aligned} 4(x|y)_H &= \|x+y\|_H^2 - \|x-y\|_H^2 && \text{if } \mathbb{K} = \mathbb{R}, \text{ and} \\ 4(x|y)_H &= \|x+y\|_H^2 + i\|x+iy\|_H^2 - \|x-y\|_H^2 - i\|x-iy\|_H^2 && \text{if } \mathbb{K} = \mathbb{C}. \end{aligned}$$

(4) If A is a subset of H, then $A \subset (A^\perp)^\perp$.
(5) The orthogonal complement $H^\perp$ agrees with $\{0\}$.

5.1. Fourier series and orthonormal bases

Definition 5.13. *Let H be a Hilbert space and $\{x_n \in X : n \in N\}$ a family of vectors. The associated series $\sum_{n\in N} x_n$ is called* **convergent to** *$x \in H$ if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$\|x - \sum_{n=1}^{N} x_n\|_H < \epsilon$$

(and **convergent** *if it is convergent to any $x \in H$).*

It is called **absolutely convergent** *if $\sum_{n\in\mathbb{N}} \|x_n\|_H < \infty$.*

[2] A vector subspace X of H is called **dense** if each $x \in H$ is limit of a suitable sequence $(x_n)_{n\in\mathbb{N}} \subset X$.

Example 5.14. *Prove that if H is a Hilbert space, then every absolutely convergent series is convergent. The converse is not true: consider e.g. $H = \ell^2$, the canonical basis $(\delta_{nk})_{n\in\mathbb{N}} : k \in \mathbb{N})$, where δ_{nm} denotes the Kronecker delta, and the sequence defined by $x_n := \frac{1}{n}e_n$, and prove that it is convergent although it is (clearly) not absolutely convergent.*

Lemma 5.15. *Let H be a Hilbert space. Let $\{x_n \in H : n \in \mathbb{N}\}$ be an orthogonal family. Then the following assertions are equivalent.*

(1) The series $\sum_{n\in\mathbb{N}} x_n$ is convergent.

(2) For all $\epsilon > 0$ there exists a finite set $J \subset \mathbb{N}$ such that

$$\Big\| \sum_{k\in J} x_k \Big\|_H < \infty.$$

(3) There holds

$$\sum_{n\in\mathbb{N}} \|x_n\|_H^2 < \infty.$$

Exercise 5.16. *Prove Lemma 5.15.*

Theorem 5.17. *Let H be a pre-Hilbert space. Let $\{e_n \in H : n \in J\}$ be an orthonormal family, $J \subset \mathbb{N}$. Then the following assertions hold.*

(1) $\sum_{n\in J} |(x|e_n)_H|^2 \le \|x\|_H^2$ for all $x \in H$.

(2) If H is complete, then the series $\sum_{n\in J}(x|e_n)_H e_n$ converges.

(3) If $x = \sum_{n\in J} a_n e_n$, then $\|x\|_H^2 = \sum_{n\in J} |a_n|^2$ and $a_n = (x|e_n)_H$ for all $n \in J$.

The assertions in (1) and (3) are usually called **Bessel's inequality** and **Parseval's identity**. The scalars a_n in (3) are called **Fourier coefficients of** x.

Beweis. (1) Upon going to the limit, it suffices to prove the claimed inequality for any *finite* orthonormal family $\{e_1, \dots, e_N\}$. Then one has

$$\begin{aligned} 0 &\le \Big\| x - \sum_{n=1}^{d} (x|e_n)_H e_n \Big\|^2 \\ &= \|x\|_H^2 - \sum_{n=1}^{d} (x|e_n)_H (e_n|x)_H - \sum_{n=1}^{d} \overline{(x|e_n)_H} (e_n|x)_H + \Big(\sum_{n=1}^{d} (x|e_n) e_n \Big| \sum_{m=1}^{d} (x|e_m) e_m \Big) \\ &= \|x\|_H^2 - 2\sum_{n=1}^{d} |(x|e_n)_H|^2 + \sum_{n=1}^{d} |(x|e_n)_H|^2. \end{aligned}$$

(2) If moreover H is complete, then convergence of $\sum_{n\in J}(x|e_n)_H e_n$ can be deduced showing its absolute convergence, i.e., applying Lemma 5.15 to the sequence $((x|e_n)_H e_n)_{n\in\mathbb{N}}$.

(3) If J is finite, say $J = \{1, \dots, N\}$, then the first assertion is follows by repeatedly applying the Theorem of Pythagoras.

If J is infinite – and hence without loss of generality $J = \mathbb{N}$ – then

$$\|x\|_H^2 = \lim_{n\to\infty} \Big\| \sum_{k=1}^{d} a_k e_k \Big\|_H^2 = \lim_{n\to\infty} \sum_{k=1}^{d} |a_k|^2 = \sum_{n\in J} |a_n|^2.$$

Moreover, since $x = \sum_{n\in J} a_n e_n$, i.e., $\lim_{n\to\infty} \|\sum_{k=1}^d a_n e_n - x\|_H = 0$, one sees that for fixed $m \in \mathbb{N}$ and all $n \geq m$

$$\Big(\sum_{k=1}^d a_n e_n | e_m\Big)_H - (x|e_m)_H = \Big(\sum_{k=1}^d a_n e_n - x | e_m\Big)_H,$$

and by the Cauchy–Schwarz inequality

$$\Big|\Big(\sum_{k=1}^d a_n e_n - x|e_m\Big)_H\Big| \leq \Big\|\sum_{k=1}^d a_n e_n - x\Big\|_H \|e_m\|_H = \Big\|\sum_{k=1}^d a_n e_n - x\Big\|_H.$$

Accordingly,

$$\lim_{n\to\infty}\Big(\sum_{k=1}^d a_n e_n|e_m\Big)_H = (x|e_m).$$

Furthermore,

$$\Big(\sum_{k=1}^d a_k e_k|e_m\Big)_H = \sum_{k=1}^d (a_k e_k|e_m)_H = \sum_{k=1}^d a_k (e_k|e_m)_H = \sum_{k=1}^d a_k \delta_{km} = a_m.$$

This concludes the proof. □

Proposition 5.18. *Let H be a Hilbert space. An orthonormal family $\{e_n : n \in \mathbb{N}\}$ is in fact total (i.e., a basis of H) if and only if*

$$(f|e_n) = 0 \text{ for all } n \in \mathbb{N} \quad \text{implies} \quad f = 0. \tag{5.2}$$

Beweis. Let $f \in H$. Let $\{e_n : n \in \mathbb{N}\}$ be a basis such that $(f|e_k)_H = 0$ for all $k \in \mathbb{N}$. Fix an $\epsilon > 0$. By totality of $\{e_n : n \in \mathbb{N}\}$ there exists a finite family $\{a_1, \dots, a_n\}$ such that $\|f - \sum_{k=1}^d a_k e_k\|_H < \epsilon$. Accordingly,

$$\begin{aligned} \|f\|_H^2 &= \Big|\|f\|_H^2 - \Big(f\Big|\sum_{k=1}^n a_k e_k\Big)_H\Big| \\ &= \Big|\Big(f\Big|f - \sum_{k=1}^n a_k e_k\Big)_H\Big| \\ &\leq \|f\|_H \Big\|f - \sum_{k=1}^n a_k e_k\Big\|_H \\ &< \epsilon\|f\|_H. \end{aligned}$$

Therefore, $\|f\|_H < \epsilon$ for all $\epsilon > 0$, i.e., $\|f\|_H = 0$, hence $f = 0$.

Conversely, let the only vector orthogonal to each e_n be 0. Define $(s_n)_{n\in\mathbb{N}} \subset H$ by

$$s_n := \sum_{k=1}^n (f|e_k)_H e_k,$$

which is a Cauchy sequence by Theorem 5.17.(2), hence convergent towards some $g := \sum_{k=1}^\infty (f|e_k)_H e_k \in H$. Thus, $(g|e_k)_H = (f|e_k)_H$ or rather $(g - f|e_k) = 0$ for all $k \in \mathbb{N}$. By Exercise 5.12.(6) this means that $f = g$. I.e., any $f \in H$ can be expressed as a Fourier series with respect to $(e_n)_{n\in\mathbb{N}}$. It follows that $(e_n)_{n\in\mathbb{N}}$ is total. □

Hence, if $\{e_k : k \in \mathbb{N}\}$ is a basis, then

$$\sum_{n\in\mathbb{N}} (f|e_n)e_n$$

actually converges to f, and it is called the **Fourier series** associated with f.

Exercise 5.19. *Let $\ell > 0$. Show that the family*

$$\left\{1, \sqrt{2}\cos\frac{2\pi}{\ell}n\cdot, \sqrt{2}\sin\frac{2\pi}{\ell}m\cdot : n, m = 1, 2, 3, \ldots\right\}$$

is orthonormal in $L^2(0, \ell; \mathbb{R})$. Finite linear combinations of elements of this family are called **trigonometric polynomials**.

Theorem 5.20. *The family*

$$\{1, \sqrt{2}\cos 2\pi n\cdot, \sqrt{2}\sin 2\pi m\cdot : n, m = 1, 2, 3, \ldots\}$$

is a basis of $L^2(0, 1; \mathbb{R})$.

In the proof we will exploit completeness of the space $L^2(0, 1; \mathbb{R})$.

Beweis. We first consider a continuous function $f : [0, 1] \to \mathbb{R}$ and observe that if $f \neq 0$, then there is $x_0 \in [0, 1]$ such that $f(x_0) \neq 0$. Without loss of generality we can assume $f(x_0)$ to be a positive maximum of f. Due to continuity there exists a neighbourhood $(x_0-\delta, x_0+\delta)$ such that $2f(x) > f(x_0)$ for all $x \in [x_0-\delta, x_0+\delta]$. Consider a linear combination p of basis vectors such that

$$m \leq p(y) \quad \text{for some } m > 1 \text{ and all } y \in [x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}]$$

and

$$|p(y)| \leq 1 \quad \text{for all } y \notin [x_0 - \delta, x_0 + \delta].$$

(Such a function p surely exists, consider e.g. the trigonometric polynomial

$$p(x) := 1 - \cos 2\pi\delta + \cos 2\pi(x_0 - x)).$$

It follows from the assumption in (5.2) that f is orthogonal to p^d for each $n \in \mathbb{N}$, and hence

$$\begin{aligned} 0 &= \int_0^1 f(x)p^d(x)dx \\ &= \int_0^{x_0-\delta} f(x)p^d(x)dx + \int_{x_0-\delta}^{x_0+\delta} f(x)p^d(x)dx + \int_{x_0+\delta}^1 f(x)p^d(x)dx. \end{aligned}$$

Moreover, for all $n \in \mathbb{N}$ one has

$$\begin{aligned} \left|\int_0^{x_0-\delta} f(x)p^d(x)dx\right| + \left|\int_{x_0+\delta}^1 f(x)p^d(x)dx\right| &\leq \int_0^{x_0-\delta} |f(x)p^d(x)|dx + \int_{x_0+\delta}^1 |f(x)p^d(x)|dx \\ &\leq \int_0^{x_0-\delta} |f(x)|dx + \int_{x_0+\delta}^1 |f(x)|dx \\ &\leq \|f\|_{L^1(0,1)} < \infty, \end{aligned}$$

since in particular $f \in L^1(0,1)$. Still, one sees that

$$\int_{x_0-\delta}^{x_0+\delta} f(x)p^d(x)dx \geq \int_{x_0-\frac{\delta}{2}}^{x_0+\frac{\delta}{2}} f(x)m^d dx \geq \frac{f(x_0)}{2} m^d \frac{\delta}{2},$$

and therefore $\lim_{n\to\infty} \int_{x_0-\delta}^{x_0+\delta} f(x)p^d(x)dx = \infty$, a contradiction to the assumption that $\int_0^1 f(x)p^d(x)dx = 0$ for all $n \in \mathbb{N}$.

Let us now consider a possibly discontinuous function $g \in L^2(0,1;\mathbb{R}) \subset L^1(0,1;\mathbb{R})$ and consider the continuous function $G := \int_0^{\cdot} g(x)dx$. Since by assumptions g is orthogonal to each function $e^{2\pi ik\cdot}$, $k \in \mathbb{Z}$, integrating by parts one clearly obtains that also $G - \int_0^1 G(x)dx$ is orthogonal to each function $e^{2\pi ik\cdot}$, $k \in \mathbb{Z}$. (In fact, the corrective term $\int_0^1 G(x)dx$ is needed since G is in general not orthogonal to $1 = e^0$). Due to continuity of $G - \int_0^1 G(x)dx$, we can apply the result obtained above and deduce that $G - \int_0^1 G(x)dx \equiv 0$, i.e., $g(x) = G'(x) \equiv 0$ for a.e. $x \in (0,1)$. Here we are using the well-known fact that if a function $h : [0,1] \to \mathbb{C}$ satisfies $\int_0^t h(t)dt = 0$ for all $t \in (0,1)$, then $h \equiv 0$.

Moreover, L^2-convergence of Fourier series is a direct consequence of Bessel's inequality, as already deduced in the proof of Proposition 5.18. ☐

With respect to the above introduced basis we consider the following.

Definition 5.21. *Let $f : [0,1] \to \mathbb{R}$. Then the Fourier sequence associated with f is*

$$\int_0^1 f(x)dx + 2\sum_{k=1}^n \int_0^1 f(y)\cos(ky)dx\cos(nx) + 2\sum_{k=1}^n \int_0^1 f(y)\sin(ky)dy\sin(nx), \qquad t \in [0,1].$$

Corollary 5.22. *The family*

$$\{e_n := e^{2\pi in\cdot} : n \in \mathbb{Z}\}$$

is a basis of $L^2(0,1;\mathbb{C})$. Accordingly, for all $f \in L^2(0,1;\mathbb{C})$ the Fourier series associated with it converges to f with respect to the L^2-norm, i.e.,

$$\lim_{n\to\infty} \int_0^1 \left| f(t) - \sum_{|k|\leq n} \int_0^1 f(x)e^{2\pi ik(t-x)}dx \right|^2 dt = 0.$$

Exercise 5.23. *Deduce Corollary 5.22 from Theorem 5.20.* (Hint: observe that it suffices to check condition (5.2) for any real-valued function f.

5.2. The wave equation and the separation of variables

Let us apply the above results on Fourier series to the wave equation introduced in Section 2.1.

In order to solve the 1-dimensional wave equation assuming that the solution can be written as product of two functions X and T that only depend on the space and time variables x and t, respectively. By assumption the function $u = X \cdot T$ satisfies the wave equation, hence

$$\frac{1}{T(t)}\frac{\partial^2 T}{\partial t^2}(t) = \frac{c^2}{X(x)}\frac{\partial^2 X}{\partial x^2}(x).$$

Since the LHS and the RHS agree identically although they depend on different variables, each of them must agree with the same constant λ: this yields the two ordinary differential equations

$$T''(t) = \lambda T(t) \qquad \text{and} \qquad X''(x) = \frac{\lambda}{c^2}X(x).$$

Another educated guess suggests to try assuming $\lambda \leq 0$, say, $\lambda = -\mu^2$, which yields

$$T''(t) = -k^2 T(t) \qquad \text{and} \qquad X''(x) = -\frac{\mu^2}{c^2} X(x).$$

Solving each of them and multiplying both solutions yield the solution to the wave equation. Since they are ODEs of second order, each of them needs two boundary conditions. The conditions relevant for T impose initial condition on u, i.e., they impose the initial displacement and velocity of the string, i.e.,

$$u(0,x) = u_0(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(0,x) = u_1(x),$$

for some functions u_0, u_1 defined on the string; whereas those relevant for X impose a condition on the displacement at any time at the endpoints a, b of the string, as we already know. A necessary compatibility condition postulates that u_0, u_1 satisfy the boundary conditions, too.

Remark 5.24. *This* Ansatz *on the form of the solution, which usually goes under the name* **separation of variables***, is justified by the fact that it does lead to a function that solves the equation. In fact, since it is known that the wave equation admits at most one solution, it then necessarily agrees with the one we will obtain following the idea sketched above.*

To fix the ideas, let us solve the 1-dimensional wave equation on a string of length ℓ (i.e., we identify it with an interval $(0, \ell)$ and with Dirichlet boundary conditions. We have to divide the cases $\mu = 0$ and $\mu > 0$: they imply

- $X(x) = \alpha_1 x + \beta_1$, $T(t) = \alpha_2 t + \beta_2$ and
- $X(x) = A_1 \cos(\frac{\mu}{c} x) + B_1 \sin(\frac{\mu}{c} x)$, $T(t) = A_2 \cos(\mu t) + B_2 \sin(\mu t)$,

respectively. Now, let us impose the boundary conditions. If $\mu = 0$, then it follows from $X(0) = X(\ell) = 0$ that $\alpha_1 = \beta_1 = 0$, hence $X \equiv 0$ and also u vanishes identically. We neglect this trivial solution and hence rather consider the case $\mu > 0$. The Dirichlet boundary condition $X(0) = 0$ yields $A_1 = 0$ and hence

$$X(x) = B_1 \sin\left(\frac{\mu}{c} x\right)$$

which is satisfied if and only if $B_1 = 0$ (leading again to the trivial solution) or

$$\sin\left(\frac{\mu}{c}\ell\right) = 0,$$

i.e., if and only if

$$\frac{\mu}{c}\ell = k\pi$$

for some $k \in \mathbb{Z}$, i.e.,

$$\mu = \frac{ck\pi}{\ell}.$$

(A different choice of μ would have been required by different – e.g. Neumann – boundary conditions at ℓ). It follows that

$$\begin{aligned} u(t,x) &= X(x)T(t) \\ &= B_1 \sin\left(\frac{k\pi}{\ell} x\right) \left(A_2 \cos\left(\frac{ck\pi}{\ell} t\right) + B_2 \sin\left(\frac{ck\pi}{\ell} t\right)\right). \end{aligned}$$

We still have to impose the initial conditions in order to determine T. Since the wave equation is *linear*, i.e., each linear combination of solutions is again a solution, looking for existence of a solution to the form

$$u(t,x) = B_1 \sin\left(\frac{k\pi}{\ell}x\right)\left(A_2\cos\left(\frac{ck\pi}{\ell}t\right) + B_2 \sin\left(\frac{ck\pi}{\ell}t\right)\right)$$

is equivalent to looking for existence of a solution to the form

$$u(t,x) = \sum_{k=1}^{\infty} B_1 \sin\left(\frac{k\pi}{\ell}x\right)\left(\gamma_k\cos\left(\frac{ck\pi}{\ell}t\right) + \delta_k \sin\left(\frac{ck\pi}{\ell}t\right)\right). \tag{5.3}$$

Imposing the initial conditions yields to looking for coefficients $(\gamma_k)_{k\in\mathbb{N}}$ and $(\delta_k)_{k\in\mathbb{N}}$ such that

$$u(0,x) = \sum_{k=1}^{\infty} B_1 \sin\left(\frac{k\pi}{\ell}x\right)\gamma_k = u_0(x)$$

along with

$$\frac{\partial u}{\partial t}(0,x) = \sum_{k=1}^{\infty} B_1 \sin\left(\frac{k\pi}{\ell}x\right)\delta_k = u_1(x).$$

This is only possible if both u_0 and u_1 satisfy Dirichlet boundary conditions.

In other words, we are looking for coefficients $(\tilde{\gamma}_k)_{k\in\mathbb{N}}$ and $(\tilde{\delta}_k)_{k\in\mathbb{N}}$ such that

$$u_0(x) = \sum_{k=1}^{\infty} \tilde{\gamma}_k \sin\left(\frac{k\pi}{\ell}x\right)$$

along with

$$u_1(x) = \sum_{k=1}^{\infty} \tilde{\delta}_k \sin\left(\frac{k\pi}{\ell}x\right),$$

i.e., the wave equation with Dirichlet boundary conditions has a (unique) solution given by (5.3) whenever the initial values u_0 and u_1 can be represented as the above series of sines for suitable sequences $(\gamma_k)_{k\in\mathbb{N}}$ and $(\delta_k)_{k\in\mathbb{N}}$. Since by Theorem (5.19) the orthogonal family

$$\left\{1, \cos\frac{2\pi}{\ell}n\cdot, \sin\frac{2\pi}{\ell}m\cdot : n,m\in\mathbb{N}\right\}$$

is total in $L^2(0,\ell;\mathbb{R})$, this claimed representability holds whenever the initial values satisfy Dirichlet boundary condition.

Exercise 5.25. *Apply the above strategy to find a solution (as a series) of the wave equation with Robin boundary conditions on an interval* $(0,\ell)$*.*

Exercise 5.26. *Express the solution to the damped wave equation introduced in Remark 3.16 with Dirichlet boundary conditions on an interval* $(0,\ell)$ *by Fourier series.*

Exercise 5.27. *Solve the heat equation on an interval* $(0,\ell)$ *with Neumann boundary condition by separation of variables.*

KAPITEL 6

The Poisson equation and the theorem of Lax–Milgram

One of the most ubiquitous equations in applied mathematics is the **Laplace equation**

$$\Delta u(x,y) = 0, \qquad (x,y) \in \mathbb{R}^2,$$

and its inhomogeneous companion, **Poisson equation**

$$\Delta u(x,y) = -f(x,y), \qquad (x,y) \in \mathbb{R}^2.$$

The easiest way to look at this equation is to interpret it as describing the equilibrium reached by a heat equation. It also plays a fundamental role in electrostatics, as it yields the electric potential u in a body which has been charged – the charge being described by f, but its derivation requires knowledge of the Maxwell's equation, which we have not met yet.

However, historically the Poisson equation was first derived to describe two-dimensional elastical systems. Consider a domain $\Omega \subset \mathbb{R}^2$ whose pointwise extension $u(x,y)$, $(x,y) \in \Omega$, with respect to a reference level we want to describe. We assume Ω to be fixed at the boundary: this corresponds to Dirichlet boundary conditions. It is subject to gravity, which we model by $f : \Omega \to \mathbb{R}$. Newton's first law of motion states that *every object in a state of uniform motion tends to remain in that state of motion unless an external force is applied to it.* In the special case we are considering, this implies that an object that is not moving has to undergo forces that are mutually balancing. This implies that the total action on the system has to be a minimum among those arising under all allowed configurations. In an elastic body, as we have already mentioned in Section 2.1, each point is subject to tension: the total action is therefore the difference of tension energy and potential energy. In the present situation, the tension energy has to be computed (according to Hooke's law) as an elastic constant times surface variation of Ω. It is known from vector analysis that in general the area of the surface beneath Ω is given by

$$\int_\Omega \sqrt{1 + |\nabla u(x,y)|^2} dx\, dy.$$

Hence the (2-dimensional) surface of Ω in a quiet state is simply given by its (Lebesgue) measure

$$\int_\Omega 1 dx,$$

and their difference (which is relevant to the Hooke's law) is

$$\int_\Omega \left(\sqrt{1 + |\nabla u(x,y)|^2} - 1 \right) dx\, dy.$$

Using the first order Taylor development of $h : z \mapsto \sqrt{1+z}$ in $z = 0$ (hence implicitly assuming small extensions), i.e.,

$$\sqrt{1+z} = h(z) \approx h(0) + h'(0)z + O(z^2) \approx 1 + \frac{1}{2} z$$

we deduce that the difference is approximately given by

$$\frac{1}{2}\int_\Omega |\nabla u(x,y)|^2 dx\ dy.$$

Hence, by Hooke's law the tension applied to each point $(x,y)\in\Omega$ is

$$F(x,y)=\alpha|\nabla u(x,y)|^2,$$

for some $\alpha>0$, and the tension energy is

$$I_1=\int_\Omega F(x,y)dx\ dy=\frac{\alpha}{2}\int_\Omega |\nabla u(x,y)|^2 dx\ dy.$$

The potential energy, on the contrary, is given by the integral of gravity (which tends in the direction opposite to tension) times displacement

$$I_2=\int_\Omega f(x,y)u(x,y)dx\ dy.$$

The total action functional is therefore

$$I(u)=I_1(u)-I_2(u)=\int_\Omega \Big(\frac{\alpha}{2}|\nabla u(x,y)|^2-f(x,y)u(x,y)\Big)\,dx\ dy.$$

Considering I as a mapping from $C^1(\overline{\Omega})\to\mathbb{R}$ it is possible to consider its derivative

$$\frac{d}{d\epsilon}I(u+\epsilon v),\qquad v\in\{h\in C^1(\overline{\Omega}):h_{|\partial\Omega}=0\}.$$

A minimum attained by I (at one configuration among all the admissible ones) satisfies

$$\frac{d}{d\epsilon}I(u+\epsilon v)_{|\epsilon=0}=0\qquad\text{for all }\ v\in\{h\in C^1(\overline{\Omega}):h_{|\partial\Omega}=0\}.$$

(Observe that if v vanishes at the boundary, then u and $u+\epsilon v$ satisfy the same boundary conditions, hence are admissible. Hence

$$\begin{aligned}0 &= \frac{d}{d\epsilon}\int_\Omega \Big(\frac{\alpha}{2}|\nabla(u+\epsilon v)(x,y)|^2-f(x,y)(u+\epsilon v)(x,y)\Big)\,dx\ dy\Big|_{\epsilon=0}\\ &= \lim_{\epsilon\to 0}\left(\alpha\nabla(u+\epsilon v)\nabla v-f(x,y)v(x,y)\right)dx\ dy.\end{aligned}$$

We conclude that

$$\begin{aligned}\alpha\int_\Omega \Delta u(x,y)v(x,y)dx\ dy &= -\alpha\int_\Omega \nabla u(x,y)\nabla v(x,y)dx\ dy\\ &= -\int_\Omega f(x,y)v(x,y)dx\ dy.\end{aligned}$$

Since this holds for all $v\in\{h\in C^1(\overline{\Omega}):h_{|\partial\Omega}=0\}$, and since this space is dense in $L^2(\Omega)$, we conclude that

$$(\alpha\Delta u+f|v)_{L^2}=0$$

for all $v\in L^2(\Omega)$, hence Exercise 5.12.(5) finally yields that

$$\alpha\Delta u(x,y)=-f(x,y),\qquad (x,y)\in\Omega,$$

has to hold. In the particular case of $f\equiv 0$, one derives the Laplace equation. The tension energy

$$\frac{1}{2}\int_\Omega |\nabla u|^2 dx$$

is sometimes called **Dirichlet integral**. While this integral can be made arbitrarily large, hence it does not have a maximum, Bernhard Riemann extended an idea by Gustave Dirichlet and observed in 1851 that since I only takes positive values, the twice continuously differentiable zeroes of $\frac{d}{d\epsilon}I(u+\epsilon v)_{|\epsilon=0}$ have to be minima of I. He went on to formulate the following theorem.

Theorem 6.1 (Dirichlet's principle)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain, smooth enough that the Gauß–Green formulae hold. Let $u \in C^2(\overline{\Omega})$ such that $u|_{\partial\Omega} = 0$. Then the following assertions are equivalent.*

(i) $I(u) \le I(w)$ for all $w \in C^2(\overline{\Omega})$ such that $w|_{\partial\Omega} = 0$.
(ii) u is a critical point for I, i.e.,

$$\frac{d}{d\epsilon}\int_\Omega \Big(\frac{\alpha}{2}|\nabla(u+\epsilon w)(x)|^2 - f(x)(u+\epsilon w)(x)\Big)\,dx\Big|_{\epsilon=0} = 0$$

for all $w \in C^2(\overline{\Omega})$ such that $w|_{\partial\Omega} = 0$.
(iii) u solves the Poisson equation

$$\alpha\Delta u(x) = -f(x), \qquad x \in \Omega.$$

While this theorem does hold, Riemann's original argument (only based on boundedness from below of I) contains a small fallacy. Karl Weierstraß pointed out in 1870 that the zeroes of $\frac{d}{d\epsilon}I(u+\epsilon v)_{|\epsilon=0}$ only need to be infima of I, and that there exist infima that are not minima. He observed that when trying to minimise the new Dirichlet integral

$$I(u) := \int_{-1}^{1} \frac{x^2}{2}(u'(x))^2 dx,$$

which is clearly bounded from below, over the set of continuous, piecewise continuously differentiable functions $u : [-1,1] \to \mathbb{R}$ with boundary values $u(-1) = 0$ and $u(1) = 1$, there exists a sequence u_n such that $I(u_n)$ is arbitrarily small. However, the value 0 can never be attained considering functions in the given class, since the only function u such that $I(u) = 0$ would satisfy $xu(x) = 0$ for all $x \in [-1,1]$, i.e., $u(x) = 0$, and this function does not satisfy the boundary conditions.

Proof of Theorem 6.1 . The implication (i)⇒(ii) is clear, and we have proved above that (ii)⇒(iii) also holds.

To prove that (iii)⇒ (i), take $w \in C^2(\overline{\Omega})$ with compact support. It follows that for all $\epsilon \in \mathbb{R}$ the functions u and $u + \epsilon w$ satisfy the same boundary conditions, i.e., they are admissible configurations. Since $-\alpha\Delta u = f$, one has

$$\begin{aligned}
0 &= \int_\Omega \big(-\alpha\Delta u(x) - f(x)\big)\big(u(x) - w(x)\big)dx \\
&= -\int_\Omega \alpha\Delta u(x)\big(u(x) - w(x)\big)dx - \int_\Omega f(x)\big(u(x) - w(x)\big)dx \\
&= \int_\Omega \alpha\nabla u(x)\nabla(u(x) - w(x))dx - \int_\Omega f(x)\big(u(x) - w(x)\big)dx \\
&= \alpha\int_\Omega |\nabla u(x)|^2 dx - \int_\Omega f(x)u(x)dx - \alpha\int_\Omega \nabla u(x)\nabla w(x)dx + \int_\Omega f(x)w(x)dx.
\end{aligned}$$

Consequently

$$\begin{aligned}\alpha\int_\Omega |\nabla u(x)|^2dx - \int_\Omega f(x)u(x)dx &= \alpha\int_\Omega \nabla u(x)\nabla w(x) - \int_\Omega f(x)w(x)dx\\ &\le \frac{\alpha}{2}\int_\Omega |\nabla u(x)|^2dx + \frac{\alpha}{2}\int_\Omega |\nabla w(x)|^2dx - \int_\Omega f(x)w(x)dx,\end{aligned}$$

where the latter step follows from the Young inequality (see Remark 5.7). Rearranging the terms yields the claim. □

Remark 6.2. *Observe that our use of the Young inequality in the above proof depends critically on the fact that the Dirichlet integral is tightly related to the inner product of the pre-Hilbert space $C(\overline{\Omega})$. In particular, while the above proof can be easily extended to the case of general elliptic partial differential equations of the form*

$$\nabla(A\nabla u)(x) = -f(x), \qquad x\in\Omega,$$

where $A:\overline{\Omega}\to M_d(\mathbb{R})$ is a positive definite $(d\times d)$-matrix-valued function, it fails miserably when we remove the assumption that A be positive definite. In the special case of $d=2$ and

$$A = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

we will see that this corresponds to failure of a Dirichlet-type principle for the wave equation.

6.1. The representation theorem of Riesz–Fréchet and the lemma of Lax–Milgram

Definition 6.3. *Let H_1, H_2 be Hilbert spaces over the field $\mathbb{K}$. A mapping $T: H_1\to H_2$ such that*

$$T(\alpha x+\beta y) = \alpha T(x) + \beta T(y) \tag{6.1}$$

for all $x,y\in H_1$ and all $\alpha,\beta\in\mathbb{K}$ is called a **linear operator**, *and one usually writes Tx instead of $T(x)$. An scalar-valued operator is usually called a* **functional**.

A linear operator is called **bounded** *if there exists $M>0$ such that*

$$\|T(x)\|_{H_2} \le M\|x\|_{H_1}, \qquad x\in H.$$

Endowed with the norm defined by

$$\|T\| := \sup_{\|x\|_{H_1}\le 1} \|T(x)\|_{H_2},$$

the set of bounded linear operators from H_1 to H_2 becomes in fact a normed vector space, which we denote by $\mathcal{L}(H_1,H_2)$, or rather H_1' if $H_2=\mathbb{K}$. An invertible bounded linear operator is called an **isomorphism**. *A bounded linear operator such that $\|Tx\|_{H_2} = \|x\|_{H_1}$ for all $x\in H_1$ is called* **isometric**.

Remark 6.4. *In some occasions (in particular, in physical applications) one has to deal with operators that are not linear, but rather* **antilinear**, *i.e., such that*

$$T(x+y) = Tx+Ty, \qquad T(\alpha x) = \overline{\alpha}x, \qquad \alpha\in\mathbb{C},\ x,y\in H_1.$$

holds instead of (6.1) *(clearly, antilinearity differs from linearity only if H_1 and H_2 are complex Hilbert spaces). An bounded antilinear bijection is called an* **antiisomorphism**.

Exercise 6.5. *Let H_1,H_2 be Hilbert spaces. Show that any linear or antilinear operator $T: H_1\to H_2$ such that*

$$\|Tx\|_{H_2} \ge \alpha\|x\|_{H_1}, \qquad x\in H_1,$$

for some $\alpha>0$, is one-to-one.

Exercise 6.6. *Show that a linear operator is bounded if and only if it is Lipschitz continuous if and only if it is continuous.*

Remark 6.7. *Let H_1, H_2 be Hilbert spaces and $T \in \mathcal{L}(H_1, H_2)$. It is clear that* $\mathrm{Ker}T := \{x \in H_1 : Tx = 0\}$ *and* $\mathrm{Ran}T := \{y \in H_2 : \exists x \in H_1 \text{ s.t. } y = Tx\}$ *are vector spaces. Since T is continuous, $T^{-1}C$ is a closed subset of X for all closed subsets C of Y. In particular,* $\mathrm{Ker}T = T^{-1}\{0\}$ *is a closed subspace of X, while* $\mathrm{Ran}T$ *need not be closed. Can you find an example?*

Definition 6.8. *Let A be a subset of H and $x_0 \in H$. A vector x is said to be of* ***best approximation to x_0 in*** *A if*

$$\|x - x_0\|_H = \inf_{y \in A} \|y - x_0\|_H .$$

Points of best approximation need neither exist (think of the case of a point outside an open ball) nor be unique (as in the case of the centre of a circle). However, the following holds.

Theorem 6.9. *Let H be a Hilbert space. Let A be closed and convex subset of H and let $x_0 \in H$.*
1) Then there exists exactly one vector x of best approximation of x_0.
2) Such a best approximation x of x_0 is characterized by the inequality

$$(x_0 - x|y - x) \leq 0 \qquad \text{for all } y \in A \text{ if } \mathbb{K} = \mathbb{R}, \text{ or by} \tag{6.2}$$

$$\mathrm{Re}(x_0 - x|y - x) \leq 0 \qquad \text{for all } y \in A \text{ if } \mathbb{K} = \mathbb{C}. \tag{6.3}$$

Such an x is usually denoted by $P_A(x_0)$ and called **orthogonal projection of x_0 onto A**. The operator P_A is called **orthogonal projector of H onto A**.

Beweis. 1) One can assume without loss of generality that $x_0 = 0 \notin A$.

i) In order to prove existence of the vector of best approximation, let $z := \inf_{y \in A} \|y\|_H$ and consider a sequence $(y_n)_{n \in \mathbb{N}} \subset A$ such that $\lim_{n\to\infty} \|y_n\|_H = z$. Then by the parallelogram law (see Exercise 5.12.(2))

$$\lim_{m,n\to\infty} \left\|\frac{y_n + y_m}{2}\right\|_H^2 + \lim_{m,n\to\infty} \left\|\frac{y_n - y_m}{2}\right\|_H^2 = \lim_{m,n\to\infty} \frac{1}{2}\left(\|y_n\|_H^2 + \|y_m\|_H^2\right) = z^2.$$

Because A is convex $\frac{y_n+y_m}{2} \in A$, so that by definition $\|\frac{y_n+y_m}{2}\|_H^2 \geq z^2$. One concludes that

$$\lim_{m,n\to\infty} \left\|\frac{y_n - y_m}{2}\right\|_H^2 = 0,$$

i.e., $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. By completeness of H there exists a $x := \lim_{n\to\infty} y_n$, which belongs to A since A is closed. Clearly, $\|x\| = \lim_{n\to\infty} \|y_n\| = z$.

ii) In order to prove that a vector of best approximation is unique, assume that both x, x^* satisfy $\|x\|_H = \|x^*\|_H = z$. If $x \neq x^*$, then $\|x + x^*\|_H^2 < \|x + x^*\|_H^2 + \|x - x^*\|_H^2$, and by the parallelogram law

$$\left\|\frac{x + x^*}{2}\right\|_H^2 < \left\|\frac{x + x^*}{2}\right\|_H^2 + \left\|\frac{x - x^*}{2}\right\|_H^2 = \frac{1}{2}(\|x\|_H^2 + \|x^*\|_H^2) = z^2.$$

In other words, $\frac{x+x^*}{2}$ would be a better approximation of x. Since $\frac{x+x^*}{2} \in A$, this would contradict the construction of x as vector of best approximation of x_0 in A. Hence, $x = x^*$.

2) Let $w \in A$ and set $y_t := (1-t)x + tw$, where $t \in (0, 1]$ is a scalar to be optimized in the following. Since A is convex, $y_t \in A$ and accordingly

$$\|x_0 - x\|_H < \|x_0 - y_t\|_H,$$

since y_t is not the (unique!) best approximation of x_0 in A. Accordingly,

$$\|x_0 - x\|_H < \|x_0 - (1-t)x - tw\|_H = \|(x_0 - x) + t(x - w)\|_H,$$

and squaring both sides we obtain (using

$$(x + y|x + y)_H = (x|x)_H + 2\mathrm{Re}(x|y) + (y|y)_H, \qquad x, y \in H),$$

that

$$\|x_0 - x\|_H^2 < \|(x_0 - x)\|_H^2 + t^2\|(x - w)\|_H^2 - 2t\mathrm{Re}(x_0 - x|w - x).$$

It follows that $t^2\|(x - w)\|_H^2 > 2t\mathrm{Re}(x_0 - x|w - x)$ for all $t \in (0, 1]$. Therefore, $0 \geq 2\mathrm{Re}(x_0 - x|w - x)$ in the limit $t \to 0$ and the claimed inequality holds.

Conversely, let x satisfy (6.3). Then for all $y \in A$

$$\|x - x_0\|_H^2 - \|y - x_0\|_H^2 = 2\mathrm{Re}(x_0 - x|y - x) - \|y - x\|_H^2 \leq 0,$$

i.e., $\|x - x_0\|_H^2 \leq \|y - x_0\|_H^2$. It follows that x is the best approximation of x_0 in A. (In the case $\mathbb{K} = \mathbb{R}$ the assertions can be proved in just the same way). ☐

Exercise 6.10. *Let $(H, (\cdot|\cdot)_H)$ be a Hilbert space. Let $\{e_n \in H : n \in \mathbb{N}\}$ be an orthonormal family and denote by Y its linear span. Show that the orthogonal projection P_Y of H onto Y is given by*

$$P_Y x = \sum_{n \in \mathbb{N}} (x|e_n)_H e_n, \qquad x \in H.$$

(Observe that the series converges by the Bessel inequality.)

Exercise 6.11. *Let H be a Hilbert space. Let A_1, A_2 be closed and convex subsets of H and denote by P_1, P_2 the orthogonal projections onto A_1, A_2, respectively. Prove that $P_1 A_2 \subset A_2$ if and only if $P_2 A_1 \subset A_1$ if and only if P_1, P_2 commute, i.e., $P_1 P_2 x = P_2 P_1 x$ for all $x \in H$.*

Exercise 6.12. *Let A be a closed convex subset of H. Prove that if P_A is the orthogonal projector onto A, then the orthogonal projector onto $A^\perp$ is $I - P_A$, where I denotes the identity operator.*

Exercise 6.13. *Define $A_1, A_2 \subset L^2(\mathbb{R})$ as the sets of all square summable functions that are a.e. even and positive, respectively.*

(1) Show that A_1, A_2 are closed convex subsets of $L^2(\mathbb{R})$.
(2) Prove that the orthogonal projections P_{A_1}, P_{A_2} onto A_1, A_2 are given by

$$P_{A_1} f(x) = \frac{f(x) + f(-x)}{2} \qquad \textit{and} \qquad P_{A_2} f(x) = \frac{|f(x)| + f(x)}{2} \qquad \textit{for a.e. } x \in \mathbb{R}.$$

Exercise 6.14. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain. Find the orthogonal projection onto the closed subspace of constant functions.*

Exercise 6.15. *Let H be a Hilbert space and A be a nonempty closed convex subset of H.*

(1) Show that the orthogonal projection P_A is linear if and only if A is a closed subspace of H.
(2) Prove that P_A is Lipschitz continuous with Lipschitz constant 1.

Exercise 6.16. *Let H be a Hilbert space and Y be a closed subspace of H.*

(1) Show that if $Y \neq \{0\}$, then the orthogonal projection P_Y of H onto Y satisfies $\|P_Y\| = 1$ and $\mathrm{Ker} P_Y = Y^\perp$.
(2) Prove that each $x \in H$ admits a unique decomposition as $x = y + z$, where $y = P_Y x \in Y$ and $z = P_{Y^\perp} x \in Y^\perp$.

Recall that the linear span of two vector spaces Y, Z whose interection is $\{0\}$ is called their **direct sum**, and we write $Y \oplus Z$. Accordingly, under the assumptions of Exercise 6.16 H is the direct sum of Y and $Y^\perp$. We also denote by $x = y \oplus z$ the decomposition introduced in Exercise 6.16.(2).

Lemma 6.17. *Let A be a closed convex set of a Hilbert space H. Then the projector onto A is a contraction, i.e., it satisfies*

$$\|P_A f - P_A g\|_H \le \|f - g\|_H, \qquad f, g \in H.$$

Beweis. For all $f, g \in H$ and all $h, k \in A$ one has

$$\mathrm{Re}(f - P_A f | h - P_A f) \le 0 \qquad \text{and} \qquad \mathrm{Re}(g - P_A g | k - P_A g) \le 0.$$

Hence, setting in particular $h = P_A g$ and $k = P_A f$, respectively, one obtains

$$\mathrm{Re}(f - P_A f | P_A g - P_A f) \le 0 \qquad \text{and} \qquad \mathrm{Re}(g - P_A g | P_A f - P_A g) \le 0.$$

Summing these inequalities we obtain

$$\mathrm{Re}(f - P_A f - g + P_A g | P_A f - P_A g) \ge 0,$$

or rather

$$\mathrm{Re}(f - g | P_A f - P_A g) \ge \mathrm{Re}(P_A f - P_A g | P_A f - P_A g) = \|P_A f - P_A g\|^2.$$

By the Cauchy–Schwarz inequality

$$\mathrm{Re}(f - g | P_A f - P_A g) \le \|f - g\| \|P_A f - P_A g\|,$$

and combining the above inequalities the claim follows. □

The following has been proved in 1907 by Frigyes Riesz and independently also by Maurice René Fréchet.

Theorem 6.18 (Representation theorem of Riesz–Fréchet)**.** *Let H be a Hilbert space. For each bounded linear functional ϕ on H there exists a unique $y_\phi \in H$ such that*

$$\phi(x) = (x | y_\phi)_H \qquad \text{for all } x \in H. \tag{6.4}$$

Moreover, the mapping $H' \ni \phi \mapsto y_\phi \in H$ is an isometric isomorphism if $\mathbb{K} = \mathbb{R}$, and an isometric antiisomorphism if $\mathbb{K} = \mathbb{C}$.

Beweis. It suffices to prove that $\Phi : H \ni y \mapsto \phi_y := (\cdot | y) \in H'$ is an isometric (anti)isomorphism.

