Dr. James Kennedy
Institut für Analysis
Universität Ulm

Nonlinear Partial Differential Equations

Summer Semester 2016

Contents

Preface

Notation

$\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, \ldots\}$ is the natural numbers without and with 0, respectively;

$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ the set of integers, rational numbers, real numbers, complex numbers;

$\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$;

$x = (x_1, \ldots, x_n)$ is a point in $\mathbb{R}^n$, $n \geq 1$. If $n = 2$, we also write $(x, y) \in \mathbb{R}^2$;

$\Omega \subset \mathbb{R}^n$, $n \geq 1$, is always an open set;

$u, f : \Omega \to \mathbb{K}$ (or $\mathbb{K}^m$, $m \geq 1$) will denote arbitrary functions which, if nothing further is said, are smooth enough that all expressions in which they appear (such as derivatives) are well defined;

$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}$ is the (positive) half space in $\mathbb{R}^n$;

$\operatorname{int} A$, $\overline{A}$ denote the interior and the closure of a set A, respectively;

$A \subset\subset B$ (A, B sets) if A is contained in a compact subset of B;

$B(x, r)$ is the open ball centred at x of radius r (either in $\mathbb{R}^n$ or a general metric space depending on the context);

V generally denotes a Banach space, V' its dual, and H a Hilbert space. We write $(\,\cdot\,,\,\cdot\,) = (\,\cdot\,,\,\cdot\,)_H$ for the inner product on H and $\langle\,\cdot\,,\,\cdot\,\rangle = \langle\,\cdot\,,\,\cdot\,\rangle_{V',V}$ for the dual pairing between V and V'.

1 Introduction and classification of PDEs

1.1 Notation and basic definitions

1.1.1 Definition. The vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ is called a *multiindex* of order

$$|\alpha| := \sum_{j=1}^{n} \alpha_j.$$

If $x \in \mathbb{K}^n$, then we will use the notation x^α for the product $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \in \mathbb{K}$. For a function $u : \Omega \to \mathbb{K}$ and an arbitrary multiindex α, we denote the αth partial derivative of u by

$$D^\alpha u := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots x_n^{\alpha_n}} u = u_{\underbrace{x_1 \ldots x_1}_{\alpha_1} \cdots \underbrace{x_n \ldots x_n}_{\alpha_n}}.$$

If $\alpha = 0$, then $D^0 u := u$.

For $k \in \mathbb{N}_0$, $D^k u$ is shorthand for the vector consisting of all partial derivatives of u of order k:[1]

$$D^1 u \equiv Du = (u_{x_1}, \ldots, u_{x_n}),$$
$$D^2 u = (u_{x_1 x_1}, u_{x_1 x_2}, \ldots, u_{x_1 x_n}, \ldots, u_{x_n x_n}),$$

etc.. In case of $D^2 u$, we also use the same notation for the Hessian matrix, as it should be clear from the context whether we mean a matrix or simply the collection of derivatives.

1.1.2 Definition. A *partial differential equation* (of *order* $m \in \mathbb{N}_0$), abbreviated PDE, is a relation of the form

$$F(x, u(x), Du(x), \ldots, D^m u(x)) = f(x), \qquad x \in \Omega, \tag{1.1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a given, usually open, set, $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \ldots \times \mathbb{R}^{n^m} \to \mathbb{K}$ and $f : \Omega \to \mathbb{K}$ are given functions, and $u : \Omega \to \mathbb{K}$ is the unknown.

Typically we impose auxiliary condition(s) on u, either *boundary conditions* at the boundary $\partial\Omega$ of the set Ω where the PDE is to be solved, or conditions at infinity (if, e.g., $\Omega = \mathbb{R}^n$ or $\mathbb{R}^n_+$). We wil use the term PDE both for an equation of the form (1.1.1) and for such an equation together with given auxiliary condition(s).

An expression of the form $F(x, u, \ldots, D^m u)$ will be called a *(partial) differential expression*, or, loosely, a *(partial) differential operator*, PDO. We will often denote such an expression by

$$Lu := F(x, u, \ldots, D^m u).$$

Several PDEs in one or more unknowns constitute a *system* (which we will not focus on in this course).

[1] In opposition to the lectures, in these notes we will *only* use the notation Du for the gradient ∇u of u.

1.1.3 Example. The *Laplacian* of a function $u \in C^2(\Omega)$ is the PDO of order two given by[2]

$$\Delta u := \operatorname{div}(Du) = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2};$$

if $f \in C(\Omega)$ is a given function, then

$$\begin{cases} -\Delta u(x) = f(x) & \text{if } x \in \Omega, \\ u(x) = 0 & \text{if } x \in \partial\Omega, \end{cases}$$

is a PDE.

There is no general theory for handling – or even classifying – all PDEs, and given the rich variety of behaviour which they can exhibit (and phenomena which they model), it seems unlikely that such a theory should every exist. For most PDEs, it is not even generally possible to find an explicit formula for the solutions; instead, most theory is devoted to proving existence and some basic properties of solutions (number of solutions, positivity, symmetry, boundedness/stability in some sense), usually for a particular equation or type of equation.

In this context, we talk about the *well-posedness* of the equation we wish to consider: the PDE $Lu = f$ (plus auxiliary condition(s)) is, loosely speaking, considered to be *well posed* if its solution(s) satisfy the following three properties:

1. Existence (for each f in an appropriate space there actually exists a solution);
2. Uniqueness;
3. Continuous dependence on the data: a "small" change in f should lead to a correspondingly small change in the solution u.

In practice we have to be careful about how we define and understand "solutions": in many cases existence results are obtained from properties of L considered as an operator between well-chosen spaces of functions.

