~~39~~

Lösungen - Blatt 11

39.) Def. $f(x,y,z) := \begin{pmatrix} y \\ \cos x \\ \sin x \end{pmatrix}$. Dann ist

$f: E \to Z$ glatt (da glatt : $\mathbb{R}^3 \to \mathbb{R}^3$, vgl. A.28)

Sei $p = (p_1, p_2, 0) \in E$, $v = (v_1, v_2, 0) \in T_pE = E$.

Dann für $c(t) := p + tv$ gilt

$$df(p)(v) = \frac{d}{dt}\Big|_{t=0} (f \circ c)(t) = \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} p_2 + tv_2 \\ \cos(p_1 + tv_1) \\ \sin(p_1 + tv_1) \end{pmatrix}$$

$$= \begin{pmatrix} v_2 \\ -\sin p_1 \; v_1 \\ \cos p_1 \; v_1 \end{pmatrix}.$$

In lok. Koordinaten ist gemäß Aufg. 31 $(g^Z_{ij})_{i,j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

und $(g^E_{ij})_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, also $\forall \; \underset{\begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}}{\overset{}{v}} , \underset{\begin{pmatrix} w_1 \\ w_2 \\ 0 \end{pmatrix}}{w} \in T_pE$ gilt

$$g^Z_{f(p)}\left(df(p)v, df(p)w\right) = \begin{pmatrix} v_2 \\ -\sin p_1 \; v \\ \cos p_1 \; v_1 \end{pmatrix}^T \cdot \begin{pmatrix} w_2 \\ -\sin p_1 \; w_1 \\ \cos p_1 \; w_1 \end{pmatrix}$$

$$= v_2 \cdot w_2 + v_1 \cdot w_1 = \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}^T \begin{pmatrix} w_1 \\ w_2 \\ 0 \end{pmatrix}$$

$$= g^E_p(v, w). \Rightarrow f \text{ lok. Isometrie} \qquad \square$$

40.)

$$g_{ij}(p) = \left\langle \frac{\partial F}{\partial u^i}(F^{-1}(p)), \frac{\partial F}{\partial u^j}(F^{-1}(p)) \right\rangle$$

$$\tilde{g}_{ij}(f(p)) = \tilde{g}_{f(p)}\left(\frac{\partial (f \circ F)}{\partial u^i}(F^{-1}(p)) \frac{\partial (f \circ F)}{\partial u^j}(F^{-1}(p)) \right)$$

$$= \tilde{g}_{f(p)}\left(\frac{d}{dt}\Big|_{t=0} (f \circ c_1)(t), \frac{d}{dt}\Big|_{t=0} (f \circ c_2)(t) \right)$$

mit $c_1(t) = F(F^{-1}(p) + t e_i)$, $c_2(t) = F(F^{-1}(p) + t e_j)$

$$= \tilde{g}_{f(p)}\left(df(p)\left(\frac{\partial F}{\partial u^i}(F^{-1}(p))\right), df(p)\left(\frac{\partial F}{\partial u^j}(F^{-1}(p))\right)\right)$$

$$\overset{\text{Isom.}}{=} g_p\left(\frac{\partial F}{\partial u^i}(F^{-1}(p)), \frac{\partial F}{\partial u^j}(F^{-1}(p)) \right) = g_{ij}(p).$$

41.) i) $\frac{D}{dt}(x(t)+y(t)) = \pi_{c(t)}(\dot{x}(t)+\dot{y}(t))$

$$= \pi_{c(t)}\,\dot{x}(t) + \pi_{c(t)}\,\dot{y}(t)$$

$$= \frac{D}{dt}x(t) + \frac{D}{dt}y(t).$$

ii) $\frac{D}{dt}(f(t)\,x(t)) = \pi_{c(t)}(\dot{f}(t)\,x(t) + f(t)\,\dot{x}(t))$

$$= \dot{f}(t)\,\underbrace{\pi_{c(t)}\,x(t)}_{=x(t) \text{ da } \in T_{c(t)}S} + f(t)\,\pi_{c(t)}\,\dot{x}(t)$$

$$= \dot{f}(t)\,x(t) + f(t)\,\frac{D}{dt}x(t).$$

iii) $g_p\left(\frac{\partial F}{\partial u^i}(u), \frac{\partial F}{\partial u^j}(u)\right) \overset{u=F^{-1}(p).}{=} g_{ij}(p) = \left\langle \frac{\partial F}{\partial u^i}(u), \frac{\partial F}{\partial u^j}(u)\right\rangle$

Bilin $\Rightarrow$ $g_p(x,y) = \langle x, y\rangle \ \forall p \in S,\ x,y \in T_pS.$

Damit $\frac{d}{dt}\,g_{c(t)}(x(t), y(t)) = \frac{d}{dt}\langle x(t), y(t)\rangle$

$$= \langle \dot{x}(t), y(t)\rangle + \langle x(t), \dot{y}(t)\rangle$$

$$= \left\langle \frac{D}{dt}x(t), y(t)\right\rangle + \left\langle x(t), \frac{D}{dt}y(t)\right\rangle. \quad \blacksquare$$