To begin with, we prove that Φ is isometric (and therefore injective, too, by Exercise 6.5). Clearly, by definition and the Cauchy–Schwarz inequality $|\langle \phi_y, x \rangle| = |(x|y)_H| \le \|y\|_H \|x\|_H$, so that the norm of ϕ_y satisfies $\|\phi_y\| \le \|y\|_H$. In order to check the equality in the non-trivial case of $y \neq 0$, take $x := \frac{y}{\|y\|_H}$ and observe that $\langle \phi_y, x \rangle = \|y\|_H$ by definition.

In order to prove surjectivity of Φ, take $\phi \in H'$. If $\mathrm{Ker}\phi = H$, then $\phi = 0$ and the assertion is clear – so we can assume that $\mathrm{Ker}\phi \neq H$ and (up to rescaling) that $\|\phi\| = 1$. By Remark 6.7, one has $H = \mathrm{Ker}\phi \oplus \mathrm{Ker}\phi^\perp$. Moreover, the closed subspace $\mathrm{Ker}\phi^\perp$ has dimension 1, since the restriction of ϕ to $\mathrm{Ker}\phi^\perp$ is an isomorphism from $\mathrm{Ker}\phi^\perp$ to $\mathbb{K}$. Accordingly, there exists $\xi \in \mathrm{Ker}\phi^\perp$ – which up to rescaling can be assumed to satisfy $\langle \phi, \xi \rangle = 1$ – such that each $z \in \mathrm{Ker}\phi^\perp$ has the form $z = \lambda \xi$ for some $\lambda \in \mathbb{K}$, and in particular each $x \in H$ admits the decomposition $x = P_{\mathrm{Ker}\phi} x \oplus \lambda \xi$. Then

$$\phi(x) = \langle \phi, P_{\mathrm{Ker}\phi} x + \lambda \xi \rangle = \lambda \langle \phi, \xi \rangle = \lambda$$

as well as

$$(x|\xi)_H = (P_{\mathrm{Ker}\phi} x + \lambda \xi | \xi)_H = \lambda \|\xi\|_H^2,$$

where we have used the fact that $P_{\mathrm{Ker}\phi}x$ and ξ belong to subspaces that are orthogonal to each other. We deduce that

$$\phi(x) = \lambda = \frac{(x|\xi)_H}{\|\xi\|_H^2} =: (x|y_\phi)_H,$$

for all $x \in H$, and we conclude that Φ is surjective. This concludes the proof. □

Remark 6.19. *One can show likewise that $y \mapsto \phi_y := \overline{(\cdot|y)}$ is an isometric isomorphism between H and the vector space of antilinear bounded functionals on H.*

While the representation theorem of Riesz–Fréchet is sometimes sufficient in order to prove existence and uniqueness of solutions to some partial differential equation, in many applications we need a stronger version of it, which we present next following the approach in [**4**, §5.3].

Definition 6.20. *Let H be a Hilbert space. A mapping $a : H \times H \to \mathbb{K}$ is called a* **sesquilinear form** *if it is linear in the first and (anti)linear in the second coordinate.*

A sesquilinear form is called **coercive** *if there exists $\alpha > 0$ such that*

$$\mathrm{Re}a(f,f)| \geq \alpha \|f\|_H^2 \qquad \text{for all } f \in H.$$

It is called **continuous** *if there exists $M > 0$ such that*

$$|a(f,g)| \leq M\|f\|_H\|g\|_H \qquad \text{for all } f,g \in H.$$

Finally, a sesquilinear form is called (anti)symmetric if

$$a(f,g) = \overline{a(g,f)} \qquad \text{for all } f,g \in H.$$

The following strong result on differential inequalities related to sesquilinear forms is due to Guido Stampacchia.

Theorem 6.21 (Stampacchia)**.** *Let H be a Hilbert space and $a : H \times H \to \mathbb{K}$ be a continuous, coercive sesquilinear form. Let A be a nonempty closed convex subset of H. Then for all $\phi \in H'$ there exists $u = u_\phi \in A$ such that*

$$\mathrm{Re}a(u, v-u) \geq \mathrm{Re}\phi(v-u) \qquad \text{for all } v \in A. \tag{6.5}$$

If moreover a is (anti)symmetric, then u is the unique element of A such that

$$\frac{1}{2}a(u,u) - \mathrm{Re}\phi(u) = \min_{v \in A}\left\{\frac{1}{2}a(v,v) - \mathrm{Re}\phi(v)\right\}.$$

Beweis. Let $\phi \in H'$. Then by the representation theorem of Riesz–Fréchet there exists a unique $y \in H$ such that

$$\phi(x) = (x|y)_H \qquad \text{for all } x \in H.$$

Furthermore, for all $u \in H$ also the mapping $v \mapsto a(u,v)$ is a continuous (anti)linear functional on H, hence (again by the representation theorem of Riesz–Fréchet or by Remark 6.19, respectively) there exists some element – say, $Tu \in H$ – such that

$$a(u,v) = \overline{(v|Tu)_H} = (Tu|v)_H \qquad \text{for all } v \in H.$$

By linearity of a in its first coordinate, T turns out to be a linear operator on H. Moreover, for all $u \in H$ we have

$$\|Tu\|_H = \sup_{\|v\|_H=1} |a(u,v)| \leq \sup_{\|v\|_H=1} |M\|u\|_H\|v\|_H = M\|u\|_H$$

along with

$$\mathrm{Re}(Tu|u)_H \geq \alpha\|u\|_H^2.$$

In order to find $u \in A$ such that (6.5) – or, equivalently,

$$\mathrm{Re}(Tu|v-u)_H \geq \mathrm{Re}(v-u|y)_H \qquad \text{for all } v \in A$$

holds – we consider a (nonlinear) operator S that maps each $v \in A$ into the best approximation of $\rho(y-Tv)+v$ on A, i.e.,

$$S : H \ni v \mapsto P_A(\rho y - \rho Tv + v) \in H,$$

for some $\rho > 0$. Our goal is to pick ρ in a proper way and thus turn S into a strict contraction to which Banach's fixed point theorem (see e.g. [**10**, § IV.7]) can be applied. If we achieve this, then there exists $u_\phi \in A$ such that $u_\phi = P_A(\rho y - \rho Tu_\phi + u_\phi)$, i.e., by Theorem 6.9 u_ϕ satisfies

$$\mathrm{Re}(\rho y - \rho Tu_\phi + u_\phi - u_\phi|v-u_\phi)_H \leq 0 \qquad \text{for all } v \in K.$$

But this is simply

$$\mathrm{Re}(\rho y - \rho Tu_\phi|v-u_\phi) \leq 0 \qquad \text{for all } v \in K,$$

that is,

$$\rho\mathrm{Re}\phi(v-u_\phi) = \mathrm{Re}(\rho y|v-u_\phi)_H \leq \mathrm{Re}(\rho Tu_\phi|v-u_\phi)_H = \rho\mathrm{Re}a(u_\phi|v-u_\phi)_H \qquad \text{for all } v \in K,$$

as we wanted to show. It remains to check that S is actually a strict contraction, for some ρ suitable. This is a simple consequence of the contractivity of orthogonal projections, cf. Lemma 6.17, so that for all $x, y \in H$

$$\|Sx - Sy\|_H \leq \|x - y - \rho(Tx - Ty)\|_H,$$

and squaring both sides we obtain, by boundedness and coercivity of a,

$$\begin{aligned}
\|Sx - Sy\|_H^2 &\leq \|x-y\|^2 - 2\rho\mathrm{Re}(x-y|Tx-Ty)_H + \rho^2\|Tx-Ty\|_H^2, \\
&= \|x-y\|^2 - 2\rho\mathrm{Re}a(x-y|x-y)_H + \rho^2\|T(x-y)\|_H^2, \\
&= \|x-y\|^2 - 2\rho\alpha\|x-y\|_H^2 + M^2\rho^2\|x-y\|_H^2.
\end{aligned}$$

Now, letting e.g. $\rho = \frac{\alpha}{M^2}$ we obtain the desired estimate, as $1 > \frac{\alpha}{M}$.

Finally, we observe that

$$((u|v))_H := a(u,v), \qquad u, v \in H,$$

defines a scalar product on H, whenever a is symmetric – in particular, symmetry implies that

$$((u|u))_H = a(u,u) = \overline{a(u,u)} = \overline{((u|u))_H}, \qquad u \in H,$$

and in particular that $((u|u))_H \in \mathbb{R}$ for all $u \in H$, whereas coercivity of a ensures positive definiteness of $((\cdot|\cdot))_H$. Moreover, coercivity and continuity of a imply that the norm associated with $((\cdot|\cdot))_H$, i.e.,

$$H \ni u \mapsto ((u|u))_H^{\frac{1}{2}} \in [0, \infty),$$

is equivalent to the norm $\|\cdot\|$ associated with $(\cdot|\cdot)_H$. Accordingly, H is a Hilbert space with respect to $((\cdot|\cdot))_H$, too, to which the representation theorem of Riesz–Fréchet applies. Representing ϕ with respect to $((\cdot|\cdot))_H$, we conclude that there exists $w = w_\phi \in H$ such that

$$\phi(v) = ((v|w))_H = a(v,w), \qquad v \in H.$$

Then, (6.5) reads

$$\mathrm{Re}a(u, v-u) \geq \mathrm{Re}\phi(v-u) = \mathrm{Re}a(v-u, w) = \mathrm{Re}a(w, v-u), \qquad v \in A,$$

i.e., finding $u = u_\phi \in A$ as in the statement of the theorem amounts to finding $u \in A$ such that

$$\mathrm{Re}a(u-w, v-u) \geq 0 \qquad \text{for all } v \in H,$$

or rather

$$\mathrm{Re}((w-u|v-u))_H \leq 0, \qquad v \in H.$$

Of course, by Theorem 6.9 validity of this inequality means that u is the best approximation of w in A, i.e., $u = P_A w$, *with respect to the distance defined by means of the norm associated with* $((\cdot|\cdot))_H$. The best approximation of w in A with respect to this distance is, by definition, the element $u \in A$ in which the distance

$$\sqrt{a(w-\cdot, w-\cdot)} = \sqrt{a(w,w) - 2\mathrm{Re}\phi(\cdot) + a(\cdot,\cdot)}$$

from w attains the minimal value. Since $a(w,w)$ is constant (w is now fixed), the above distance is minimal in u if and only if

$$a(u,u) - 2\mathrm{Re}\phi(u)$$

is minimal. This concludes the proof. □

A direct corollary of the above result has been obtained by Peter David Lax and Arthur Norton Milgram. It is commonly known as the **Lax–Milgram Lemma**.

Corollary 6.22 (Lax–Milgram 1954)**.** *Let H be a Hilbert space and $a : H \times H \to \mathbb{K}$ a continuous and coercive sesquilinear function. Then, for any $\phi \in H'$ there is a unique solution $u =: T\phi \in H$ to*

$$a(u,v) = \overline{\phi(v)}, \qquad v \in H.$$

If moreover a is symmetric, then u is the unique element of H such that

$$\frac{1}{2}a(u,u) - \phi(u) = \min_{v\in H}\left\{\frac{1}{2}a(v,v) - \phi(v)\right\}.$$

Following the approach in [**4**, Chapter 5], the proof is based on the above theorem of Stampacchia. Another proof is more common in the literature, see e.g. [**5**]. This one, being based on the fixed point theorem of Banach, which is constructive, is more suitable for numerical applications.

Beweis. Apply the theorem of Stampacchia with $A = H$. If a is symmetric, deduce the claimed characterisation of u using the fact that by Theorem 6.9

$$u = P_Y x \qquad \Leftrightarrow \qquad u \in Y \text{ and } (f - x|v)_H = 0 \text{ for all } v \in Y$$

characterises the orthogonal projector onto a closed subspace Y of H. □

The most interesting application of the Lax–Milgram Lemma is the possibility of proving solvability of partial differential equations of elliptic type, like the Poisson equation

$$\Delta u(x) = -f(x), \qquad x \in \Omega, \tag{6.6}$$

with – say – Dirichlet boundary conditions

$$u(z) = 0, \qquad z \in \partial\Omega, \tag{6.7}$$

for given inhomogeneous terms f. The strategy is easy: first, weaken the notion of solution to an elliptic problem by re-writing it in integral form; then, interpret such an integral equation as an equation involving a bounded linear functional; third, apply the theorem of Lax–Milgram to get existence and uniqueness of solutions.

Definition 6.23. *A function $u : \Omega \to \mathbb{K}$ is called a* **weak solution** *of the elliptic problem* (6.6)–(6.7) *if*

$$\int_\Omega \nabla u(x)\overline{\nabla h(x)}dx = \int_\Omega f(x)\overline{h(x)}dx \qquad \text{for all } h \in C_c^1(\Omega). \tag{6.8}$$

This definition does not come out of the blue. In fact, each "classical" solution u of (6.6)–(6.7), once integrated "against a C^1-function" (i.e., upon multiplying it by any $h \in C_c^1(I)$ and the integrating over I) satisfies (6.8) after integration by parts.

6.2. Sobolev spaces

Unfortunately, if we try to carry over the above program, it turns out that we have to consider functionals acting on functions by differentiation, i.e., we define them on $C^1(\overline{\Omega})$ – which, however, is a pre-Hilbert space but not a Hilbert one, so that the theorems of Riesz–Fréchet and Lax–Milgram do not apply to it. Hence, we would like to consider a Hilbert space that is *large enough to be complete but at the same time small enough to allow differentiation.* To this aim, in Section 6.2 we are going to introduce a special class of function spaces, the so-called Sobolev spaces. As we are going to see, the main idea is to *define* a weak form of differentiation by one of classical differentiation's outmost useful consequences, namely validity of the integration by parts. Since the latter can be defined as an integral property, it turns out to be more general than the usual differential formulation.

A brief, old but still unsurpassed introduction to Sobolev spaces can be found in [**4**, Chapters 8–9].

Definition 6.24. *Let $I \subset \mathbb{R}$ be an open interval. A function $f \in L^2(I)$ is said to be* **weakly differentiable** *if there exists $g \in L^2(I)$ such that*

$$\int_I f(x)\overline{h'(x)}dx = -\int_I g(x)\overline{h(x)} \qquad \text{for all } h \in C_c^1(I). \tag{6.9}$$

The set of weakly differentiable functions $f \in L^2(I)$ such that g is in $L^2(I)$ is denoted[1] *by $H^1(I)$ and called* **first Sobolev space**. *They were introduced in 1936 by Sergei Lvovich Sobolev.*

For an open interval $I \subset \mathbb{R}$ we have here denoted by $C_c^1(I)$ the vector space of continuously differentiable functions with compact support, i.e., continuously differentiable functions $f : I \to \mathbb{K}$ such that $f(x) = f'(x) = 0$ for all x outside some compact subset of I.

Exercise 6.25. *Let $I \subset \mathbb{R}$ be an open interval. Let $f \in L^2(I)$. Show that if a function g satisfying* (6.9) *exists, then it is unique.*

Exercise 6.26. *Consider the operator $S : f \mapsto f'$. Show that S is a linear operator that is not bounded on $L^2(0,1)$, but indeed bounded from $H^1(0,1)$ to $L^2(0,1)$. Find an example of an operator on $L^2(0,1)$ that is not linear.*

Remark 6.27. *Clearly, there exist nonlinear functionals – any nonlinear function $\mathbb{R} \to \mathbb{R}$ is an example. However, showing that there exist linear functionals that are unbounded is much more delicate and needs the axiom of choice.*

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[1] Or sometimes by $W^{1,2}(I)$, due to historical reasons. The number 2 suggests that the basic space is L^2, because – yes – given the definition of weak derivative, the same construction can be repeated starting from L^p-spaces. Since our focus lie in the theory of Hilbert spaces – which Sobolev spaces are not, unless $p = 2$ – we neglect this extension and refer the reader to the already mentioned references [**4, 5**].

Example 6.28. *Let $I = (-1, 1)$. The prototypical case of a weakly differentiable function that does not admit a classical derivative in some point is given by*

$$f(x) := \frac{|x| + x}{2} = \begin{cases} 0 & \text{if } x \le 0, \\ x & \text{if } x > 0. \end{cases}$$

Take some function $h \in C_c^1((-1, 1))$ and observe that

$$\begin{aligned} \int_{-1}^{1} f(x)\overline{h'(x)}dx &= \int_{-1}^{0} f(x)\overline{h'(x)}dx + \int_{0}^{1} f(x)\overline{h'(x)}dx \\ &= \int_{0}^{1} x\overline{h'(x)}dx \\ &= \Big[xh(x)\Big]_0^1 - \int_0^1 \overline{h(x)}dx \\ &= -\int_0^1 \overline{h(x)}dx, \end{aligned}$$

where the last equality follows from compactness of support of h (whence $h(1) = 0$). In other words,

$$\int_{-1}^{1} f(x)\overline{h'(x)}dx = \int_{-1}^{1} H(x)\overline{h(x)}dx$$

for all $h \in C_c^1(I)$, where H is defined by

$$H(x) := \begin{cases} 0 & \text{if } x \le 0, \\ 1 & \text{if } x > 0. \end{cases}$$

This shows that f is weakly differentiable with $f' = H$, where H is called the **Heaviside function** *after Oliver Heaviside.*

Remark 6.29. *Let I be an interval of $\mathbb{R}$. By Exercise 12.4 we can deduce that if f is a function in $L^2(I)$ and its primitive F is in $L^2(I)$ as well (e.g., if I is bounded, since F is continuous by the fundamental theorem of calculus), then $F \in H^1(I)$.*

The definition of Sobolev space can be promptly extended to the higher dimensional case.

Definition 6.30. *Let $\Omega \subset \mathbb{R}^d$ be an open domain. A function $f \in L^2(\Omega)$ is said to be* **weakly differentiable** *if there exist $g := (g_1, \ldots, g_d) \in \left(L^2(\Omega)\right)^d$ such that*

$$\int_\Omega f(x)\overline{\frac{\partial h}{\partial x_i}(x)}dx = -\int_\Omega g_i(x)\overline{h(x)} \qquad \text{for all } h \in C_c^1(\Omega) \text{ and all } i = 1, \ldots, d. \tag{6.10}$$

The set of weakly differentiable functions $f \in L^2(\Omega)$ such that g is in $\left(L^2(\Omega)\right)^d$ is denoted by $H^1(\Omega)$, the **first Sobolev space** *on Ω.*

Definition 6.31. *Let $I \subset \mathbb{R}$ be an open interval. Let $f \in L^2(I)$ be weakly differentiable. The unique function g introduced in Definitions 6.24 and 6.30 is called the* **weak derivative** *of f and with an abuse of notation we write $f' = g$.*

Remark 6.32. *Observe that since any two continuosly differentiable functions f, h satisfy* (6.9) *(which is nothing but the usual formula of integration by parts), by definition $C^1(\overline{\Omega}) \subset H^1(\Omega)$ – i.e., each function in $C^1(\overline{\Omega})$ is representative of a weakly differentiable L^2-function whose weak derivative is again in L^2.*

Moreover, integrating by parts one clearly sees that each continuously differentiable function is also weakly differentiable, i.e., $C^1(\overline{\Omega}) \subset H^1(\Omega)$. In general a function that is merely in $C^1(\Omega)$ need not be in $L^2(\Omega)$, but in fact each $u \in C^1(\Omega) \cap L^2(\Omega)$ such that $u' \in \left(L^2(\Omega)\right)^d$ also belongs to $H^1(\Omega)$.

Lemma 6.33. *Let Ω be an open domain of $\mathbb{R}^d$. Then the set $H^1(\Omega)$ is a Hilbert space with respect to the inner product*

$$(f|g)_{H^1} := (f|g)_{L^2} + (f'|g')_{(L^2)^d} = \int_\Omega f(x)\overline{g(x)}dx + \int_\Omega \nabla f(x) \cdot \overline{\nabla g(x)}dx.$$

Observe that

$$(f|g)_{H^1} = \sum_J (f^J|g^J)_{L^2}, \qquad f, g \in H^1(\Omega),$$

where the sum is taken over all multi-indices J of length 0 or 1 (recall the notation introduced in (4.5)).

Beweis. It is easy to see that $(\cdot|\cdot)_{H^1}$ is an inner product, since in particular $(\cdot|\cdot)_{L^2}$ is. In order to show completeness, take a Cauchy sequence in $H^1(\Omega)$, i.e., a sequence $(f_n)_{n\in\mathbb{N}}$ of weakly differentiable functions such that both $(f_n)_{n\in\mathbb{N}}$ and $(f_n')_{n\in\mathbb{N}}$ are Cauchy in $L^2(\Omega)$ and in $(L^2(\Omega))^d$, respectively. By completeness of $L^2(I)$ both sequences converge, say to ϕ, ψ respectively. Furthermore, for all $i = 1, \ldots, d$ we see that

$$\int_\Omega f_n(x)\overline{\frac{\partial h}{\partial x_i}}(x)dx = -\int_\Omega \frac{\partial f}{\partial x_i}_n(x)\overline{h(x)} \qquad \text{for all } h \in C_c^1(I),$$

so that

$$\int_\Omega \phi(x)\overline{\frac{\partial h}{\partial x_i}}(x)dx = \lim_{n\to\infty}\int_\Omega f_n(x)\frac{\partial h}{\partial x_i}(x)dx = -\int_\Omega \frac{\partial f}{\partial x_i}_n(x)\overline{h(x)}dx = -\int_\Omega \psi_i(x)\overline{h(x)}dx \qquad \text{for all } h \in C_c^1(\Omega).$$

This shows that $\phi' = \psi$, so that $H^1(\Omega)$ is in fact complete, i.e., a Hilbert space. ☐

Exercise 6.34. *Let I be an interval.*

(1) Let $f \in L^2(I)$ such that $\int_I f(x)\overline{h'(x)}dx = 0$ for all $h \in C_c^1(\overline{I})$. Show that there exists a constant $c \in \mathbb{K}$ such that $f(x) = c$ for a.e. $x \in I$.
(2) Let $g \in L^2(I)$ and $x_0 \in I$. Define $G : I \ni x \mapsto \int_{x_0}^x g(t)dt \in \mathbb{K}$. Show that $G \in C(I)$ and moreover $\int_I G(x)\overline{h'(x)}dx = -\int_I g(x)\overline{h(x)}dx$ for all $h \in C_c^1(I)$.
(3) Conclude that each $f \in H^1(I)$ has a representative $f^* \in C(\overline{I})$ such that $f(x) = f^*(x)$ for a.e. $x \in I$. Moreover, $\|f^*\|_{C(\overline{I})} \le \|f\|_{H^1(I)}$ for all $f \in C([0,1])$.

Remark 6.35. *Accordingly, it is common (although a slight abuse of language) to say that* in 1-dim, weakly differentiable functions are continuous. *It is worthwile to emphasize that this important property is exclusive of the* 1*-dimensional case. In particular, this allows to talk about point evaluation of functions in $H^1(I)$.*

The converse is not true, i.e., there exist continuous functions that are not weakly differentiable. A continuous but nowhere differentiable, like the Weierstraß function, yields a counterexample. In fact, it can be proved that a weakly differentiable function has to be differentiable (in the classical sense) almost everywhere.

Higher order Sobolev spaces can be introduced recursively, like in the case of classical derivatives.

Definition 6.36. *Let $\Omega \subset \mathbb{R}^d$ be an open domain and $k \in \mathbb{N}$. The k^{th}* **Sobolev space** *is defined recursively by setting*

$$H^{k+1}(\Omega) := \left\{ f \in H^k(\Omega) : \frac{\partial f}{\partial x_i} \in H^k(\Omega) \text{ for all } x \in \Omega \right\}, \qquad k \in \mathbb{N}.$$

Reasoning as in the proof of Lemma 6.33, we see that for any k the k^{th} Sobolev space is a Hilbert space, when endowed with the scalar product

$$(f|g)_{H^k} := \sum_J (f^J|g^J)_{L^2}, \qquad f, g \in H^k(\Omega),$$

where the sum is taken over all multi-indices J of length between 0 and k. We denote the associated norm by $\|\cdot\|_{H^k}$.

The following is a fundamental result of the theory of Sobolev spaces (which we are not going to prove, but cf. [**5**, §5.3]).

Theorem 6.37 (Meyers–Serrin 1964)**.** *Let Ω be an open domain of $\mathbb{R}^d$. Then the Sobolev space $H^k(\Omega)$ agrees with the closure of $C^\infty(\Omega) \cap H^k(\Omega)$ with respect to $\|\cdot\|_{H^k}$. If Ω is bounded and has C^1-boundary, then $H^k(\Omega)$ even agrees with the closure of $C^\infty(\overline{\Omega})$ with respect to $\|\cdot\|_{H^k}$.*

The main feature of the theorem of Meyers–Serrin is the possibility of extending results that are known to hold for continuous functions to more general functions in certain Sobolev spaces, by simple density arguments. For example, the Gauß–Green formulae hold for functions that are merely in H^k-spaces, instead of C^k. Among all such extensions we would like to point out the following one. Its proof is not difficult, but it relies heavily on the technical tool of *convolutions*, which we wish to avoid.

Corollary 6.38. *Let Ω be a bounded open domain of $\mathbb{R}^d$. Then each $u \in H^1(\Omega)$ is limit of a sequence $(u_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^d)$ with respect to the $H^1(\Omega)$-norm.*

Clearly, taking the restriction in an open domain Ω of a function $u \in H^1(\mathbb{R}^d)$ yields a function $u|_\Omega$ that is again of class H^1: in fact, this defines a linear contraction operator R by

$$Ru := u|_\Omega, \qquad u \in H^1(\mathbb{R}^d).$$

The following shows that in some sense the converse holds, too.

Theorem 6.39. *Let $\Omega \subset \mathbb{R}^d$ be an open domain with C^1-boundary. Then there exists a linear operator $\Theta : H^1(\Omega) \to H^1(\mathbb{R}^d)$ and $C > 0$ such that for all $u \in H^1(\Omega)$*

- $(\Theta u)|_\Omega) = u$,
- $\|\Theta u\|_{L^2(\mathbb{R}^d)} \leq C\|u\|_{L^2(\Omega)}$,
- $\|\Theta u\|_{H^1(\mathbb{R}^d)} \leq C\|u\|_{H^1(\Omega)}$.

In the proof the above theorem we will need the following.

Exercise 6.40. *Consider $u \in H^1(0,\infty)$ and $\eta \in C^1(0,\infty)$ such that $\eta|_{\{x \geq 1-\epsilon\}} = 0$ for some $\epsilon \in (0,1)$. Prove that then $\eta u \in H^1(0,\infty)$ and the product rule*

$$(\eta u)' = \eta' u + \eta u'$$

holds for its weak derivative. (Hint: After taking a test function, apply the usual rule for products of continuously differentiable functions to its product with η.)

Proof of Theorem 6.39. We only prove the assertion in the case of $d = 1$, the general case being definitely more technical (we refer to [**4**, Chapter 9] for a detailed proof). For $d = 1$ we only have to consider, without loss of generality, the cases of

(1) $\Omega = (0, \infty)$,
(2) $\Omega = (0, 1)$.

The three most natural ways of extending a function are

- extension by 0,
- even extension,
- odd extension.

Each one of them is better suited to some particular situations. We will meet all three in this proof.

(1) If $\Omega = (0, \infty)$, take $u \in H^1(0, \infty)$ and let Eu be its even extension, i.e.,

$$(Eu)(x) := \begin{cases} u(x), & \text{if } x \geq 0, \\ u(-x), & \text{if } x < 0, \end{cases}$$

(Here we have defined $u(0)$ considering the boundary value of the continuous representative of u). One has

$$\|Eu\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{0} |f(x)|^2 dx + \int_0^{\infty} |f(x)|^2 dx = 2\|Eu\|^2_{L^2(0,\infty)}, \tag{6.11}$$

by variable substitution. Moreover, Eu is weakly differentiable: this amounts to finding a function $g \in L^2(\mathbb{R})$ that turns out to be its weak derivative. An educated guess suggests us to try with the odd extension of u', i.e., with

$$(Ou')(x) := \begin{cases} u'(x), & \text{if } x \geq 0, \\ -u'(-x), & \text{if } x < 0, \end{cases}$$

which is in $L^2(\mathbb{R})$, as one sees reasoning as we have done to prove that Eu is. By Remark 6.29 it suffices to show that Eu actually is the primitive of Ou': and in fact

$$(Eu)(x) - (Eu)(0) = \begin{cases} u(x) \\ u(-x) \end{cases} - u(0) = \int_0^x (Ou')(s) ds \qquad \text{for all } x \in \mathbb{R},$$

which holds because u is weakly differentiable and it is the primitive of u'. (Here we could perform point evaluation because $u \in H^1(0, \infty)$, and hence it has a continuous representative).

Moreover, a direct computation shows that $\|Eu\|_{H^1} \leq 2\|u\|_{H^1}$: in fact,

$$\|(Eu)'\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{0} |f'(x)|^2 dx + \int_0^{\infty} |f'(x)|^2 dx = 2\|Eu\|^2_{L^2(0,\infty)},$$

and combining with (6.11) yields the claim.

(2) Let us now consider the case of $\Omega = (0, 1)$. Consider a function $\eta : \mathbb{R} \to \mathbb{R}$ with range contained in the interval $[0, 1]$ that is a smoothed version of $H(\frac{1}{2} + \cdot)$, H the Heaviside function: to fix the ideas, we will assume that $\eta \equiv 1$ on $(\frac{1}{2} + \epsilon, \infty)$ and $\eta \equiv 0$ on $(-\infty, \frac{1}{2} - \epsilon)$, for some small $\epsilon > 0$. We remark that neither η nor $1 - \eta$ are $L^2(\mathbb{R})$-functions, hence in particular they are not $H^1(\mathbb{R})$-functions.

For any $u \in H^1(0, 1)$, we can write $u = \eta u + (1 - \eta)u$. The idea is to apply Lemma 6.40 to the functions ηu and $(1 - \eta)u$, whose support is contained in two (disjoint) halflines. First of all, observe that $\big(\eta u\big)(x) = 0$ and $\big((1 - \eta)u\big)(x) = 0$ for all $x \leq 0$ and all $x \geq 1$, respectively. In the case of ηu (or rather, of its restriction to $(0, 1)$), extend it to $(-\infty, 1)$ by 0 on $(-\infty, 0)$: such an extension, which we denote by $\widetilde{\eta u}$, we then extend to the whole $\mathbb{R}$ as we did in (1) (*almost* as we did in (1), because now we are reflecting in 1 and not in 0), i.e., we consider $u_1 := E(\widetilde{\eta u})$. Similarly, $(1-\eta)u$ can be extended to $\mathbb{R}$ by first extending it to $(0,\infty)$ by 0 on $(1,\infty)$, obtaining $\widetilde{(1-\eta)}u$, and then considering the even extension u_2 of $\widetilde{(1-\eta)}u$.

It is straightforward to check that the three claimed conditions hold for both u_1 and u_2. We complete the proof by setting $\Theta u := u_1 + u_2$. □

One of the main features of Sobolev spaces is that while they are defined on the top of L^2, their relation to other Lebesgue spaces and to spaces of continuously differentiable functions is quite well-behaved, as the following **Sobolev embedding theorems** show.

Theorem 6.41 (First Sobolev embedding theorem)**.** *Let Ω be an open domain of $\mathbb{R}^d$. Then there exists a constant $C > 0$, only depending on Ω – and in fact only on the* measure *of Ω – and d, such that for all $u \in H^1(\Omega)$*

(1) $\|u\|_{L^\infty} \le \|u\|_{H^1}$ if $d = 1$ (and in particular $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$),
(2) $\|u\|_{L^p} \le C\|u\|_{H^1}$ if $d = 2$ (and in particular $H^1(\mathbb{R}^2) \subset L^p(\mathbb{R}^2)$), for all $p \in [2,\infty)$, and finally
(3) $\|u\|_{L^{\frac{2d}{d-2}}} \le C\|u\|_{H^1}$ if $d \ge 3$ (and in particular $H^1(\mathbb{R}^2) \subset L^{\frac{2d}{d-2}}(\mathbb{R}^3)$).

This explains in particular the special role played by $L^6(\mathbb{R}^3)$ in many partial differential equations of physical relevance, where energy functionals are defined on $H^1(\mathbb{R}^3)$.

Beweis. The 1-dimensional is case is elementary, while the higher dimensional ones rely more upon properties of L^p-spaces. Hence, we are only going to prove the assertion (1) – in fact, it suffices to check it for the case of $\Omega = \mathbb{R}$, since the case of a bounded or semibounded interval can be recovered via Theorem 6.39.

In view of Corollary 6.38, it suffices to prove the assertion (1) for $u \in C^1_c(\mathbb{R})$; for if $u \in H^1(\mathbb{R})$, then there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset C^1_c(\mathbb{R})$ approximating u in H^1-norm, hence by (1) also in L^∞-norm, and since (1) is satisfied by each u_n, the claim follows.

To begin with, define a function $G \in C^1(\mathbb{R})$ by

$$G(x) := \begin{cases} x^2 & \text{if } x \ge 0, \\ -x^2 & \text{otherwise.} \end{cases}$$

Let now $u \in C^1_c(\mathbb{R})$ and consider $w := G \circ u$, with $|w| = |u|^2$. The function w is composition of two continuously differentiable functions, hence it is of class C^1, and in fact it has compact support because u does. Observe that by the chain rule

$$w'(x) = (G \circ u)'(x) = G'(u(x))u'(x) = 2|u(x)|u'(x), \qquad x \in \mathbb{R}.$$

By the fundamental theorem of calculus, the chain of inequalities

$$w(x) = \int_{-\infty}^x w'(x)dx = 2\int_{-\infty}^x |u(x)|u'(x)dx \le 2\int_{-\infty}^x |u(x)||u'(x)|dx \le 2\int_{\mathbb{R}} |u(x)||u'(x)|dx \le 2\|u\|_{L^2}\|u'\|_{L^2}$$

holds for all $x \in \mathbb{R}$. Hence, the absolute value of u satisfies

$$|u(x)|^2 = |w(x)| \le 2\|u\|_{L^2}\|u'\|_{L^2} \le \|u\|^2_{L^2} + \|u'\|^2_{L^2} = \|u\|^2_{H^1},$$

i.e.,

$$|u(x)| \le \|u\|_{H^1}.$$

Finally, taking the maximum yields the claim.

We refer to [**4**, Cor. 9.11 and Thm. 9.9] for the proofs of the remaining cases. □

The following is a generalisation of the assertion in Exercise 6.34.

Theorem 6.42 (Second Sobolev embedding theorem)**.** *Let Ω be an open bounded domain of $\mathbb{R}^d$ with C^1-boundary. Then each $u \in H^k(\Omega)$ has a continuous representative u^* if $2k > d$. Moreover, in this case there exists a constant C, only depending on Ω, such that $\|u^*\|_{L^\infty} \le C\|u\|_{H^k}$ for all $u \in H^k(\Omega)$.*

The above theorem can be in fact strenghtened a bit if one introduces the spaces of *Hölder continuous functions*, a generalisation of Lipschitz continuous ones. We omit the details and refer to [**5**, Thm. 5.6.5] for the details and for the proof.

Definition 6.43. *Let $\Omega \subset \mathbb{R}^d$ be an open domain. The closure of $C_c^\infty(\overline{\Omega})$ with respect to the $H^k(\Omega)$-norm is called the* **Sobolev space of order k with vanishing boundary values***, which we denote by $H_0^k(\Omega)$.*

If the reason for its name is a little obscure in view of Definition 6.30, take into account the theorem of Meyers–Serrin and observe that $H_0^k(\Omega)$ is constructed in a way that is analogous to $H^k(\Omega)$, with the only difference that we are taking the H^k-closure of $C_c^\infty(\Omega)$, rather than of the (larger) space $C^\infty(\Omega) \cap H^k(\Omega)$. Accordingly, $H_0^k(\Omega)$ is generally smaller than $H^k(\Omega)$, although they may occasionally agree (examples?). The explanation for the second part of the name of the spaces H^k is more delicate. Informally speaking, one may e.g. expect functions in $H_0^1(\Omega)$ to vanish at the boundary. But what does this mean? If $\Omega \subset \mathbb{R}$, then by Exercise 6.34 $H^1(\Omega)$-functions have a continuous representative on $\overline{I}$, so that we can perform point evaluation. But what about the higher dimensional case?

Theorem 6.44. *Let Ω be an open domain of $\mathbb{R}^d$ with C^1-boundary. If $u \in H_0^1(\Omega) \cap C(\overline{\Omega})$, then $u(z) = 0$ for all $z \in \partial\Omega$.*

Beweis. The proof is based on a technique that is useful in many instances in the theory of Sobolev spaces, e.g. for proving Sobolev embeddings theorems. In the present case, the idea is that the boundary of any C^1-domain can be locally flattened: more precisely, at each point $z \in \partial\Omega$ there is a neighbourhood ω of z that is diffeomorphic to the cylinder $(-1,1) \times B_1(0)$, where $B_1(0)$ is the unit ball of $\mathbb{R}^{d-1}$, such that $\omega \cap \Omega$, $\omega \cap \partial\Omega$ and $\omega \cap (\mathbb{R}^d \setminus \overline{\Omega})$ are mapped into $(0,1) \times B_1(0)$, $\{0\} \times B_1(0)$, and $(-1,0) \times B_1(0)$, respectively, cf. [**5**, Appendix C.1].

Thus, it suffices to prove that if

$$u \in H_0^1\big((0,1) \times B_1(0)\big) \cap C\big([0,1] \times \overline{B_1(0)}\big),$$

then $u(z) = 0$ for all $z \in \{0\} \times B_1(0)$. By definition of H_0^1 we can choose a sequence $(u_n)_{n\in\mathbb{N}} \subset C_c^1\big((0,1) \times B_1(0)\big)$ such that $\lim_{n\to\infty} u_n = u$ with respect to the H^1-norm. Take $x_1 \in (0,1)$ and $y \in B_1(0)$. We have for all n

$$|u_n(x_1, y)| \le \int_0^{x_1} \left|\frac{\partial u_n}{\partial x_1}(s, y)\right| ds$$

(there is no boundary term because u_n has compact support), hence an analogous inequality holds for the average of u. Thus, for all $\epsilon \in (0,1)$

$$\frac{1}{\epsilon}\int_0^\epsilon |u_n(x_1, y)| dx_1 = |u_n(x_1, y)| \le \int_0^\epsilon \left|\frac{\partial u_n}{\partial x_1}(s, y)\right| ds.$$

Integrating both sides over the $(d-1)$-dimensional unit ball $B_1(0)$ we obtain

$$\frac{1}{\epsilon}\int_{B_1(0)}\int_0^\epsilon |u_n(x_1, y)| dx_1 dy \le \int_{B_1(0)}\int_0^\epsilon \left|\frac{\partial u_n}{\partial x_1}(s, y)\right| ds dy$$

Taking the limit as $n \to \infty$ we deduce that

$$\frac{1}{\epsilon}\int_{B_1(0)}\int_0^\epsilon |u(x_1,y)|dx_1 dy \le \int_{B_1(0)}\int_0^\epsilon \left|\frac{\partial u}{\partial x_1}(s,\tilde{x})\right| ds dy,$$

and letting finally $\epsilon \to 0$ we obtain by Lebesgue's differentiation theorem

$$\int_{B_1(0)} |u(0,y)|dy \le 0,$$

i.e., $u(z) = 0$ for all $z \in \{0\} \times B_1(0)$, corresponding to the $\omega \cap \partial\Omega$, as we wanted to show. $\square$

Remark 6.45. *Conversely, it is also true that if $u \in H^1(\Omega) \cap C(\overline{\Omega})$ and $u(z) = 0$ for all $z \in \partial\Omega$, then $u \in H_0^1(\Omega)$, but the proof is more delicate, cf.* [**4**, Thm. 9.17].

Remark 6.46. *By density of $C_c^1(\Omega)$ in $H_0^1(\Omega)$, the definition of weak solution to an elliptic problem can be equivalently reformulated as follows: For any $A \in L^\infty(\Omega; M_d(\mathbb{K})$ A function $u : \Omega \to \mathbb{K}$ is called a* **weak solution** *of the elliptic problem*

$$\nabla(A\nabla u)(x) = f(x), \qquad x \in \Omega, \tag{6.12}$$

with Dirichlet boundary conditions

$$u(z) = 0, \qquad z \in \partial\Omega, \tag{6.13}$$

if

$$\int_\Omega A(x)\nabla u(x)\overline{\nabla h(x)}dx = \int_\Omega f(x)\overline{h(x)}dx \qquad \text{for all } h \in H_0^1(\Omega). \tag{6.14}$$

Exercise 6.47. *Prove that if $\Omega = \mathbb{R}^d$, then $H^1(\Omega) = H_0^1(\Omega)$.*

Theorem 6.48 (Poincaré inequality)**.** *Let Ω be a bounded open domain of $\mathbb{R}^d$ that is contained in a strip of width 2δ, i.e., such that there exist $\delta > 0$ and $i \in \{1, \dots, d\}$ for which*

$$x_i \in (-\delta, \delta) \quad \text{for all } x = (x_1, \dots, x_d) \in \Omega. \tag{6.15}$$

Then for all $u \in H_0^1(\Omega)$ there holds

$$\|f\|_{L^2}^2 \le 4d^2 \|\nabla f\|_{L^2}^2.$$

In particular, $u \mapsto \|\nabla u\|_{L^2}$ defines a norm on $H_0^1(\Omega)$ which is equivalent to the one induced by $H^1(\Omega)$.

Beweis. We first begin by proving the assertion for $d = 1$ and $u \in C^1[-\delta, \delta]$ such that u vanishes at the boundary (in fact, we will only need that u vanishes at $-\delta$ *or* at δ). There holds

$$\begin{aligned}\int_{-\delta}^{\delta} |u(x)|^2 dx &= \int_{-\delta}^{\delta}\left|\int_{-\delta}^{x} u'(y)dy\right|^2 dx \\ &= \int_{-\delta}^{\delta}\left|\int_{-\delta}^{x} u'(y)\cdot 1 dy\right|^2 dx.\end{aligned}$$

By the Cauchy–Schwarz inequality applied to the inner integral we deduce

$$\begin{aligned}
\int_{-\delta}^{\delta} |u(x)|^2 dx &\leq \int_{-\delta}^{\delta} \left(\int_{-\delta}^{x} \left|u'(y)\right|^2 dy \cdot \int_{-\delta}^{x} 1 dy \right) dx \\
&\leq \int_{-\delta}^{\delta} \int_{-\delta}^{x} \left|u'(y)\right|^2 dy (x+\delta) dx \\
&\leq 2\delta \int_{-\delta}^{\delta} \int_{-\delta}^{x} \left|u'(y)\right|^2 dy dx \\
&\leq 4\delta^2 \int_{-\delta}^{\delta} \left|u'(y)\right|^2 dy.
\end{aligned}$$

In the d-dimensional case, it suffices to consider $u \in C_c^\infty(\Omega)$: the general assertion will follow by density. Assume without loss of generality that it (6.15) holds with respect to the first coordinate. Applying the 1-dimensional Poincaré inequality to u and $\frac{\partial u}{\partial x_1}$ and identifying u with its extension by 0 to the whole $\mathbb{R}^d$ we obtain that

$$\begin{aligned}
\int_\Omega |u(x)|^2 dx &= \underbrace{\int_{\mathbb{R}} \ldots \int_{\mathbb{R}}}_{d-1 \text{ times}} \int_{-\delta}^{\delta} |u(x_1, \ldots, x_d)|^2 dx_1 \ldots dx_d \\
&\leq 4d^2 \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} \int_{-\delta}^{\delta} \left| \frac{\partial u}{\partial x_1}(x_1, \ldots, x_d) \right|^2 dx_1 \ldots dx_d \\
&\leq 4d^2 \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} \int_{-\delta}^{\delta} |\nabla u(x_1, \ldots, x_d)|^2 dx_1 \ldots dx_d \\
&= 4d^2 \int_\Omega |\nabla u(x)|^2 dx,
\end{aligned}$$

since u is not defined outside Ω. This concludes the proof. □

Exercise 6.49. *In the literature it is common to find a different formulation of the Poincaré inequality, namely that*

$$\left\| u - \frac{1}{|\Omega|} \int_\Omega u(x) dx \right\|_{L^2} \leq 4\delta^2 \|\nabla u\|_{L^2} \qquad \textit{for all } u \in H^1(\Omega).$$

Find out what is the connection of this inequality to the one in the above theorem. (Hint: Take into account the fact that $\frac{1}{|\Omega|} \int_\Omega u(x) dx$ agrees with the best approximation of any function u on the subspace $\langle \mathbf{1} \rangle$ of constant functions, cf. Exercise 6.14, and hence by Exercise 6.12

$$u - \frac{1}{|\Omega|} \int_\Omega u(x) dx$$

turns out to be the best approximation of u onto the subspace orthogonal to $\langle \mathbf{1} \rangle$.)

As we have already mentioned, Sobolev spaces were first applied in order solve elliptic equations, like the Poisson equation. Applying the Lemma of Lax–Milgram we are finally in the position to formulate an existence and uniqueness result.

Corollary 6.50. *Let $\Omega \subset \mathbb{R}^d$ be an open domain. For all $f \in L^p(\Omega)$ the elliptic problem* (6.6)–(6.7) *has a unique weak solution, where*

$$p = \begin{cases} 1 & \text{if } d = 1, \\ 1 + \epsilon \text{ for any } \epsilon \in (0, 1] & \text{if } d = 2, \\ \frac{2d}{d+2} & \text{if } d \geq 3. \end{cases}$$

Moreover, the energy functional

$$u \mapsto \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int f \overline{u} dx$$

attains in u the minimal value among those attained at all $v \in H^1_0(\Omega)$.

While the former assertion can (and will) be proved simply by means of the representation theorem of Riesz–Fréchet, the latter is a direct consequence of the approach we have chosen to prove the theorem of Lax–Milgram. Observe that we recover Dirichlet's principle, which we have first discussed in Theorem 6.1. Also observe that we have formulated the above result only for the Poisson equation exclusively for the sake of simplicity: it can be immediately extended to the case of general elliptic operators $u \mapsto \nabla(A\nabla u) + B\nabla u + cu$, for suitable coefficients $A : \Omega \to M_d(\mathbb{K})$, $B : \Omega \to \mathbb{R}^d$, $c : \Omega \to \mathbb{K}$.

Beweis. Consider the mapping $\phi : H^1_0(\Omega) \ni h \mapsto \int_I h\overline{f} \in \mathbb{K}$, which is clearly linear and bounded, since $h \in L^{p^*}(\Omega)$ by the Sobolev embedding theorem and therefore by the Hölder inequality

$$|\phi(h)| \leq \|h\overline{f}\|_{L^1} \leq \|h\|_{L^{p^*}} \|f\|_{L^p} \leq C\|h\|_{H^1} \|f\|_{L^2}$$

for all $h \in H^1(\Omega)$. Then by the representation theorem of Riesz–Fréchet there exists a unique $u \in H^1_0(\Omega)$ (continuously depending on f) such that $\phi(h) = (h|u)_{H^1_0(\Omega)} = (h'|u')_{L^2(\Omega)}$. By definition of weak solution, this completes the proof, since we have already remarked that the characterisation of solutions via minimal energy is a consequence of the theorem of Lax–Milgram for symmetric forms. ☐

Remark 6.51. *The above theorem only applies to an elliptic equation with Dirichlet boundary conditions and not, say, with Neumann ones. This is not a problem of the method based on the theorem of Lax–Milgram, but an intrinsic feature of elliptic equations. To see this, observe that even in the case of $f \equiv 0$, the Poisson equation (or rather, the Laplace equation) is not uniquely solvable, since if u is a solution, then so is $u+c$ for any constant c. This can be reformulated as follows: for any $\lambda \in \mathbb{K}$, unique solvability of the elliptic equation*

$$\nabla(A\nabla u)(x) - \lambda u(x) = -f(x), \qquad x \in \Omega, \tag{6.16}$$

with Dirichlet, Neumann, Robin or even more general boundary conditions is equivalent to invertibility of the linear (unbounded on $L^2(\Omega)$) operator $\lambda - \nabla(A\nabla)$. However, with Neumann boundary conditions this operator is not *invertible for $\lambda = 0$. It turns out that in fact there exist infinitely many (countably many, if Ω is bounded) values λ, called the* **eigenvalues**, *such that $\lambda - \nabla(A\nabla)$ is not invertible. What we have just seen is that 0 is an eigenvalue if we impose Neumann boundary conditions, but not if Dirichlet boundary conditions are considered. Whenever $A(x)$ is a hermitian matrix for all $x \in \Omega$, all such eigenvalues are real, but this is generally false. The equation* (6.16) *is called* **Helhmholtz equation**..

The weak formulation of an elliptic problem (say, with Dirichlet boundary conditions) is an interesting method for obtaining *some* kind of solution, via the Theorem of Lax–Milgram, but its application would be of limited interest if one could not link weak solutions to classical solutions. Fortunately, this is not the case. On one hand, we have already remarked that each classical solution – i.e., any C^2-function solving the elliptic problem pointwise – is also a weak solution. On the other hand, if one already knows that a weak solution (i.e., a $H^1_0(\Omega)$-function) is a $C^2(\overline{\Omega})$-function, then it is a solution in classical sense. In fact, u has vanishing boundary data, and moreover $(\Delta u + f|v)_{L^2(\Omega)} = 0$ for all $v \in C^1_c(\Omega)$ – or rather for all $v \in L^2(\Omega)$, by density. Therefore, u satisfies $\Delta u(x) = -f(x)$ for a.e. $x \in \Omega$, and hence for *all* $x \in \Omega$, since it is a C^2-function.

What is more important, there is an extensive *regularity theory* developed with the aim of yielding stronger (that is, stronger than the theorem of Lax–Milgram) assertions on the regularity of weak solutions, provided that the inhomogeneous term in the Poisson equation is "better" than L^2. A favorite approach is based on the so-called method of incremental quotients developed in the 1950s by Louis Nirenberg and others. Its technical core lies in the following result, where we denote by τ_ϵ the linear *translation operator* τ_ξ defined for all $\xi \in \mathbb{R}^d$ by

$$\tau_\xi u := u(\cdot + \xi).$$

Niremberg's idea was to use integrability properties of images of such operators to characterise H^1-functions, just like strong differentiability is characterized by convergence of incremental quotients.

Lemma 6.52. *Let Ω be an open domain of $\mathbb{R}^d$. Let $u \in L^2(\Omega)$. Then the following assertions are equivalent.*

(i) $u \in H^1(\Omega)$.

(ii) The inequality

$$\left| \int_\Omega u \overline{\frac{\partial h}{\partial x_i}} dx \right| \le C \|h\|_{L^2}$$

holds for all $h \in C^1_c(\Omega)$ and all $i = 1, \ldots, d$.

(iii) The inequality

$$\|\tau_\xi u - u\|_{L^2(\omega)} \le C|\xi|$$

holds for some $C > 0$, all open subsets ω whose closure is contained in Ω and all $\xi \in \mathbb{R}^d$ such that $|\xi| < \operatorname{dist}(\omega, \partial\Omega)$.

Observe that by definition of $H^1(\Omega)$, for all $h \in C^1_c(\Omega)$ and all $i = 1, \ldots, d$ we have

$$\left| \int_\Omega u \overline{\frac{\partial h}{\partial x_i}} dx \right| = \left| \int_\Omega \frac{\partial u}{\partial x_i} \overline{h} dx \right| \le C \|h\|_{L^2},$$

where the last step follows from the Cauchy–Schwarz inequality letting $C := \|\frac{\partial u}{\partial x_i}\|_{L^2}$ or, more generally, $C := \|u'\|_{(L^2)^d}$.

Beweis. (i)$\Rightarrow$(iii) To begin with, assume $u \in C^1_c(\mathbb{R}^d)$ and let $\xi \in \mathbb{R}^d$. Then,

$$u(x+\xi) - u(x) = \int_0^1 \xi \nabla u(x + t\xi) dt \qquad \text{for all } x \in \mathbb{R}^d :$$

this can be seen e.g. introducing the function

$$v : \mathbb{R} \ni t \mapsto u(x + t\xi) \in \mathbb{R},$$

which permits to write $u(x+\xi) - u(x) = v(1) - v(0) = \int_0^1 v'(s) ds$, which implies the claim. Therefore,

$$|u(x+\xi) - u(x)|^2 \le |\xi|^2 \int_0^1 |\nabla u(x + t\xi)|^2 dt \qquad \text{for all } x \in \mathbb{R}^d,$$

and integrating over any compact subset ω of $\mathbb{R}^d$ yields

$$\begin{aligned}\int_\omega |u(x+\xi)-u(x)|^2 dx &\leq |\xi|^2 \int_\omega \int_0^1 |\nabla u(x+t\xi)|^2 dt dx \\ &\leq |\xi|^2 \int_0^1 \int_{\omega+t\xi} |\nabla u(y)|^2 dy dt,\end{aligned}$$

where in the last step we have applied Fubini and an obvious variable substitution.

If now $|\xi| < \operatorname{dist}(\omega, \partial\Omega)$, then $\tilde{\omega} := \bigcup_{t\in[0,1]} \omega + t\xi$ is a compact subset of $\mathbb{R}^d$. Accordingly,

$$\|\tau_\xi u - u\|^2_{L^2} = \int_\omega |u(x+\xi)-u(x)|^2 dx \leq |\xi|^2 \int_0^1 \int_{\tilde{\omega}} |\nabla u(y)|^2 dy dt = |\xi|^2 \int_{\tilde{\omega}} |\nabla u(y)|^2 dy,$$

as we wanted to prove. The general case follows by a density argument based on a technical result, cf. [**4**, Thm. 9.2].

(iii)$\Rightarrow$(ii) Let $h \in C^1_c(\Omega)$. Take an open subset ω that contains the support of h whose closure is contained in Ω. By a variable substitution and due to the assumption we have

$$\left|\int_\Omega u(y)\left(h(y-\xi)-h(y)\right) dy\right| = \left|\int_\Omega \left(u(x+\xi)-u(x)\right) h(x) dx\right| = \left|\int_\Omega (\tau_\xi u - u) h dx\right| \leq C|\xi| \|h\|_{L^2},$$

i.e.,

$$\left|\int_\Omega u(y)\left(\frac{h(y-\xi)-h(y)}{|\xi|}\right) dy\right| \leq C\|h\|_{L^2}.$$

Now, the assertion follows taking $\xi = te_i$ (for all $i = 1,\dots,d$) and then letting $\epsilon \to 0$.

(ii)$\Rightarrow$(i) Since for all $i = 1,\dots,d$ the linear functional

$$C^1_c(\Omega) \ni h \mapsto \int_\Omega \frac{\partial h}{\partial x_i} u dx \in \mathbb{K}$$

is continuous with respect to $\|\cdot\|_{L^2}$, and since $C^1_c(\Omega)$ is dense in $L^2(\Omega)$, it can be extended by continuity to a bounded linear functional $\Phi_i : L^2(\Omega) \to \mathbb{K}$. Now, by the representation theorem of Riesz–Fréchet there exists $g_i \in L^2(\Omega)$ such that

$$\Phi_i(h) = \int_\Omega h\overline{g_i} dx \qquad \text{for all } h \in L^2(\Omega),$$

and in particular

$$\int_\Omega \frac{\partial h}{\partial x_i} u dx = \Phi_i(h) = \int_\Omega h\overline{g_i} dx \qquad \text{for all } h \in C^1_c(\Omega).$$

Since this holds for all $i = 1,\dots,d$, $u \in H^1(\Omega)$ by definition of weak differentiability. □

Remark 6.53. *Adopting the notation*

$$D_\xi u := \frac{\tau_\xi u - u}{|\xi|}, \tag{6.17}$$

what Lemma 6.52 says is that if $u \in L^2(\Omega)$, then $u \in H^1(\Omega)$ if and only if the estimate $\|D_\xi u\|_{L^2(\omega)} \leq C$ holds uniformly in ξ on open subsets ω with $\overline{\omega} \subset \Omega$. In fact, the proof yields even $C = \|\nabla u\|_{L^2(\Omega)}$, i.e.,

$$\|D_\xi v\|_{L^2(\omega)} \leq \|\nabla v\|_{L^2(\Omega)} \qquad \text{for all } \xi \in \mathbb{R}^d \setminus \{0\} \text{ small and all } v \in H^1(\Omega). \tag{6.18}$$

Observe also that

$$(D_\xi u|v)_{L^2} = (u|D_{-\xi} v)_{L^2}, \qquad u, v \in L^2(\Omega). \tag{6.19}$$

We are now in the position to discuss the promised regularity results. We follow the approach in [**4**, § 9.6] and, in particular, share Brezis' mind on the utility of well understanding the cases of Ω without boundary and then of Ω with a flat boundary.

Theorem 6.54. *Let $\Omega = \mathbb{R}^d$. Let $k \in \mathbb{N}$ and $f \in H^k(\Omega)$. Then the weak solution*[2] *$u \in H^1(\Omega)$ of*

$$\Delta u(x) - u(x) = f(x), \qquad x \in \mathbb{R}^d, \tag{6.20}$$

is of class

$$H^{k+2}(\Omega) := \{u \in L^2(\Omega) : u \in H^2(\omega) \text{ for all open } \omega \text{ with } \overline{\omega} \subset \Omega\}.$$

Observe that since $\mathbb{R}^d$ is (of course) not contained in any strip, Poincare's inequality does not apply, $H^1(\mathbb{R}^d)$ cannot be endowed with the H^1_0-norm and one has indeed to consider the corrective term $-u$ in (6.20).

Beweis. Let $\xi \in \mathbb{R}^d \setminus \{0\}$ and define $D_\xi u$ as in (6.17). Let moreover

$$\phi := D_{-\xi}(D_\xi u) = \frac{\tau_{-\xi}(\tau_\xi u - u) - (\tau_\xi u - u)}{|\xi|^2} = \frac{-\tau_{-\xi} u - \tau_\xi u}{|\xi|^2},$$

i.e.,

$$\phi(x) := \frac{-u(x-\xi) - u(x+\xi)}{|\xi|^2},$$

Since $u \in H^1(\mathbb{R}^d)$, $\phi \in H^1(\mathbb{R}^d)$ and we can plug it in (6.14), obtaining by (6.19)

$$\|D_\xi u\|^2_{H^1} = \int_\Omega \left(|\nabla D_\xi u(x)|^2 + |D_\xi u(x)|^2\right) dx = \int_\Omega \overline{f} D_{-\xi}(D_\xi u) dx \le \|f\|_{L^2} \|D_{-\xi}(D_\xi u)\|_{L^2},$$

where we have used the fact that for H^1-functions ∇ and D_ξ commute. Taking into account (6.18) and applying it to $v := D_\xi u$ we can estimate the last term by

$$\|f\|_{L^2} \|D_{-\xi}(D_\xi u)\|_{L^2} \le \|f\|_{L^2} \|\nabla(D_\xi u)\|_{L^2},$$

i.e., we obtain

$$\|D_\xi u\|^2_{H^1} \le \|f\|_{L^2} \|\nabla(D_\xi u)\|_{L^2} \le \|f\|_{L^2} \|(D_\xi u)\|_{H^1}.$$

Summing up, we deduce that

$$\|D_\xi \frac{\partial u}{\partial x_i}\|_{L^2} \le \|D_\xi u\|_{H^1} \le \|f\|_{L^2} \qquad i = 1, \ldots, d.$$

By Lemma 6.52 this means that each weak partial derivative of u is in H^1, and finally we get $u \in H^2(\mathbb{R}^d)$, by definition of second Sobolev space.

The proof of the general assertion is based on induction. In order to get a clue on how to perform the inductive step, we show that if $f \in H^1(\mathbb{R}^d)$, then $u \in H^3(\mathbb{R}^d)$, i.e., we have to prove that each weak partial derivative $\frac{\partial^2 u}{\partial x_i \partial x_j}$ of u is in $H^2(\mathbb{R}^d)$. Since the method presented above for in order to show that $u \in H^2$ is solely based on the fact that u satisfies the weak formulation of the elliptic equation, our goal is to replicate the same argument. To this aim we want to show that each weak partial derivative $\frac{\partial u}{\partial x_i}$ satisfies

$$\int_{\mathbb{R}^d} \left(\nabla \frac{\partial u}{\partial x_i} \overline{\nabla h} + \frac{\partial u}{\partial x_i} \overline{h}\right) dx = \int_{\mathbb{R}^d} \frac{\partial f}{\partial x_i} \overline{h} dx \qquad \text{for all } h \in H^1(\mathbb{R}^d),$$

[2] whose existence (and uniqueness) is ensured by the theorem of Lax–Milgram.

or rather, by density (cf. Exercise 6.47), that

$$\int_{\mathbb{R}^d} \left(\nabla \frac{\partial u}{\partial x_i} \overline{\nabla h} + \frac{\partial u}{\partial x_i} \overline{h} \right) dx = \int_{\mathbb{R}^d} \frac{\partial f}{\partial x_i} \overline{h} dx \qquad \text{for all } h \in C_c^\infty(\mathbb{R}^d). \tag{6.21}$$

This would yield that $v := \frac{\partial u}{\partial x_i}$ is a weak solution to

$$\Delta v(x) - v(x) = \frac{\partial f}{\partial x_i}(x), \qquad x \in \mathbb{R}^d,$$

which would suffice in order to deduce the claim.

Let therefore h and hence $\frac{\partial h}{\partial x_i} \in C_c^\infty(\mathbb{R}^d)$. Since u is a weak solution, by definition

$$\int_{\mathbb{R}^d} \left(\nabla u \nabla \overline{\frac{\partial h}{\partial x_i}} + u \overline{\frac{\partial h}{\partial x_i}} \right) dx = \int_{\mathbb{R}^d} f \overline{\frac{\partial h}{\partial x_i}} dx \qquad \text{for all } h \in C_c^\infty(\mathbb{R}^d),$$

and integrating by parts we see that (6.21) holds. □

The above kind of assertions goes under the name of *interior regularity results*. The reason for this name is that alone imposing boundary values (we will as usual consider Dirichlet boundary conditions, but Neumann and Robin would cause the same effect) ensures regularity up to the boundary.

Theorem 6.55. *Let $\Omega = \mathbb{R}^d_+$. Let $k \in \mathbb{N}$ and $f \in H^k(\Omega)$. Then the weak solution $u \in H^1_0(\Omega)$ of*

$$\Delta u(x) - u(x) = f(x), \qquad x \in \mathbb{R}^d, \tag{6.22}$$

with boundary conditions

$$u(z) = 0, \qquad z \in \partial\mathbb{R}^d_+, \tag{6.23}$$

is of class

$$H^{k+2}(\Omega).$$

The interesting feature of this result lies in the – somewhat surprising – possibility of deducing from the assumption $\Delta u \in L^2(\Omega)$ that *every* partial derivative of order 2 is in L^2, too.

In the proof we will repeatedly use the fact that if $\xi \in \mathbb{R}^{d-1} \times \{0\}$, i.e., if ξ is a vector that is parallel to the boundary of Ω, then

$$\tau_\xi u \in H^1_0(\Omega) \qquad \text{whenever } u \in H^1_0(\Omega),$$

i.e., $H^1_0(\Omega)$ is invariant under translations in directions that are parallel to the boundary of $\mathbb{R}^d_+$, i.e., in all directions e_j, $j \in \{1, \ldots, d-1\}$.

Furthermore, we will need the following two auxiliary results. The former can be proved following the proof of Lemma 6.52 almost verbatim, using invariance of $H^1_0(\Omega)$ under translations along directions parallel to $\partial\mathbb{R}^d_+$. The proof of the latter needs some (elementary) functional analytical tool, and we omit it.

Lemma 6.56. *Let $\Omega = \mathbb{R}^d_+$. Let ξ be a vector that is parallel to the boundary of Ω. Then*

$$\|D_\xi u\|_{L^2} \leq \|\nabla u\|_{L^2} \qquad \textit{for all } u \in H^1(\Omega).$$

Lemma 6.57. *Let $\Omega = \mathbb{R}^d_+$ and $u \in H^2(\Omega) \cap H^1_0(\Omega)$ be a weak solution to (6.22). Then $\frac{\partial u}{\partial x_i} \in H^1_0(\Omega)$ and*

$$\int_{\mathbb{R}^d} \left(\nabla \frac{\partial u}{\partial x_i} \overline{\nabla h} + \frac{\partial u}{\partial x_i} \overline{h} \right) dx = \int_{\mathbb{R}^d} \frac{\partial f}{\partial x_i} \overline{h} dx \qquad \textit{for all } h \in H^1_0(\Omega).$$

Proof of Theorem 6.55. Arguing as in the proof of Theorem 6.52 one sees that (6.18) holds in this context, too. More precisely, if ξ is a vector that is parallel to the boundary of Ω – say, $\xi = |\xi|e_j$, $j \in \{1, \ldots, d-1\}$, then

$$\|D_\xi u\|_{H^1}^2 \leq \|f\|_{L^2}\|D_{-\xi}(D_\xi u)\|_{L^2} \qquad \text{for all } u \in H^1(\Omega).$$

Accordingly, by Lemma 6.56

$$\begin{aligned}\|D_\xi u\|_{H^1}^2 &\leq \|f\|_{L^2}\|D_{-\xi}(D_\xi u)\|_{L^2}\\ &\leq \|f\|_{L^2}\|\nabla(D_\xi u)\|_{L^2}\\ &\leq \|f\|_{L^2}\|(D_\xi u)\|_{H^1} \qquad \text{for all } u \in H^1(\Omega),\end{aligned}$$

i.e.,

$$\|D_\xi u\|_{H^1} \leq \|f\|_{L^2} \qquad \text{for all } u \in H^1(\Omega). \tag{6.24}$$

Let now $h \in C_c^\infty(\Omega)$ and $i \in \{1, \ldots, d\}$. It follows – applying (6.19) and then integrating by parts – that

$$\begin{aligned}\left|\int_\Omega u\overline{D_{-\xi}\frac{\partial h}{\partial x_i}}dx\right| &= \left|\int_\Omega D_\xi u\overline{\frac{\partial h}{\partial x_i}}dx\right|\\ &= \left|\int_\Omega \frac{\partial}{\partial x_i}D_\xi u\overline{h}dx\right|\\ &\leq \left\|\frac{\partial}{\partial x_i}D_\xi u\right\|_{L^2}\|h\|_{L^2}\\ &\leq \|D_\xi u|_{H^1}\|h\|_{L^2}\\ &\leq \|f\|_{L^2}\|h\|_{L^2},\end{aligned}$$

where we have used (6.24) in the last step. Passing to the limit as $\xi \to 0$, Lebesgue's dominated convergence theorem yields

$$\left|\int_\Omega u\overline{\frac{\partial^2 h}{\partial x_i \partial x_j}}dx\right| \leq \|f\|_{L^2}\|h\|_{L^2} \qquad \text{for all } h \in C_c^\infty(\Omega), \tag{6.25}$$

which – we repeat it – holds for all $i \in \{1, \ldots, d\}$ and all $j \in \{1, \ldots, d-1\}$. Our aim is to show this inequality for *all* partial derivatives of second order of h, i.e., to show also

$$\left|\int_\Omega u\overline{\frac{\partial^2 h}{\partial x_d^2}}dx\right| \leq \|f\|_{L^2}\|h\|_{L^2} \qquad \text{for all } h \in C_c^\infty(\Omega).$$

This estimate can be obtained taking into account the definition of weak solution, which (applying the formulae of Gauß–Green) yields

$$\int_\Omega u\overline{\Delta h}dx = \int_\Omega (u+f)\overline{h}dx \qquad \text{for all } h \in C_c^\infty(\Omega),$$

or rather

$$\int_\Omega u\overline{\frac{\partial^2 h}{\partial x_d^2}}dx = -\sum_{j=1}^{d-1}\int_\Omega u\overline{\frac{\partial^2 h}{\partial x_j^2}}dx + \int_\Omega (u+f)\overline{h}dx \qquad \text{for all } h \in C_c^\infty(\Omega),$$

Accordingly,

$$\left|\int_\Omega u\overline{\frac{\partial^2 h}{\partial x_d^2}}dx\right| \leq \sum_{j=1}^{d-1}\left|\int_\Omega u\overline{\frac{\partial^2 h}{\partial x_j^2}}dx\right| + \left|\int_\Omega (u+f)\overline{h}dx\right| \leq K\|u+f\|_{L^2}\|h\|_{L^2} \leq \tilde{K}\|f\|_{L^2}\|h\|_{L^2},$$

for some constants $K, \tilde{K} > 0$, where the term $\|u+f\|_{L^2}$ has been estimated by the triangle inequality and by (6.24). This finally shows that for all $i,j = 1,\ldots,d$ the linear functional

$$C_c^\infty(\Omega) \ni h \mapsto \int_\Omega u \overline{\frac{\partial^2 h}{\partial x_i \partial x_j}} dx \in \mathbb{K}$$

is continuous, hence applying the representation theorem of Riesz–Fréchet as in the proof of (ii)⇒ (i) in Lemma 6.52 yields the existence of functions $g_{ij} \in L^2(\Omega)$ such that

$$\int_\Omega u \overline{\frac{\partial^2 h}{\partial x_i \partial x_j}} dx = \int_\Omega g_{ij} \overline{h} dx \qquad \text{for all } h \in C_c^\infty(\Omega).$$

By definition of weak derivative, this means that $u \in H^2(\Omega)$.

Let us finally perform the induction step, i.e., let us assume validity of the implication $f \in H^k(\Omega) \Rightarrow u \in H^{k+2}(\Omega)$ and let us take $f \in H^{k+1}$. Since $f \in H^k$, by assumption $u \in H^{k+2}$, i.e., for all $j \in \{1,\ldots,d-1\}$ Lemma 6.57 yields $\frac{\partial u}{\partial x_j} \in H_0^1$. Moreover, again by Lemma 6.57, $\frac{\partial u}{\partial x_j}$ is the weak solution to an elliptic problem analogous to ours, with the inhomogeneous term f replaced by $\frac{\partial f}{\partial x_j} \in H^k$. By the induction assumption, $\frac{\partial u}{\partial x_j} \in H^{k+2}$, and the claim finally follows if we only can prove that the unique missing partial derivative, $\frac{\partial^2 u}{\partial x_d^2}$, also belongs to H^{k+1}. But this follows re-writing the elliptic equation: since $u \in H^2$,

$$\Delta u(x) + u(x) = -f(x)$$

is satisfied for a.e. $x \in \Omega$, hence in particular

$$\frac{\partial^2 u}{\partial x_d^2} = -\sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2} - u + f \in H^{k+1}.$$

This concludes the proof. □

The case of general Ω (with regular boundary) is tackled essentially by boundary flattening – like in the proof of Lemma 6.44 – but the method is made overly technical by the necessity of performing several variable substitutions. The interested reader is referred to Brezis' book or to [**5**, § 6.3] for the generalization of these results to the case of elliptic equations involving operators that are more general than Δ.

Theorem 6.58. *Let Ω be an open bounded domain of $\mathbb{R}^d$. Let $k \in \mathbb{N}$ and $f \in H^k(\Omega)$. Then the weak solution $u \in H_0^1(\Omega)$ of*

$$\Delta u(x) - u(x) = f(x), \qquad x \in \mathbb{R}^d, \tag{6.26}$$

with boundary conditions

$$u(z) = 0, \qquad z \in \partial\mathbb{R}^d_+, \tag{6.27}$$

is of class

$$H^{k+2}(\Omega).$$

Remark 6.59. *The above regularity results show in particular, together with the second Sobolev embedding theorem, that if $f \in H^m(\Omega)$ for some $m > d/2$, then the weak solution u to the elliptic problem is in fact in $C^2(\overline{\Omega})$, and in fact even in $C^\infty(\overline{\Omega})$ if $f \in C^\infty(\overline{\Omega})$ (e.g. if $f \equiv 0$).*

One of the most interesting features of the method based on weak formulation of partial differential equations is the possibility of easily obtaining invariance results.

Definition 6.60. *Let V, H be Hilbert spaces, such that $V \subset H$, $\overline{V} = H$, and such that the canonical injection $V \to H$ is continuous. Let $a : H \times H \to \mathbb{K}$ be a continuous and coercive sesquilinear form*[3]*. The* **operator** A **on** H **associated with** *a is the linear unbounded operator defined by*

$$\begin{aligned} D(A) &:= \{f \in V : \exists g \in H : a(f,h) = (Af|h)_H \ \forall h \in V\}, \\ Af &:= -g. \end{aligned}$$

Exercise 6.61. *Consider the Laplace equation with Robin boundary conditions on the interval $(0,1)$. Find a weak formulation of the problem by presenting a sesquilinear form on $H^1(0,1)$ whose associated operator on $L^2(0,1)$ is the Laplacian with Robin boundary conditions.*

Proposition 6.62 (Ouhabaz's lemma)**.** *Let V, H be Hilbert spaces, such that $V \subset H$, $\overline{V} = H$, and such that the canonical injection $V \to H$ is continuous. Let $a : H \times H \to \mathbb{K}$ be a continuous and coercive sesquilinear form and denote by A the associated operator on H. Let finally C be a closed convex set of H and denote by P_C the projector of H onto C.*

Then the following assertions are equivalent.

(i) If $f \in C$, then for all $\lambda > 0$ also the weak solution[4] *u_λ of the Helmholtz equation*

$$\lambda u - Au = f$$

satisfies $\lambda u_\lambda \in C$.

(ii) $P_C V \subset V$ and $\mathrm{Re}(v|v - P_C v)_H \geq 0$ for all $v \in V$.

This result has been obtained by ElMaati Ouhabaz in 1992. The proof of the implication $(i) \Rightarrow (ii)$ is more delicate and we omit it. It can be found as part of [**8**, Thm. 2.1 and Thm. 2.2]. In the proof of $(ii) \Rightarrow (i)$ we use repeatedly the fact that

$$f = \lambda u_\lambda - Au_\lambda,$$

at least in weak sense.

Beweis. (ii)⇒(i) Let $\lambda > 0$. Then setting $v := \lambda u_\lambda$ we obtain by assumption

$$\begin{aligned} 0 &\leq \mathrm{Re}a(\lambda u_\lambda | \lambda u_\lambda - \lambda P_C u_\lambda)_H \\ &= \lambda \mathrm{Re} - (Au_\lambda | \lambda u_\lambda - \lambda P_C u_\lambda)_H \\ &= \lambda \mathrm{Re}(f - \lambda u_\lambda | \lambda u_\lambda - \lambda P_C u_\lambda)_H \\ &= \lambda \mathrm{Re}(f - \lambda P_C u_\lambda | \lambda u_\lambda - \lambda P_C u_\lambda)_H + \mathrm{Re}(\lambda P_C u_\lambda - \lambda u_\lambda | \lambda u_\lambda - \lambda P_C u_\lambda)_H \\ &= \lambda \mathrm{Re}(f - \lambda P_C u_\lambda | \lambda u_\lambda - \lambda P_C u_\lambda)_H - \lambda \|P_C u_\lambda - u_\lambda\|_H^2 \\ &\leq \lambda \mathrm{Re}(f - \lambda P_C u_\lambda | \lambda u_\lambda - \lambda P_C u_\lambda)_H \\ &\leq 0, \end{aligned}$$

where the last inequality follows from Theorem 6.9. This inequality chain can only hold if

$$\|P_C u_\lambda - u_\lambda\|_H = 0,$$

[3] Actually, slightly less restrictive assumption on a are needed: it suffices that a is closed and accretive – i.e., that

$$\mathrm{Re}a(u,u) + \omega \|u\|_H^2 \geq \alpha \|u\|_V^2 \qquad \text{and} \qquad \mathrm{Re}a(u,u) \geq 0$$

for some $\alpha > 0$, some $\omega \in \mathbb{R}$, and all $u \in V$ – rather than coercive. These two assumptions are only slightly weaker than coercivity of a, but this relaxation is enough to accomodate the case of a associated with the Laplace operator with Neumann boundary conditions.

[4] whose existence is yielded by the lemma of Lax–Milgram.

i.e., if u_λ agrees with its best approximation in C – in other words, if $u_\lambda \in C$. □

Exercise 6.63. *Use Sobolev spaces of second order to solve the following.*

(1) *Find a weak formulation of the elliptic equation*

$$\frac{d^4u}{dx^4}(x) + u(x) = f(x), \qquad x \in (0,1),$$

with Dirichlet and *Neumann boundary conditions (a fourth order equation needs four boundary conditions).*

(2) *Use the implication (a)⇒(b) in the Proposition 6.62 to prove that the above elliptic equation does* not *satisfy the maximum principle.*

Exercise 6.64. *Let Ω be an open domain of $\mathbb{R}^d$. Denote by P_+ the orthogonal projector of $L^2(\Omega)$ onto the closed convex set $\{f \in L^2(\Omega) : f(x) \geq 0$ for a.e. $x \in \Omega\}$. Consider the sesquilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$, whose associated operator is the Laplace operator with Neumann boundary conditions. Show that a and P_+ satisfy the assumptions of Proposition 6.62.*

Exercise 6.65. *Let V be a Hilbert space and $b : V \times V \to \mathbb{R}$ a continuous symmetric bilinear form. Show that for each function $u \in C^1(\mathbb{R}_+, V)$ the chain rule*

$$\frac{d}{dt} b(u(t), u(t)) = 2b\left(\frac{\partial u}{\partial t}(t), u(t)\right), \qquad t \geq 0,$$

holds. How can this help to relax the hypotheses of Theorem 2.18?

Exercise 6.66. *Prove the estimate in Remark 4.1.*

KAPITEL 7

The telegraph equation and Noether's theorem

The main aim of this chapter is to extend the results of Section 4.1. If a (partial or ordinary)differential equation has a *variational structure*, in a sense that will be defined soon, then it is possible to turn its symmetries into conservation laws. This piece of mathematical magic is the essential feature of the theory developed by Emmy Noether in the second half of the 1910s. We will mostly follow the approach of [**5**, § 8.6] and [**7**, § 4.2].

Traditionally, Noether's theorem is developed in the framework of the calculus of variations, but one may just as well rely upon other methods to obtain existence of solutions to differential equations, and only afterwards perform a symmetry analysis on them, just like we have done in Section 4.1.

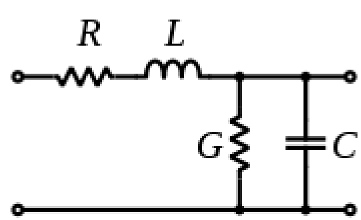
The telegraph equation has been introduced by Oliver Heaviside in the 1920s. It is derived by a simple model of electric circuit. Consider a thin electric wire of infinite length that can be approximated as a diffuse mix of **resistors** and **coils**, with a given resistance Rdx ohm and inductance Ldx henry, respectively. The current flowing through resistors and coils in the wire between the points x and $x + \Delta x$ at time t is

$$i(t,x)R\Delta x \qquad \text{and} \qquad \frac{\partial i}{\partial t}(t,x)L\Delta x,$$

respectively, hence the total voltage between the points x and $x + \Delta x$ at time t is

$$u(t,x+\Delta x) - u(t,x) = -\int_x^{x+\Delta x} i(t,\xi)Rd\xi - \int_x^{x+\Delta x} \frac{\partial i}{\partial t}(t,\xi)Ld\xi.$$

With the aim of describing the behaviour of a (1-dimensional) telegraph (or railway, or telephone...) wire, one wants the assume the possibility that – because the wire is not grounded – the electric current may flow out of the system through **insulators** or **capacitors** of conductance $G^{-1}dx$ mho or capacitance Cdx farad – modelling telegraph poles and wires, respectively.



(from Wikipedia)

Now, the telegraph equation arises as one tries to describe the time evolution of the current i and the voltage u along the wire. The total current that escapes from the wire through the resistor is

$$\int u(t,\xi)G^{-1}d\xi. \tag{7.1}$$

In order to compute the current escaping through the capacitor, observe that the total charge on a piece of the capacitor of length Δx is

$$\int_x^{x+\Delta x} u(t,\xi)Cd\xi,$$

hence its time variation is

$$\int_x^{x+\Delta x} \frac{\partial u}{\partial t}(t,\xi)Cd\xi :$$

combining this expression with (7.1) we obtain the total current flowing through resistors and coils in the wire between the points x and $x+\Delta x$ at time t is

$$i(t,x+\Delta x)-i(x) = -\int_x^{x+\Delta x} \frac{\partial u}{\partial t}(t,\xi)Cd\xi - \int_x^{x+\Delta x} u(t,\xi)G^{-1}d\xi.$$

Taking the limit $\Delta x \to 0$ we obtain the system of partial differential equations

$$\begin{cases} \frac{\partial u}{\partial x}(t,x) &= -i(t,x)R - \frac{\partial i}{\partial t}(t,x)L, \\ \frac{\partial i}{\partial x}(t,x) &= -u(t,x)G^{-1} - \frac{\partial u}{\partial t}(t,x)C. \end{cases}$$

Taking the partial derivative with respect to x of the former equation yields

$$\frac{\partial^2 u}{\partial x^2}(t,x) = -\frac{\partial i}{\partial x}(t,x)R - \frac{\partial^2 i}{\partial t \partial x}(t,x)L,$$

while taking the partial derivative with respect to t of the latter one yields

$$\frac{\partial^2 i}{\partial t \partial x}(t,x) = -\frac{\partial u}{\partial t}(t,x)G^{-1} - \frac{\partial^2 u}{\partial t^2}(t,x)C.$$

Combining them yields the partial differential equations

$$L\left(-C\frac{\partial^2 u}{\partial t^2}(t,x) - G^{-1}\frac{\partial u}{\partial t}(t,x)\right) + R\left(-C\frac{\partial u}{\partial t}(t,x) - G^{-1}u(t,x)\right) + \frac{\partial^2 u}{\partial x^2} = 0.$$

(It is clear that a similar argument yields an analogous "dual" equation for the current i.)

Thus, renaming some constants we finally obtain the telegraph equation

$$\frac{\partial^2 u}{\partial t^2}(t,x) + h\frac{\partial u}{\partial t}(t,x) + k^2 u = c^2\frac{\partial^2 u}{\partial x^2}(t,x), \qquad t,x \in \mathbb{R}, \tag{7.2}$$

for $h, k^2, c^2 \in \mathbb{R}$ (or its d-dimensional version

$$\frac{\partial^2 u}{\partial t^2}(t,x) + h\frac{\partial u}{\partial t}(t,x) + k^2 u = c^2\Delta u(t,x), \qquad t \in \mathbb{R},\ x \in \Omega, \tag{7.3}$$

for some open domain $\Omega \subset \mathbb{R}^d$, which describes wave propagation phenomena in differerent media).

This equation has an interesting dependence on these parameters: it is an ODE if $c = 0$, it behaves like a (damped) wave equation for $h = 0$ but mimics a heat equation for $h \to \infty$ (and it becomes an elliptic equation for $c \in i\mathbb{R}$). In fact, it can be seen as a kind of hybrid between parabolic and hyperbolic equations.

7.1. Noether's theorem

Let $U \subset \mathbb{R}^d$ be open and bounded with ∂U so smooth that the Gauß–Green formulae hold. For $k \in \mathbb{N}$ we consider again the jet bundle $J^k(U)$ introduced in Chapter 4. Since $J^k(U)$ is a finite-dimensional vector space, one can consider the second Sobolev spaces of (for simplicity) real-valued functions $H^2(J^k(U))$, $k \in \mathbb{N}$. Consider $L \in H^2(J^k(U)) \cap L^1(J^k(U))$ and define $I : C^k_{BC}(U) \to \mathbb{R}$ by

$$I(u) := \int_U L(x, j_k u(x))dx.$$

where $C^k_{BC}(U)$ is a set of $C^k(U)$-functions that satisfy certain boundary conditions – for $k = 1$, these are typically Dirichlet, Neumann or Robin (possibly inhomogeneous) boundary conditions depending on the specifical problems we consider, i.e.

$$u|_{\partial U} = g|_{\partial U} \qquad \text{or} \qquad \frac{\partial u}{\partial n} + p_{\partial U} u|_{\partial U} = g|_{\partial U}$$

for some $p, g \in C^(\overline{U})$.

For the sake of simplicity we restrict our discussion to the case of Lagrangians defined on $J^1(U)$, i.e., we are thinking of applications to partial differential equations whose (weak) solutions are defined as H^1-solutions. The general case is treated in [**7**, §4.4] and is only notationally more demanding, but not much harder to discuss.

Proposition 7.1. *Let the value $I(u)$ be a minimum among all those attained by I on the set $C^1_{BC}(U)$. Then u satisfies the partial differential equation*

$$\sum_{k=1}^{d} \frac{\partial^2 L}{\partial x_k \partial u_{x_k}}(x, j_1 u(x)) + \frac{\partial L}{\partial u}(x, j_1 u(x)) = 0 \qquad \text{for a.e. } x \in U. \tag{7.4}$$

If Theorem 7.1 applies, then (7.4) is called **Euler–Lagrange equation** associated with the **Lagrangian** L and its **action functional** I.

The reason why we are assuming that the Lagrangian L is a H^2-function is that we wish to derivate it twice in the Euler–Lagrange equation, while we want it to be an L^1-function in order to have a well-defined action functional I. These requirements do not depend on the specific differential equation we are considering.

Remark 7.2. *Observe that the converse implication, which holds in the context of Dirichlet's principle, is not always valid in the much more general context of arbitrary Euler–Lagrange equations. In fact, the proof of Theorem 6.1 depends much on the particular structure of the problem. This becomes clear if we write it (quite informally) as*

$$D_x D_{\nabla u} L + D_u L = 0.$$

Three main reasons for the general failure of a Dirichlet-type principle are that on one hand, when it comes to apply the Gauß–Green formulae, only the term D_x (the divergence with respect to the space variables) is well-behaved; and on the other hand that, even when $D_u L \equiv 0$, the term $D_{\nabla u} L$ is in general very nonlinear, cf. Example 7.3 below. In particular, if one tries to mimic the proof in Theorem 6.1 naively when trying to compare the action $I(u)$ of a solution with any other action $I(w)$, one gets stuck with

$$0 = \int_U D_{\nabla u} L(j_1 u) D_x(j_1 u - j_1 w)dx.$$

But even when neither of the above problems arise, failure of a Dirichlet-type principle may depend on the failure of the basic argument in Weierstraß' (incomplete) argument: namely, that $D_u L$ is related to an inner product on $C(\overline{U})$. Generally speaking, an action functional does not have to reflect a physical energy. An example of this kind can be found in Exercise 7.6 below.

PROOF OF PROPOSITION 7.1. The starting point is an argument similar to that used when we have derived the Poisson equation by minimising its total energy. Also in this case, we fix $u \in C^1_{BC}(U)$ and for all $\epsilon \in \mathbb{R}$ consider the functions

$$u + \epsilon v \qquad \text{for all } v \in C^\infty_c(U).$$

Observe that since v is smooth also $u + \epsilon v \in C^1(U)$, and since it has compact support it does not contribute to the trace and normal derivative of $u + \epsilon v$, which therefore agree with the trace and normal derivative of u – i.e., $u + \epsilon v \in C^1_{BC}(U)$.

A minimum of I satisfies

$$\frac{d}{d\epsilon} I(u + \epsilon v)_{|\epsilon=0} = 0,$$

hence

$$\begin{aligned}
0 &= \frac{d}{d\epsilon} \int_U L(x, j_1(u+\epsilon v)(x))dx\Big|_{\epsilon=0} \\
&= \frac{d}{d\epsilon} \int_U L(x, u(x)+\epsilon v(x), \nabla u(x) + \epsilon \nabla v(x))dx\Big|_{\epsilon=0} \\
&= \int_U \left(\sum_{k=1}^d \frac{\partial L}{\partial u_{x_i}} \Big(x, u(x)+\epsilon v(x), \nabla u(x) + \epsilon \nabla v(x)\Big) v_i(x) \right. \\
&\qquad \left. + \frac{\partial L}{\partial u} \Big(x, u(x)+\epsilon v(x), \nabla u(x) + \epsilon \nabla v(x)\Big) v(x) \right) dx\Big|_{\epsilon=0} \\
&= \int_U \left(\sum_{k=1}^d \frac{\partial L}{\partial u_{x_i}} \Big(x, u(x), \nabla u(x)\Big) \frac{\partial v}{\partial x_i}(x) + \frac{\partial L}{\partial u} \Big(x, u(x), \nabla u(x)\Big) v(x) \right) dx.
\end{aligned}$$

Applying the Gauß–Green formulae we finally obtain[1]

$$\begin{aligned}
0 &= \int_U \left(\sum_{k=1}^d \frac{\partial^2 L}{\partial x_i \partial u_{x_i}} \Big(x, u(x), \nabla u(x)\Big) v(x) + \frac{\partial L}{\partial u} \Big(x, u(x), \nabla u(x)\Big) v(x) \right) dx \\
&= \left(\sum_{k=1}^d \frac{\partial^2 L}{\partial x_i \partial u_{x_i}} (j_1 u) + \frac{\partial L}{\partial u}(j_1 u) \Big| v \right)_{L^2(U)},
\end{aligned}$$

which – as we have seen – holds for all $v \in C^\infty_c(U)$ and hence, by density, also for all $v \in L^2(U)$. By Exercise 5.12.(5) this concludes the proof. □

Example 7.3. *Define for $u : \mathbb{R}^2 \to \mathbb{R}$ the Lagrangian*

$$L(x, y, j_1 u(x,y)) := \sqrt{1 + |\nabla u(x,y)|^2}.$$

We have already met this Lagrangian at the beginning of Chapter 6, in the derivation of the Poisson equation. For $i = 1, 2$ one has

$$\frac{\partial L}{\partial u_{x_i}}(x, y, j_1 u(x,y)) = \frac{\partial}{\partial u_{x_i}} \sqrt{1 + |\nabla u(x,y)|^2} = \frac{1}{2} \frac{u_{x_i}}{\sqrt{1 + |\nabla u(x,y)|^2}},$$

[1] Here, no boundary term appears since $v|_{\partial U} = 0$.

and accordingly the associated Euler–Lagrange equation is

$$\nabla\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \frac{\partial}{\partial x}\frac{u_x}{\sqrt{1+|\nabla u(x,y)|^2}} + \frac{\partial}{\partial y}\frac{u_y}{\sqrt{1+|\nabla u(x,y)|^2}} = 0,$$

the so-called **mimimal surface equation**. *As soon as we impose (generally speaking, inhomogeneous) boundary conditions, a solution to this equation yields the surface with minimal area among all those with prescribed boundary.*

Usually, however, one proceeds the other way round: one considers a partial differential equation and – typically by partial integration – is led to introduce a suitable Lagrangian that realizes the given equation as associated Euler–Lagrange-equation.

Example 7.4. *Let $f : \mathbb{R} \to \mathbb{R}$ and consider the nonlinear Poisson equation*

$$\Delta u(x,y) = -f(u(x,y)), \qquad (x,y) \in U \subset \mathbb{R}^d.$$

Take $F := \int_0^{\cdot} f(y)dy$. This equation can be recovered as a Euler–Lagrange equation by introducing the Lagrangian

$$L(x, j_1 u) := \frac{1}{2}|\nabla u(x)|^2 - F(u(x)).$$

Now, for all $i = 1, \ldots, d$

$$\frac{\partial L}{\partial u_{x_i}}(x, j_1 u(x)) = \frac{1}{2}\frac{\partial}{\partial u_{x_i}}|\nabla u(x)|^2 = \frac{1}{2}\frac{\partial}{\partial u_{x_i}}\sum_{k=1}^d u_k(x)^2 = u_{x_i}(x)$$

and accordingly

$$\frac{\partial^2 L}{\partial x_i \partial u_{x_i}}(x, j_1 u(x)) = \frac{\partial u_{x_i}}{\partial x_i}(x) = \frac{\partial^2 u}{\partial x_i^2}(x).$$

Similarly,

$$\frac{\partial L}{\partial u}(x, j_1 u(x)) = -\frac{\partial F}{\partial u}(u(x)) = -\frac{\partial}{\partial u}\int_0^{u(x)} f(w(x))dw(x) = -f(u(x)).$$

This shows that the Euler–Lagrange equation associated with the Lagrangian L is exactly the above nonlinear Poisson-type equation.

While many equations are actually not associated with a Lagrangian, in some cases an equation that does not look like a Euler–Lagrange one only needs a tricky transformation.

Exercise 7.5. *Consider the* **convective Poisson equation**

$$\Delta u(x) + \nabla\phi(x)\cdot\nabla u(x) = f(x), \qquad x \in U \subset \mathbb{R}^d,$$

for some $\phi, f : U \to \mathbb{R}$.

(1) *For $d = 1$, find a Lagrangian for the above equation by considering a multiplicative perturbation of the Dirichlet integral by a suitable exponential function related to ϕ.*
(2) *Extend the above result to general d using Lebesgue's differentiation theorem.*
(3) *Find an inspiration to represent the* **elliptic regularisation of the heat equation**

$$\frac{\partial u}{\partial t} - \epsilon\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad (t,x) \in (0,\infty)\times\mathbb{R},$$

as Euler–Lagrange equations, where $\epsilon > 0$.

Exercise 7.6. *Let U be an open domain of $\mathbb{R}^{d+1}$ and $f : \mathbb{R} \to \mathbb{R}$. Consider the Lagrangian*

$$L(x, y, j_1 u) := \frac{1}{2} \sum_{k=1}^{d} u_{x_k}(x, y)^2 - \frac{1}{2} u_y(x, y)^2 + F(u(x, y),$$

where $F := \int_0^{\cdot} f(z)dz$. Observe that even in the case $f \equiv 0$, its action functional differs from the total energy introduced in Definition 2.19.

(1) Check that the Euler–Lagrange equation associated with L is a d-dimensional nonlinear wave equation.
(2) Show that a solution to the wave equation need not minimise the energy: *Let $U = (0, 1) \times \mathbb{R}$ and $f \equiv 0$ and find initial conditions u_0, u_1 and a smooth functions $w : U \to \mathbb{R}$ with $w(0, y) = w(1, y) = 0$, $y \in \mathbb{R}$, satisfying them, not solving the wave equation and nevertheless such that $I(w) < I(u)$, where u is the (unique) solution to the wave equation with Dirichlet boundary conditions and given initial values. (Hint: apply the Dirichlet principle to an affine function).*

In the following we are going to consider not only additive variations of the solution u, but also more general ones, and even variations of the domain where the Euler–Lagrange equations take place. All this up will show up in the statement of Lemma 7.11. Let us consider a one-parameter point transformation group $(T_\epsilon(\mathbf{x}))_{\epsilon \in I_{\mathbf{x}}, \mathbf{x} \in J^0(U)}$ with generator A – or rather, their 1-jets. For the sake of notational simplicity, for ϵ small enough we write

$$(x_\epsilon, u_\epsilon) := (j_0 T_\epsilon)(x, u).$$

and

$$\omega_\epsilon := \{x_\epsilon \in \mathbb{R}^d : x \in \omega\}.$$

Definition 7.7. *Let U be an open domain of $\mathbb{R}^d$ and L be a Lagrangian. A one-parameter point transformation group $(T_\epsilon(\mathbf{x}))_{\epsilon \in I_{\mathbf{x}}, \mathbf{x} \in J^0(U)}$ satisfying*

$$\int_\omega L(j_1 T_\epsilon(x, j_1 u))dx = \int_{\omega_\epsilon} L(x, j_1 u)dx \tag{7.5}$$

for all $|\epsilon|$ small enough, all bounded open domains $\omega \subset \mathbb{R}^d$ and all $u \in H^1(U)$ is called **variational symmetry group** *of the Euler–Lagrange equation associated with L.*

Remark 7.8. *By the above definition, the Lagrangian involves derivatives of order 1 at most, whereas typically the associated Euler–Lagrange equations is a differential equation of order 2. Therefore, checking that a given point transformation group is a variational symmetry group of the given Euler–Lagrange equation – i.e., checking that* (7.5) *holds – may be more convenient then checking that the same group is a point symmetry group – i.e., checking* (4.12) *– since typically we only have to determine a lower order jet of the generator A: as we know, this is usually the lengthiest part of the assignment.*

Remark 7.9. *Moreover, it is true that each one-parameter point symmetry groups of a Euler–Lagrange differential equation is also a* variational *symmetry group (but the converse does not hold). We are not going to prove this result, and refer instead to* [**7**, §4.2].

In the following we need the following notation: if $(T_\epsilon(\mathbf{x}))_{\epsilon \in I_{\mathbf{x}}, \mathbf{x} \in J^0(U)}$ is a one-parameter transformation group with generator

$$A := (\xi_1, \ldots, \xi_d, \phi),$$

then we denote by $(\tilde{\xi}_1, \ldots, \tilde{\xi}_d, \tilde{\phi})$ the mappings defined by

$$(\tilde{\xi}_1, \ldots, \tilde{\xi}_d, \tilde{\phi})(x, f) := \left(\lim_{\epsilon \to 0} \frac{x_\epsilon - x}{\epsilon}, \lim_{\epsilon \to 0} \frac{f_\epsilon(x_\epsilon) - f(x)}{\epsilon} \right). \tag{7.6}$$

Remark 7.10. *We stress the difference from the definition of the generator A, cf. Remark 4.18. In particular, observe that in general $(\tilde{\xi}_1, \ldots, \tilde{\xi}_d, \tilde{\phi})(x, f)$ may also involve terms related to derivatives of f (unlike the generator A). In the setting of Example 4.14, for instance,*

$$\begin{aligned}(\tilde{\xi}, \tilde{\phi})(x, f) &:= \lim_{\epsilon \to 0} \frac{(x + \epsilon e_j, f(x + \epsilon e_j)) - (x, f(x))}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{(\epsilon e_j, f(x + \epsilon e_j) - f(x))}{\epsilon} \\ &= \left(e_j, \frac{\partial f}{\partial x_j}(x)\right),\end{aligned}$$

i.e.,

$$(\tilde{\xi}_1, \ldots, \tilde{\xi}_d, \tilde{\phi})(x, f) = \frac{\partial}{\partial x_j} + \frac{\partial f}{\partial x_j}(x) \frac{\partial}{\partial f}.$$

Lemma 7.11. *Consider a Lagrangian $L : H^2(J^1(U)) \cap L^1(J^1(U))$ and a one-parameter variational symmetry group $(T_\epsilon(\mathbf{x}))_{\epsilon \in I_{\mathbf{x}}, \mathbf{x} \in J^0(U)}$ with generator*

$$A := (\xi_1, \ldots, \xi_d, \phi).$$

Assume each ξ_i to depend on the independent variables only, i.e. $\xi_i \neq \xi_i(u)$, $i = 1, \ldots, d$. Then, one has

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k} \left(\tilde{\phi}(x, j_1 u) \frac{\partial L}{\partial u_{x_k}}(x, j_1 u) - L(x, j_1 u) \xi_k(x, j_1 u) \right) = \tilde{\phi}(x, j_1 u) \left(\sum_{k=1}^{d} \frac{\partial^2 L}{\partial x_k \partial u_{x_k}}(x, j_1 u) - \frac{\partial L}{\partial u}(x, j_1 u) \right)$$

for a.e. $(x, u) \in J^1(U)$, where $\tilde{\phi}$ is defined as in (7.6).

Remark 7.12. *The component ϕ of the generator A of $(T_\epsilon(\mathbf{x}))_{\epsilon \in I_{\mathbf{x}}, \mathbf{x} \in J^0(U)}$ is sometimes called* **multiplier** *of the Euler–Lagrange equation associated with L, since by Lemma 7.11 simply pointwise multiplying the Euler–Lagrange equation by ϕ turns it into a new equivalent equation of the form*

$$\nabla(\ldots) = 0.$$

Such kind of equations have several pleasant features: they are called equations **in divergence form**. *In easy cases the divergence form can be easily obtained, but in some cases one needs to apply the Gauß–Green formulae not without ingenuity.*

In the proof we will apply (without proving it) an extension of Gauß–Green formulae for integration by parts on variable domains. Let (ω_ϵ) be a family of open bounded domains of $\mathbb{R}^d$ smoothly depending on ϵ and such that their union is contained in a reference open domain U. Denoting by $\partial\omega_\epsilon$ the boundary of ω_ϵ, by $\xi(z)$ its velocity field and (as usual) by $n(z)$ the unit normal at $z \in \partial\omega_\epsilon$, we obtain for all $f \in H^1(U)$ that

$$\frac{d}{d\epsilon} \int_{\omega_\epsilon} f(x)dx = \int_{\partial\omega_\epsilon} f(z)\xi(z) \cdot n(z) d\sigma(z) + \int_{\omega_\epsilon} \frac{\partial f}{\partial \epsilon}(x)dx, \tag{7.7}$$

cf. [**5**, App. C.4].

Proof of Lemma 7.11. It suffices to differentiate both sides of (7.5) with respect to ϵ, i.e., to consider

$$\int_\omega \frac{\partial}{\partial \epsilon} L(x, j_1 u_\epsilon) dx \quad = \quad \frac{\partial}{\partial \epsilon} \int_{\omega_\epsilon} L(x, j_1 u) dx$$

and to apply the above Gauß–Green formula (7.7) for variable domains, to obtain (since $L(x, j_1u)$ does not depend on ϵ) that

$$\int_\omega \left(\sum_{k=1}^d \frac{\partial L}{\partial u_{x_k}}(x, j_1u_\epsilon) \frac{\partial^2 u_\epsilon}{\partial \epsilon \partial x_k}(x, j_1u_\epsilon) + \frac{\partial L}{\partial u}(x, j_1u_\epsilon) \frac{\partial u_\epsilon}{\partial \epsilon}(x, j_1u_\epsilon) \right) dx = \int_{\partial \omega_\epsilon} L(z, j_1u) \frac{\partial x_\epsilon}{\partial \epsilon}(z) \cdot n(z) d\sigma(z)$$

where we have used the fact that the velocity field of $\partial \omega_\epsilon$ is exactly the component ξ of the generator A of the transformation group. Evaluating[2] the above expression at $\epsilon = 0$ we obtain

$$\int_\omega \left(\sum_{k=1}^d \frac{\partial L}{\partial u_{x_k}}(x, j_1u) \frac{\partial \tilde{\phi}}{\partial x_k}(x, j_1u) + \frac{\partial L}{\partial u}(x, j_1u) \tilde{\phi}(x, j_1u) \right) dx = \int_{\partial \omega} L(z, j_1u) \xi(z, j_1u) \cdot n(z) d\sigma(z). \tag{7.8}$$

In order to conclude the proof we want to get rid of the partial derivatives of ϕ: the Gauß–Green formula applied to the function ϕ and to the vector field $D_{\nabla u} L$ yields

$$\begin{aligned} \int_\omega \left(\sum_{k=1}^d \frac{\partial L}{\partial u_{x_k}}(x, j_1u) \frac{\partial \tilde{\phi}}{\partial x_k}(x, j_1u) \right) dx \quad &= \quad - \int_\omega \left(\sum_{k=1}^d \frac{\partial^2 L}{\partial x_k \partial u_{x_k}}(x, j_1u) \tilde{\phi}(x, j_1u) \right) dx \\ &\quad + \int_{\partial \omega} \left(\sum_{k=1}^d \frac{\partial L}{\partial u_{x_k}}(z, j_1u) \cdot n_k(z) \tilde{\phi}(x, j_1u) \right) d\sigma(z). \end{aligned}$$

Plugging this expression into (7.8) we obtain

$$\begin{aligned} &\int_\omega \left(- \sum_{k=1}^d \frac{\partial^2 L}{\partial x_k \partial u_{x_k}}(x, j_1u) \tilde{\phi}(x, j_1u) + \frac{\partial L}{\partial u}(x, j_1u) \tilde{\phi}(x, j_1u) \right) dx \\ &\qquad = \int_{\partial \omega} \sum_{k=1}^d \left(- \frac{\partial L}{\partial u_{x_k}}(z, j_1u) \tilde{\phi}(x, j_1u) + L(z, j_1u) \xi(z, j_1u) \right) n_k(z) d\sigma(z) \\ &\qquad = \int_\omega \sum_{k=1}^d \frac{\partial}{\partial x_k} \left(- \frac{\partial L}{\partial u_{x_k}}(z, j_1u) \tilde{\phi}(x, j_1u) + L(z, j_1u) \xi_k(z, j_1u) \right) d\sigma(z). \end{aligned}$$

where the last identity follows from the divergence theorem. This concludes the proof, since this integral expression holds on all open sets, hence a.e. (again because the Lagrangian is an H^2-function. □

We can finally state the main result of this section, the fundamental theorem obtained by Emmi Noether in 1915.

Theorem 7.13 (Noether's theorem, 1918). *Under the assumptions of Lemma 7.11, for each one-parameter variational symmetry group $(T_\epsilon(\mathbf{x}))_{\epsilon \in I_{\mathbf{x}}, \mathbf{x} \in J^0(U)}$ with generator*

$$A := (\xi_1, \ldots, \xi_d, \phi)$$

of the corresponding Euler–Lagrange equation the divergence identity

$$\sum_{k=1}^d \frac{\partial}{\partial x_k} \left(\tilde{\phi}(x, j_1u) \frac{\partial L}{\partial u_{x_k}}(x, j_1u(x)) - L(x, j_1u(x)) \xi_k(x, j_1u) \right) = 0$$

[2] We can do this because by definition every one-parameter point transformation group, and hence also every variational symmetry group, depends in a continuously differentiable fashion on its arguments, and in particular on ϵ.

holds for a.e. $(x,u) \in J^1(U)$ *in the solution manifold of the same equation, where* $\tilde{\phi}$ *is defined as in* (7.6).

Beweis. By assumption

$$\sum_{k=1}^{d} \frac{\partial^2 L}{\partial x_k \partial u_{x_k}}(x, j_1 u(x)) + \frac{\partial L}{\partial u}(x, j_1 u(x)) = 0 \qquad \text{a.e.} \tag{7.9}$$

Plugging it into the identity in Lemma 7.11 yields the claim. □

Remark 7.14. *Another way to express Noether's theorem is to say that*

$$\phi D_{\nabla u} L - L\xi$$

is a solenoidal vector field on $J^1(U)$.

Sometimes the Lagrangian does not depend on x. For example, this is true for the linear wave or Laplace equations, but also – more interestingly – for the minimal surface equation. Then, Noether's theorem can be promptly applied.

Corollary 7.15. *Let* $j \in \{1, \dots, d\}$ *and assume* L *not to depend explicitly on the independent variable* x_j. *Then, under the assumptions of Theorem 7.13 for all solutions* u *of the Euler–Lagrange equation there holds*

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k}\left(\frac{\partial u}{\partial x_j}(x)\frac{\partial L}{\partial u_{x_k}}(j_1 u(x))\right) = \frac{\partial L}{\partial x_j}(j_1 u(x)) \qquad \textit{for a.e. } (x, j_1 u) \in J^1(U).$$

Beweis. First of all, consider the one-parameter point transformation group

$$T_{\epsilon,j} : (x,u) \mapsto (x + \epsilon e_j, u), \qquad \epsilon \in \mathbb{R},$$

with infinitesimal generator $A := \frac{\partial}{\partial x_j}$, for the basis vector $e_j \in \mathbb{R}^d$ follows by a direct application of the substitution formula for the integral, since L does not depend on x.

Moreover, we know from Remark 7.10 that

$$j_0 A(x,f) := \frac{\partial}{\partial x_j} + \frac{\partial f}{\partial x_j}(x)\frac{\partial}{\partial u},$$

i.e.,

$$\xi(x,f) = e_j \qquad \text{and} \qquad \phi(x,f) := \frac{\partial f}{\partial x_j}(x).$$

Noether's theorem then states that if u is a solution to the Euler–Lagrange equation, then for all j

$$\begin{aligned}
0 &= \sum_{k=1}^{d} \frac{\partial}{\partial x_k}\left(\phi(j_1 u(x))\frac{\partial L}{\partial u_{x_k}}(j_1 u(x)) - L(j_1 u(x))\xi_k(j_1 u(x))\right) \\
&= \sum_{k=1}^{d} \frac{\partial}{\partial x_k}\left(\frac{\partial u}{\partial x_j}(x)\frac{\partial L}{\partial u_{x_k}}(j_1 u(x)) - L(j_1 u(x))\delta_{jk}\right) \\
&= \sum_{k=1}^{d} \frac{\partial}{\partial x_k}\left(\frac{\partial u}{\partial x_j}(x)\frac{\partial L}{\partial u_{x_k}}(j_1 u(x))\right) - \frac{\partial L}{\partial x_j}(j_1 u(x)).
\end{aligned}$$

This concludes the proof. □

Remark 7.16. *As we will see, the identities*

$$\nabla\left(\frac{\partial u}{\partial x_j} D_{\nabla u} L\right) = \frac{\partial L}{\partial x_j}$$

can be often given a physical intepretation. In the special case the independent variable can be identified with time, $x_j = t$, then

$$\nabla\left(u_t D_{\nabla u} L(t,x,j_1 u)\right) = \frac{\partial L}{\partial t}(t,x,j_1 u),$$

i.e., the so-called **current density** J **for the Lagrangian** L, *defined by*

$$J(u) := -u_t D_{\nabla u} L(u),$$

satisfies

$$\frac{\partial L}{\partial t}(u(t,x)) + \nabla J(u(t,x)) = 0, \qquad t \in \mathbb{R},\ x \in U,$$

which is called the **continuity equation**.

(If the Lagrangian does not depend explicitly on any *indpendent variables, we promptly obtain a family of d equations.)*

By putting the Euler–Lagrange equation in divergence form, Noether's theorem shows that some vector field related to the problem is divergence-free. In many interesting cases this can be interpreted as the existence of some invariant of the system.

Exercise 7.17. *Let $p \in (1,\infty)$ and consider the action*

$$I(u) = \int_U |\nabla u(x)|^p dx, \qquad u \in H^1_0(U).$$

*Prove that the associated Euler–Lagrange equation is the p-***Laplace equation**

$$\nabla\left(|\nabla u|^{p-2}\nabla u\right) = 0.$$

Show that

$$T_\epsilon : (x,u) \mapsto (e^\epsilon x, e^{\epsilon\frac{n-p}{n}} u), \qquad \epsilon \in \mathbb{R},$$

defines a variational symmetry group for this problem and that the point transformation group

$$S_\epsilon : (x,u) \mapsto (x, e^\epsilon u), \qquad \epsilon \in \mathbb{R},$$

does not, even in the linear case of $p = 2$ (the usual Laplace equation). Derive the corresponding divergence identity from Noether's theorem and show that it also follows directly from the Euler–Lagrange equation.

Example 7.18. *Let Ω be an open domain of $\mathbb{R}^d$. The nonlinear wave equation of Exercise 7.6 is an example of Euler–Lagrange equation with an action that does not explicitly depend on the independent variables. If in particular we apply Corollary 7.15 to the Lagrangian*

$$L(x,j_1 u) := \sum_{k=1}^d \frac{1}{2} u_{x_k}(x)^2 - \frac{1}{2} u_{x_{d1}}(x)^2 + F(u(x)), \qquad (x,j_1 u) \in J^1(U),$$

where $U := \Omega\times\mathbb{R}$, then we obtain (in the $d+1$-dimensional notation, setting $t := x_{d+1}$) in particular the identity

$$\sum_{k=1}^{d+1} \frac{\partial}{\partial x_k}\left(\frac{\partial u}{\partial t}(x)\frac{\partial L}{\partial u_{x_k}}(j_1 u)\right) = \frac{\partial L}{\partial t}(j_1 u(x)),$$

corresponding to the last identity in Corollary 7.15. In other words,

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k}(u_t u_{x_k}) - \frac{\partial}{\partial t}(u_t^2) = \frac{\partial L}{\partial t}(j_1 u(x,t)).$$

This yields

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k}(u_t u_{x_k}) = \frac{\partial}{\partial t}(u_t^2 + L(j_1 u(x,t)) = \frac{\partial}{\partial t}\left(\frac{u_t^2}{2} + \sum_{k=1}^{d} \frac{u_{x_k}^2}{2} + F(u)\right) =: \eta'(t).$$

In the RHS of the above expression, $\eta(t)$ represents the total **energy density** *of the system. To obtain the total energy $E(t)$ it suffices to integrate $\eta(t)$ over the whole space $\mathbb{R}^d$, cf. Definition 2.19. On the other hand, the LHS of the above expression is the divergence (with respect to the spacial variables only!) of the vector field $u_t \nabla u$. If we therefore integrate both terms over $\mathbb{R}^d$ we obtain*

$$\begin{aligned} E'(t) &= \frac{d}{dt}\int_{\mathbb{R}^d} \eta(t)dt \\ &= \int_{\mathbb{R}^d} \nabla \cdot (u_t \nabla u)dx \\ &= \lim_{R\to\infty} \int_{B_R(0)} \nabla \cdot (u_t \nabla u)dx, \end{aligned}$$

and using the divergence theorem

$$E'(t) = \lim_{R\to\infty} \int_{\partial B_R(0)} u_t \nabla u \cdot n d\sigma.$$

Now, assume the term on the RHS to vanish. For example, this is the case if the wave equation takes place on a bounded domain, so that the wave function u has compact support; or if we already know that it enjoys a strong (spacial) asymptotic decay. Then we conclude that $E'(t) = 0$, i.e., the nonlinear elastic system has constant energy.

Exercise 7.19. *Let $d = 2$. Show in a similar way, using the fact that also space translations define variational symmetry groups, that*

$$\int_{\mathbb{R}^2} \frac{\partial u}{\partial x}\frac{\partial u}{\partial t} dx \quad \text{and} \quad \int_{\mathbb{R}^2} \frac{\partial u}{\partial y}\frac{\partial u}{\partial t} dx$$

are conserved quantities. In physics they are identified with the **linear momenta** *in the x- and y-directions of an elastic system.*

Exercise 7.20. *Consider again the setting of Exercise 7.6, with $f \equiv 0$. Show that the rotation, i.e., the group generated by*

$$A := x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

is a variational symmetry and determine the associated conserved quantity, which in physics is identified with the **angular momentum**.

Are any of the further point symmetry groups variational *symmetry groups, too?*

Exercise 7.21. *Apply Corollary 7.15 to the minimal surface equation.*

Exercise 7.22. *The nonlinear partial differential equation*

$$\frac{\partial u}{\partial t}(t,x) + 6u(t,x)\frac{\partial u}{\partial x}(t,x) + \frac{\partial^3 u}{\partial x^3}(t,x) = 0, \qquad t \geq 0,\ x \in [0,1]$$

has been introduced by Diederik Korteweg and Gustav de Vries in 1895 as a model of waves on shallow water surfaces. It is usually known as **KdV-equation**. *Show that the KdV-equation can be put in variational form upon setting* $u =: \frac{\partial v}{\partial x}$ *(which implies a certain regularity of* u*, of course) for a generic function* v*. Apply Noether's theorem.*

Exercise 7.23. *A modification of the KdV-equation has been proposed in 1972 by Thomas Benjamin, Jerry Bona and J.J. Mahony and is therefore known as the* **BBM-equation**. *It reads*

$$\frac{\partial u}{\partial t}(t,x) + \frac{\partial u}{\partial x}(t,x) + u(t,x)\frac{\partial u}{\partial x}(t,x) - \frac{\partial^3 u}{\partial x^2 \partial t}(t,x) = 0, \qquad t \geq 0,\ x \in [0,1], \tag{7.10}$$

and it has proved mathematically more tractable than the KdV equation. After performing the same transformation as in Exercise 7.22, are you able to represent the BBM equation as the Euler–Lagrange equation associated to a suitable Lagrangian? What does Noether's theorem imply?

7.2. Variational symmetries of the telegraph equation

If $h = 0$, then the telegraph equation on an open domain $\Omega \subset \mathbb{R}^d$ turns into a damped wave equation that can be studied by introducing the Lagrangian

$$L_0(t,x,j_1 u) := \frac{c^2}{2}|\nabla u(t,x)|^2 + \frac{k^2}{2}|u(t,x)|^2 - \frac{1}{2}|u_t(t,x)|^2, \qquad (t,x,j_1 u) \in J^1(U),$$

where $U := \mathbb{R} \times \Omega$. At a first glance, the general telegraph equation seems not to have a variational structure, due to the lower order term that arises if $h \neq 0$. This can be accomodated by introducing a multiplicative term as follows: define

$$L(t,x,j_1 u) := L_0(t,x,j_1 u)e^{-th} = \frac{c^2}{2}|\nabla u(t,x)|^2 e^{-th} + \frac{k^2}{2}|u(t,x)|^2 e^{-th} - \frac{1}{2}|u_t(t,x)|^2 e^{-th}. \tag{7.11}$$

Then the vector field $D_{\nabla u} L$ agrees with

$$D_{\nabla u} L = (-u_t e^{-th}, c^2 \nabla u e^{-th})$$

and therefore its divergence is given by

$$\nabla D_{\nabla u} L = (-u_{tt} e^{-th} - h u_t e^{-th} + c^2 \Delta u e^{-th}).$$

On the other hand

$$D_u L = k^2 u e^{-th},$$

and summing up we obtain the associated Euler–Lagrange equation

$$\left(-u_{tt} - h u_t e^{-th} + c^2 \Delta u e^{-th} - k^2 u\right) e^{-th} = 0,$$

whose solutions are also solutions to (7.2), and viceversa.

In comparison to its two "parents" – the heat and the wave equation – the telegraph equation presents some properties that make it interesting in applications.

The telegraph equation can be seen as a variational regularisation of the heat equations: the heat equations is not a Euler–Lagrange equation, but adding to it a term $\frac{1}{m}u_{tt}$ it turns into a telegraph equation. It is then possible to discuss properties (e.g., symmetries) of such a regularisation by variational methods, and then let $m \to \infty$.

Example 7.24. *While the solutions to the wave equation have constant energy, as we have seen, the solutions to the telegraph equation may have exponentially decaying energy. This is both more realistic and more desirable in applications*[3]*. To see this, consider the special case of $h = 2k$, i.e., the equation*

$$\frac{\partial^2 u}{\partial t^2}(t,x) + 2k\frac{\partial u}{\partial t}(t,x) + k^2 u(t,x) = c^2 \Delta u(t,x), \qquad t \in \mathbb{R},\ x \in \Omega.$$

Since this equation can be written as

$$\left(\frac{\partial}{\partial t} + kI\right)^2 u(t,x) := \left(\frac{\partial}{\partial t} + kI\right)\left(\frac{\partial}{\partial t} + kI\right) u(t,x) = c^2 \Delta u(t,x), \qquad t \in \mathbb{R},\ x \in \Omega,$$

where I denotes the identity operator, the kinetic energy of the system[4] *is*

$$E_k(t) := \frac{1}{2}\int_\Omega \left|\frac{\partial u}{\partial t}(t,x) + ku(t,x)\right|^2 dx, \qquad t \in \mathbb{R},$$

hence the total energy is

$$E(t) := \frac{1}{2}\int_\Omega \left|\frac{\partial u}{\partial t}(t,x) + ku(t,x)\right|^2 dx + \frac{1}{2}\int_\Omega |\nabla u(t,x)|^2, \qquad t \in \mathbb{R}.$$

Introducing the function

$$w(t,x) := u(t,x)e^{tk}, \qquad t \in \mathbb{R},\ x \in \Omega,$$

yields a solution to the wave equation (2.12). To see this, compute

$$w_t(t,x) = \big(u_t(t,x) + ku(t,x)\big)e^{tk} \qquad \text{and} \qquad w_{tt}(t,x) = \big(u_t(t,x) + 2ku(t,x) + k^2 u(t,x)\big)e^{tk}$$

and

$$w_x(t,x) = u_x(t,x)e^{tk} \qquad \text{and} \qquad w_{xx}(t,x) = u_{xx}(t,x)e^{tk}.$$

Now, the energy associated with the solution w of the wave equation is

$$\begin{aligned}
\tilde{E}(t) \quad &:= \quad \frac{1}{2}\int_\Omega \left|\frac{\partial w}{\partial t}(t,x)\right|^2 dx + \frac{1}{2}\int_\Omega |\nabla w(t,x)|^2 \\
&= \quad \frac{1}{2}\int_\Omega \left|\left(\frac{\partial u}{\partial t}(t,x) + ku(t,x)\right) e^{kt}\right|^2 dx + \frac{1}{2}\int_\Omega |\nabla u(t,x)e^{tk}|^2 dx \\
&= \quad \frac{e^{2tk}}{2}\int_\Omega \left|\left(\frac{\partial u}{\partial t}(t,x) + ku(t,x)\right)\right|^2 dx + \frac{e^{2tk}}{2}\int_\Omega |\nabla u(t,x)|^2 dx \\
&= \quad \tilde{E}(t)e^{2tk}.
\end{aligned}$$

Assume finally that Ω is bounded, or that we already know suitable decay estimates for the solution u as $x \to \infty$. Since $\tilde{E}(t)$ is constant by Example 7.18, i.e.,

$$\tilde{E}(t) = \tilde{E}(0), \qquad t \in \mathbb{R},$$

it follows that

$$E(t) = e^{-2tk}\tilde{E}(0), \qquad t \in \mathbb{R}.$$

[3] An engineer could, say, forge elastic components of a mechanical systems in such a way that they obey a certain telegraph equation.

[4] Observe that this corresponds to the potential energy of the system described by the dual telegraph equation in i.

This shows in particular that the time translation is a point symmetry group of the telegraph equation, but not a variational one (for otherwise Noether's theorem would yield a conservation law for the energy).

Example 7.25. *On the other hand, the Lagrangian defined in* (7.11) *does not depend on the spacial variables, hence in particular the space translations are variational symmetries. Corollary 7.15 then implies that the identity*

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k}\left(\frac{\partial u}{\partial x_j}(t,x)\frac{\partial L}{\partial u_{x_k}}\right) - \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x_j}(t,x)\frac{\partial u}{\partial t}(t,x)e^{-th}\right) = \frac{\partial L}{\partial x_j}$$

holds for any solution u of the telegraph equation. To derive another nontrivial conservation law, turn back to the case of $d = 1$, which is more specifical for the original telegraph equation of electromagnetism. Then in particular, for $j := 1$,

$$\begin{aligned}\frac{\partial L}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}(t,x)\frac{\partial L}{\partial u_x}\right) + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}(t,x)\frac{\partial L}{\partial u_t}\right) \\ &= \frac{\partial}{\partial x}\left(u_x(t,x)c^2 u_x(t,x)e^{-th}\right) - \frac{\partial}{\partial t}\left(u_x(t,x)u_t(t,x)e^{-th}\right),\end{aligned}$$

whence the linear momentum P satisfies

$$\begin{aligned}\frac{\partial}{\partial t}P(t)e^{-th} &= \frac{\partial}{\partial t}\int_{\mathbb{R}}\left(u_x(t,x)u_t(t,x)e^{-th}\right)dx \\ &= \int_{\mathbb{R}}\frac{\partial}{\partial x}\left(c^2u_x^2(t,x)e^{-th} - L\right)dx \\ &= \int_{\mathbb{R}}\frac{\partial}{\partial x}\left(\frac{c^2}{2}u_x^2(t,x) - \frac{k^2}{2}u^2(t,x) + \frac{1}{2}u_t^2(t,x)\right)e^{-th}dx,\end{aligned}$$

where the last term vanishes whenever u has compact support (i.e., for telegraph equations over finite wires).

KAPITEL 8

Nonlinear elliptic equations and compactness methods

In this last chapter we finally discuss a class of nonlinear partial differential equations related to so-called **reaction-diffusion systems**. Such equations arise in a number of different fields, they involve a lot of different nonlinear terms and it is hardly possible to propose a unified derivation[1]. However, all of them are justified by the following heuristical observation:

If a system with initial condition u_0 is governed by two competing and simultaneous phenomena, each described by a certain (finite or infinite) dynamical system governed by a flow Φ and Ψ, respectively, then the whole dynamics is given by a flow Ξ that is given by

$$\Xi(t)u_0 = \lim_{n\to\infty} \left(\Phi\left(\frac{t}{n}\right)\Psi\left(\frac{t}{n}\right)\right)^n u_0.$$

This *product formula*, which generalises a result for matrix exponentials that goes back to Lie, means to suggest that the evolution of a system with two simultaneous ongoing dynamich processes can be described by switching on one dynamics, let it work for a time step, then switching it off and switching on the the second dynamics, let it work for a time step, then switching it off and so on..., and repeating this procedure for time steps becoming ever shorter. It can be precisely and formally proved in a few cases [2] that the above flow Ξ governs the system given by the linear overlapping of the two dynamical systems.

A typical example of this approach consists in the modelling of chemical processes, or population dynamics: two solutions diffuse in a medium Ω and *simultaneously* (i.e., on a comparable time scale) they react; or else, two populations (say, prey and predator) diffuse in a region while they reproduce themselves and die. This is typically modelled by a system of partial differential equations

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) = c_1^2\Delta v(t,x) + f(v(t,x),w(t,x)), & t\geq 0,\ x\in\Omega, \\ \frac{\partial w}{\partial t}(t,x) = c_1^2\Delta w(t,x) + f(v(t,x),w(t,x)), & t\geq 0,\ x\in\Omega, \end{cases}$$

or rather,

$$\frac{\partial u}{\partial t}(t,x) = c\Delta u(t,x) + f(u(t,x)), \qquad t\geq 0,\ x\in\Omega,$$

for vector-valued functions u. (Of course, such equations also need to be endowed with further initial and boundary conditions.)

Generally speaking, all this equations represent phenomena in which appearance or disappearance of material is involved: molecules are created or destroyed, animals are born or die, cars move on a traffic network, current waves diffuse through a cardiac tissue. In the case of the Hodgkin–Huxley equation 8, for instance, a spike (i.e., an excitation wave) diffuses along a neuron while certain biochemical reactions keep the wave going or damp

[1] Generally speaking, however, such systems can be derived in a way similar to diffusion or transport equations, if we allow for a nonlinear flow function.

[2] But beware that this assertion fails sometimes dramatically, even in the purely linear case.

it up, depending on the convenience of the cell. More precisely, the nonlinear term of the Hodgkin–Huxley equation,

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) - \frac{1}{2}u(t,x)\Big(u(t,x)-1\Big)\Big(u(t,x)-\alpha\Big),$$

i.e.,

$$\frac{\partial u}{\partial t}(t,x) = u(t,x)\Big(u(t,x)-1\Big)\Big(u(t,x)-\alpha\Big),$$

where $\alpha \in (0,1)$, represent a system in which the *membrane potential* u is pushed towards the values 0 or 1, depending on whether the initial data is larger or smaller than α.

Now, one is usually interested in equilibria of such systems, since it is usually impossible to find a general solution formula. But if u is an equilibrium solution, i.e., if u is constant (and therefore

$$\frac{\partial u}{\partial t}(t,x) = 0)$$

then

$$c\Delta u(t,x) + f(u(t,x)) = 0, \qquad x \in \Omega. \tag{8.1}$$

In other words, a necessary condition for an equilibrium to exist is that (8.1) has a solution.

8.1. Fixed point theorems

To fix the ideas, we first consider a simple example – a nonlinear evolution equation

$$\frac{\partial u}{\partial t}(t,x) = F(x, u(t,x), \nabla u(t,x), \Delta u(t,x)),$$

with initial data

$$u(0,x) = u_0(x),$$

with t and x in suitable domains[3]. While of course we cannot represent any arbitrary nonlinear equation in this way, this is still general enough to accommodate the BBM equation introduced in (7.10) or the nonlinear wave equation considered in Exercise 7.6 (how?).

Integrating with respect to t we then have

$$u = u_0 + \int_0^t F(x, u, \nabla u, \Delta u)dt =: K(u).$$

Now, a possible strategy to show well-posedness of the above nonlinear differential equation is to find a solution of its integrated form and then - hopefully - to show that it is differentiable and satisfies the equation in its differential form, too. Now, the first step amounts to finding u in a certain function space such that $u = K(u)$. This is where *fixed point theorems* come into play.

Similar methods are particularly popular when considering nonlinear elliptic equations. As we have seen while discussing the Poisson equation, elliptic equations can typically be thought of as describing equilibria of time dependent equations – i.e., solutions with vanishing derivative with respect to time. The same holds for *nonlinear* elliptic equations. A favourite strategy to obtain existence and uniqueness of solutions to such problems is to apply fixed point theorems. We will meet Brouwer's fixed point theorem and its infinite dimensional generalisations – Schauder's and Schaefer's theorems.

Theorem 8.1 (Brouwer's fixed point theorem). *Let Ω be an open bounded domain of $\mathbb{R}^d$ that is homeomorphic to $B_1(0)$. Then each continuous function $f : \Omega \to \Omega$ has a fixed point, i.e., there exists $x \in \Omega$*

[3] Unlike in the linear case, global well-posedness in time of nonlinear evolution equations is quite unusual.

The above theorem has been first proved by Luitzen Brouwer in 1912. Nowadays several proofs of Brouwer's theorem are known, most of which analytical – and some of which even based on the variational methods developed in Chapter 7, see e.g. [**5**, §8.1]. Instead, we are going to present an elegant one, based on the following result obtained by Emanuel Sperner in 1928 and celebrated in [**1**, § 25.6]. Its main ingredients are a few elementary ideas and methods from discrete mathematics.

Definition 8.2. *Let $v_1, \ldots, v_{d+1} \in \mathbb{R}^{d+1}$ be linearly independent. A* **d-dimensional simplex** *with vertices $v_1, \ldots, v_{d+1}$ is the closed set*

$$\left\{ \sum_{k=1}^{d+1} x_k v_k \in \mathbb{R}^{d+1} : 0 \leq x_k \text{ and } \sum_{k=1}^{d+1} x_k = 1 \right\},$$

i.e., the **closed convex hull** *of the set $\{v_1, \ldots, v_{d+1}\}$ (that is, the intersection of all closed convex subsets of X containing the set $\{v_1, \ldots, v_{d+1}\}$).*

The **canonical** *d-dimensional simplex is the d-dimensional simplex whose vertices are the elements of the canonical basis of $\mathbb{R}^{d+1}$.*

Observe that the boundary of any d-dimensional simplex is the union of $(d-1)$-dimensional simplices – more precisely, of $d+1$ of them – which we call **hyperfaces**. The intersection of two hyperfaces is either empty or a $(d-2)$-dimensional simplex, which we call a **hyperedge**. In the 2-dimensional case, a simplex is a triangle, a hyperface is actually one of its edges and a hyperedge is a point.

Definition 8.3. *A* **simplicial subdivision** *of a d-dimensional simplex Σ is a family $(\omega_1, \ldots, \omega_N)$ of d-dimensional (sub)simplices whose union agrees with Σ, whose interiors are pairwise disjoint, and whose intersections are either empty or agree with one of the hyperfaces constituting the boundary of both.*

Remark 8.4. *Equivalently, the canonical d-dimensional simplex can be defined as the polyhedron contained in the hyperplane of $\mathbb{R}^{d+1}$ whose extremal points are $e_1, \ldots, e_{d+1}$ – the vectors of the canonical basis of $\mathbb{R}^{d+1}$. Also observe that in fact, a 2-dimensional simplicial subdivision is just a triangulation, as usually encountered in numerical analysis and in graph theory.*

Definition 8.5. *A* **Sperner colouring** *of a d-dimensional simplicial subdivision Ω of the canonical d-dimensional simplex Sigma with vertex set $\mathcal{V}$ is any mapping $c : \mathcal{V} \to \{1, \ldots, d\}$ such that*

- *$c(e_i) = i$, $i = 1, \ldots, d$, and*
- *$c(V) \in \{i_1, \ldots, i_d\}$ for any vertex of the $(d-1)$-dimensional face of Ω whose extremal points are $V_{i_1}, \ldots, V_{i_d}$.*

The values 1,...,d are referred to as colours.

Lemma 8.6 (Sperner's lemma)**.** *Let $\Omega = (\omega_1, \ldots, \omega_N)$ be a simplicial subdivision of a d-dimensional simplex Σ. Then for each Sperner colouring of Ω there is an odd number of* **rainbow (sub)simplices***, i.e., of (sub)simplices ω_i such that their $d+1$ vertices are assigned $d+1$ different colours.*

The proof we perform is slightly redundant – but the 2-dimensional case is particularly elucidating and we perform it in detail.

Beweis. We will prove – by induction – a slightly stronger result, namely that the number of rainbow (sub)simplices in a simplicial subdivision is always odd.

In the case $d = 1$, the canonical simplex is simply the interval connecting the points $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Consider a Sperner colouring of it. Since all the intermediate points have to be coloured by either

1 or 2, then it clearly exists a $1-2$-edge, i.e., a subinterval whose extremal points are coloured by 1 and 2, respectively. Furthermore, the number of such $1-2$-edges has to be odd: to see this, observe that walking away from e_1 towards e_2 through the first $2m$ $1-2$-edges we necessarily encounter a vertex coloured by 1.

Now, consider a Sperner colouring of the 2-dimensional canonical simplex and single out all the N subsimplices (i.e., the triangles) that have a $1-2$-edge (i.e., an edge whose extremal points are coloured 1 and 2). Among them, there are M rainbow simplices and $N-M$ where either of the colours 1, 2 is used twice. Similarly, if H is the total number of $1-2$-edges, then J of them lie on the boundary of Σ and $H-J$ will consist (with the possible exception of the extremal points) of interior points of Σ. *The proof is now based on* double counting*, a classical proof technique in discrete mathematics, based on the simple idea of counting the elements of a set in two different ways (according to two different features, that is): this yields to different formulae that necessarily have to agree, yielding a desired relation.* Now, let us count the $1-2$-edges on the border of all subsimplices: the M rainbow triangles yield M $1-2$-edges, while the remaining $N-M$ triangles yield $2(N-M)$ edges. Of course, in this way we have counted some edges twice: to be more precise, we have counted twice exactly those $H-J$ edges lying in the interior of Σ, while we have counted once the J boundary edges. Thus, this elementary double counting yields the formula

$$2(H-J)+J=2(N-M)+M.$$

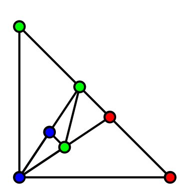
We do not know yet that there is an odd number of rainbow triangles, i.e., that M is an odd number. Proving this is equivalent to proving that J is an odd number. But how many $1-2$-edges lie on the boundary of Σ? By definition of Sperner colouring, $1-2$-edges can only possibly lien on the interval connecting e_1 and e_2. But then, we know from the first step that an odd number of $1-2$-edges has to appear, i.e., J is odd [4].

Finally, consider a Sperner colouring of the d-dimensional canonical simplex, which by definition uses the colours $1,2,\dots,d+1$. By inductive assumption we know that the number of rainbow simplices is odd in the Sperner colouring of each $d-1$-dimensional simplex (hence in particular of each hyperface). We single out all the N subsimplices that have a $1-2-\ldots-d$-hyperface (i.e., an edge whose extremal points are coloured using all the colours $1,2,\dots,d$). Among them, there are M rainbow simplices and $N-M$ where (by the pigeonhole principle) each but one of the colours $1,2,\dots,d$ is used once, and the exceptional colour is used twice. Similarly, if H is the total number of $1-2-\ldots-d$-hyperfaces, then J of them lie on the boundary of Σ and $H-J$ will consist (with the possible exception of the extremal hyperedges) of interior points of Σ. Now, the same double counting as in the 2-dimensional case yields the formula

$$2(H-J)+J=2(N-M)+M,$$

and as in the 2-dimensional case the inductive process yields that J, hence M are odd. □

Remark 8.7. *One might be tempted to conjecture that Sperner's lemma also holds if a weaker notion of simplicial subdivision is considered, namely, that of a family of subsimplices whose union yields Σ, regardless of how may ("alien") vertices can be contained in the boundary of each subsimplex. This is wrong, as one can see considering the example*



[4] For those who have some familiarity with graph theory or numerical analysis, it will be clear that this is nothing but the hand-shaking lemma applied to a certain subgraph of the Delaunay triangulation of Σ.

Remark 8.8. *Observe that a d-dimensional simplicial subdivision can be made finer and finer by an iterative process. Namely, consider*

$$\Omega_0 := \Sigma$$

and for a subdivision $\Omega_n = (\omega_1, \ldots, \omega^n_{d+1})$ consider a new subdivision Ω_{n+1} by subdividing each of its (smaller) simplices ω_i – say, having vertices $v^i_1, \ldots, v^i_{d+1}$ – by adding the **centre of gravity** *$v_c(\omega_i)$ of ω_i and considering the $d+1$ smaller simplices with vertices*

- $v_c(\omega_i), v^i_2, \ldots, v^i_{d+1}$,
- $v^i_1, v_c(\omega_i), \ldots, v^i_{d+1}$,
- $\ldots$,
- $v^i_1, v^i_2, \ldots, v_c(\omega_i)$.

of ω_i, yielding a new subdivision in $(d+1)^{n+1}$ subsimplices.

Remark 8.9. *While a simplicial subdivision can obviously be thought of as a graph, we emphasize that a Sperner colouring is in general not a proper colouring of it in the graph theoretical sense. In particular, already the first simplicial subdivision of Σ in the 2-dimensional case yields what in graph theoretical language is called the complete graph K^4. Such a graph is obviously seen to be* not *3-colourable. In fact, the n^{th} simplicial subdivision Ω_n of the d-dimensional canonical simplex is a proper colouring if and only if*

- *$d = 1$ and n is an odd number, or*
- *$d > 1$ and $n = 0$.*

An obvious corollary of Sperner's lemma is that any simplicial subdivision of Σ contains *at least one* rainbow (sub)simplex. This is all we need in the following.

Proof of Theorem 8.1. First of all, observe that it suffice to prove the assertion for $\Omega = \Sigma$, the canonical d-dimensional simplex, since any Ω that is homeomorphic to $B_1(0)$ is homeomorphic to Σ, too, and fixed points remain such under homeomorphisms.

Construct a sequence of simplicial subdivisions $(\Omega_n)_{n\in\mathbb{N}}$ that are finer and finer, as in Remark 8.8. In particular, denoting

$$|\Omega_n| := \max\{\mathrm{vol}(\omega) : \omega \text{ in } \Omega_n\}$$

we obtain[5] that $\lim_{n\to\infty} |\Omega_n| = 0$.

If a continuous function $f : \Sigma \to \Sigma$ without fixed points would exist, then it would be possible to define a colouring c_n of any Ω_n as follows: since on one hand for any $x \in \Sigma$ (and in particular for any vertex of the subdivision) one has $f(x) \neq x$, and since on the other hand $\sum_{k=1}^{d+1} x_k = 1$, one sees that for each vertex $V = (y_1, \ldots, y_{d+1})$ in Ω_n there must exist at least one index i such that $f(V)_i < y_i$. Choose the smallest such $i \in \{1, \ldots, d+1\}$ and assign the corresponding colour to V, i.e., $c_n(V) := i$. Repeating this procedure for all vertices we obtain a colouring: let us check that it is in fact one of Sperner type. Since the k^{th} extremal points of Σ is e_k, the k^{th} vector of the canonical basis, and since $f(e_k) \neq e_k$, then at least one coordinate of $f(e_k) \in \mathbb{R}^{d+1}$ has to differ from the corresponding coordinate of e_k. Since however all coordinates $f(e_k)_j$ of $f(e_k)$ have to stay nonnegative, and since by definition of simplex $\sum_{k=1}^{d+1} f(e_k)_j = 1$, the only way to balance this increase in the other coordinates is to have $f(e_k)_k < (e_k)_k = 1$, i.e., $c(e_k) = k$, for all $k \in \{1, \ldots, d+1\}$. Similarly, if V is a vertex on the $(d-1)$-dimensional face Ω whose extremal points are $V_{i_1}, \ldots, V_{i_d}$, such that $k = \{1, \ldots, d+1\} \setminus \{i_1, \ldots, i_d\}$, then the k^{th} coordinate of V is 0. Therefore, $f(V)_k \geq 0$, and this can only be

[5] In fact, using the same simplicial subdivisions as in Remark 8.8, all ω in Ω_n have the same volume.

balanced by having (at least) one of the remaining d coordinates satisfy $f(V)_i < y_i$: since all such i's belong to the set $\{1,\dots,d+1\}\setminus\{k\}$, also the second defining condition of a Sperner colouring is satisfied.

Summing up, we have defined a Sperner colouring of each simplicial subdivision Ω_n, and by Sperner's lemma we deduce the existence of a rainbow subsimplex ω^n in Ω_n, i.e., of a subsimplex with vertices $v_1^n = (y^n_{1\,1},\dots,y^n_{1\,d+1})$ of colour 1, $v_2^n = (y^n_{2\,1},\dots,y^n_{2\,d+1})$ of colour 2, etc., which therefore satisfy $f(v_i^n)_i < y^n_{i\,i}$.

Consider now the sequence $(v_c(\omega_n))_{n\in\mathbb{N}}$ of centres of gravity of ω_n. Such a sequence is bounded, since $v_c(\omega_n)\in\Sigma$ for all $n\in\mathbb{N}$, hence by the theorem of Bolzano–Weierstraß it admits a convergent subsequence $(z^{n_h})_{h\in\mathbb{N}}$. Denote

$$\lim_{h\to\infty} z^{n_h} =: \tilde z.$$

The corresponding subsimplices become smaller and smaller, hence the sequence of i^{th} vertices of the corresponding (rainbow) subsimplices converge to the same point, regardless of i, i.e.,

$$\lim_{h\to\infty} v_1^{n_h} = \ldots = \lim_{h\to\infty} v_{d+1}^{n_h} =: \tilde z \in \Sigma.$$

It follows that

$$f(\tilde z)_i = f\left(\lim_{h\to\infty} v^{n_h}\right)_i = \lim_{h\to\infty} f\left(v^{n_h}\right)_i \le \lim_{h\to\infty} v_i^{n_h} = \tilde z_i,$$

for all $i\in\{1,\dots,d+1\}$. Since however both $\tilde z$ and $f(\tilde z)$ belong to Σ, and hence

$$\sum_{i=1}^{d+1} \tilde z_i = \sum_{i=1}^{d+1} f(\tilde z)_i = 1,$$

one sees that necessarily

$$f(\tilde z)_i = \tilde z_i,$$

for all $i\in\{1,\dots,d+1\}$, i.e., $f(\tilde z)=\tilde z$. This yields a contradiction to our standing assumption and therefore a proof of the desired assertion. □

While Brouwer's theorem has a broad spectrum of consequences, ranging from the theory of ordinary differential equations to game theory, it is typically hopeless to try to apply it to partial differential equations – with the possible, remarkable exception of elliptic partial differential equations can be turned into ordinary ones by some symmetry argument (remember the reduction of 3-dimensional wave equation to the Euler–Poisson–Darboux equation and think of the associated eigenvalue problems). Its extension to infinite dimensional vector spaces, which is due to Juliusz Schauder, has therefore been a breakthrough back in 1930.

Theorem 8.10 (Schauder's fixed point theorem)**.** *Let X be a complete and normed vector space over $\mathbb{R}$. Let K be a compact and convex subset of X. Then each continuous function $\Phi : K\to K$ has a fixed point in K.*

Beweis. Let $n\in\mathbb{N}$. By the theorem of Heine–Borel, K has a finite open covering of coarseness $\frac{1}{n}$, i.e., there exist finitely many open balls $B_{\frac{1}{n}}(x_1),\dots,B_{\frac{1}{n}}(x_{N_n})$ whose union contains K. Taking the closed convex hull of their centers $x_1,\dots,x_{N_n}$ yields a convex set K_n: since all these points also belong to K and K_n is minimal among all the closed convex sets that contain $x_1,\dots,x_{N_n}$, we deduce that $K_n\subset K$. Moreover, K_n is homeomorphic to the (closed) unit ball of $\mathbb{R}^{n_0}$, for some $n_0\le N_n$. (The equality holds if and only if the points $x_1,\dots,x_{N_n}$ are linearly independent.)

Now, define a mapping P_n by

$$P_n(x) := \sum_{i=1}^{N_n} \frac{\lambda_i}{\|\lambda\|_1} x_i,$$

where[6]

$$\lambda_i := \operatorname{dist}(x, K \setminus B_{\frac{1}{n}}(x_i)) \geq 0, \qquad \text{hence} \quad \|\lambda\|_1 = \sum_{i=1}^{N_n} \lambda_i.$$

Observe that each P_n is continuous because the norm and hence distance function are. (Because $B_{\frac{1}{n}}(x_1), \ldots, B_{\frac{1}{n}}(x_{N_n})$ is an open covering of K, for all $x \in K$ there exists i_0 with $x \in B_{\frac{1}{n}}(x_{i_0})$, i.e., with $x \notin K \setminus B_{\frac{1}{n}}(x_{i_0})$, or rather such that the distance between x and $K \setminus B_{\frac{1}{n}}(x_{i_0})$ is strictly positive. This shows that the denominator in the definition of P_n never vanishes.)

Since $P_n(x)$ is for all $x \in K$ a linear combination of $x_1, \ldots, x_{N_n}$, with positive coefficients that sum up to 1, it follows that $P_n(x) \in K_n$, i.e., P_n maps K into K_n. Moreover, we observe the inequality

$$\|P_n(x) - x\|_X \leq \sum_{i=1}^{N_n} \frac{\lambda_i \|x_i - x\|_X}{\|\lambda\|_1} \leq \frac{1}{n}, \qquad x \in K. \tag{8.2}$$

Here, the former inequality follows directly from the triangle inequality for the norm of $\|\cdot\|_X$ while the latter from the observation that x may be either element of $B_{\frac{1}{n}}(x_i)$, and in this case

$$\|x_i - x\|_X \leq \frac{1}{n}, \qquad \text{hence} \quad \lambda_i \|x_i - x\|_X \leq \frac{1}{n}\lambda_i,$$

or not, and in this case $\lambda_i = 0$ and again (trivially)

$$\lambda_i \|x_i - x\|_X \leq \frac{1}{n}\lambda_i.$$

In order to apply Brouwer's fixed point theorem we need a continuous function mapping on K_n and somehow related to Φ: the right choice is $P_n \circ \Phi$. Then, by Brouwer's theorem there exists $x_n \in K_n$ such that $P_n(\Phi(x_n)) = x_n$. In order to get a fixed point for Φ, observe that by compactness of K the sequence $(x_n)_{n \in \mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k \in \mathbb{N}}$. Now, it suffices to prove that its limit x is a fixed point of Φ. In fact,

$$\|\Phi(x_{n_k}) - x_{n_k}\| = \|\Phi(x_{n_k}) - P_n(\Phi(x_{n_k}))\| \leq \frac{1}{n_k},$$

by (8.2), i.e.,

$$\Phi(x) = \Phi\left(\lim_{k\to\infty} x_{n_k}\right) = \lim_{k\to\infty} \Phi(x_{n_k}) = \lim_{k\to\infty} x_{n_k} = x.$$

This concludes the proof. $\square$

Exercise 8.11. *Let X be a complete and normed vector space over $\mathbb{R}$. Prove the following simple corollary of Schauder's theorem:* Let K be a bounded, closed and convex subset of X. Then each continuous function $\Phi : K \to K$ such that $T(K)$ is relatively compact has a fixed point in K.

As an application of Schauder's fixed point theorem we mention the following result by Giuseppe Peano. It is one of the earliest existence theorems for ordinary differential equations, originally proved (of course, by different methods) in the 1880s.

[6] Recall that the distance between a point x and a set is defined as the infimum of all distances between x and any point in the set.

Proposition 8.12 (Peano's theorem)**.** *Let Ω be a bounded open domain of $\mathbb{C}^n$, $T > 0$, and $f : [0,T] \times \Omega \to \mathbb{C}^n$ be continuous. Then the Cauchy problem*

$$\begin{cases} \frac{du}{dt}(t) &= f(t, u(t)), \qquad t \in [0,T], \\ u(0) &= u_0, \end{cases}$$

is locally well-posed, i.e., it admits a *solution on some interval $[0, t_0]$, $t_0 \leq T$.*

It is well-known that the assertion in Peano's theorem cannot be sharpened to yield uniqueness (just set $n = 1$ and $f(t, u(t)) := \sqrt{u(t)}$.) In our approach, this limitation is a direct consequence of the non-constructive method used to prove Brouwer's and Schauder's fixed point theorems.

In the proof we will have to check relatively compactness of a certain set: this will be accomplished applying the theorem of Ascoli–Arzelà, which we briefly recall.

Theorem 8.13 (Theorem of Ascoli–Arzelà)**.** *Let $d, m \in \mathbb{N}$. Let Ω be a bounded domain of $\mathbb{R}^d$. A subset of $C(\Omega; \mathbb{R}^m)$ is relatively compact if and only if it is bounded and equicontinuous.*

Proof of Theorem 8.12. Consider the integral formulation

$$u(t) = u_0 + \int_0^t f(s, u(s))ds =: (K(u))(t)), \qquad t \in [0,T],$$

of the given Cauchy problem: K is by definition an operator mapping continuous functions into continuous functions. (A priori this formulation is only weaker, but it will eventually turn out that to our purposes they are equivalent.)

Consider $t_0 \in (0, T]$ and $M > 0$ such that

$$|f(t, x)| \leq M, \qquad t \in [-t_0, t_0],\ |x| \leq Mt_0 :$$

such constants exist due to continuity of f. Let

$$X := B_M(u_0) = \{g \in C([0, t_0]; \mathbb{R}^d) : \max_{t \in [0,t_0]} |g(t) - u_0| \leq Mt_0\}.$$

The closed set X is bounded and convex. Moreover, $K : X \to C([0, t_0]; \mathbb{R}^d)$ is continuous because if $(g_n)_{n\in\mathbb{N}} \subset X$ and $g \in X$, and if $\lim_{n\to\infty} g_n = g$ with respect to the $\|\cdot\|_\infty$-norm, then

$$\lim_{n\to\infty} |Kg_n(t) - Kg(t)| \leq \lim_{n\to\infty} \int_0^t |f(s, g_n(s)) - f(s, g(s))|ds \leq t_0 \lim_{n\to\infty} \max_{s\in[0,t_0]} |f(s, g_n(s)) - f(s, g(s))| = 0, \qquad t \in [0,T],$$

due to continuity of f.

In order to apply Exercise 8.11, it suffices to prove that that $K(X) \subset X$ and that $K(X)$ is relatively compact. The former assertion is a direct consequence of the estimate

$$|Kg(t) - u_0| \leq \int_0^{t_0} |f(s, g(s))|ds \leq Mt_0, \qquad t \in [0, t_0],\ g \in X.$$

The latter follows applying the theorem of Ascoli–Arzelà: in order to show equicontinuity of the bounded set X, observe that for all $t_1, t_2 \in [-t_0, t_0]$ and for all $g \in X$ one has

$$|Kg(t_1) - Kg(t_2)| \leq \int_{t_1}^{t_2} |f(s, g(s))|ds \leq M|t_2 - t_1|.$$

Therefore, by Exercise 8.11 there exists a fixed point u of K.

Finally, the fundamental theorem of calculus implies that Ku is continuously differentiable for all $u \in X$, hence in particular for the fixed point(s) of K. This concludes the proof. □

In the following, we need the notion of compact (nonlinear) operator.

Definition 8.14. *Let X be a complete and normed vector space. A function $\Phi : X \to X$ is called* **compact** *if for any bounded sequence $(x_n)_{n\in\mathbb{N}} \subset X$ the sequence of values $(\Phi(x_n))_{n\in\mathbb{N}}$ has a convergent subsequence.*

In other words, Φ is compact if it maps bounded sets in relatively compact sets.

The following result has been obtained by Helmut Schaefer in 1955, one of his last results obtained in the field of integral equations before fleeing to Western Germany and devoting himself to abstract functional analysis. Schaefer's assertion is essentially a corollary of Schauder's fixed point theorem, and in fact in the literature it often goes under the name of theorem of Leray–Schauder. Though, it is often more easily applied to partial differential equations than original Schauder's theorem[7].

Theorem 8.15 (Schaefer's fixed point theorem)**.** *Let X be a complete and normed vector space over $\mathbb{R}$. Then each continuous and compact function $\Phi : X \to X$ has a fixed point, provided that the set*

$$\{x \in X : \exists \lambda \in [0,1] \text{ s.t. } x \text{ is a fixed point of } \lambda\Phi\}$$

is bounded.

Beweis. By assumption, there exists $M > 0$ such that

$$\|x\|_X \le M$$

whenever $x = \lambda\Phi(x)$ for some $\lambda \in [0,1]$. We introduce a "cut-off" version $\tilde{\Phi}$ of Φ by setting

$$\tilde{\Phi}(x) := \begin{cases} \Phi(x) & \text{if } \|\Phi(x)\|_X \le M, \\ \frac{M}{\|\Phi(x)\|}\Phi(x) & \text{if } \|\Phi(x)\|_X \ge M. \end{cases}$$

Observe that $\|\tilde{\Phi}(x)\| \le M$ for all $x \in X$, hence in particular $\tilde{\Phi}$ maps $B_M(0) := \{x \in X : \|x\|_X \le M\}$ into itself. Our aim is to apply Schauder's fixed point theorem to $\tilde{\Phi}$ taking K to be closed convex hull of the image of $B_M(0)$ under $\tilde{\Phi}$. It is clear that $\tilde{\Phi}$ is continuous, since $\|\cdot\|_X : X \to \mathbb{R}$ is continuous. Moreover, $\tilde{\Phi}$ maps K into itself.

Finally, observe that since Φ is compact, so is $\tilde{\Phi}$, since $\|\tilde{\Phi}(x)\|_X \le M$ for all $x \in X$, and since closed balls are compact in $\mathbb{R}^d$ (i.e., sequences in closed balls of $\mathbb{R}^d$ have convergent subsequences by the theorem of Bolzano–Weierstraß). Accordingly, $\tilde{\Phi}(B_M(0))$ is relatively compact, hence K is compact. By Schauder's theorem we deduce the existence of a fixed point x_0 of $\tilde{\Phi}$. It remains to show that x_0 is also a fixed point of Φ. If this would not be the case, i.e., if it would hold

$$\Phi(x_0) \ne x_0 = \tilde{\Phi}(x_0) = \frac{M}{\|\Phi(x_0)\|}\Phi(x_0),$$

then necessarily $\|\Phi(x_0)\| > M$. But then defining

$$\lambda := \frac{M}{\|\Phi(x_0)\|_X} \in (0,1)$$

this would yield

$$x_0 = \lambda\Phi(x_0),$$

[7] Identifying a compact convex set left invariant under Φ is seldom an easy task. One may of course try to apply Proposition 6.62, under whose assumptions however the theorem of Lax–Milgram already yields a stronger result – not only existence, but also uniqueness of solutions.

i.e., x_0 would be a fixed point of Φ for some $\lambda \in (0,1)$, and by assumption $\|x_0\|_X \leq M$. Now, on one hand x_0 is a fixed point of $\tilde{\Phi}$, hence $\|\Phi(x_0)\|_X \leq M$. On the other hand, this implies that $\Phi(x_0) = \tilde{\Phi}(x_0)$, hence x_0 is a fixed point of Φ, a contradiction. □

Exercise 8.16. *Let X be a complete and normed vector space over $\mathbb{R}$. Prove the following corollary of Schaefer's theorem.* If $\Phi : X \to X$ is continuous and compact, then $\alpha\Phi$ has a fixed point for at least one $\alpha \in (0,1]$.

We have mentioned the possibility of applying fixed point theorems to evolution equations. A more elementary example is given by nonlinear elliptic equations, as follows. Most applications of Schauder's or Schaefer's theorems are based on the following theorem, for whose proof we refer to [**4**, Thm. 9.16].

Proposition 8.17 (Rellich–Kondrachov's embedding theorem)**.** *Let Ω be an open bounded domain of $\mathbb{R}^d$ with C^1-boundary. Then the unit ball of $H^1(\Omega)$ is relatively compact in*

- *$C(\overline{\Omega})$ if $d = 1$,*
- *$L^p(\Omega)$ if $d = 2$, for all $p \in [1,\infty)$, or finally*
- *$L^{\frac{2d}{d-2}}(\Omega)$ if $d \geq 3$.*

In particular, the unit ball $H^1(\Omega)$ is relatively compact in $L^{\frac{2d}{d-2}}(\Omega)$, and hence in $L^2(\Omega)$, for all $d \in \mathbb{N}$. Observe that this assertion is equivalent to saying that the identity is compact as a mapping from $H^1(\Omega)$ to $L^2(\Omega)$.

Remark 8.18. *For applications of Schaefer's fixed point theorem is also useful to recall that for each compact subset K of $\mathbb{R}^d$, the unit ball of each subspace $C^k(K)$ is relatively compact in $C(K)$, $k \in \mathbb{N}$.*

8.2. Semilinear elliptic problems

Following [**5**, § 9.2] we wish to discuss the nonlinear elliptic problem

$$-\Delta u(x) + \mu u(x) = b(\nabla u(x)), \qquad x \in \Omega, \tag{8.3}$$

with Dirichlet boundary conditions.

Proposition 8.19. *Let Ω be an open bounded domain of $\mathbb{R}^d$ and consider a continuous function $b : \mathbb{R}^d \to \mathbb{R}$ with sublinear growth, i.e., such that there exists $C > 0$ such that*

$$|b(x)| \leq C(|x| + 1), \qquad x \in \mathbb{R}^d. \tag{8.4}$$

Then for all $\mu \geq C^2$ there exists a weak solution of class $H^2(\Omega)$ to (8.3) with Dirichlet boundary conditions, i.e., $u \in H^2(\Omega) \cap H^1_0(\Omega)$ such that

$$\int_\Omega (\nabla u(x) \nabla v + \mu u(x) v(x))\, dx = \int_\Omega b(\nabla u(x)) v(x) dx \qquad \textit{for all } v \in H^1_0(\Omega).$$

Beweis. The proof is based on the following trick: we let $u \in H^1_0(\Omega)$ (arbitrary, so far) and set

$$f_u(x) := b(\nabla u(x)) \in \mathbb{R}.$$

For this function we consider the Helmholtz equation

$$-\Delta w(x) + \mu w(x) = f_u(x), \qquad x \in \Omega, \tag{8.5}$$

with Dirichlet boundary conditions. By the theorem of Lax–Milgram (more precisely, by Corollary 6.50) this equation has a unique solution $w \in H^1_0(\Omega)$ which, in turn, by Theorem 6.58 also belongs to $H^2(\Omega)$, as $f_u \in L^2(\Omega)$, and in fact the estimate

$$\|w\|_{H^2} \leq M\|f\|_{L^2}$$

holds, for some $M > 0$. This construction defines a mapping $\Phi : H_0^1(\Omega) \ni u \mapsto w \in H_0^1(\Omega)$, to which we are going to apply Theorem 8.15. If we succeed in doing so, then by Schaefer's theorem there exists u such that $\Phi(u) = u$, i.e., such that

$$-\Delta u(x) + \mu u(x) = f_u(x) = b(\nabla u(x)).$$

This would then conclude the proof.

To this aim, we first have to prove that Φ is continuous and compact. First of all, let us observe the estimate

$$\|\Phi(u)\|_{H^2} \leq M\|f\|_{L^2} = M\|b(\nabla u)\|_{L^2} \leq MC(\|\nabla u\|_{L^2} + 1) = MC(\|u\|_{H_0^1} + 1). \tag{8.6}$$

Moreover, let us take a sequence $(u_n)_{n\in\mathbb{N}} \subset H_0^1(\Omega)$ such that $\lim_{n\to\infty} u_n = u$ with respect to the norm of $H_0^1(\Omega)$. Then obviously the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$, and therefore so is $(\Phi(u_n))_{n\in\mathbb{N}}$ in $H^2(\Omega)$, by (8.6). By the embedding theorem of Rellich–Kondrachov $(\Phi(u_n))_{n\in\mathbb{N}}$ has a subsequence $(\Phi(u_{n_k}))_{k\in\mathbb{N}}$ that converges in $H_0^1(\Omega)$, say to a function w. If we can show that $\Phi(u) = w$, then the continuity of Φ is completely proved. This can be seen as follows: since each $\Phi(u_{n_k})$ is by construction weak solution of (8.5), i.e., it satisfies

$$\int_\Omega (\nabla\Phi(u_{n_k})(x)\nabla v + \mu\Phi(u_{n_k})(x)v(x))\,dx = \int_\Omega f_{u_{n_k}}(x)v(x)dx = \int_\Omega b(\nabla u_{n_k}(x))v(x)dx, \qquad v \in H_0^1(\Omega). \tag{8.7}$$

Passing to the limit for $k \to \infty$, we know that

$$\Phi(u_{n_k}) \to w \qquad \text{and} \qquad u_{n_k} \to u,$$

both with respect to the norm of $H_0^1(\Omega)$, whence in particular (by continuity of b)

$$\nabla\Phi(u_{n_k}) \to \nabla w \qquad \text{and} \qquad b(\nabla u_{n_k}) \to b(\nabla u)$$

with respect to the norm of $L^2(\Omega)$. Summing up, (8.7) implies

$$\int_\Omega (\nabla w(x)\nabla v + \mu w(x)v(x))\,dx = \int_\Omega b(\nabla u(x))v(x)dx = \int_\Omega f_u(x)v(x)dx, \qquad v \in H_0^1(\Omega), \tag{8.8}$$

i.e., w weak solution of (8.5), hence $w = \Phi(u)$.

Similarly, Φ is compact because if a sequence $(u_n)_{n\in\mathbb{N}}$ is bounded with respect to the norm of $H_0^1(\Omega)$, then (as observed above) $(\Phi(u_n))_{n\in\mathbb{N}}$ has a subsequence $(\Phi(u_{n_k}))_{k\in\mathbb{N}}$ that is convergent with respect to the norm of $H_0^1(\Omega)$.

Finally, we have to check that the set

$$\{v \in H_0^1(\Omega) : \exists\lambda \in [0,1] \; s.t. \; v \text{ is a fixed point of } \lambda\Phi\}$$

is bounded, at least for $\mu > 0$ large enough. To do so, consider $v \in H_0^1(\Omega)$ such that $\frac{v}{\lambda} = \Phi(v)$ for some $\lambda \in (0,1]$ (if $\lambda = 0$, then the only fixed point is the 0-function, hence we can neglect this trivial case): this means that $\frac{v}{\lambda}$ is weak solution to (8.3) with Dirichlet boundary conditions, hence in particular integrating (8.3) against the same $v \in H_0^1(\Omega)$ yields

$$\begin{aligned}
\int_\Omega \left(|\nabla v(x)|^2 + \mu|v(x)|^2\right) dx &= \int_\Omega \lambda b(\nabla v(x))v(x)dx \\
&\leq \int_\Omega C(|\nabla v(x)| + 1)|v(x)|dx \\
&= \int_\Omega C|\nabla v(x)||v(x)|dx + \int_\Omega C|v(x)|dx \\
&\leq \frac{1}{2}\int_\Omega |\nabla v(x)|^2dx + \frac{C^2}{2}\int_\Omega |v(x)|^2dx + \frac{C^2}{2}\int_\Omega |v(x)|^2dx + \int_\Omega dx,
\end{aligned}$$

where the last step follows from the Young inequality. Summing up,

$$\frac{1}{2}\int_\Omega |\nabla v(x)|^2 dx \leq (C^2-\mu)\int_\Omega |v(x)|^2 dx + |\Omega|.$$

Hence, for $\mu \geq C^2$ we deduce that

$$\|v\|^2_{H^1_0(\Omega)} = \int_\Omega |\nabla v(x)|^2 dx \leq 2|\Omega|.$$

This completes the proof. □

Remark 8.20. *The viscid Burger's equation can be put in the form*

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(x) + b(\nabla u(x)), \qquad x \in \Omega.$$

for $b(x) := x^2$, $x \in \mathbb{R}$. Hence, when looking for stationary solutions, the corresponding time-independent problem is the Helmoltz equation (8.3)*. However, if we try to apply the above theorem, we see that the assumptions on b are not satisfied by the square function. A possible workaround is to show some* a priori *estimates on solutions u, so that at least*

$$|b(v(x))| = |v(x)|^2 \leq C(|v(x)|+1), \qquad x \in \Omega,$$

is satisfied for all solutions v, but in general this reveals an ubiquitous feature of nonlinear elliptic equations: there is such a large manifold of them that they cannot be tackled by a unitary method – or perhaps it is just the current techniques that are still much too weak.

Exercise 8.21. *Let Ω an open bounded domain of $\mathbb{R}^d$. Carefully check the proof of Proposition 8.19 and show that the assertion there is still valid if we replace in Proposition 8.19 the term $b(\nabla u)$ by a term $p(u)$, where $p : \mathbb{R} \to \mathbb{R}$ is a polynomial of arbitrary order, provided $d \leq 2$. What does this mean for the Hodgkin–Huxley equation* (8)*?*

You can use the fact that the Gagliardo–Niremberg inequalities

$$\|u\|_{L^\infty} \leq C\|u\|^{\frac{1}{2}}_{H^1}\|u\|^{\frac{1}{2}}_{L^2}, \qquad u \in H^1(\Omega),$$

and

$$\|u\|_{L^p} \leq C\|u\|^{1-2/p}_{H^1}\|u\|^{2/p}_{L^2}, \qquad u \in H^1(\Omega),$$

for all $p \geq 2$ and $\alpha := 1-2/p$, hold for $d=1$ and $d=2$, respectively, cf. [**4**, Comment 8.1.(iii), page 233 and Comment 9.3.C, page 313].

Index

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ANHANG A

Übungsaufgaben

Sei $u : \mathbb{R}^n \mapsto \mathbb{R}$ mit $x := (x_1, \ldots, x_n) \mapsto u(x)$. Wir werden für die partiellen Ableitungen der Funktion u auch folgende Notation verwenden: $\quad u_{x_i} := \frac{\partial u}{\partial x_i}, \qquad u_{x_i x_j} := \frac{\partial^2 u}{\partial x_i \partial x_j} \qquad$ etc.

(1) Gegeben sei die folgende partielle Differenzialgleichung ("gedämpfte Diffusionsgleichung")

$$u_t = u_{xx} + u, \quad t \geq 0,\ x \in \mathbb{R}.$$

Eine Lösung u der gedämpften Diffusionsgleichung heißt "*travelling wave*", falls eine Funktion $\phi : \mathbb{R} \mapsto \mathbb{R}$ und eine Konstante $c \in \mathbb{R}$ existieren, so dass $u(t, x) = \phi(x - ct)$ für alle $t \geq 0$, $x \in \mathbb{R}$.

Bestimmen Sie alle "*travelling wave*" Lösungen.

(2) Gegeben sei die folgende partielle Differenzialgleichung ("Laplace-Gleichung")

$$\Delta u = 0, \quad \text{wobei } u : \mathbb{R}^n \mapsto \mathbb{R} \text{ und } \Delta u(x) := \sum_{i=1}^{n} u_{x_i x_i}(x).$$

Zeigen Sie, dass die Laplace-Gleichung rotationsinvariant ist, d.h ist A eine orthogonale $n \times n$-Matrix ($AA^T = I$) und u eine Lösung der Gleichung, so ist auch die Funktion $v : \mathbb{R}^n \mapsto \mathbb{R}$ mit $v(x) := u(Ax)$ eine Lösung.

(3) Die Methode der Charakteristiken eignet sich auch für Transportgleichungen mit nicht konstanten Koeffizienten. Wir betrachten das Anfangswertproblem

$$\begin{aligned} u_t(t, x) &= -x u_x(t, x), \qquad t \geq 0,\ x \in \mathbb{R}, \\ u(0, x) &= u_0(x). \end{aligned}$$

Bestimmen Sie die Charakteristiken für diese Transportgleichung und finden Sie die Lösung des Anfangswertproblems.

(4) Betrachte das Anfangs-Randwertproblem für die Transportgleichung auf dem Intervall $[0, 1]$

$$u_t(t, x) = -c u_x(t, x), \qquad t \geq 0,\ x \in [0, 1]$$

mit Anfangswert $u(0, x) = u_0(x)$, $x \in [0, 1]$.
Bestimmen Sie die Lösungen für folgende Randbedingungen. Welche Forderungen sind an den Anfangswert u_0 bei diesen Randbedingungen zu stellen?

$a)$ Periodische Randbedingungen : $u(t, 0) = u(t, 1),\ t \geq 0;$

$b)$ Dirichlet-Randbedingungen $(c > 0)$: $u(t, 0) = 0,\ t \geq 0;$

$c)$ Periodische und Dirichlet-Randbedingungen : $u(t, 0) = u(t, 1) = 0,\ t \geq 0.$

(5) Finden Sie die Lösungen der inhomogenen Transportgleichung

$$u_t(t, x) = 2u_x(t, x) + tx^2, \quad t \geq 0,\ x \in \mathbb{R}.$$

(6) Leiten Sie, den eindimensionalen Fall imitierend, die Lösungsformel für die Transportgleichung im $\mathbb{R}^d$

$$\begin{aligned} u_t(t,x) &= -b \cdot \nabla u(t,x) + f(t,x), \quad t \geq 0,\, x \in \mathbb{R}^d, \\ u(0,x) &= u_0(x) \end{aligned}$$

her, wobei $b \in \mathbb{R}^d$ ein konstanter Vektor und $\nabla u(t,x) := (u_{x_1}, \ldots, u_{x_d})(t,x))$ ist.

(7) Wir definieren wie in der Vorlesung die Operatoren

$$(C(t)f)(x) := \frac{1}{2}(f(x+ct) + f(x-ct)) \quad \text{und} \quad (S(t)f)(x) := \frac{1}{2}\int_{x-ct}^{x+ct} f(y)\,dy.$$

Zeige $C(0)f \equiv f$ und $2C(t)(C(s)f) \equiv C(t+s)f + C(t-s)f$ sowie $S(t+s)f \equiv C(s)(S(t)f) + S(s)(C(t)f)$ für alle $s,t \geq 0$ und alle Funktionen $f \in C^1$. Warum wird S Sinus und C Cosinus genannt?

(8) Sei $A, c \in \mathbb{R} \setminus \{0\}$. Zeige, dass die beiden *travelling waves*

$$u(t,x) := A\sin(x-ct) \qquad \text{und} \qquad v(t,x) := A\sin(x+ct)$$

sowie auf Grund von Linearität auch deren Summe $u+v$ Lösungen der eindimesionalen Wellengleichung ($u_{tt} = c^2 u_{xx}$) sind, $u+v$ aber keine *travelling wave*, sondern eine *stehende Welle* ist, d.h. $u+v$ läßt sich mittels zweier Funktionen $\eta : \mathbb{R} \mapsto \mathbb{R}$ und $\xi : \mathbb{R} \mapsto \mathbb{R}$ in der Form $(u+v)(t,x) = \eta(t)\xi(x)$ schreiben.

(9) Sei $u(t,x)$ eine Lösung der eindimensionalen Wellengleichung $u_{tt} = c^2 u_{xx}$.
Führe die Variablentransformation $\lambda := x - ct$ und $\mu := x + ct$ durch und definiere $v(\lambda,\mu) := u(t(\lambda,\mu), x(\lambda,\mu))$.
Zeige, dass $v_{\lambda\mu} = 0$ äquivalent zu $u_{tt} = c^2 u_{xx}$ ist. Schließe daraus, dass die allgemeine Lösung der eindimensionalen Wellengleichung die Gestalt $F(x-ct) + G(x+ct)$ hat, wobei $F : \mathbb{R} \mapsto \mathbb{R}$ und $G : \mathbb{R} \mapsto \mathbb{R}$ zweimal stetig differenzierbare Funktionen sind. Leite dann aus dieser Darstellung der Lösung d'Alemberts Formel für das Anfangswertproblem

$$u_{tt}(t,x) = c^2 u_{xx} \quad \text{mit} \quad u(0,x) = u_0(x) \text{ und } u_t(0,x) = u_1(x)$$

her.

(10) Sei $u_0 \in C^2(\mathbb{R})$ und $u_1 \in C^1(\mathbb{R})$. Löse das Anfangswertproblem

$$\begin{aligned} u_{tt}(t,x) &= x^2 u_{xx}(t,x) + x u_x(t,x), \qquad t \geq 0,\, x \in \mathbb{R}, \\ u(0,x) &= u_0(x), \\ u_t(0,x) &= u_1(x). \end{aligned}$$

Hinweis: Finde eine Zerlegung des Differentialoperators $\left(\frac{\partial^2}{\partial t^2} - x^2\frac{\partial^2}{\partial x^2} - x\frac{\partial}{\partial x}\right)$, ähnlich derjenigen, die in der Vorlesung bei der Herleitung von d'Alemberts Formel für die Wellengleichung benutzt wurde.

(11) Sei u eine Lösung des Anfangswertproblems für die Wellengleichung

$$u_{tt}(t,x) = u_{xx}(t,x), \quad t \geq 0,\, x \in \mathbb{R}, \qquad u(0,x) = u_0(x) \quad \text{und} \quad u_t(0,x) = u_1(x),$$

wobei $u_0 \in C^2(\mathbb{R})$ and $u_1 \in C^1(\mathbb{R})$ kompakten Träger haben sollen, d.h. $u_0(x) = 0$ und $u_1(x) = 0$ für alle x mit $|x| \geq R$ für ein $R \in \mathbb{R}$.
Zeige, dass für t groß genug die kinetische Energie gleich der potentiellen Energie ist

$$\mathrm{E}_{kin}(t) := \int_{\mathbb{R}} \frac{1}{2}(u_t(t,x))^2\,dx = \int_{\mathbb{R}} \frac{1}{2}(u_x(t,x))^2\,dx =: \mathrm{E}_{pot}(t).$$

Hinweis: Benutze d'Alemberts Formel.

(12) Sei $u_0 \in C^2([0,1])$ mit $u_0(0) = u_0(1) = u_0''(0) = u_0''(1) = 0$ (warum?) und $u_1 \in C^1([0,1])$ mit $u_1(0) = u_1(1) = 0$. Löse das Anfangs-Randwertproblem für die Wellengleichung auf dem Interval $[0,1]$

$$u_{tt}(t,x) = u_{xx}(t,x), \quad t \geq 0,\, x \in [0,1], \qquad u(0,x) = u_0(x) \quad \text{und} \quad u_t(0,x) = u_1(x),$$

mit Dirichletrandbedingungen $u(t,0) = u(t,1) = 0$ für $t \geq 0$.

(13) Sei $u_0 \in C^2([0,1])$ mit $u_0'(0) = u_0'(1) = 0$ und $u_1 \in C^1([0,1])$ mit $u_1'(0) = u_1'(1) = 0$. Löse das Anfangs-Randwertproblem für die Wellengleichung auf dem Interval $[0,1]$

$$u_{tt}(t,x) = u_{xx}(t,x), \quad t \geq 0,\, x \in [0,1], \qquad u(0,x) = u_0(x) \quad \text{und} \quad u_t(0,x) = u_1(x),$$

mit Neumannrandbedingungen $u'(t,0) = u'(t,1) = 0$ für $t \geq 0$.
Kann die Lösung u negative Werte annehmen, wenn $u_0(x) \geq 0$ und $u_1(x) \geq 0$ für alle $x \in [0,1]$?

(14) Sei Ω ein Gebiet im $\mathbb{R}^d$, $d \geq 1$, mit C^1 Rand (für eine Definition siehe z.B im Buch von Arendt und Urban auf Seite 220), insbesondere gilt Gauß-Green für $\Omega \cap B_R(0)$ ($B_R(0)$ der Ball mit Radius R um 0). Seien $u_0 \in C^2(\bar{\Omega})$ und $u_1 \in C^1(\bar{\Omega})$. Zeige, dass höchstens eine Lösung $u \in C^{2,2}(\mathbb{R}_+ \times \bar{\Omega})$ für das Anfangswertproblem

$$u_{tt}(t,x) = \Delta u(t,x), \quad (t,x) \in \mathbb{R}_+ \times \Omega \quad \text{und} \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x)$$

existiert, wobei u auf dem Rand Dirichlet- oder Neumannrandbedingungen genügen und im Falle eines unbeschränkten Gebietes ferner gelten soll: Für jedes kompakte Zeitintervall $[t_1, t_2]$ findet man eine auf Ω quadratintegrierbare Funktion $g : \Omega \mapsto \mathbb{R}_+$ ($\int_\Omega |g(x)|^2\, dx < \infty$) mit

$$|u_{x_i}(t,x)|, |u_{x_i t}(t,x)|, |u_t(t,x)|, |u_{tt}(t,x)| \leq g(x),\ i = 1, \dots, d\,, \quad \text{für alle } t \in [t_1, t_2].$$

(15) Sei $k \in \mathbb{N}$ und $r > 0$. Zeige, dass für alle $\phi \in C^{k+1}(\mathbb{R})$ gilt

$$\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right) = \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k}\frac{d\phi}{dr}(r)\right).$$

(16) Sei $k \in \mathbb{N}$. Für $u_0 \in C^{k+2}(\mathbb{R}^{2k+1})$ und $u_1 \in C^{k+1}(\mathbb{R}^{2k+1})$ definiere die Funktion u durch

$$\begin{aligned} u(t,x) \quad &:= \quad \frac{1}{(2k-1)!!}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(\frac{t^{2k-1}}{|\partial B_t|}\int_{\partial B_t(x)} u_0(z)d\sigma(z)\right) \\ &\qquad + \frac{1}{(2k-1)!!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(\frac{t^{2k-1}}{|\partial B_t|}\int_{\partial B_t(x)} u_1(z)d\sigma(z)\right), \qquad t > 0,\ x \in \mathbb{R}^{2k+1}. \end{aligned}$$

Zeige, dass u die $(2k+1)$-dimensionale Wellengleichung $u_{tt}(t,x) = \Delta u(t,x)$ löst.

(17) Es sollen $u_0 \in C^3(\mathbb{R}^3)$ und $u_1 \in C^2(\mathbb{R}^3)$ kompakten Träger besitzen (d.h. es existiert $R > 0$, so dass $u_0(x) = 0$ bzw. $u_1(x) = 0$ für alle x mit $\|x\| \geq R$). Sei $u \in C^{2,2}(\mathbb{R}_+ \times \mathbb{R}^3)$ die Lösung des Anfangswertproblems

$$u_{tt}(t,x) = \Delta u(t,x) \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \quad \text{und} \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x).$$

Zeige $|u(t,x)| \leq C/t$ für alle $x \in R^3$ und $t > 0$, wobei $C > 0$ eine Konstante ist, die nicht von x abhängt.

(18) Betrachte die Wellengleichung $u_{tt}(t,x) = \Delta u(t,x)$ auf einem beschränkten Gebiet Ω mit C^1-Rand $\partial\Omega$ und der Randwertbedingung $\frac{\partial u}{\partial n}(t,z) = \pm\frac{\partial u}{\partial t}(t,z)$ für alle $t \geq 0$ und $z \in \partial\Omega$. Für welches Vorzeichen ist die wie in der Vorlesung definierte Energie $E(t)$ für jede Lösung $u \in C^{2,2}(\bar{\Omega}$ eine fallende bzw. wachsende Funktion in der Zeit?

(19) Sei Ω ein beschränktes Gebiet mit C^1-Rand. In der Physik betrachtet man auch die Wellengleichung mit "akustischen" Randbedingungen

$$\begin{cases} \frac{\partial^2\phi}{\partial t^2}(t,x) &= c^2\Delta\phi(t,x), & t \geq 0,\ x \in \Omega, \\ m\frac{\partial^2\delta}{\partial t^2}(t,z) &= -d\frac{\partial\delta}{\partial t}(t,z) - k\delta(t,z) - \rho\frac{\partial\phi}{\partial t}(t,z), & t \geq 0,\ z \in \partial\Omega, \\ \frac{\partial\delta}{\partial t}(t,z) &= \frac{\partial\phi}{\partial n}(t,z), & t \geq 0,\ z \in \partial\Omega. \end{cases}$$

Hierbei bezeichnet ϕ das Gechwindigkeitspotential einer Flüssigkeit, die im Gebiet Ω eingeschlossen ist, und δ die Auslenkung des Randes in Normalenrichtung aus seiner Ruhelage; $m > 0$, $d > 0$, $k > 0$ sind Konstanten, die die Masse pro Fläche, den Widerstand und die Federkonstante des Randes beschreiben; $c > 0$ und $\rho > 0$ stehen für die Schallgeschwindigeit in der Flüssigkeit bzw. die Dichte der Flüssigkeit im ungestörten Zustand.

Führe ein geeignetes Energiefunktional ein und zeige, dass diese Energie mit der Zeit abnimmt.

Hinweis: Beachte, dass auch der Rand durch die Auslenkung δ zur Gesamtenergie beiträgt.

(20) Betrachte folgende Transformationen $T_\epsilon : \mathbb{R}^3 \mapsto \mathbb{R}^3$ mit $T_\epsilon(t,x,z) = (t, x+2\epsilon t, e^{-\epsilon x-\epsilon^2 t}z)$, $\epsilon \in \mathbb{R}$. Zeige, dass die Familie von Transformationen $\mathscr{T} := (T_\epsilon)_{\epsilon\in\mathbb{R}}$ eine eindimensionale Transformationsgruppe bildet und bestimme deren infinitisimalen Erzeuger.

(21) Sei u eine Lösung der eindimensionalen Wärmeleitungsgleichung

$$u_t(t,x) = u_{xx}(t,x), \qquad (t,x) \in \mathbb{R}^2.$$

Betrachte den Graphen der Funktion u

$$G_u := \{(t,x,u(t,x)) : (t,x) \in \mathbb{R}^2\} \subset \mathbb{R}^3.$$

Zeige, dass das Bild des Graphen G_u unter den in Aufgabe 3. definierten Transformationen T_ϵ wieder der Graph einer Funktion $\tilde{u}_\epsilon : \mathbb{R}^2 \mapsto \mathbb{R}$ ist und diese Funktionen $\tilde{u}_\epsilon$ ebenfalls Lösungen der Wärmeleitungsgleichung sind.

(22) Die algebraische Gleichung $H(\mathbf{x}) = 0$, $\mathbf{x} \in J^K$, habe maximalen Rang, d.h. für alle $\mathbf{x} \in J^K$ mit $H(\mathbf{x}) = 0$ ist $\nabla H(\mathbf{x}) \neq \mathbf{0}$.
Zeige, dass dann für jede Lösung $(x_0, j_K u_0)$, $x_0 \in \mathbb{R}^d$, der Differentialgleichung $H(x, j_K u)$ ein lokaler Koordinatenwechsel $\mathbf{y} = (y_1, \ldots, y_{d+d_K})$ existiert, der $(x_0, j_K u_0)$ auf $\mathbf{0} \in J^K$ abbildet und für den die Gleichung $H(x, j_K u) = 0$ in $y_1(x, j_K u) = 0$ übergeht, d.h. $\mathbf{y}$ ist ein C^∞-Diffeomorphismus

$$\mathbf{y} : U((x_0, j_K u_0)) \mapsto V(\mathbf{0}), \qquad \mathbf{y}((x, j_K u)) = (y_1, \ldots, y_{d+d_K}),$$

wobei $U((x_0, j_K u_0)) \subseteq J^K$ eine offene Umgebung von $(x_0, j_K u_0)$ und $V(\mathbf{0}) \subseteq J^K$ eine offene Umgebung der Null in J^K ist.

Hinweis: Satz über implizite Funktionen.

(23) Zeige, dass die Wärmeleitungsgleichung $u_t(t,x) = cu_{xx}(t,x)$ für jedes $c \in \mathbb{R}$ maximalen Rang hat.

(24) Proposition 5.28. aus der Vorlesung gibt eine Methode zur Berechnung der K-Jets $j_K A(x, j_K u)$ der infinitesimalen Erzeuger von einparametrischen Transformationsgruppen.
  a) Sei $A = \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u}$ der infinitesimale Erzeuger einer einparametrischen Transformationsgruppe auf $J^0 = \mathbb{R}^2$. Berechne den 3-Jet $j_3 A(t, j_3 u)$ von A.
  b) Sei $A = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u}$ der infinitesimale Erzeuger einer einparametrischen Transformationsgruppe auf $J^0 = \mathbb{R}^3$. Berechne den Koeffizienten ϕ^{tx} von $\frac{\partial}{\partial u_{tx}}$ in $j_2 A(t, x, j_2 u)$.

(25) Betrachte den infinitesimalen Erzeuger einer einparametrischen Transformationsgruppe auf J^0

$$A(x,u) = \sum_{j=1}^{d} \xi_j(x,u)\frac{\partial}{\partial x_j} + \phi(x,u)\frac{\partial}{\partial u}.$$

Zeige, dass der 1-Jet von A die Form

$$j_1A(x, j_1u) = A + \sum_{i=1}^{d} \phi^i(x, j_1u)\frac{\partial}{\partial u_{x_i}}$$

hat, wobei die ϕ^i gegeben sind durch

$$\phi^i = D_i\left(\phi - \sum_{l=1}^{d} \xi_l(x,u)u_{x_i}\right) + \sum_{l=1}^{d} \xi_l(x,u)\frac{\partial u_{x_i}}{\partial x_l};$$

d.h. beweise Proposition 5.28. aus der Vorlesung für 1-Jets.

(26) Betrachte die einparametrische Transformationsgruppe $\mathcal{T} = (T_\epsilon)_{\epsilon\in I_{(x,u)}}$ auf $J^0 = \mathbb{R}^d \times R^l$,

$$T_\epsilon(x,u) =: (X_\epsilon(x,u), U_\epsilon(x,u)) \in J^0 = \mathbb{R}^d \times R^l,$$

wobei $x = (x_1,\ldots,x_d)$ die unabhängigen und $u = (u_1,\ldots,u_l)$ die von x abhängigen Variablen umfasst. Sei $x_0 \in \mathbb{R}^d$, $f : U(x_0) \mapsto \mathbb{R}^l$ und $(\epsilon, x, u) \mapsto T_\epsilon(x,u)$ eine C^∞-Abbildung von $W := \cup_{(x,u)\in J^0}(I_{(x,u)} \times \{(x,u)\}) \subseteq \mathbb{R} \times J^0$ nach J^0 ($\{0\} \times J^0 \subset W$).

Zeige: Es existiert ein $\delta' > 0$ und für jedes $0 \le \delta \le \delta'$ eine offene Umgebung $V_\delta(x_0) \subseteq U(x_0)$, so dass der lokale Graph $G^f_\delta := \{(x, f(x)) : x \in V_\delta(x_0)\}$ der Funktion f unter der Transformation T_δ übergeht in den lokalen Graphen $G^{f^\delta} = T_\delta(G^f_\delta)$ einer Funktion $f^\delta : X_\delta(V_\delta(x_0)) \mapsto \mathbb{R}^l$.

Hinweis: $f^\delta(x^\delta) = (U_\delta \circ [\mathrm{Id} \times f]) \circ (X_\delta \circ [\mathrm{id} \times f])^{-1}(x^\delta)$.

(27) Es gelten die Bezeichnungen aus Aufgabe 1. Zeige, dass der k-Jet der einparametrische Transformationsgruppe $\mathcal{T}$

$$(j_kT_\delta)(x_0, j_kf(x_0)) := (x_0^\delta, j_kf^\delta(x_0^\delta))$$

nur von x_0, $f(x_0)$ und den Ableitungen der Ordnung $\le k$ von f an der Stelle x_0 abhängt, also wohldefiniert ist.

Hinweis: Für $k = 1$ benutze die explizite Darstellung von f^δ (siehe Aufgabe 1) und schließe dann induktiv, d.h. betrachte $j_{(k-1)}u$ als die neuen abhängigen Variablen.

(28) Betrachte die Transformationsgruppe $\mathcal{T}$ aus Beispiel 5.13

$$T_\theta(x,u) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ u \end{pmatrix}, \quad (x,u) \in \mathbb{R}^2 = J^0,\ \theta \in \mathbb{R}.$$

Berechne den 1-Jet $(j_1T_\theta)(x,u,u_x)$, $\theta \in I_{(x,u,u_x)}$, der Transformationsgruppe $\mathcal{T}$, den 1-Jet $(j_1A)(x,u,u_x)$ des infinitesimalen Erzeugers A von $\mathcal{T}$ und bestimme $I_{(x,u,u_x)}$. (Siehe Beispiel 5.18)

(29) Benutze die Formeln aus Korollar 5.29, um die Punktsymmetriegruppen der zweidimensionalen Laplace-Gleichung herzuleiten

$$\frac{\partial^2 u}{\partial x^2}(t,x) + \frac{\partial^2 u}{\partial y^2}(t,x) = 0.$$

(30) Sei $A = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u}$ der infinitesimale Erzeuger einer einparametrischen Transformationsgruppe auf $J^0 \approx \mathbb{R}^3$. Berechne den Koeffizienten ϕ^{xx} von $\frac{\partial}{\partial u_{xx}}$ in $j_2A(t,x,j_2u)$.

(31) Zeige, dass

$$\Phi(t,x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{\|x\|^2}{4t}}, \qquad t > 0,\ x \in \mathbb{R}^d,$$

die d-dimensionale Wärmeleitungsgleichung $\Delta\Phi(t,x) - \Phi_t(t,x) = 0$ löst.

(32) Sei Ω ein beschränktes Gebiet im $\mathbb{R}^d$. Zeige, dass höchstens eine Funktion u in $C^1(\mathbb{R}_+ \times \bar{\Omega})$ existiert, die die Wärmeleitungsgleichung mit Neumann- oder Dirichletranbedingungen bei beliebig, aber fest vorgegebenem Anfangswert ($u(0,x) = u_0(x)$, $x \in \Omega$) löst.

Hinweis: Finde eine geeignete Energiefunktion und schließe ähnlich wie im Fall der Wellengleichung.

(33) Seien $\mathbf{v} = \xi_1(x)\frac{\partial}{\partial x_1} + \cdots + \xi_n(x)\frac{\partial}{\partial x_n}$ und $\mathbf{w} = \eta_1(x)\frac{\partial}{\partial x_1} + \cdots + \eta_n(x)\frac{\partial}{\partial x_n}$ zwei Vektorfelder auf dem $\mathbb{R}^n$; ein Vektorfeld kann aufgefasst werden als Ableitungsoperator von $C^\infty(\mathbb{R}^n)$ nach $C^\infty(\mathbb{R}^n)$ mit $\mathbf{v}(f(x)) = \xi_1(x)\frac{\partial f(x)}{\partial x_1} + \cdots + \xi_n(x)\frac{\partial f(x)}{\partial x_n}$.
Zeige, dass auch der Kommutator $[\mathbf{v},\mathbf{w}]$, definiert durch

$$[\mathbf{v},\mathbf{w}](f(x)) = \mathbf{v}(\mathbf{w}(f(x))) - \mathbf{w}(\mathbf{v}(f(x))), \quad f \in C^\infty(\mathbb{R}^n),$$

ein Vektorfeld auf dem $\mathbb{R}^n$ bildet und bestimme die Koeffizienten ϕ_i von $[\mathbf{v},\mathbf{w}] = \phi_1(x)\frac{\partial}{\partial x_1} + \cdots + \phi_n(x)\frac{\partial}{\partial x_n}$.

(34) Sei H ein Prä-Hilbertraum. Beweise die folgenden Aussagen.
(1) Falls x, y orthogonal aufeinander stehen, gilt $\|x\|_H^2 + \|y\|_H^2 = \|x+y\|_H^2$.
(2) Allgemeiner $2\|x\|_H^2 + 2\|y\|_H^2 = \|x+y\|_H^2 + \|x-y\|_H^2$ für alle $x, y \in H$.
(3) Ferner gilt für alle $x, y \in H$

$$\begin{aligned} 4(x|y)_H &= \|x+y\|_H^2 - \|x-y\|_H^2 && \text{falls } \mathbb{K} = \mathbb{R}, \text{ und} \\ 4(x|y)_H &= \|x+y\|_H^2 + i\|x+iy\|_H^2 - \|x-y\|_H^2 - i\|x-iy\|_H^2 && \text{falls } \mathbb{K} = \mathbb{C}. \end{aligned}$$

(4) Ist A eine Teilmenge von H, dann ist $A \subset (A^\perp)^\perp$.
(5) Das orthogonale Komplement $H^\perp$ ist $\{0\}$.

(35) Zeige, dass die Familie von Funktionen $\{1, \sqrt{2}\cos(2\pi n\cdot), \sqrt{2}\sin(2\pi m\cdot) : n, m = 1, 2, 3, \ldots\}$ orthonormal in $L^2(0,1;\mathbb{R})$ ist und dass die Familie $\{e^{2n\pi i\cdot} : n \in \mathbb{Z}\}$ orthonomal in $L^2(0,1;\mathbb{C})$ ist.
Zeige ferner, dass $\{e^{2n\pi i\cdot} : n \in \mathbb{Z}\}$ auch total ist, also eine Hilbertraumbasis von $L^2(0,1;\mathbb{C})$ bildet.

Hinweis: Benutze, dass $\{1, \sqrt{2}\cos(2\pi n\cdot), \sqrt{2}\sin(2\pi m\cdot) : n, m = 1, 2, 3, \ldots\}$ eine Hilbertraumbasis von $L^2(0,1;\mathbb{R})$ ist, und zerlege in Reell- und- Imaginärteil. (s.a. Proposition 6.15)

(36) Finde mit dem Ansatz der Trennung der Variablen $u(t,x) = v(t)w(x)$ eine nicht konstante Lösung der Poröse-Medien-Gleichung

$$u_t(t,x) - \Delta(u^\gamma)(t,x) = 0, \qquad (t,x) \in (0,\infty) \times \mathbb{R}^d,$$

wobei $u(\cdot,\cdot) \geq 0$ eine positive Funktion sein soll und $\gamma > 1$ eine Konstante ist. Existiert die Lösung für alle $t \geq 0$ oder explodiert sie in endlicher Zeit t^* (d.h. $\lim_{t \to t^*} u(t,x) = \infty,\ x \neq 0$).

Hinweis: Mache für w den Ansatz $w(x) = (\|x\|_2)^\alpha$, wobei α ein noch zu bestimmender Exponent ist.

(37) Betrachte das Eigenwertproblem des Laplace-Operators auf dem Intervall $I = (0,l)$ mit Robin-Randbedingungen, d.h.

$$\begin{aligned} -f''(x) &= \lambda f(x), && x, \in (0,1), \quad f \in C^2(I) \cap C^1(\bar{I}), \quad \lambda \in \mathbb{C} \\ b_0 f(0) &= f'(0), && b_0 > 0, \\ -b_l f(l) &= f'(l), && b_l > 0. \end{aligned}$$

Zeige, dass $\lambda \in \mathbb{C}$ nur dann ein Eigenwert sein kann, falls $\lambda > 0$, und dass zwei Eigenfunktionen (d.h. nicht triviale Lösungen obiger Gleichung) zu verschiedenen Eigenwerten $\lambda \neq \mu$ bezüglich des L^2-Skalarprodukts orthogonal aufeinander stehen.

(38) Sei H ein Hilbertraum und $\{x_n \in H : n \in \mathbb{N}\}$ eine orthonomale Familie. Zeige, dass die Reihe $\sum_{n\in\mathbb{N}} x_n$ genau dann in H konvergiert, wenn $\sum_{n\in N} \|x_n\|^2 < \infty$.

(39) Es seien $\mathbf{v}$ und $\mathbf{w}$ Vektorfelder auf $J^0 \approx \mathbb{R}^d \times \mathbb{R}$. Ferner seien $\mathbf{v}$ und $\mathbf{w}$ infinitesimale Erzeuger von einparametrischen Transformationsgruppen von Symmetrien der Differentialgleichung $H(x, j_k u) = 0$. Zeige, dass dann auch das Vektorfeld $[\mathbf{v}, \mathbf{w}]$ eine Symmetriegruppe von $H(x, j_k u) = 0$ erzeugt. (Siehe Aufgabe 4 von Blatt 9)

(40) Die infinitesimalen Erzeuger der eindimensionalen Wärmeleitungsgleichung sind gegeben durch (siehe Vorlesung)

$$A_1 = \frac{\partial}{\partial t}, \qquad A_2 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \qquad A_3 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - (2t + x^2)u\frac{\partial}{\partial u},$$
$$A_4 = \frac{\partial}{\partial x}, \qquad A_5 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}, \qquad A_6 = u\frac{\partial}{\partial u}.$$

Bestimme die Kommutatoren $[A_i, A_j]$, $i, j = 1, \dots, 6$.

(41) Es habe $H(x, j_k u)$ vollen Rang für alle $(x, j_k u) \in J^k$ mit $H(x, j_k u) = 0$ und sei A der Erzeuger einer einparametrischen Transformationsgruppe auf $J^0 \approx \mathbb{R}^d \times \mathbb{R}$. Zeige, dass die Bedingung $((j_k A)H)(x, j_k u) = 0$ für alle $(x, j_k u) \in J^k$ mit $H(x, j_k u) = 0$ äquivalent ist zur Existenz einer Funktion $Q : J^k \mapsto \mathbb{R}$, die $((j_k A)H)(x, j_k u) = Q(x, j_k u)H(x, j_k u)$ für alle $(x, j_k u) \in J^k$ erfüllt.

Hinweis: Zeige die Behauptung zuerst lokal (siehe dazu Theorem 5.21 aus der Vorlesung) und benutze dann eine Zerlegung der Eins.

(42) Sei H ein Hilbertraum und A_1, A_2 abgeschlossene, konvexe Teilmengen von H. Bezeichne mit P_1 bzw. P_2 die orthogonalen Projektionen auf A_1 bzw. A_2. Beweise, dass folgende Aussagen äquivalent sind:
i) $P_1 A_2 \subset A_2$,
ii) $P_2 A_1 \subset A_1$,
iii) P_1 und P_2 kommutieren, d.h. $P_1 P_2 x = P_2 P_1 x$ für alle $x \in H$.
(Allgemein: Warum betrachtet man stets abgeschlossene und konvexe Mengen?)

(43) Definiere A_1 bzw. $A_2 \subset L^2(\mathbb{R})$ als die Menge aller quadratintegrierbaren Funktionen, die fast überall gerade bzw. positiv sind.
(1) Zeige, dass A_1, A_2 abgeschlossene, konvexe Teilmengen von $L^2(\mathbb{R})$ sind.
(2) Zeige, dass die orthogonalen Projektionen P_{A_1}, P_{A_2} auf A_1, A_2 gegeben sind durch

$$P_{A_1} f(x) = \frac{f(x) + f(-x)}{2} \qquad \text{und} \qquad P_{A_2} f(x) = \frac{|f(x)| + f(x)}{2} \qquad \text{für fast alle } x \in \mathbb{R}.$$

(44) Sei H ein Hilbertraum und Y ein abgeschlossener Unterraum von H.
(1) Sei $Y \neq \{0\}$. Zeige, dass dann für die orthogonale Projektion P_Y von H auf Y gilt: $\|P_Y\| = 1$ und $\mathrm{Ker} P_Y = Y^\perp$.
(2) Zeige, dass jedes $x \in H$ eine eindeutige Zerlegung $x = y + z$ hat, wobei $y = P_Y x \in Y$ und $z = P_{Y^\perp} x \in Y^\perp$.

(45) Finde mittels der Methode der Trennung der Variablen eine Reihendarstellung der Lösung der Wärmeleitungsgleichung auf dem Intervall $(0, l)$ mit den Randbedingungen $f(0) = 0$ und $-f(l) = f'(l)$.

(46) Sei H ein Hilbertraum und A eine abgeschlossene, konvexe Teilmenge von H.
(1) Zeige, dass die Projektion P_A genau dann linear ist, wenn A ein Unterraum von H ist.

(2) Zeige, dass P_A Lipschitz-stetig ist mit Lipschitz-Konstante 1.

(47) Sei $I \subseteq \mathbb{R}$ ein offenes Intervall. Sei $f \in L^2(I)$. Zeige, dass die schwache Ableitung von f eindeutig ist, d.h. es existiert höchstens ein $g \in L^2(I)$ mit

$$\int_I f(x)\overline{h'(x)}\,dx = \int_I g(x)\overline{h(x)}\,dx \qquad \text{für alle } h \in C_c^1(I).$$

Hinweis: Benutze, dass $C_c^1(I)$ dicht in $L^2(I)$ liegt.

(48) Zeige, dass $C^0(0,1)$, versehen mit der L^2-Norm $\|f\|^2 = \int_0^1 f^2(x)\,dx$, und $C^1(0,1)$, versehen mit der H^1-Norm $\|f\|^2 = \int_0^1 f^2(x) + (f'(x))^2\,dx$, keine Hilberträume sind.

(49) Sei $I = (a,b)$ ein Intervall.
(1) Sei $f \in L^2(I)$ derart, dass $\int_I f(x)h'(x)dx = 0$ für alle $h \in C_c^1(\overline{I})$. Zeige, dass $f(x) = c$ für fast alle $x \in I$, wobei $c \in \mathbb{K}$ eine Konstante ist.
(2) Sei $g \in L^2(I)$. Definiere $G : I \ni x \mapsto \int_a^x g(t)dt \in \mathbb{K}$. Zeige, dass $G \in C(I)$ und $\int_I G(x)h'(x)dx = -\int_I g(x)h(x)dx$ für alle $h \in C_c^1(I)$.
(3) Folgere, dass jedes $f \in H^1(I)$ einen stetigen Repräsentanten $\tilde{f} \in C(\overline{I})$ hat, d.h. $f(x) = \tilde{f}(x)$ für fast alle $x \in I$.
Ferner gilt für $I = (0,1)$: $\|f\|_{C(\overline{I})} \leq \sqrt{2}\|f\|_{H^1(I)}$ für alle $f \in C([0,1]) \cap H^1(0,1)$.

(50) Seien H_1, H_2 Hilberträume und $T : H_1 \mapsto H_2$ ein linearer Operator.
(a) Zeige, dass T genau dann beschränkt ist, wenn T Lipschitz-stetig ist, und T genau dann Lipschitz-stetig ist, wenn T stetig ist.
Sei $\Omega \subseteq \mathbb{R}^d$ und betrachte den Hilbertraum $H = L^2(\Omega)$.
(b) Finde die orthogonale Projektion in H auf den Unterraum der konstanten Funktionen.

(51) Sei $\eta \in C^1(0,\infty)$. Sei $u \in H^1(0,1)$ und es existiere ein $\epsilon > 0$, so dass $\eta(x) = 0$ für $x \geq 1-\epsilon$. Bezeichne mit $\tilde{u}$ die Erweiterung von u auf $(0,\infty)$ durch $\tilde{u}(x) = 0$ für $x \geq 1$. Zeige $\eta\tilde{u} \in H^1(0,\infty)$, und beweise die Produktregel $(\eta\tilde{u})' = \eta'\tilde{u} + \eta\tilde{u}'$.

(52) Sei $\Omega \subseteq \mathbb{R}^d$ und $u \in H^1(\Omega)$. Zeige $u^+ := \frac{|u|+u}{2} \in H^1(\Omega)$.
Hinweis: Die schwache Ableitung $\frac{\partial u^+}{\partial x_j}$ ist gegeben durch $\chi_{\{u>0\}}\frac{\partial u}{\partial x_j}$. Definiere $f \in C^1(\mathbb{R})$ durch $f_n(r) := \begin{cases} (r^2 + n^{-2})^{1/2} - n^{-1} & \text{für } r > 0, \\ 0 & \text{für } r \leq 0 \end{cases}$ und betrachte $-\int_\Omega (f_n \circ u(x))\frac{\partial \phi}{\partial x_j}(x)\,dx$ mit $\phi \in C_c^1(\Omega)$.

(53) Sei Ω eine offene Teilmenge des $\mathbb{R}^d$. Definiere $C_0^2(\Omega)_+ := \{f \in C_c^2(\Omega) : f(x) \geq 0 \ \forall x \in \Omega\}$ und $H_0^1(\Omega)_+ := \{f \in H_0^1 : f(x) \geq 0 \text{ f.ü.}\}$. Eine Funktion $f \in L^2(\Omega)$ heißt subharmonisch, falls gilt $-\int_\Omega f(x)\Delta\phi(x) \leq 0$ für alle $\phi \in C_0^2(\Omega)_+$.
Zeige: Ist $u \in H^1(\Omega)$ subharmonisch und $(u-c)^+ \in H_0^1(\Omega)$, $c \in \mathbb{R}$, so gilt $u(x) \leq c$ für fast alle $x \in \Omega$.
Hinweis: Benutze, dass $C_c^2(\Omega)_+$ dicht in $H_0^1(\Omega)_+$ liegt und dass aus $\nabla f = 0$ f.ü. folgt $f = 0$ f.ü., falls $f \in H_0^1(\Omega)$.

Für die nächste Aufgabe benutze das folgende **Lemma**.

Sei Ω ein offenes Gebiet im $\mathbb{R}^d$ und $u \in H^1(\Omega)$. Dann existiert eine Folge $u_n \in C_c^1(\Omega)$, so dass $u_n \to u$ in $L^2(\Omega)$ und $\frac{\partial u_n}{\partial x_j} \to \frac{\partial u}{\partial x_j}$ in $L^2(\omega)$, $j = 1,\dots,d$, für alle $\omega \subset \Omega$ offen mit kompaktem Abschluss $\overline{\omega} \subset \Omega$ (Abschluss bezüglich $\mathbb{R}^d$).

(54) (a) (Kettenregel) Sei $f \in C^1(\mathbb{R})$ mit $f'(r) \leq M < \infty$ (und $f(0) = 0$, falls $|\Omega| = \infty$). Sei $u \in H^1(\Omega)$.
Zeige: $f \circ u \in H^1(\Omega)$ und $\frac{\partial(f\circ u)}{\partial x_j}(x) = f'(u(x))\frac{\partial u}{\partial x_j}(x), \quad j = 1,\dots,d.$

(b) (Produktregel) Seien $u, v \in H^1(\Omega)$ und $v, \frac{\partial v}{\partial x_j} \in L^\infty(\Omega), \quad j = 1, \ldots, d.$

Zeige: $uv \in H^1(\Omega)$ und $\frac{\partial(uv)}{\partial x_j} = u\frac{\partial v}{\partial x_j} + \frac{\partial u}{\partial x_j}v, \quad j = 1, \ldots, d.$

(55) (a) Sei H ein Hilbertraum und $a : H \times H \mapsto \mathbb{K}$ eine stetige und koerzive Sesquilinearform. Sei C eine abgeschlossene und konvexe Teilmenge von H und bezeichne mit P die orthogonale Projektion auf C.

Zeige, dass $Re a(u|u - Pu) \geq 0$ für alle $u \in H$ äquivalent ist zu $Re a(Pu|u - Pu) \geq 0$ für alle $u \in H$.

(b) Sei $a(\cdot|\cdot)$ jetzt zudem symmetrisch (d.h $a(u|v) = \overline{a(v|u)}$).

Zeige, dass $a(Pu|Pu) \leq a(u|u)$ für alle $u \in H$ aquivalent ist zu $Re a(u|u - Pu) \geq 0$ für alle $u \in H$.

Welche Äquivalenz lässt sich also in diesem Fall zu Proposition 7.60 aus der Vorlesung hinzufügen?

(c) Sei Ω ein offenes Gebiet im R^d, für das die Poincaré-Ungleichung gilt, und bezeichne mit P_+ die orthogonale Projektion auf $L^2(\Omega)_+$. Sei $a : H_0^1 \times H_0^1 \mapsto \mathbb{R}$ mit $a(u|v) = \int_\Omega \nabla u(x) \nabla v(x)\, dx$. Der zu $a(\cdot|\cdot)$ gehörige Operator A ist der Laplace-Operator mit Dirichlet-Randbedingungen.
Zeige, dass für jedes $f \in L^2(\Omega)_+$ die Lösung von $\lambda u - Au = f$, $\lambda > 0$, ebenfalls Element von $L^2(\Omega)_+$ ist.

(56) (a) Sei H ein Hilbertraum und $b : H \times H \mapsto \mathbb{R}$ eine stetige und symmetrische Bilinearform.

Zeige, dass für jede Funktion $u \in C^1(\mathbb{R}_+, H)$ gilt $\frac{d}{dt} b(u(t)|u(t)) = 2b(\frac{du}{dt}(t)|u(t)), \quad t \geq 0.$

(b) Sei $a : V \times V \mapsto \mathbb{R}$ eine stetige und symmetrische Bilinearform mit $a(u|u) \geq 0$ für alle $u \in V$ und A der zugehörige Operator, wobei der Hilbertraum V dicht und mit stetiger Einbettung in H liege.

Zeige, dass das Anfangswertproblem auf $C^2(\mathbb{R}_+, H)$

$$w_{tt}(t) - Aw(t) = 0 \quad \text{für } t \geq 0, \quad w(0) = u_0 \in V,\ w_t(0) = u_1 \in V,$$

höchstens eine Lösung besitzt, und wende dieses Resultat auf die Wellengleichung an.

(57) Finde die Lagrange-Funktion (a) zur konvektiven Poisson-Gleichung und (b) zur modifizierten Wärmeleitungsgleichung

$$(a) \quad \Delta u(x) + \nabla\phi(x)\nabla u(x) = f(x) \qquad x \in \Omega \subset \mathbb{R}^d, \quad \phi, f : \Omega \mapsto \mathbb{R},$$

$$(b) \quad \frac{\partial u}{\partial t} - \epsilon\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad (t, x) \in (0, \infty) \times \Omega \subset \mathbb{R}^2, \quad \epsilon > 0.$$

(58) (a) Überprüfe, dass die mit der Lagrange-Funktion

$$L(x, y, z, j_1 u) := u_x^2(x, y, z) + u_y^2(x, y, z) - u_z^2(x, y, z) + F(u(x, y, z)), \quad F(\cdot) = \int_0^{\cdot} f(z)\, dz,$$

assoziierte Euler-Lagrange-Gleichung eine nichtlineare Wellengleichung ist.

(b) Zeige, dass die Lösung $u : (0, \infty) \times (0, 1) =: \Omega \mapsto \mathbb{R}$ der eindimensionalen Wellengleichung $(u_{xx} - u_{tt} = 0)$ mit Neumann-Randbedingungen $(u_x(t, 0) = u_x(t, 1) = 0)$ und Anfangswerten $u(0, \cdot) = u_0(\cdot)$, $u_t(0, \cdot) = u_1(\cdot)$ nicht notwendiger Weise ein Minimum des entsprechenden Lagrange-Funktionals $I(u) = \int_\Omega (u_x^2(t, x) - u_t^2(t, x))\, d(t, x)$ ist, d.h. finde Anfangswerte u_0, u_1 und eine Funktion w, die Rand- und Anfangsbedingungen erfüllt und für die $I(w) < I(u)$ gilt.

)

(59) Sei $p \in (1, \infty)$ und betrachte das Lagrange-Funktional

$$I(u) = \int_\Omega |\nabla u(x)|^p \, dx, \qquad u \in H_0^1(\Omega).$$

Zeige, dass die assoziierte Euler-Lagrange-Gleichung die p-Laplace-Gleichung

$$\nabla(|\nabla u|^{p-2} \nabla u) = 0$$

ist und dass die Abbildungen

$$T_\epsilon : \quad x \mapsto e^\epsilon x, \qquad u(x) \mapsto e^{\epsilon \frac{n-p}{p}} u(e^\epsilon x), \qquad \epsilon \in \mathbb{R},$$

eine variationelle Symmetrie erzeugen.
Leite ferner die zu T_ϵ gehörige Noethersche Divergenzgleichung

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k} \left(\phi(x, j_1 u) \frac{\partial L}{\partial u_{x_k}}(x, j_1 u(x)) - L(x, j_1 u(x)) \xi_k(x, j_1 u) \right) = 0,$$

$$\phi(x, j_1 u) := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \left(e^{\epsilon \frac{n-p}{p}} u(e^\epsilon x) \right), \quad \xi_k(x, j_1 u) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} e^\epsilon x,$$

her und überprüfe auch direkt, dass diese aus der p-Laplace-Gleichung folgt.

(60) Betrachte die zweidimensionale Wellengleichung $u_{tt} - u_{xx} - u_{yy} = 0$. Die Vektorfelder $\mathbf{r}_{xy} := -y\partial_x + x\partial_y$ (Rotation) und $\mathbf{d} := x\partial_x + y\partial_y + t\partial_t$ (Dilatation) erzeugen einparametrische Symmetriegruppen der Wellengleichung.

Untersuche, ob sie auch variationelle Symmetrien erzeugen.

ANHANG B

Lösungen

(1) Dass $u(t,x) = \phi(x-ct)$ die partielle Differenzialgleichung löst, ist äquivalent dazu, dass $\phi(y)$ ($y = x-ct$) der gewöhnlichen Differenzialgleichung $\phi''(y) + c\phi'(y) + \phi(y) = 0$ genügt. Die Nullstellen des charakteristischen Polynoms dieser gewöhnlichen Differenzialgleichung sind $\lambda_{1/2} = \frac{1}{2}(-c \pm \sqrt{c^2-4})$. Deren allgemeine Lösung lautet:

$$\begin{aligned} \phi(y) &= C_1 e^{\lambda_1 y} + C_2 e^{\lambda_2 y}, && \text{falls} \quad c^2 - 4 > 0 \\ \phi(y) &= (C_1 + C_2 y) e^{\lambda_1 y}, && \text{falls} \quad c^2 - 4 = 0 \\ \phi(y) &= e^{\alpha y}(C_1 \cos(\beta y) + C_2 \sin(\beta y)), && \text{falls} \quad c^2 - 4 < 0. \end{aligned}$$

mit $\alpha = -\frac{c}{2}$ und $\beta = \frac{1}{2}\sqrt{4-c^2}$. Man muss nun nur noch y durch $x - ct$ ersetzen.

(2) Schreib $y = Ax$. Dann ist $v_{x_i} = \sum_{k=1}^n a_{k,i} u_{y_k}$ und

$$\sum_{i=1}^n v_{x_i x_i} = \sum_{i=1}^n \sum_{k=1}^n a_{k,i} \sum_{l=1}^n a_{l,i} u_{y_k y_l} = \sum_{k,l=1}^n \left(\sum_{i=1}^n a_{l,i} a_{k,i} \right) u_{y_k y_l} = \sum_{k=1}^n u_{y_k y_k} = 0.$$

(3) Gleichung für die Charakteristiken: $\gamma'(s) = \gamma(s)$, also $\gamma(s) = \alpha e^s$.
Charakteristik durch den Punkt (t,x): $\gamma(s) = xe^{s-t}$.
Lösung des AWP: $u(t,x) = u(0,\gamma(0)) = u_0(xe^{-t})$.

(4) Nach Definition einer (klassischen) Lösung, muss u_0 zumindest stetig differenzierbar sein.

a) Setze u_0 1-periodisch auf ganz $\mathbb{R}$ zu $\tilde{u}_0$ fort. $\tilde{u}(t,x) = \tilde{u}_0(x-ct)$ ist die Lösung der Transportgleichung auf ganz $\mathbb{R}$ zum Anfangswert $\tilde{u}_0$. Setze für $0 \le x \le 1$

$$u(t,x) = \tilde{u}(t,x) = \tilde{u}_0(x-ct) = u_0(x - ct - \lfloor x - ct \rfloor).$$

$u(t,x)$ ist dann die Lösung des Anfangs-Randwertproblems mit periodischen Randbedingungen, vorausgesetzt $\tilde{u}_0$ ist stetig differenzierbar, d.h. $u_0(0) = u_0(1)$ und $u_0'(0) = u_0'(1)$.

b) Unter der Bedingung, dass $u_0(0) = u_0'(0) = 0$, ist die Lösung gegeben durch

$$u(t,x) = \begin{cases} u_0(x-ct), & \text{falls } x - ct \ge 0, \\ 0, & \text{falls } x - ct < 0. \end{cases}$$

c) Sei o.B.d.A $c > 0$. Aus $0 = u(t,1) = u_0(1-ct)$ folgt $u_0 \equiv 0$.

(5) Die Lösungen sind gegeben durch

$$u(t,x) = u_0(x+2t) + \int_0^t s(x - 2(s-t))^2 \, ds = u_0(x+2t) + \frac{1}{3}t^4 + \frac{2}{3}xt^3 + \frac{1}{2}x^2t^2,$$

wobei u_0 eine stetig differenzierbare Funktion ist.

(6) Angenommen $u(t,\bar{x})$ löst $u_t(t,\bar{x}) = -\bar{b}\cdot\nabla u(t,\bar{x})$. Man sucht die Kurven (Charakteristiken) $\{(s,\bar{\gamma}(s)) : s\in(a,b)\}\subset\mathbb{R}\times\mathbb{R}^d$, entlang derer u konstant ist, also

$$0 = \frac{\partial u(s,\bar{\gamma}(s))}{\partial s} = u_t(s,\bar{\gamma}(s)) + \bar{\gamma}'(s)\cdot\nabla u(s,\bar{\gamma}(s)),$$

und erhält die Bedingung $\bar{\gamma}'(s) = \bar{b}$. Die Charakteristik durch den Punkt $(t,\bar{x})$ ist somit $\bar{\gamma}(s) = \bar{x}+(s-t)\bar{b}$ und die Lösung zum Anfangswert $u_0 = u(0,\bar{x})$ gegeben durch $u_0(\bar{x}-t\bar{b})$. Sei u eine Lösung der inhomogenen Gleichung $u_t(t,\bar{x}) = -\bar{b}\cdot\nabla u(t,\bar{x}) + f(t,\bar{x})$, dann ist

$$\frac{\partial u(s,\bar{\gamma}(s))}{\partial s} = u_t(s,\bar{\gamma}(s)) + \bar{b}\cdot\nabla u(s,\bar{\gamma}(s)) = f(t,\bar{x})$$

und $u(t,\bar{x}) = u_0(\bar{x}-t\bar{b}) + \int_0^t f(s,\bar{x}+(s-t)\bar{b})\,ds$.

(7)

$$C(0)f(x) = \frac{1}{2}\left(f(x)+f(x)\right) = f(x)$$

$$\begin{aligned}
2C(t)(C(s)f)(x) &= 2C(t)(\frac{1}{2}(f(x+cs)+f(x-cs)))\\
&= \frac{1}{2}\left(f(x+cs+ct)+f(x+cs-ct)+f(x-cs+ct)+f(x-cs-ct)\right)\\
&= C(s+t)f(x)+C(t-s)f(x)
\end{aligned}$$

$$\begin{aligned}
(C(s)(S(t)f)+S(s)(C(t)f))(x) &= \frac{1}{4}\int_{x-ct+cs}^{x+ct+cs} f(y)\,dy + \frac{1}{4}\int_{x-ct-cs}^{x+ct-cs} f(y)\,dy\\
&\quad + \frac{1}{4}\int_{x-cs}^{x+cs} f(z+ct)+f(z-ct)\,dz\\
&= S(t+s)f(x)
\end{aligned}$$

Vergleiche mit $\cos(0) = 1$, $\cos(s+t)+\cos(t-s) = 2\cos(t)\cos(s)$ und $\sin(s+t) = \cos(s)\sin(t)+\sin(s)\cos(t)$.

(8)

$$\frac{\partial^2}{\partial t^2}(A\sin(x-ct)+A\sin(x-ct)) = Ac^2\sin(x-ct)+Ac^2 sin(x+ct) = c^2\frac{\partial^2}{\partial x^2}(A\sin(x-ct)+A\sin(x+ct))$$

$$A\sin(x-ct)+A\sin(x+ct) = 2A\sin(x)\cos(ct)$$

Aber $2A\sin(x)\cos(ct)$ läßt sich nicht durch $\phi(x+bt)$ für ein $b\in\mathbb{R}$ und eine Funktion ϕ darstellen (setzt man z.B $t=\pi/2c$, so würde folgen $\phi\equiv 0$).

(9)

$$t(\lambda,\mu) = \frac{\mu-\lambda}{2},\ x(\lambda,\mu) = \frac{\mu+\lambda}{2};\quad 0 = v_{\lambda\mu} = \frac{1}{4}u_{xx} - \frac{1}{4c^2}u_{tt}\quad\Leftrightarrow\quad u_{tt} = c^2u_{xx}$$

Aus $v_{\lambda\mu} = 0$ folgt $v_\lambda(\lambda,\mu) = f(\lambda)$ und daraus $v(\lambda,\mu) = F(\lambda)+G(\mu)$, d.h. die allgemeine Lösung der Wellengleichung ist $F(x-ct)+G(x+ct)$ mit $F,G\in C^2(\mathbb{R})$.

Aus den Anfangsbedingungen $F(x)+G(x) = u_0(x)$, bzw. $F'(x)+G'(x) = u_0'(x)$ und $c(G'(x)-F'(x)) = u_1(x)$ erhält man

$$G'(x) = \frac{1}{2}u_0'(x)+\frac{1}{2c}u_1(x),\quad \text{bzw.}\quad G(x+ct)-G(0) = \frac{1}{2c}\int_0^{x+ct} u_1(y)\,dy + \frac{1}{2}u_0(x+ct) - \frac{1}{2}u_0(0)$$

und

$$F'(x) = \frac{1}{2}u_0'(x) - \frac{1}{2c}u_1(x), \quad \text{bzw.} \quad -F(x-ct) + F(0) = -\frac{1}{2c}\int_{x-ct}^{0} u_1(y)\,dy - \frac{1}{2}u_0(x-ct) + \frac{1}{2}u_0(0)$$

) und somit d'Alemberts Formel.

(10) Sei u eine Lösung des Anfangswertproblems, dann gilt

$$0 = \left(\frac{\partial^2}{\partial t^2} - x^2\frac{\partial^2}{\partial x^2} - x\frac{\partial}{\partial x}\right)u = \left(\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}\right)u.$$

Setze $v := \left(\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}\right)u$. Es folgt $v(t,x) = v(0, xe^{-t}) =: \phi(xe^{-t})$ für eine Funktion $\phi \in C^1(\mathbb{R})$ und $\left(\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}\right)u = \phi(xe^{-t})$. Die Charakterisik für letztere Gleichung ist $\gamma(s) = xe^{t-s}$ und damit

$$u(t,x) = u_0(xe^t) + \int_0^t \phi(xe^{t-s}e^{-s})\,ds.$$

Aus $u_1(x) = u_t(0,x) = xu_0'(x) + \phi(x)$ bzw. $\phi(x) = u_1(x) - xu_0'(x)$ folgt

$$u(t,x) = u_0(xe^t) + \int_0^t -xe^{t-2s}u_0'(xe^{t-2s}) + u_1(xe^{t-2s})\,ds = \frac{1}{2}\left(u_0(xe^t) + u_0(xe^{-t}\right) + \frac{1}{2}\int_{xe^{-t}}^{xe^t}\frac{1}{y}u_1(y)\,dy$$

und man rechnet leicht nach, dass die so definierte Funktion tatsächlich eine Lösung ist.

(11) Sei $t > R$. Es ist $u(t,x) = \frac{1}{2}(u_0(x-t) + u_0(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)\,dy$ und

$$\begin{aligned} u_t(t,x) &= -\frac{1}{2}u_0'(x-t) + \frac{1}{2}u_0'(x+t) + \frac{1}{2}u_1(x+t) + \frac{1}{2}u_1(x-t), \\ u_x(t,x) &= \frac{1}{2}u_0'(x-t) + \frac{1}{2}u_0'(x+t) + \frac{1}{2}u_1(x+t) - \frac{1}{2}u_1(x-t), \end{aligned}$$

wobei

$$\begin{aligned} &u_0'(x-t) = 0 \quad \text{und} \quad u_1(x-t) = 0 \quad \text{für} \quad x-t < -R \quad \text{und} \quad x-t > R, \\ &u_0'(x+t) = 0 \quad \text{und} \quad u_1(x+t) = 0 \quad \text{für} \quad x+t < -R \quad \text{und} \quad x+t > R, \end{aligned}$$

also

$$\begin{aligned} E_{kin} &= \frac{1}{2}\left(\int_{-R-t}^{R-t}(\frac{1}{2}u_0'(x+t) + \frac{1}{2}u_1(x+t))^2\,dx + \int_{-R+t}^{R+t}(-\frac{1}{2}u_0'(x-t) + \frac{1}{2}u_1(x-t))^2\,dx\right) \\ &= \frac{1}{2}\left(\int_{-R-t}^{R-t}(\frac{1}{2}u_0'(x+t) + \frac{1}{2}u_1(x+t))^2\,dx + \int_{-R+t}^{R+t}(\frac{1}{2}u_0'(x-t) - \frac{1}{2}u_1(x-t))^2\,dx\right) = E_{pot}. \end{aligned}$$

(12) Definiere

$$\tilde{u}_{0/1} := \begin{cases} u_{0/1}(x - \lfloor x \rfloor) & \text{falls } \lfloor x \rfloor = 2k, \\ -u_{0/1}(1 - (x - \lfloor x \rfloor)) & \text{falls } \lfloor x \rfloor = 2k+1, \end{cases} \qquad k \in \mathbb{Z}.$$

Dann ist $\tilde{u}_{0/1}(n) = 0$ für alle $n \in \mathbb{Z}$ und $\tilde{u}_{0/1}(-x) = -\tilde{u}_{0/1}(x)$ sowie $\tilde{u}_{0/1}(1-x) = -\tilde{u}_{0/1}(1+x)$.

$$u(t,x) := \frac{1}{2}(\tilde{u}_0(x-t) + \tilde{u}_0(x+t)) + \frac{1}{2}\int_{x-t}^{x+t}\tilde{u}_1(y)\,dy$$

löst also das Anfangswertproblem mit Dirichletrandbedingungen.

Da $\lim_{h\to+0} u_0''(n+h) = -\lim_{h\to+0} u''(n-h)$ muss $u_0''(0) = u_0''(1) = 0$ gelten.

(13) Definiere

$$\tilde{u}_{0/1} := \begin{cases} u_{0/1}(x - \lfloor x \rfloor) & \text{falls } \lfloor x \rfloor = 2k, \\ u_{0/1}(1 - (x - \lfloor x \rfloor)) & \text{falls } \lfloor x \rfloor = 2k+1, \end{cases} \qquad k \in \mathbb{Z}.$$

Dann ist $\tilde{u}_{0/1}(-x) = \tilde{u}_{0/1}(x)$ und $\tilde{u}_{0/1}(1-x) = \tilde{u}_{0/1}(1+x)$, woraus folgt $\tilde{u}_0'(-x) = -\tilde{u}_0(x)$ und $\tilde{u}_0'(1-x) = -\tilde{u}_0'(1+x)$.

$$u(t,x) := \frac{1}{2}(\tilde{u}_0(x-t) + \tilde{u}_0(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} \tilde{u}_1(y)\, dy$$

löst also das Anfangswertproblem mit Neumannrandbedingungen.
Aus der Lösungsformel für u folgt unmittelbar, dass $u(t,x) \geq 0$, falls $\tilde{u}_0(x) \geq 0$ und $\tilde{u}_1(x) \geq 0$ für alle $x \in \mathbb{R}$, was äquivalent zu $u_0(x) \geq 0$ und $u_1(x) \geq 0$ für alle $x \in [0,1]$ ist.

(14) Seien u und v Lösungen des gegebenen Anfangswertproblems, die die genannten Eigenschaften bzw. Randbedingungen besitzen, dann hat $w := u - v$ ebenfalls die gewünschten Eigenschaften bzw. Randbedingungen und

$$w_{tt}(t,x) = \Delta w(t,x), \quad (t,x) \in \mathbb{R}_+ \times \Omega \quad \text{und} \quad w(0,x) = 0, \quad w_t(0,x) = 0.$$

Die Energiefunktion von w läßt sich in diesem Fall auch bei nicht beschränktem Ω ableiten, und man erhält

$$\begin{aligned}
\frac{d}{dt}E(t) &= \frac{d}{dt}\left(\int_\Omega \frac{1}{2}|w_t(t,x)|^2 + \frac{1}{2}|\nabla w(t,x)|^2\, dx\right) \\
&= \int\limits_{\Omega\cap B_R} (w_t(t,x)w_{tt} + \nabla w(t,x)\cdot\nabla w_t(t,x))\, dx + \int\limits_{\Omega\cap B_R^C} (w_t(t,x)w_{tt} + \nabla w(t,x)\cdot\nabla w_t(t,x))\, dx \\
&= \int\limits_{\Omega\cap B_R} (w_t(t,x)(w_{tt} - \Delta w(t,x)))\, dx + \int\limits_{\partial(\Omega\cap B_R)} w_t(t,y)\frac{\partial w}{\partial n}(t,y)\, d\sigma(y) \\
&\quad + \int\limits_{\Omega\cap B_R^C} (w_t(t,x)w_{tt} + \nabla w(t,x)\cdot\nabla w_t(t,x))\, dx \\
&= \int\limits_{\partial(\Omega\cap B_R)} w_t(t,y)\frac{\partial w}{\partial n}(t,y)\, d\sigma(y) + \int\limits_{\Omega\cap B_R^C} (w_t(t,x)w_{tt}(t,x) + \nabla w(t,x)\cdot\nabla w_t(t,x))\, dx,
\end{aligned}$$

wobei B_R den Ball mir Radius R um den Nullpunkt und B_R^C dessen Komplement bedeutet. Der letzte Ausdruck geht mit $R \to \infty$ gegen 0, da $w_{x_i}(t,\cdot), w_{x_it}(t,\cdot), w_t(t,\cdot), w_{tt}(t,\cdot)$ quadrat integrierbar sind bzw. w_t und $\frac{\partial w}{\partial n}$ auf $\partial\Omega$ verschwinden. D.h. die Energie von w ist wegen $E(0) = 0$ konstant Null und damit auch $w_t(t,x)$, weshalb w die Nullfunktion ist, woraus $u \equiv v$ folgt.

(15) Induktionsanfang:

$$\frac{d^2}{d\,r^2}\,(r\phi(r)) = 2\phi'(r) + r\phi''(r) = \left(\frac{1}{r}\frac{d}{d\,r}\right)\left(r^2\frac{d}{d\,r}\phi(r)\right).$$

Induktionsschritt:

$$\begin{aligned}
\frac{d^2}{d\,r^2}\left(\frac{1}{r}\frac{d}{d\,r}\right)^k\left(r^{2k+1}\phi(r)\right) &= \frac{d^2}{d\,r^2}\left(\frac{1}{r}\frac{d}{d\,r}\right)^{k-1}\left((2k+1)r^{2k-1}\phi(r)+r^{2k}\frac{d}{d\,r}\phi(r)\right)\\
&= \frac{d^2}{d\,r^2}\left(\frac{1}{r}\frac{d}{d\,r}\right)^{k-1}\left(r^{2k-1}\left((2k+1)\phi(r)+r\frac{d}{d\,r}\phi(r)\right)\right)\\
&= \left(\frac{1}{r}\frac{d}{d\,r}\right)^{k}\left(r^{2k}\frac{d}{d\,r}\left((2k+1)\phi(r)+r\frac{d}{d\,r}\phi(r)\right)\right)\\
&= \left(\frac{1}{r}\frac{d}{d\,r}\right)^{k}\left((2k+2)r^{2k}\frac{d}{d\,r}\phi(r)+r^{2k+1}\frac{d^2}{d\,r^2}\phi(r)\right)\\
&= \left(\frac{1}{r}\frac{d}{d\,r}\right)^{k+1}\left(r^{2k+2}\frac{d}{d\,r}\phi(r)\right).
\end{aligned}$$

(16) Anfangsbedingungen:
Sei x fest. Schreibe $\phi_0(t) := \frac{1}{|\partial B_t|}\int_{\partial B_t(x)} u_0(z)\,d\sigma(z)$ und $\phi_1(t) := \frac{1}{|\partial B_t|}\int_{\partial B_t(x)} u_1(z)\,d\sigma(z)$. Dann gilt

$$\phi'_{0/1}(t) = \frac{t}{(2k+1)|B_t|}\int_{B_t(x)}\Delta u_{0/1}(y)\,dy \quad \text{und} \quad \frac{d^j}{dt^j}\phi_{0/1}(t) = \frac{t^{2-j}}{|B_t|}\int_{B_t(x)} f(y)\,dy$$

mit $f \in C^{k+2-(j+1)}$, $j = 1,\dots,k+1$, für ϕ_0 und $f \in C^{k+1-(j+1)}$, $j = 1,\dots,k$, für ϕ_1.
Lemma 4.6 aus dem Skript liefert

$$\begin{aligned}
u(t,x) &= \frac{1}{(2k-1)!!}\frac{d}{d\,t}\left((2k-1)!!t\phi_0(t)+\sum_{j=1}^{k-1}\beta_j^k t^{j+1}\frac{d^j}{dt^j}\phi_0(t)\right)\\
&\quad+\frac{1}{(2k-1)!!}\left((2k-1)!!t\phi_1(t)+\sum_{j=1}^{k-1}\beta_j^k t^{j+1}\frac{d^j}{dt^j}\phi_1(t)\right)\\
&= \phi_0(t)+t\phi_0'(t)+\frac{1}{(2k-1)!!}\sum_{j=1}^{k-1}\left((j+1)\beta_j^k t^j\frac{d^j}{dt^j}\phi_0(t)+\beta_j^k t^{j+1}\frac{d^{j+1}}{d\,t^{j+1}}\phi_0(t)\right)\\
&\quad+t\phi_1(t)+\frac{1}{(2k-1)!!}\left(\sum_{j=1}^{k-1}\beta_j^k t^{j+1}\frac{d^j}{dt^j}\phi_1(t)\right),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t}u(t,x) &= 2\phi_0'(t)+t\phi_0''(t)+\phi_1(t)\\
&\quad+\frac{1}{(2k-1)!!}\left(\sum_{j=1}^{k-1}c_1t^{j+1}\frac{d^j}{dt^j}\phi_0(t)+c_2t^j\frac{d^{j+1}}{dt^{j+1}}+c_3t^{j+1}\frac{d^{j+2}}{dt^{j+2}}\phi_0(t)+c_4t^j\frac{d^j}{dt^j}\phi_1(t)+c_5t^{j+1}\frac{d^{j+1}}{dt^{j+1}}\right).
\end{aligned}$$

Es folgt $\lim_{t\to 0} u(t,x) = \lim_{t\to 0}\phi_0(t) = u_0(x)$ und $\lim_{t\to 0}\frac{\partial}{\partial t}u(t,x) = \lim_{t\to 0}\phi_1(t) = u_1(x)$.
Dass $u(t,x)$ tatsächlich eine Lösung ist, wurde schon in der Vorlesung gezeigt.

(17) Sei M das Maximum von $|u_0(x)|$, $|\Delta u_0(x)|$ und $|u_1(x)|$ über alle $x \in \mathbb{R}^3$ und sei $t \geq 1$. Dann gilt

$$\begin{aligned}
|u(t,x)| &= \Big|\frac{\partial}{\partial t}\Big(\frac{t}{|\partial B_t|}\int\limits_{\partial B_t(x)} u_0(z)\,d\sigma(z)\Big) + \frac{t}{|\partial B_t|}\int\limits_{\partial B_t(x)} u_1(z)\,d\sigma(z)\Big| \\
&= \Big|\frac{1}{|\partial B_t|}\int\limits_{\partial B_t(x)\cap B_R} u_0(z)\,d\sigma(z) + \frac{t}{|\partial B_t|}\int\limits_{B_t(x)\cap B_R} \Delta u_0(x)\,dx + \frac{t}{|\partial B_t|}\int\limits_{\partial B_t(x)\cap B_R} u_1(z)\,d\sigma(z)\Big| \\
&\leq \frac{1}{|\partial B_1|t^2}\int\limits_{\partial B_t(x)\cap B_R} M\,d\sigma(z) + \frac{t}{|\partial B_1|t^2}\int\limits_{B_t(x)\cap B_R} M\,dx + \frac{t}{|\partial B_1|t^2}\int\limits_{\partial B_t(x)\cap B_R} M\,d\sigma(z)| \\
&\leq \frac{M}{|\partial B_1|t}(2|\partial B_R| + |B_R|) =: \frac{C_1}{t}
\end{aligned}$$

Ferner sei $u(t,x) \leq C_2$ für $0 \leq t \leq 1$ und $\|x\| \leq R+1$. Mit $C := \max\{C_1, C_2\}$ folgt dann $u(t,x) \leq C/t$ für alle $t \geq 0$ und $x \in \mathbb{R}^3$.

(18)

$$\begin{aligned}
\frac{d}{dt}E(t) &= \frac{\partial}{\partial t}\frac{1}{2}\int_\Omega |u_t(t,x)|^2 + \|\nabla u(t,x)\|^2\,dx = \int_\Omega u_t u_{tt} + \nabla u_t \cdot \nabla u\,dx \\
&= \int_\Omega u_t(u_{tt} - \Delta u)\,dx + \int_{\partial\Omega} \frac{\partial u}{\partial n}\frac{\partial u}{\partial t}\,d\sigma = \int_{\partial\Omega} \frac{\partial u}{\partial n}\frac{\partial u}{\partial t}\,d\sigma
\end{aligned}$$

Die Energie $E(t)$ wächst also für $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial t}$ und nimmt ab für $\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial t}$.

(19) Mit $E(t) := \frac{1}{2}\int_\Omega \frac{\rho}{c^2}|\phi_t(t,x)|^2 + \rho\|\nabla\phi(t,x)\|^2\,dx + \frac{1}{2}\int_{\partial\Omega} k\delta(t,y)^2 + m\delta_t(t,y)^2\,d\sigma(y)$ gilt

$$\begin{aligned}
\frac{d}{dt}T(t) &= \int_\Omega \rho\phi_t(\frac{1}{c^2}\phi_{tt} - \Delta\phi)\,dx + \int_{\partial\Omega} \rho\frac{\partial\phi}{\partial n}\frac{\partial\phi}{\partial t} + k\delta\delta_t + m\delta_t\delta_{tt}\,d\sigma \\
&= \int_{\partial\Omega} \delta_t(\rho\phi_t + k\delta - d\delta_t - k\delta - \rho\phi_t)\,d\sigma = \int_{\partial\Omega} -d\delta_t^2\,d\sigma \leq 0.
\end{aligned}$$

(20) Man hat $T_0(t,x,z) = (t,x,z)$ und

$$\begin{aligned}
T_\delta(T_\epsilon(t,x,z)) &= T_\delta(t, x+2\epsilon t, e^{-\epsilon x - \epsilon^2 t}z) = (t, x+2\epsilon t + 2\delta t, e^{-\delta(x+2\epsilon t) - \delta^2 t}e^{-\epsilon x - \epsilon^2 t}z) \\
&= (t, x+2(\epsilon+\delta)t, e^{-(\epsilon+\delta)x - (\epsilon+\delta)^2 t}z) = T_{\epsilon+\delta}(t,x,z).
\end{aligned}$$

Ferner ist $A = (\frac{d}{d\epsilon}T_\epsilon)|_{\epsilon=0} = (0, 2t, -xz)$.

(21) Setze $\tilde t = t$, $\tilde x = x + 2\epsilon t$ bzw. $x = \tilde x - 2\epsilon\tilde t$, dann ist die Funktion $\tilde u_\epsilon$ gegeben durch

$$\tilde u(\tilde t, \tilde x) = e^{-\epsilon(\tilde x - 2\epsilon\tilde t) - \epsilon^2\tilde t}u(\tilde t, \tilde x - 2\epsilon\tilde t) = e^{-\epsilon\tilde x + \epsilon^2\tilde t}u(\tilde t, \tilde x - 2\epsilon\tilde t)$$

und

$$\begin{aligned}
\tilde u_{\tilde t} &= e^{-\epsilon\tilde x + \epsilon^2\tilde t}(\epsilon^2 u(\tilde t, \tilde x - 2\epsilon\tilde t) + u_t(\tilde t, \tilde x - 2\epsilon\tilde t) - 2\epsilon u_x(\tilde t, \tilde x - 2\epsilon\tilde t)) \\
&= e^{-\epsilon\tilde x + \epsilon^2\tilde t}(\epsilon^2 u(\tilde t, \tilde x - 2\epsilon\tilde t) + u_{xx}(\tilde t, \tilde x - 2\epsilon\tilde t) - 2\epsilon u_x(\tilde t, \tilde x - 2\epsilon\tilde t)) = \tilde u_{\tilde x\tilde x}(\tilde t, \tilde x).
\end{aligned}$$

(22) Da H maximalen Rang am Punkt $(x_0, j_K u_0) =: z_0 = (z_0^1, \ldots, z_0^n)$, $n = d + d^K$, hat, existiert $0 \leq i \leq n$ mit $\frac{\partial H}{\partial z^i}(z_0) \neq 0$. O.B.d.A. können wir annehmen, dass $z_0 = 0$ und $i = 1$ ist ($z \mapsto (z^i \leftrightarrow z^1)(z - z_0)$ ist ein C^∞-Diffeomorphismus). Nach dem Satz über implizite Funktionen existiert daher eine offene Umgebung $U(z_0)$ und eine C^∞-Funktion $f(z^2, \ldots, z^n) =: f(\hat z)$ (H ist C^∞), so dass $H(z) = 0 \;\Leftrightarrow z^1 = f(\hat z)$ (also

insbesondere $z_0^1 = f(\hat{z}_0))$ und $0 \neq \frac{\partial H(z)}{\partial z^1} =: h(z)$ für $z \in U(z_0)$. Die partiellen Ableitungen von f sind gegeben durch $f_i := \frac{\partial f}{\partial z_i}(\hat{z}) = -\frac{1}{h(z)}\frac{\partial H(z)}{\partial z^i}$, $i = 2, \ldots, n$ und $z = (f(\hat{z}), \hat{z})$. Definiere

$$\mathbf{y}(z) := h(z)(z^1 - f(\hat{z}), z^2, \ldots, z^n) = (y^1, \ldots, y^n) = y.$$

Da die Determinante der Jacobimatrix

$$J\mathbf{y}(z) = \begin{pmatrix} h_{z^1}z^1 + h - h_{z^1}f & h_{z^2}z^1 - f_2h - h_{z^2}f & \cdots & h_{z^n}z^1 - f_nh - h_{z^n}f \\ h_{z^1}z^2 & h_{z^2}z^2 + h & \ldots & h_{z^n}z^2 \\ \vdots & & \ddots & \vdots \\ h_{z^1}z^n & \ldots & \ldots & h_{z^n}z^n + h \end{pmatrix}$$

an der Stelle $z = z_0 (= 0)$ gleich $h(0)^n \neq 0$ ist, existiert nach dem Satz über inverse Abbildungen eine Umgebung $U'(0)$, so dass $\mathbf{y}$ ein C^∞-Diffeomorphismus von $U'(0)$ nach $V(0) := \mathbf{y}(U'(0))$ ist. Insbesondere ist $\mathbf{y}(0) = 0$ und $H(z)$ geht über in $\tilde{H}(y) = H(\mathbf{y}^{-1}(y))$ mit $\tilde{H}(y) = 0$ genau dann, wenn $y^1 = 0$. Wegen $h_{z^i}z^1 - f_ih - h_{z^i}f = -f_ih = \frac{\partial H(z)}{\partial z^i}$ für $i = 2\ldots, n$ folgt

$$\nabla\tilde{H}(y) = \nabla H(z) \cdot (J\mathbf{y}(z))^{-1} = (1, 0, \ldots, 0),$$

also $\tilde{H}(y) = y_1$ für alle $y \in V(0)$.

(23) $u_t(t,x) = cu_{xx}(t,x)$ ist äquivalent zu $H(t, x, u, u_t, u_x, -u_{tt}, u_{xx}, u_{tx}) := u_t - cu_{xx} = 0$ und $\frac{\partial H}{\partial u_t} = 1$, d.h. die Wärmeleitungsgleichung hat vollen Rang.

(24) a) $j_3A(t, j_3u) = \tau(t,u)\frac{\partial}{\partial t} + \phi(t,u)\frac{\partial}{\partial u} + \phi^t(t, j_1u)\frac{\partial}{\partial u_t} + \phi^{tt}(t, j_2u)\frac{\partial}{\partial u_{tt}} + \phi^{ttt}\frac{\partial}{\partial u_{ttt}}$

$$\phi^t = D_t(\phi - \tau u_t) + \tau u_t t = (\phi_t + \phi_u u_t - \tau_t u_t - \tau_u u_t^2 - \tau u_{tt}) + \tau u_{tt} = \phi_t + (\phi_u - \tau_t)u_t - \tau_u u_t^2$$

$$\begin{aligned} \phi^{tt} &= D_t(\phi_t + \phi_u u_t - \tau_t u_t - \tau_u u_t^2 - \tau u_{tt}) + \tau u_{ttt} \\ &= (\phi_{tt} + (2\phi_{ut} - \tau_{tt})u_t + (\phi_u - 2\tau_t)u_{tt} + (\phi_{uu} - 2\tau_{tu})u_t^2 - \tau_{uu}u_t^3 - 3\tau_u u_t u_{tt} - \cancel{\tau u_{ttt}}) + \cancel{\tau u_{ttt}} \end{aligned}$$

$$\begin{aligned} \phi^{ttt} &= D_t(D_t(D_t(\phi - \tau u_t))) + \tau u_{tttt} \\ &= \phi_{ttt} + (3\phi_{ttu} - \tau_{ttt})u_t + (3\phi_{uut} - 3\tau_{ttu})u_t^2 + (3\phi_{ut} - 3\tau_{tt})u_{tt} + (3\phi_{uu} - 9\tau_{tu})u_t u_{tt} \\ &\quad + (\phi_u - 3\tau_t)u_{ttt} + (\phi_{uuu} - 3\tau_{tuu})u_t^3 - 6\tau_{uu}u_t^2 u_{tt} - 4\tau_u u_t u_{ttt} - \tau_{uuu}u_t^4 \end{aligned}$$

b)

$$\begin{aligned} \phi^{tx} &= D_x D_t(\phi - \tau u_t - \xi u_x) + \tau u_{txt} + \xi u_{txx} \\ &= D_x(\phi_t + \phi_u u_t - \tau_u u_t^2 - \tau u_{tt} - \xi_t u_x - \xi_u u_t u_x - \xi u_{xt}) + \tau u_{txt} + \xi u_{txx} \\ &= \phi_{tx} + (\phi_{tu} - \xi_{tx})u_x + (\phi_{ux} - \tau_{tx})u_t + (\phi_{uu} - \tau_{tu} - \xi_{ux})u_t u_x + (\phi_u - \tau_t - \xi_x)u_{tx} \\ &\quad - \tau_{ux}u_t^2 - \tau_{uu}u_x u_t^2 - 2\tau_u u_t u_{tx} - \tau_x u_{tt} - \tau_u u_x u_{tt} - \xi_{tu}u_x^2 - \xi_t u_{xx} - \xi_{uu}u_x^2 u_t - 2\xi_u u_{tx}u_x \end{aligned}$$

(25) Betrachte die eindimensionale Transformationsgruppe $\mathcal{T} = (T_\epsilon)_{\epsilon \in I_{(x,u)}}$ auf $J^0 = \mathbb{R}^d \times \mathbb{R}$. Sei $(\bar{x}, j_1\bar{u}) = (\bar{x}, \bar{u}, \bar{u}_{x_1}, \ldots, \bar{u}_{x_d}) \in J^1$ und $f(x) : U(\bar{x}) \mapsto \mathbb{R}$ mit $f(\bar{x}) = \bar{u}$, $\nabla f(\bar{x}) = (\bar{u}_{x_1}, \ldots, \bar{u}_{x_d})$. Schreibe $T_\epsilon(x,u) = (X_\epsilon(x,u), U_\epsilon(x,u))$. Dann geht für kleine ϵ die Funktion $f(x)$, $x \in V(\bar{x}) \subseteq U(\bar{x})$, unter T_ϵ über in die Funktion $f^\epsilon(x^\epsilon) = (U_\epsilon \circ [\mathrm{Id} \times f]) \circ (X_\epsilon \circ [\mathrm{Id} \times f])^{-1}(x^\epsilon)$. Nach Definition ist

$$j_1A(\bar{x}, j_1\bar{u}) = \frac{d}{d\,\epsilon}(\bar{x}^\epsilon, f^\epsilon(\bar{x}^\epsilon), \nabla f^\epsilon(\bar{x}^\epsilon))|_{\epsilon=0} = A(\bar{x}, \bar{u}) \times \frac{d}{d\,\epsilon}\nabla f^\epsilon(\bar{x}^\epsilon)|_{\epsilon=0}$$

und nach Anwendung von Ketten- und Produktregel (J bezeichne die Jacobi-Matrix)

$$\begin{aligned}
\frac{d}{d\epsilon}\nabla f^\epsilon(\bar{x}^\epsilon)|_{\epsilon=0} &= \frac{d}{d\epsilon}\left(J(U_\epsilon \circ [\mathrm{Id}\times f])(\bar{x}) \cdot (J(X_\epsilon \circ [\mathrm{Id}\times f]))^{-1}(\bar{x})\right)_{\epsilon=0} \\
&= \frac{d}{d\epsilon}\left(J(U_\epsilon \circ [\mathrm{Id}\times f])(\bar{x})\right)|_{\epsilon=0} \cdot (J(X_0 \circ [\mathrm{Id}\times f])^{-1}(\bar{x}) \\
&\quad + J(U_0 \circ [\mathrm{Id}\times f])(\bar{x}) \cdot \frac{d}{d\epsilon}\left(J(X_0 \circ [\mathrm{Id}\times f]))^{-1}(\bar{x})\right)|_{\epsilon=0} \\
&= \frac{d}{d\epsilon}\left(J(U_\epsilon \circ [\mathrm{Id}\times f])(\bar{x})\right)|_{\epsilon=0} - \nabla f(\bar{x}) \cdot \frac{d}{d\epsilon}\left(J(X_\epsilon \circ [\mathrm{Id}\times f])(\bar{x})\right)|_{\epsilon=0} \\
&= J(\frac{d}{d\epsilon}(U_\epsilon \circ [\mathrm{Id}\times f])(\bar{x})|_{\epsilon=0}) - \nabla f(\bar{x}) \cdot J(\frac{d}{d\epsilon}(X_\epsilon \circ [\mathrm{Id}\times f])(\bar{x})|_{\epsilon=0}) \\
&= \left(D_{x_i}\phi(\bar{x}, f(\bar{x})) - \sum_{l=1}^{d} u_{x_l} D_{x_i}\xi_l(\bar{x}, f(\bar{x}))\right)_{i=1,\dots,d},
\end{aligned}$$

wobei benutzt wurde, dass $J(X_0 \circ [\mathrm{Id}\times f]))^{-1}(\bar{x})$ die Einheitsmatrix ist und für jede invertierbare Matrix $M(\epsilon)$ gilt $\frac{d}{d\epsilon}M^{-1}(\epsilon)|_{\epsilon=0} = -M^{-1}(0)\cdot\frac{d}{d\epsilon}M(\epsilon)|_{\epsilon=0}M^{-1}(0)$. Ferner hat man

$$D_{x_i}\phi(\bar{x}, f(\bar{x})) - \sum_{l=1}^{d} u_{x_l} D_{x_i}\xi_l(\bar{x}, f(\bar{x})) = D_{x_i}(\phi(\bar{x}, f(\bar{x})) - \sum_{l=1}^{d}\xi_l(\bar{x}, f(\bar{x}))u_{x_l}) + \sum_{l=1}^{d}\xi_l(\bar{x}, f(\bar{x}))u_{x_i x_l}.$$

(26) Sei $x_0 \in \mathbb{R}^d$, $u_0 \in \mathbb{R}^l$ und $f = (f_1, \dots, f_d) : U(x_0) \mapsto \mathbb{R}^l$ mit $f(x_0) = u_0$, $(T_\epsilon)_{\epsilon\in I_{(x,f(x)}}$ eine einparametrische Transformationsgruppe auf $J^0 \approx \mathbb{R}^d \times \mathbb{R}^l$. Schreibe $T_\epsilon =: (X_\epsilon, U_\epsilon)$. Dann gilt

$$U_\epsilon(x, f(x)) = (U_\epsilon \circ [Id\times f])(x) \quad \text{und} \quad X_\epsilon(x, f(x)) = (X_\epsilon \circ [Id\times f])(x) \quad \text{für alle } x \in U(x_0).$$

Ferner ist $X_0(x, f(x)) = (X_0 \circ [Id\times f])(x) = x$ und daher $\det J(X_0(x, f(x))) = 1$. Auf Grund der stetigen Abhängigkeit von $\det J(X_\epsilon(x, f(x)))$ bezüglich ϵ existiert $\delta' > 0$, so dass $\det J(X_\epsilon(x, f(x))) \neq 0$ für alle $\delta \leq \delta'$. Nach dem Satz über inverse Funktionen ist $(X_\epsilon \circ [Id\times f])$, $\delta \leq \delta'$, invertierbar auf einer offenen Umgebung $V_\delta(x_0)$, und man erhält $f^\delta(x^\delta) = (U_\delta \circ [Id\times f]) \circ (X_\delta \circ [Id\times f])^{-1}(x^\delta)$.

(27) Sei $x_0 \in \mathbb{R}^d$, $u_0 \in \mathbb{R}^l$ und $f = (f_1, \dots, f_d) : U(x_0) \mapsto \mathbb{R}^l$ mit $f(x_0) = u_0$, $(T_\epsilon)_{\epsilon\in I_{(x,f(x)}}$ eine einparametrische Transformationsgruppe auf $J^0 \approx \mathbb{R}^d \times \mathbb{R}^l$.
Schreibe $T_\epsilon =: (X_\epsilon, U_\epsilon) = (X^1_\epsilon, \dots, X^d_\epsilon, U^1_\epsilon, \dots, U^l_\epsilon)$.
$(j_1 T_\epsilon)(x_0, j_1 f(x_0))$ besteht neben den Komponenten von x_0^ϵ und $f^\epsilon(x_0^\epsilon)$ aus den Einträgen $\frac{\partial f_i^\epsilon}{\partial x_j^\epsilon}(x_0^\epsilon)$, $j = 1, \dots, d$ und $i = 1, \dots, l$, d.h. mit Aufgabe 1 aus den Einträgen der Matrix

$$J((U_\delta \circ [\mathrm{Id}\times f]) \circ (X_\delta \circ [\mathrm{id}\times f])^{-1}(x^\delta)) = J(U_\epsilon(x_0, f(x_0))) \cdot J^{-1}(X_\epsilon(x_0, f(x_0))), \tag{B.1}$$

wobei die Komponenten der ersten Matrix auf der rechten Seite von der Form $\frac{\partial U_\epsilon^i}{\partial x_j}(x_0, f(x_0)) + \sum_{m=1}^{l}\frac{\partial U_\epsilon^i}{\partial f_m}(x_0, f(x_0))\frac{\partial f_m}{\partial x_j}(x_0)$ sind, also nur von x_0, $f(x_0)$ und den Ableitungen erster Ordnung von f an der Stelle x_0 abhängen, und genauso die Komponenten der zweiten Matrix.

Es hänge nun $j_k T_\epsilon$ nur von x_0, $f(x_0)$ und den Ableitungen der Ordnung $\leq k$ von f an der Stelle x_0 ab. Fasse $j_k f =: \bar{f}$ als Abbildung von $U(x_0)$ nach $\mathbb{R}^{ld_k}$ auf und gemäß Induktionsvoraussetzung $j_k T_\epsilon =: \bar{T}_\epsilon$ als Transformationsgruppe auf $\bar{J}^0 := J^k$. Dann hängt $j_1\bar{T}_\epsilon(x_0, j_1\bar{f}(x_0))$ nur von den Ableitungen ≤ 1 von $\bar{f}$ an der Stelle x_0 ab, also von den Ableitungen $\leq k+1$ von f an der Stelle x_0. Da sich jede Komponente von $(j_{k+1}T_\epsilon)(x_0, f(x_0))$ auch in $(j_1\bar{T}_\epsilon)(x_0, \bar{f}(x_0))$ wiederfindet, ist die Behauptung bewiesen.

(28) Aus (1) folgt sofort

$$(j_1 T_\theta)(x,u,u_x) = (x\cos\theta - u\sin\theta, x\sin\theta + u\cos\theta, (\sin\theta + u_x\cos\theta)(\cos\theta - u_x\sin\theta)^{-1})),$$

also $I = (-|\operatorname{arccot} u_x|, |\operatorname{arccot} u_x|)$, und nach Ableiten bezüglich θ an der Stelle $\theta = 0$

$$(j_1 A)(x,u,u_x) = (-u, x, 1+u_x^2).$$

(29) Sei $\mathbf{v} = \tau\frac{\partial}{\partial y} + \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial u}$ ein Vektorfeld auf J^0. $\mathbf{v}$ erzeugt genau dann eine Symmetriegruppe, wenn $(j_2(\mathbf{v})(u_{xx}+u_{yy}) = \phi^{yy} + \phi^{xx} = 0$ gilt, wann immer $u_{yy} = -u_{xx}$. Man erhält

1. $(1):\quad \phi_{xx} = -\phi_{yy},$
2. $(u_x):\quad 2\phi_{xu} - \xi_{xx} = \xi_{yy},$
3. $(u_y):\quad -\tau_{xx} = -2\phi_{yu} + \tau_{yy},$
4. $(u_x^1):\quad \phi_{uu} - 2\xi_{xu} = 0,$
5. $(u_y^2):\quad 0 = -\phi_{uu} + 2\tau_{yu},$
6. $(u_x u_y):\quad -2\tau_{ux} = 2\xi_{uy},$
7. $(u_x^3):\quad -\xi_{uu} = 0,$
8. $(u_y^3):\quad 0 = \tau_{uu},$
9. $(u_x^2 u_y):\quad -\tau_{uu} = 0,$
10. $(u_x u_y^2):\quad \xi_{uu} = 0,$
11. $(u_{xx}):\quad \phi_u - 2\xi_x = \phi_u - 2\tau_y,$
12. $(u_{xy}):\quad -2\tau_x = 2\xi_y,$
13. $(u_x u_{xx}):\quad -3\xi_u = -\xi_u,$
14. $(u_y u_{xx}):\quad -\tau_u = -3\tau_u,$
15. $(u_x u_{yx}):\quad -2\tau_u = 0,$
16. $(u_y u_{xy}):\quad -2\xi_u = 0.$

Aus 15. oder 14. bzw. 16. oder 13. folgt, dass $\tau(y,x)$ bzw. $\xi(y,x)$ nicht von u abhängen.
6., 7., 8., 9. und 10. geben keine weiteren Informationen.
4. und 5. werden zu $\phi_{uu} = 0$, was bedeutet $\phi = b(y,x)u + a(y,x)$.
11. ergibt $\xi_x = \tau_y$ und 12. $-\tau_x = \xi_y$, woraus folgt $\xi xx = \tau_{yx} = -\xi_{yy}$ und $\tau_{yy} = \xi_{xy} = -\tau_{xx}$, d.h. ξ und τ müssen die Laplace-Gleichung erfüllen.
2. verwandelt sich damit in $0 = \phi_{ux} = b_y$ und 3. in $0 = \phi_{uy} = b_y$, weshalb b eine Konstante ist.
Aus 1. erhält man noch, dass $a(y,x)$ eine beliebige Lösung der Laplace-Gleichung ist.

Z.B. liefert die Wahl τ bzw. ξ konstant Translationen, die Wahl $\tau = y$ und $\xi = x$ Streckungen und die Wahl $\tau = -x$ und $\xi = y$ Drehungen, während die Vektorfelder $(u + a(y,x))\partial_u$ die Linearität der Laplace-Gleichung widerspiegeln.

(30)

$$\begin{aligned}
\phi^{xx} &= D_x D_x(\phi - \tau u_t - \xi u_x) + \tau u_{xxt} + \xi u_{xxx} \\
&= D_x\big(\phi_x + \phi_u u_x - (\tau_x + \tau_u u_x)u_t - \tau u_{tx} - (\xi_x + \xi_u u_x)u_x - \xi u_{xx}\big) + \tau u_{xxt} + \xi u_{xxx} \\
&= \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\phi_u - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_x u_t - \xi_{uu}u_x^3 - \tau_{uu}u_x^2 u_t \\
&\quad + (\phi_u - 2\xi_x)u_{xx} - 2\tau_x u_{xt} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{tx}.
\end{aligned}$$

(31) $\Phi_{x_i} = \frac{-x_i}{2t}\frac{1}{(4\pi t)^{\frac{d}{2}}}e^{\frac{-\|x\|^2}{4t}}, \qquad \Phi_{x_i x_i} = \left(\frac{x_i^2}{4t^2} - \frac{1}{2t}\right)\frac{1}{(4\pi t)^{\frac{d}{2}}}e^{\frac{-\|x\|^2}{4t}}$

$\Delta\Phi = \left(\frac{\|x\|^2}{4t^2} - \frac{d}{2t}\right)\frac{1}{(4\pi t)^{\frac{d}{2}}}e^{\frac{-\|x\|^2}{4t}} = \Phi_t.$

(32) Seien u und w zwei Lösungen der Wärmeleitungsgleichung zu gegebenen Rand- und Anfangsbedingungen. Dann ist $v := u - w$ eine Lösung mit Anfangswert $v(0,\cdot) \equiv 0$ und Randbedingung $v(t,y) = 0$ bzw. $\frac{\partial v}{\partial n}(t,y) = 0$ für alle $t \geq 0$ und $y \in \partial\Omega$.

Definiere $E(t) := \frac{1}{2}\int_\Omega (\nabla v(t,x))^2\,dx$. Man erhält

$$\begin{aligned}\frac{d}{dt}E(t) &= \int_\Omega \nabla v(t,x)\cdot\nabla v_t(t,x)\,dx = -\int_\Omega (\Delta v(t,x))v_t(t,x)\,dx + \int_{\partial\Omega}\frac{\partial v(t,y)}{\partial n}v_t(t,y)\,d\sigma(y)\\ &= -\int_\Omega v_t^2(t,x)\,dx \le 0.\end{aligned}$$

Es folgt $0 \le E(t) \le E(0) = 0$, was bedeutet, dass $v(t,\cdot)$ konstant in x und wegen $v_t = \Delta v$ auch konstant in t ist, weshalb gilt $v(\cdot,\cdot) \equiv$ Konstante. Aus $v(0,\cdot) \equiv 0$ ergibt sich die Behauptung.

(33)

$$\begin{aligned}\mathbf{v}(\mathbf{w}f) - \mathbf{w}(\mathbf{v}f) &= \sum_{j=1}^d\sum_{i=1}^d\left(\xi_j\frac{\partial\eta_i}{\partial x_j}\frac{\partial f}{\partial x_i} + \xi_j\eta_i\frac{\partial^2 f}{\partial x_i\partial x_j}\right) - \sum_{j=1}^d\sum_{i=1}^d\left(\eta_j\frac{\partial\xi_i}{\partial x_j}\frac{\partial f}{\partial x_i} + \eta_j\xi_i\frac{\partial^2 f}{\partial x_i\partial x_j}\right)\\ &= \sum_{i=1}^d\left(\sum_{j=1}^d \xi_j\frac{\partial\eta_i}{\partial x_j} - \eta_j\frac{\partial\xi_i}{\partial x_j}\right)\frac{\partial f}{\partial x_i}.\end{aligned}$$

(34) (a) $\|x+y\|^2 = \langle x+y\,,\,x+y\rangle = \langle x\,,\,x\rangle + \langle x\,,\,y\rangle + \langle y\,,\,x\rangle + \langle y\,,\,y\rangle = \|x\|^2 + \|y\|^2$.

(b) $\|x+y\|^2 + \|x-y\|^2 = 2\langle x\,,\,x\rangle + 2\langle y\,,\,y\rangle = 2\|x\|^2 + 2\|y\|^2$.

(c) $\|x+y\|^2 - \|x-y\|^2 = 2\langle x\,,\,y\rangle + 2\langle y\,,\,x\rangle = 4\langle x\,,\,y\rangle + \langle y\,,\,x\rangle$ falls $\mathbb{K} = \mathbb{R}$.

$\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 = 2\langle x\,,\,y\rangle + 2\langle y\,,\,x\rangle + 2i\langle x\,,\,iy\rangle + 2i\langle iy\,,\,x\rangle$
$= 2\langle x\,,\,y\rangle + 2\langle y\,,\,x\rangle + 2\langle x\,,\,y\rangle - 2\langle y\,,\,x\rangle = 4\langle x\,,\,y\rangle$ falls $\mathbb{K} = \mathbb{C}$.

(d) $(A^\perp)^\perp = \{y \in H : \langle x\,,\,y\rangle = 0\ \forall x \in A^\perp\} \supseteq A$.

(e) $x \in H^\perp \quad\Rightarrow\quad \langle x\,,\,x\rangle = 0 \quad\Rightarrow\quad x = 0$.

(35) Einfaches Berechnen der jeweiligen Stammfunktion liefert:

$\int_0^1 1\cdot\cos(2\pi nx)\,dx = 0;\qquad \int_0^1 \cos(2\pi nx)sin(2\pi mx)\,dx = 0,\quad n,m \ge 1;$
$\int_0^1 \cos(2\pi nx)\cos(2\pi mx)\,dx = 0 = \int_0^1 \sin(2\pi nx)\sin(2\pi mx)\,dx,\quad n,m \ge 1,\ n \ne m;$
$\int_0^1 \sin(2\pi nx)\sin(2\pi nx)\,dx = \frac{1}{2} = \int_0^1 \cos(2\pi nx)\cos(2\pi nx)\,dx,\quad n \ge 1;$
$\int_0^1 e^{2n\pi x}e^{-2m\pi x}\,dx = \int_0^1 e^{2(n-m)\pi x}\,dx$, was 0 für $n \ne m$ und 1 für $n = m$ ergibt, $n,m \in \mathbb{Z}$.

Sei nun f orthogonal zu jedem $e_n \in \{e^{2n\pi i\cdot} : n \in \mathbb{Z}\}$, dann ist wegen $0 = \left(\int_0^1 f(x)e^{-2n\pi x}\,dx\right)^- = \int_0^1 \bar{f}(x)e^{2n\pi x}\,dx$ auch die Konjugierte $\bar{f}$ orthogonal zu jedem e_n, also insbesondere auch $\frac{1}{2}(f+\bar{f}) = \mathrm{Re}(f) =: g \in L^2(0,1;\mathbb{R})$ und $\frac{1}{2i}(f-\bar{f}) = \mathrm{Im}(f) =: h \in L^2(0,1;\mathbb{R})$. Es folgt

$$0 = \int_0^1 g(x)e^{-2n\pi x}\,dx = \int_0^1 g(x)\cos(-2n\pi x)\,dx + i\int_0^1 g(x)\sin(-2n\pi x)\,dx,\quad n \in \mathbb{Z}$$

und insbesondere $\int_0^1 g(x)\cos(2n\pi x)\,dx = 0$ und $\int_0^1 g(x)\sin(2n\pi x)\,dx = 0$ für alle $n \ge 0$. Da $\{1, \sqrt{2}\cos(2\pi n\cdot), \sqrt{2}$
$n,m = 1,2,3,\ldots\}$ eine Basis von $L^2(0,1;\mathbb{R})$ ist, muss $g = 0$ sein und ebenso $h = 0$, also $f = 0$, woraus sich die Behauptung ergibt.

(36) Es existiert genau dann eine Lösung der Form $v(t)w(x)$, wenn $v_t(t)w(x) - v^\gamma(t)\Delta w^\gamma(x) = 0$. D.h. es muss gelten

$$\frac{v_t(t)}{v^\gamma(t)} = \mu = \frac{\Delta w^\gamma(x)}{w(x)}\quad \text{für alle } t,x \text{ mit } v(t), w(x) \ne 0.$$

Für v erhält man die gewöhnliche Differentialgleichung $v_t(t) = \mu v^\gamma(t)$ mit Lösung $v(t) = \big((1-\gamma)\mu t + \lambda\big)^{\frac{1}{1-\gamma}}$.
Für w erhält man die Gleichung $\mu w(x) = \Delta w^\gamma(x)$ und mit dem Ansatz $w(x) = |x|^\alpha$ ($|\cdot|$ euklidische Norm)

$$\mu|x|^\alpha = \Delta|x|^{\alpha\gamma} = \gamma\alpha\big(d+\gamma\alpha-2\big)|x|^{\alpha\gamma-2},$$

woraus folgt $\alpha = \frac{2}{\gamma-1} > 0$ und $\mu = \alpha\gamma(d+\gamma\alpha-2) > 0$. Man wähle also $\lambda > 0$. Als Lösung der Porösen-Medien-Gleichung ergibt sich $u(t,x) = \big((1-\gamma)\mu t+\lambda\big)^{\frac{1}{1-\gamma}}|x|^{\frac{2}{\gamma-1}}$, d.h. für $x \neq 0$ strebt die Lösung gegen ∞ wenn t gegen $\frac{\lambda}{(\gamma-1)\mu}$ geht.

(37) Alle Eigenwerte sind echt größer Null wegen

$$\begin{aligned}\lambda\int_0^l f(x)\bar f(x)\,dx = -\int_0^l f''(x)\bar f(x)\,dx &= \int_0^l f'(x)\bar f'(x)\,dx - f'(l)\bar f(l) + f'(0)\bar f(0)\\ &= \int_0^l f'(x)\bar f'(x)\,dx + b_l f(l)\bar f(l) + b_0 f(0)\bar f(0) > 0 \quad \text{für } f\neq 0,\end{aligned}$$

und zwei Eigenvektoren zu verschiedenen Eigenwerten stehen orthogonal aufeinander wegen

$$\begin{aligned}\lambda\int_0^l f(x)\bar g(x)\,dx = \int_0^l -f''(x)\bar g(x)\,dx &= \int_0^l f'(x)\bar g'(x)\,dx + b_l f(l)\bar g(l) + b_0 f(0)\bar g(0)\\ &= -\int_0^l f(x)\bar g''(x)\,dx = \mu\int f(x)\bar g(x)\,dx.\end{aligned}$$

Es folgt $(\lambda-\mu)\int_0^l f(x)\bar g(x)\,dx = 0$ und damit auch $\int_0^l f(x)\bar g(x)\,dx = 0$.

(38) Wenn $\sum_{n\in\mathbb{N}} x_n$ konvergiert, gilt $\|\sum_{i=1}^N x_i\|^2 \leq M$ für alle $N\geq 1$ und $M \geq 0$ konstant. Also $\langle\sum_{i=1}^N x_i, \sum_{i=1}^N x_i\rangle = \sum_{i=1}^N \|x_i\|^2 \leq M$ für alle $N \geq 1$. Umgekehrt erkennt man aus voranstehender Gleichung, dass $(\sum_{i=1}^N x_i)_{N\in\mathbb{N}}$ eine Cauchyfolge in H ist, falls $(\sum_{i=1}^N \|x_i\|^2)_{N\in\mathbb{N}}$ eine Cauchyfolge in $\mathbb{K}$ ist.

(39) Man zeigt zuerst $j_k[\mathbf{v},\mathbf{w}] = [j_k\mathbf{v}, j_k\mathbf{w}]$: Für $k=1$ läßt sich die Aussage (auch im Falle mehrerer abhängiger Variablen) direkt nachrechnen (sehr langwierig) und für $k>1$ kann man per Induktion schließen, in dem man die Komponenten der Form u_J, $|J|\geq 1$, von J^{k-1} als neue, abhängige Variablen auffäßt, dann die Behauptung für $k=1$ anwendet und schließlich die auf $J^1(J^{k-1})$ lebenden Vektorfelder $j_1(j_{k-1}(\mathbf{v}))$ usw. auf den Unterraum $J^k \subset J^1(J^{k-1})$ einschränkt.

Damit erhält man $(j_k[\mathbf{v},\mathbf{w}])H(\cdot) = (j_k\mathbf{v})(j_k\mathbf{w}H(\cdot)) - (j_k\mathbf{w})(j_k\mathbf{v}H(\cdot))$. Nun sind $(j_k\mathbf{w})H =: H_w$ und $(j_k\mathbf{v})H =: H_v$ konstant Null auf der Menge $\mathcal{H} := \{y\in J^k : H(y) = 0\}$. Da $\mathbf{v}$ und $\mathbf{w}$ Symmetriegruppen von H erzeugen, verläuft der durch $j_k\mathbf{v}$ bzw. $j_k\mathbf{w}$ gegebene Fluß ganz in $\mathcal{H}$, falls der Startwert in $\mathcal{H}$ liegt, weshalb $(j_k\mathbf{v})H_w(y) = 0$ und $(j_k\mathbf{w})H_v(y) = 0$, wann immer $y\in\mathcal{H}$. Also $(j_k[\mathbf{v},\mathbf{w}])H(y) = 0$ für alle y mit $H(y) = 0$.

(40) $[A_i, A_i] = 0$ und $[A_i, A_j] = -[A_j, A_i]$.
$[A_1,A_2] = 2\partial_t = 2A_1$, $\quad [A_1,A_3] = 8t\partial_t + 4x\partial_x - 2u\partial_u = 4A_2 - 2A_6$, $\quad [A_1,A_4] = 0$,
$[A_1,A_5] = 2\partial_x = 2A_4$, $\quad [A_1,A_6] = 0$,
$[A_2,A_3] = 8t^2\partial_t + 8xt\partial_x - 2(2t+x^2)u\partial_u = 2A_3$, $\quad [A_2,A_4] = -\partial_x = -A_4$,
$[A_2,A_5] = 2t\partial_x - xu\partial_u = A_5$, $\quad [A_2,A_6] = 0$,
$[A_3,A_4] - 4t\partial_x + 2xu\partial_u = -2A_5$, $\quad [A_3,A_5] = 0$, $\quad [A_3,A_6] = 0$,

$[A_4, A_5] = -u\partial_u = -A_6, \quad [A_4, A_6] = 0, \quad [A_5, A_6] = 0.$

(41) Falls $H(x, j_k u) =: H(y) \neq 0$, existiert eine Umgebung U_y von y mit $H(\cdot) \neq 0$ auf U_y und $(j_k A)H(\cdot) = Q_y(\cdot)H(\cdot)$ auf U_y mit $Q_y(\cdot) = (j_k A)H(\cdot)/H(\cdot)$.
Falls $H(y) = 0$, existiert eine Umgebung U_y und ein lokaler Koordinatenwechsel $\mathbf{z}$, so dass $\mathbf{z}(y) = 0$ und $\tilde{H}(z) = H(\mathbf{z}^{-1}(z)) = z_1$ für alle $z \in \mathbf{z}(U_y)$. In diesen Koordinaten geht $(j_k A)H(y')$ über in $(j_k A)\tilde{H}(z') = q_1(z')\frac{\partial \tilde{H}}{\partial z_1} + q_2(z')\frac{\partial \tilde{H}}{\partial z_2} + \cdots = q_1(z')$, $z' = \mathbf{z}(y')$, was nach Voraussetzung gleich Null ist für alle $z' \in \mathbf{z}(U_y)$ mit $z_1' = 0$, weshalb die Funktion $q_1(z')/z_1' = (j_k A)\tilde{H}(z')/\tilde{H}(z')$ auf $\mathbf{z}(U_y)$ (q_1 ist stetig differenzierbar) wohldefiniert ist. Setze $\tilde{Q}_y = (j_k A)\tilde{H}(z')/\tilde{H}(z')$, d.h.$(j_k A)H(y') = Q_y(y')H(y')$ für alle $y' \in U_y$ mit $Q_y(y') = \tilde{Q}_y(\mathbf{z}(y'))$.
Wähle nun eine der offenen Überdeckung $(Q_y)_{y \in J^k}$ von J^k untergeordnete Zerlegung der Eins $(f_y)_{y \in J^k}$. Dann ist $(j_k A)H(y') = \sum_{y \in J^k} f_y(y')(j_k A)H(y') = \sum_{y \in J^k} f_y(y')Q_y(y')H(y') = Q(y')H(y')$ für alle $y' \in J^k$ mit $Q(y') := \sum_{y \in J^k} f_y(y')Q_y(y')$.

(42) $iii) \Rightarrow i), ii)$: Sei $y \in A_2$. Dann ist $P_1 y = P_1 P_2 y = P_2 P_1 y \in A_2$. $ii)$ folgt auf gleiche Weise.
$i) \Rightarrow ii)$: Sei $x \in A_1$, also $x = P_1 x$. Es gilt:

$$\begin{aligned} 0 &\leq \|P_1 P_2 x - P_2 x\|^2 = \langle P_1 P_2 x - P_2 x, P_1 P_2 x - P_2 x \rangle \\ &= \langle P_1 P_2 x - P_2 P_1 x, P_1 x - P_2 P_1 x \rangle + \langle P_1 P_2 x - P_2 x, P_1 P_2 x - P_1 x \rangle \leq 0. \end{aligned}$$

Also $P_2 x = P_1 P_2 x \in A_1$. Und ebenso $ii) \Rightarrow i)$.
$i), ii) \Rightarrow iii)$ ist falsch. Gegenbeispiel: Sei $A_1 := \{(x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1\}$ und $A_2 := A_1 \cup \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 1 \leq y \leq 2 - |x|\}$, $z := (1, 2)$. Dann sind $i)$ und $ii)$ erfüllt, aber $P_2 P_1 z = (1, 1)$ und $P_1 P_2 z = (1/2, 1)$.

(43) (a) Jede Konvexkombination von Elementen f und g aus A_1 bzw. A_2 ist mindestens an den Stellen gerade bzw. positv, an denen sowohl f als auch g gerade bzw. positiv sind, also fast überall. Eine Cauchyfolge $(f_n)_{n \in \mathbb{N}}$ in $L^2(\mathbb{R})$ konvergiert fast überall punktweise gegen eine Funktion $f \in L^2(\mathbb{R})$. Sei nun $f_n \in A_1$ bzw. $f_n \in A_2$ für alle $n \in \mathbb{N}$. Sei N_f die Ausnahmemenge, auf der f_n nicht punktweise konvergiert, und seien N_n, $n \in \mathbb{N}$, die Mengen, auf denen f_n nicht gerade bzw. nicht positiv ist, dann hat $N := N_f \cup (\cup_{n \in \mathbb{N}} N_n)$ Maß Null und die Grenzfunktion f ist zumindest außerhalb von n gerade bzw. positiv.
(b) Sei $f \in L^2(\mathbb{R})$ und $g \in A_1$, o.B.d.A. sei g überall gerade. Dann ist $\int_{\mathbb{R}} (g(x) - f(x))^2 \, dx = \int_0^\infty (g(x) - f(x))^2 + (g(x) - f(-x))^2 \, dx$ und der Ausdruck $(g(x) - f(x))^2 + (g(x) - f(-x))^2$ wird punktweise durch $g = P_{A_1} f(x)$ minimiert.
Sei $f \in L^2(\mathbb{R})$ und $g \in A_2$, o.B.d.A. sei g überall positiv. $(g(x) - f(x))^2$ wird punktweise durch $g(x) = P_{A_2} f(x)$ minimiert.

(44) (b) Definiere $y = P_y x$ und $z = x - y$. Sei $u \in Y$ beliebig. Dann gilt $Re\langle z, u \rangle = Re\langle x - y, (y + u) - y \rangle \leq 0$ und $Re\langle z, u \rangle = Re\langle x - y, y - (y - u) \rangle \geq 0$, also ist $Re\langle z, u \rangle = 0$ und ebenso $Im\langle z, u \rangle = Re\langle z, iu \rangle = 0$, weshalb $z \in Y^\perp$. Weiter gilt für alle $w \in Y^\perp$: $Re\langle x - z, w - z \rangle = Re(\langle y, w \rangle + \langle y - 0, y - x \rangle) \leq 0$, also $z = P_{Y^\perp} x$.
Aus $x = y' + z'$ mit $y' \in Y$ und $z' \in Y^\perp$ folgt $0 = \|y + z - y' - z'\|^2 = \|y - y'\|^2 + \|z - z'\|^2$, also Eindeutigkeit.
(a) Sei $0 \neq x \in H$ beliebig und $0 \neq a \in Y$. Dann ist $\frac{\|P_Y x\|}{\|x\|} = \frac{\|P_Y x\|}{\|P_Y x + P_{Y^\perp} x\|} \leq 1$ und $\frac{\|P_Y a\|}{\|a\|} = 1$, also $\|P_Y\| = 1$. Ferner folgt aus (b) sofort: $P_Y x = 0 \Leftrightarrow x \in Y^\perp$.

(45) Wir suchen eine Lösung der Gestalt $v(t)w(x)$. Nach Einsetzen in die Wärmeleitungsgleichung erhält man die Bedingung $\frac{v_t(t)}{v(t)} = -\mu = \frac{w_{xx}(x)}{w(x)}$, was zu den gewöhnlichen Dfgl. $v_t(t) = -\mu v(t)$ mit allgemeiner

Lösung $v(t) = \alpha_\mu e^{-\mu t}$ und $w_{xx} = -\mu w(x)$ mit Randbedingungen $w(0) = 0$ und $-w(l) = w'(l)$ führt. Nach Aufgabe 4 von Blatt 10 muss $\mu > 0$ sein ($\mu = 0$ liefert die triviale Lösung). D.h. die allgemeine reelle Lösung ist $w(x) = A\sin(\sqrt{\mu}x) + B\cos(\sqrt{\mu}x)$. Aus $w(0) = 0$ folgt $B = 0$ und aus $-w(l) = w'(l)$ folgt $-\sin(\sqrt{\mu}l) = \sqrt{\mu}\cos(\sqrt{\mu}l)$, also $-\sqrt{\mu} = \tan(\sqrt{\mu}l)$. Letztere Gleichung hat genau eine Lösung $\sqrt{\mu_k}$ in jedem Intervall $[k(\frac{\pi}{2l}), (k+2)(\frac{\pi}{2l})]$ mit $k \geq 1$ ungerade. D.h. man erhält die Reihendarstellung $\sum_k \alpha_{\mu_k} e^{-\mu_k t}\sin(\sqrt{\mu}x)$.

(46) a) Seien $a, b \in A$ und $\lambda, \mu \in \mathbb{K}$. Sei P_A linear. Dann ist $\lambda a + \mu b = \lambda P_A a + \mu P_A b = P_A(\lambda a + \mu b) \in A$.
Sei A ein Unterraum, $\lambda c = b \in A$ beliebig, $\lambda \in \mathbb{K}$, und $x, y \in H$. Dann ist

$$Re\langle \lambda x - \lambda P_A x, b - \lambda P_A x\rangle = |\lambda|^2 Re\langle x - P_A x, c - P_A x\rangle \leq 0,$$

also $P_A(\lambda x) = \lambda P_A x$; und

$$Re\langle x + y - P_A x - P_A y, b - P_A x - P_A y\rangle = Re\langle x - P_A x, (b - P_A y) - P_A x\rangle + Re\langle y - P_y, (b - P_A x) - P_y\rangle \leq 0,$$

also $P_A(x + y) = P_A x + P_A y$.

b)
$$\begin{aligned}\|P_A x - P_A y\|^2 &= Re\langle P_A x - P_A y, x - y\rangle + Re\langle P_A x - P_A y, P_A x - x\rangle + Re\langle P_A x - P_A y, y - P_A y\rangle \\ &\leq Re\langle P_A x - P_A y, x - y\rangle \leq \|P_A x - P_A y\|\|x - y\|\end{aligned}$$

(47) Sein $g_1, g_2 \in L^2(I)$ und $\int_I g_1(x)h(x)\,dx = \int_I f(x)h'(x)\,dx = \int_I g_2(x)h(x)\,dx$, also $\int_I (g_1 - g_2)(x)h(x)\,dx = 0$ für alle $h \in C_c^1(I)$. Sei $(h_n)_{n\in\mathbb{N}}$ eine Folge in $C_c^1(I)$, die gegen $g_1 - g_2$ konvergiert. Dann gilt $0 = \lim_{n\to\infty}\langle g_1 - g_2, h_n\rangle_{L^2} = \langle g_1 - g_2, g_1 - g_2\rangle_{L^2}$, also $g_1 = g_2$.

(48) Betrachte die Folge $(f_n)_{n\geq 2}$ in $C^0(0,1)$ mit

$$f_n(x) = \begin{cases} 0 & \text{für } 0 < x \leq 1/2 - 1/n, \\ n/2(x - 1/2 + 1/n) & \text{für } 1/2 - 1/n \leq x \leq 1/2 + 1/n, \\ 1 & \text{für } 1/2 + 1/n \leq x < 1. \end{cases}$$

(f_n) konvergiert in $L^2(0,1)$ gegen die Funktion $f \notin C^0(0,1)$ mit $f(x) = 0$ für $0 < x \leq 1/2$ und $f(x) = 1$ für $1/2 < x < 1$.

Betrachte die Folge $(g_n)_{n\in\mathbb{N}}$ in $C^1(0,1)$ mit $g_n(x) = |x - 1/2|^{1+1/n}$. (g_n) konvergiert bezüglich der H^1-Norm gegen $g \notin C^1(0,1)$ mit $g(x) = |x - 1/2|$.

(49) a) Sei $\psi \in C_c^1(I)$ fest mit $\int_a^b \psi(x)\,dx = 1$. Sei $w \in C_c^1(I)$ beliebig. Definiere $v(x) = \int_a^x w(z) - \psi(z)\left(\int_a^b w(y)\,dy\right)dz$. Dann ist wegen $\int_a^b \psi(x)\,dx = 1$ auch $v(x) \in C_c^1(I)$; und $v'(x) = w(x) - \psi(x)\int_a^b w(y)\,dy$. Man erhält mit dem Satz von Fubini

$$\begin{aligned} 0 = \int_a^b f(x)v'(x)\,dx &= \int_a^b f(y)w(y)\,dy - \int_a^b f(x)\psi(x)\left(\int_a^b w(y)\,dy\right)dx \\ &= \int_a^b \left(f(y) - \int_a^b f(x)\psi(x)\,dx\right) w(y)\,dy. \end{aligned}$$

Da $w \in C_c^1(I)$ beliebig war, folgt $f(y) = \int_a^b f(x)\psi(x)\,dx$ konstant fast überall.

b) Aus $g \in L^2(I)$ folgt $G \in C(\bar{I})$. Definiere die Indikatorfunktion $\chi(x,t) := \begin{cases} 1 & \text{für } x \geq t, \\ 0 & \text{für } x < t \end{cases}$ Dann gilt mit dem Satz von Fubini

$$\int_a^b G(x)h'(x)\,dx = \int_a^b \left(\int_a^x g(t)\,dt\right) h'(x)\,dx = \int_a^b \int_a^b \chi(x,t)g(t)h'(x)\,dt\,dx$$
$$= \int_a^b g(t)\left(\int_t^b h'(x)\,dx\right)\,dt = -\int_a^b g(t)h(t)\,dt.$$

c) Sei $f \in H^1$. Definiere $F := \int_a^x f'(x)\,dx \in C(\bar{I})$. Dann ist $\int_a^b f(x)h'(x)\,dx = -\int_a^b f'(x)h(x)\,dx = \int F(x)h'(x)\,dx$, also $\int_a^b (F-f)(x)h'(x)\,dx = 0$ für alle $h \in C_c^1(I)$. Mit a) folgt dann $f - F = c$ f.ü., $c \in \mathbb{R}$, also $f = F + c \in C(\bar{I})$ fast überall.
Sei nun $f \in C([0,1]) \cap H^1(0,1)$. Es ist $\|f\|_{C([0,1])} = \max_{x\in[0,1]} |f(x)| =: M$. Betrachte auf $H^1(0,1)$ die äquivalente Norm $\|f\| := \sqrt{\int_0^1 f^2(x)\,dx} + \sqrt{\int_0^1 (f'(x))^2\,dx} \le \sqrt{2}\|f\|_{H^1}$. Sei $y \in [0,1]$ mit $|f(y)| = M$ und o.B.d.A. $f(y) = M$. Sei $m := \max\{\min_{x\in[0,1]} f(x), 0\}$ und $z \in [0,1]$ mit $f(z) = m$. Dann folgt mit der Hölder-Ungleichung

$$\|f\| = \sqrt{\int_0^1 f^2(x)\,dx}\sqrt{\int_0^1 1^2\,dx} + \sqrt{\int_0^1 (f'(x))^2\,dx}\sqrt{\int_0^1 1^2\,dx} \ge \int_0^1 |f'(x)|\,dx + \int_0^2 |f(x)|\,dx$$
$$\ge |\int_z^y f'(x)\,dx| + m = M - m + m = M.$$

Also $\|f\|_{C([0,1])} \le \|f\| \le \sqrt{2}\|f\|_{H^1}$.

(50) a) T beschränkt, d.h $\|Tx\|_{H_2}/\|x\|_{H_1} \le M < \infty$, $\forall x \ne 0$. $\Rightarrow$ $\|Tx_1 - Tx_2\|_{H_2} = \|T(x_1 - x_2)\|_{H_2} \le M\|x_1 - x_2\|_{H_1}$ $\Rightarrow$ Lipschitz-stetig $\Rightarrow$ stetig, d.h. $\exists \delta > 0 : \|Tx - T\vec{0}\|_{H_2} \le 1$, $\forall x : \|x\|_{H_1} \le \delta$ $\Rightarrow$ $\|Tx\|_{H_2} \le 1/\delta$, $\forall x : \|x\|_{H_1} \le 1$.
b) Der Abstand in der L^2-Norm zu einer Funktion $f \in L^2(\Omega)$ wird minimiert durch diejenige konstante Funktion, für die

$$g(c) := \int_\Omega (f(x) - c)^2\,dx = \int_\Omega f^2(x)\,dx - 2c\int_\Omega f(x)\,dx + |\Omega|c^2$$

minimal ist. Es folgt $P_{konst.}f = 1/|\Omega| \int_\Omega f(x)\,dx$.

(51) ($\eta \in C_c^1(0,\infty) \cap H^1(0,\infty)$) Sei $\hat{\eta}$ die Einschränkung von η auf $(0,1)$. Mit $\phi \in C_c^1(0,1)$ ist auch $\hat{\eta}\phi \in C_c^1(0,1)$. Es folgt

$$-\int_0^1 u'(x)(\hat{\eta}(x)\phi(x))\,dx = \int_0^1 u(x)\hat{\eta}'(x)\phi(x)\,dx + \int_0^1 u(x)\hat{\eta}(x)\phi'(x)\,dx$$

für alle $\phi \in C_c^1(0,1)$ und damit $(u\hat{\eta})' = u'\hat{\eta} + u\hat{\eta}'$ und $u\hat{\eta} \in H^1(0,1)$.
Sei nun $\phi \in C_c^1(0,\infty)$ und $\tilde{\phi} \in C_c^1(0,1)$ mit $\hat{\phi}(x) = \phi(x)$ für $x \le 1-\epsilon$ und $\hat{\phi}(x) = 0$ für $x \ge 1 - \epsilon/2$ (dazwischen z.B. durch ein Polynom vom Grad ≥ 3 interpoliert). Dann ist $\int_0^\infty \tilde{u}(x)\eta(x)\phi'(x)\,dx = \int_0^1 u(x)\hat{\eta}(x)\hat{\phi}(x)\,dx = -\int_0^1 (u'(x)\hat{\eta}(x) + u(x)\hat{\eta}'(x))\hat{\phi}(x)\,dx = -\int_0^\infty (\tilde{u}'(x)\eta(x) + u(x)\eta'(x)\,dx$.

(52) (u^+ und $\nabla u^+ \in L^2(\Omega)$ ist klar). Da $f_n \in \mathbb{C}^1(\mathbb{R})$, $f_n(0) = 0$ und $f_n'(r) = r(r^2 + n^{-2})^{-1/2} \le 1$ gilt, ist mit $u \in H^1(\Omega)$ auch $f_n \circ u \in H^1(\Omega)$ (Kettenregel). Es folgt

$$-\int_\Omega f_n(u(x))\frac{\partial\phi}{\partial x_j}(x)\,dx = \int_\Omega \chi_{u>0}u(x)(u^2(x) + n^{-2})^{-1/2}\frac{\partial u}{\partial x_j}(x)\phi(x)\,dx$$

und mit dem Satz von Lebesgue nach Grenzübergang auf beiden Seiten

$$-\int_\Omega u(x)\frac{\partial\phi}{\partial x_j}(x)\,dx = \int_\Omega \chi_{u>0}\frac{\partial u}{\partial x_j}(x)\phi(x)\,dx \quad \text{für alle } \phi\in C_c^1(\Omega).$$

(53) Es gilt für alle $\phi\in C_c^1(\Omega)_+$:

$$0\geq -\int_\Omega u(x)\Delta\phi(x)\,dx = \int_\Omega \nabla u(x)\nabla\phi(x)\,dx = \int_\Omega \nabla(u-c)(x)\nabla\phi(x)\,dx.$$

Da $C_c^1(\Omega)_+$ dicht in $H_0^1(\Omega)_+$ liegt, gilt obige Gleichung auch für $(u-c)^+\in H_0^1(\Omega)_+$, d.h.

$$0\geq \int_\Omega \nabla(u-c)(x)\nabla(u-c)^+(x)\,dx = \int_\Omega \nabla(u-c)^+(x)\nabla(u-c)^+(x)\,dx,$$

was bedeutet $\nabla(u-c)^+=0$ f.ü., woraus folgt $(u-c)^+=0$ f.ü., also $u\leq c$ fast überall.

(54) (a) Es gilt $|f(r)|\leq f(0)+Mr$. Es folgt

$$\int_\Omega (f\circ u)^2(x)\,dx \leq \int_\Omega |f(0)+Mu(x)|^2dx \leq 2\int_\Omega f^2(0)\,dx + 2M^2\int_\Omega u^2(x)\,dx<\infty,$$

also $(f\circ u)\in L^2(\Omega)$; und

$$\int_\Omega (f'(u(x))\frac{\partial u(x)}{\partial x_j}dx \leq M^2\int_\Omega (\frac{\partial u(x)}{\partial x_j}dx<\infty, \qquad j=1,\dots,d.$$

Sei nun $u_n\in C_c^1(\Omega)$ eine Folge wie im Lemma. Dann erhält man

$$\int_\Omega (f\circ u_n)\frac{\partial\phi(x)}{dx_j}dx = -\int_\Omega f'(u_n(x))\frac{\partial u_n(x)}{\partial x_j}\phi(x)\,dx \quad \text{für alle } \phi\in C_c^1(\Omega),\ j=1,\dots,d,$$

und nach Grenzübergang auf beiden Seiten mit dem Satz von Lebesgue die Behauptung.

(b) Es gilt

$$\int_\Omega (uv)^2(x)\,dx \leq \|v\|_\infty^2\int_\Omega u^2(x)\,dx<\infty \quad \text{und}$$

$$\int_\Omega (u\frac{\partial v}{\partial x_j}+\frac{\partial u}{\partial x_j})^2(x)\,dx \leq 2\|v\|_\infty^2\int_\Omega u^2(x)\,dx + 2\|v\|_\infty^2\int_\Omega (u'(x))^2\,dx<\infty, \quad j=1,\dots,d.$$

Sei nun $u_n\in C_c^1(\Omega)$ eine Folge wie im Lemma. Dann erhält man

$$-\int_\Omega \frac{\partial v(x)}{\partial x_j}u_n(x)\phi\,dx = \int_\Omega v(x)\frac{\partial(u_n\phi)}{\partial x_j}(x)\,dx = \int_\Omega v(x)u_n(x)\frac{\partial\phi(x)}{\partial x_j}dx + \int_\Omega v(x)\frac{\partial u_n(x)}{\partial x_j}\phi(x)\,dx,$$

also $\int_\Omega (vu_n)(x)\frac{\partial\phi(x)}{\partial x_j}dx = -\int_\Omega\left(\frac{\partial v(x)}{\partial x_j}u_n(x)+v(x)\frac{\partial u_n(x)}{\partial x_j}\right)\phi(x)\,dx$ und nach Grenzübergang auf beiden Seiten mit dem Satz von Lebesgue die Behauptung.

(55) (a) Sei $Rea(Pu|u-Pu)\geq 0$. Dann gilt wegen der Koerzivität von $a(\cdot|\cdot)$

$0\leq Rea(Pu-u|u-Pu)+Rea(u|u-Pu) = -Rea(u-Pu|u-Pu)+Rea(u|u-Pu)\leq Rea(u|u-Pu)$.

Sei nun $Rea(u|u-Pu)\geq 0$ für alle $u\in H$. Dann folgt auf Grund von $P(Pu+\lambda(u-Pu))=Pu$ für alle $\lambda>0$

$0\leq \frac{1}{\lambda}Rea(Pu+\lambda(u-Pu)|Pu+\lambda(u-Pu)-Pu) = Rea(Pu+\lambda(u-Pu)|(u-Pu))$

und nach Grenzübergang $\lambda\to 0$ die Behauptung, da $a(\cdot|\cdot)$ stetig ist.

(b) Da $a(\cdot|\cdot)$ eine stetige, symmetrische und koerzive Sesquilinearform ist, gilt für $a(\cdot|\cdot)$ natürlich die Cauchy-Schwarz-Ungleichung.
Sei $a(Pu|Pu) \leq a(u,u)$. Dann gilt

$$a(u|u-Pu) = a(u|u) - a(u|Pu) \geq a(u|u) - a(u|u)^{1/2}a(Pu|Pu)^{1/2} \geq 0.$$

Sei $a(u|u-Pu) \geq 0$, Dann gilt mit Teil (a)

$$a(Pu|Pu) = a(Pu-u|Pu) + a(u|Pu) \leq a(u|Pu) \leq a(u|u)^{1/2}a(Pu|Pu)^{1/2}.$$

Also lässt sich in diesem Fall zu Proposition 7.60 aus der Vorlesung noch die Äquivalenz $PV \subseteq V$ und $a(Pu|Pu) \leq a(u|u)$ für alle $u \in V$ hinzufügen.

(c) Da $a(\cdot|\cdot)$ stetig auf H_0^1, koerziv und symmetrisch ist und $P_+H_0^1 \subset H_0^1$, folgt die Behauptung mit Teil (b) und Proposition 7.60 aus $\int_\Omega (\nabla(P_+u)(x))^2\,dx = \int_\Omega \chi_{u>0}(\nabla u(x))^2\,dx \leq \int_\Omega (\nabla u(x))^2\,dx$.

(56) (a) Es gilt für $t \geq 0$

$$\begin{aligned} b\left(u(t+h)|u(t+h)\right) - b\left(u(t)|u(t)\right) &= b\left(u(t+h)-u(t), u(t+h)\right) \\ &\quad + b\left(u(t)|u(t+h)\right) - b\left(u(t)|u(t)\right) \\ &= b\left(u(t+h)-u(t)|u(t+h)\right) + b\left(u(t)|u(t+h)-u(t)\right), \end{aligned}$$

woraus folgt

$$\begin{aligned} \frac{d}{dt}b(u(t),u(t)) &= \lim_{h\to 0} b\left(\frac{u(t+h)-u(t)}{h}, u(t+h)\right) \\ &\quad + b\left(u(t), \frac{u(t+h)-u(t)}{h}\right) = 2b\left(\frac{\partial u}{\partial t}(t), u(t)\right), \qquad t \geq 0. \end{aligned}$$

(b) Definiere die Energie einer Lösung des gegebenen Anfangswertproblems durch

$$E(t) := a(w(t)|w(t)) + \|w_t(t)\|_H^2 \geq 0.$$

Dann ist

$$\begin{aligned} \frac{dE}{dt}(t) &= 2a(w_t(t)|w(t)) + 2\langle w_{tt}(t), w_t(t)\rangle_H = 2\langle -Aw(t), w_t(t)\rangle_H + 2 + 2\langle w_{tt}(t), w_t(t)\rangle_H \\ &= 2\langle -w_{tt}(t), w_t(t)\rangle_H + 2\langle w_{tt}(t), w_t(t)\rangle_H = 0, \end{aligned}$$

d.h. die Energie ist konstant für jede Lösung. Existieren nun zwei Lösungen u und v, dann löst deren Differenz das Anfangswertproblem mit $(u-v)(0) = 0$ und $(u-v)_t(0) = 0$, d.h. $E(u-v) \equiv 0$. Daraus folgt $(u-v)_t(t) \equiv 0$, also $(u-v)(t) \equiv 0$.
Das obige Resultat liefert Eindeutigkeit der Lösung der Wellengleichung mit Neumann- oder Dirichlet-Ranbedingungen, weil die zugehörige Bilinearform $a : H^1(\Omega) \times H^1(\Omega) \mapsto L^2(\Omega)$ bzw. $a : H_0^1(\Omega) \times H_0^1(\Omega) \mapsto L^2(\Omega)$, gegeben durch $a(u|v) = \int_\Omega \nabla u(x)\nabla v(x)\,dx$, die Bedingungen an $a(\cdot|\cdot)$ erfüllt.

(57) (a) $L(x,u,\nabla u) = e^{\phi(x)}\left(\frac{1}{2}\nabla^2 u(x) - f(x)u(x)\right)$

(b) $L(x,u,u_x,u_t) = e^{-t/\epsilon}\left(\frac{1}{\epsilon}u_x^2 + u_t^2\right)$.

(58) (a) Euler-Lagrange-Gleichung: $2u_{xx}(x,y,z) + 2u_{yy}(x,y,z) - 2u_{zz}(x,y,z) + f(u(x,y,z)) = 0$ oder $u_{xx}(x,y,z) + u_{yy}(x,y,z) - u_{zz}(x,y,z) + \frac{1}{2}f(u(x,y,z)) = 0$.

(b) Sei $u_0 = u_1 \equiv 0$. Dann ist die eindeutige Lösung der Wellengleichung gegeben durch $u \equiv 0$ und $I(u) = 0$. Wähle z.B. $w(x,t) = t^2$. Dann erfüllt w Rand- und Anfangsbedingungen, aber $I(w) = \int_\Omega (u_x^2(t,x) - u_t^2(t,x))\, d(t,x) = -\infty$.

(59) Euler-Lagrange-Gleichung: $0 = \sum_{i=1}^d p(u_{x_i}|\nabla u|^{p-2})_{x_i} = p\nabla(|\nabla u|^{p-2}\nabla u)$.

Variationelle Symmetrie: Mit $\omega^\epsilon = \{y = xe^\epsilon : x \in \omega\}$ folgt

$$\int_\omega |\nabla(e^{\epsilon\frac{n-p}{p}} u(e^\epsilon x))|^p \, dx = \int_\omega e^{\epsilon(n-p)} |\nabla u(e^\epsilon x)|^p \, dx = \int_{\omega^\epsilon} |\nabla u(y)|^p \, dy.$$

Divergenzgleichung: Mit $\phi(x, j_1 u) = \frac{n-p}{p} u(x) + \nabla u(x) \cdot x$ und $\xi_k = x_k$ folgt

$$\begin{aligned}
0 &= \sum_{i=1}^d \left(\left(\frac{n-p}{p} u(x) + \nabla u(x) \cdot x \right) p u_{x_i} |\nabla u(x)|^{p-2} - x_i |\nabla u(x)|^p \right)_{x_i} \\
&= \cancel{((n-p)u + p\nabla u \cdot x) \sum_{i=1}^d (u_{x_i}|\nabla u|^{p-2})_{x_i}} + \sum_{i=1}^d \Big(\big((n-p)u_{x_i}^2 + p u_{x_i}^2 + p \sum_{j=1}^d u_{x_j x_i} x_j u_{x_i}\big) |\nabla u|^{p-2} \Big) \\
&\quad - \sum_{i=1}^d \big(|\nabla u|^p + p |\nabla u|^{p-2} \sum_{j=1}^d u_{x_j x_i} u_{x_j} x_i \big) = 0.
\end{aligned}$$

(60) Das Vektorfeld $-y\partial_x + x\partial_y$ erzeugt die Tranformationsgruppe $(x\sin\theta + y\cos\theta, x\cos\theta + y\sin\theta, t, u)$. Da die Determinante der Variablentransformation $(x,y,t) \to (x\sin\theta + y\cos\theta, x\cos\theta + y\sin\theta, t)$ betragsmäßig gleich 1 ist, folgt

$$\begin{aligned}
&\int_\omega \big(u_t^2(-x\sin\theta + y\cos\theta, x\cos\theta + y\sin\theta, t) - u_x^2(-x\sin\theta + y\cos\theta, x\cos\theta + y\sin\theta, t) \\
&\qquad - u_y^2(-x\sin\theta + y\cos\theta, x\cos\theta + y\sin\theta, t)\big) d(x,y,t) \\
&= \int_{\omega^\theta} \big(u_t^2(x,y,t) - (-u_x(x,y,t)\sin\theta + u_y(x,y,t)\cos\theta)^2 - (u_x(x,y,t)\cos\theta + u_y(x,y,t)\sin\theta)^2 \big) d(x,y,t) \\
&= \int_{\omega^\theta} \big(u_t^2(x,y,t) - u_x^2(x,y,t) - u_y^2(x,y,t) \big) d(x,y,t),
\end{aligned}$$

d.h. es liegt eine variationelle Symmetrie vor.

Das Vektorfeld $x\partial_x + y\partial_y + t\partial_t$ erzeugt die Transformationsgruppe $(e^\epsilon x, e^\epsilon y, e^\epsilon t, u)$ und die entsprechende Variablentransformation $(x,y,t) \to (e^\epsilon x, e^\epsilon y, e^\epsilon t)$ hat Determinante $e^{3\epsilon}$. Es folgt

$$\begin{aligned}
&\int_\omega \big(u_t^2(e^\epsilon x, e^\epsilon y, e^\epsilon t) - u_x^2(e^\epsilon x, e^\epsilon y, e^\epsilon t) - u_y^2(e^\epsilon x, e^\epsilon y, e^\epsilon t) \big) d(x,y,t) \\
&= \int_{\omega^\epsilon} \big(e^{-3\epsilon} e^{2\epsilon} (u_t^2(x,y,t) - u_x^2(x,y,t) - u_y^2(x,y,t)) \big) d(x,y,t),
\end{aligned}$$

d.h. es liegt keine variationelle Symmetrie vor.