1.2 Some important classes of PDEs

There is no universal classification scheme for all PDEs; nevertheless, there are several types of equations which are important in practice, either because they appear in many applications, or because they are particularly amenable to mathematical analysis (or hopefully both).

1.2.1 Definition. (a) A PDE is called *linear* if it is linear in the unknown function(s) and derivatives, with coefficients depending only on the independent variable(s) x: a general linear PDE of order m has the form

$$\sum_{|\alpha|=0}^{m} a_\alpha(x) D^\alpha u(x) = f(x), \qquad x \in \Omega, \tag{1.2.1}$$

for given functions $a_\alpha, f : \Omega \to \mathbb{R}$; it is called *homogeneous* if $f \equiv 0$.

[2]Note the sign! In some places Δu stands for $-\sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}$.

(b) A PDE is called *semilinear* if the coefficients a_α of the highest order derivatives, i.e., when $|\alpha| = m$, depend only on x:

$$\sum_{|\alpha|=m} a_\alpha(x) D^\alpha u(x) + a_0(x, u(x), Du(x), \dots, D^{m-1}u(x)) = f(x), \qquad x \in \Omega.$$

(c) A PDE is called *quasilinear* if the highest order derivatives appear linearly with respect to each other, but their coefficients may depend on (lower order) derivatives of u:

$$\sum_{|\alpha|=m} a_\alpha(x, u(x), \dots, D^{m-1}u(x)) D^\alpha u(x) + a_0(x, u(x), Du(x), \dots, D^{m-1}u(x)) = f(x),\ x \in \Omega.$$

(d) A PDE is *nonlinear* if it is not linear and *fully* nonlinear if it depends nonlinearly on the highest order derivatives.

1.2.2 Definition. A (linear) PDE is in *divergence form* if it has the form

$$\sum_{\substack{|\alpha|+|\beta|\leq m \\ |\beta|\leq m-1}} D^\alpha(a_{\alpha\beta}(x) D^\beta u(x)) = f(x), \qquad x \in \Omega$$

(i.e., the coefficients of the highest order derivatives appear inside the corresponding differential expressions) and in non-divergence form otherwise.

1.2.3 Remark. (a) The same definitions hold for systems (suitably adjusted).

(b) Linear $\implies$ semilinear $\implies$ quasilinear.

(c) If we have a linear PDE $\sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha u(x) = f(x)$, then $L := \sum_{|\alpha|\leq m} a_\alpha D^\alpha$ genuinely defines a *linear operator*, for example from the space $C^\infty(\Omega)$ to itself.

(d) A general rule of thumb is: an equation is harder to solve/study if it has more independent variables (n), if it is of higher order (m), or, most importantly, if it is "more" nonlinear.

1.2.4 Example. Here are three prototypical examples, which you have probably seen before:

(a) Laplace's equation

$$-\Delta u = 0$$

and the inhomogeneous variant, Poisson's equation

$$-\Delta u = f. \tag{1.2.2}$$

(b) The heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0,$$

with initial condition $u(0, x) = u_0(x)$, where now $u : \mathbb{R}_+ \times \Omega \to \mathbb{R}$, $(t, x) \mapsto u(t, x)$, and where $\Delta = \Delta_x$ denotes the Laplacian with respect to the x-variables.

(c) The wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$$

with $u(0, x) = u_0(x)$, where $u : \mathbb{R} \times \Omega \to \mathbb{R}$ and $\Delta = \Delta_x$ again.

In each case, to obtain a well-posed equation, we need a boundary condition. The most common types for equations such as (1.2.2) are:

$$\begin{aligned} u &= 0 && \text{on } \partial\Omega && \text{(Dirichlet/1st kind)} \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial\Omega && \text{(Neumann/2nd kind)} \\ \frac{\partial u}{\partial \nu} + \alpha u &= 0 && \text{on } \partial\Omega && \text{(Robin/3rd kind)} \end{aligned}$$

where ν is the outer unit normal to Ω (assuming $\partial\Omega$ is sufficiently smooth), and α is some function on $\partial\Omega$ in the Robin condition.

The equations (a), (b) and (c) are examples of *elliptic*, *parabolic* and *hyperbolic* equations, respectively; each type has its own body of theory.

1.3 The Cauchy problem, characteristics and symbols

To motivate where these names come from, and how we expect solutions in each case to be different, we will investigate the question of well-posedness of general two-dimensional quasilinear equations of second order. Writing $(x, y) \in \mathbb{R}^2$, these have the form

$$a u_{xx} + b u_{xy} + c u_{yy} = d, \tag{1.3.1}$$

where a, b, c, d are in general given functions of x, y, u, u_x, u_y and for the meantime we are considering the problem on the whole of $\mathbb{R}^2$. We are also given a (sufficiently smooth) curve $\gamma \subset \mathbb{R}^2$ and conditions that u should satisfy on γ (called *Cauchy data*):

$$u|_\gamma = f, \qquad u_x|_\gamma = g, \qquad u_y|_\gamma = h. \tag{1.3.2}$$

The (classical) *Cauchy problem* consists in determining u from the equations (1.3.1) and (1.3.2). To this end we need to determine conditions on $a, b, c, d, \gamma, f, g, h$ in order to ensure the existence and uniqueness of a (smooth) solution u (i.e. well-posedness).

Compatibility conditions.

Suppose γ is given parametrically by

$$x = \gamma_1(s), \qquad y = \gamma_2(s), \qquad s \in I \subset \mathbb{R}.$$

Then *any* differentiable function $v = v(x, y) = v(\gamma_1(s), \gamma_2(s))$ must satisfy

$$\frac{dv}{ds} = v_x \cdot \gamma_1'(s) + v_y \cdot \gamma_2'(s). \tag{1.3.3}$$

If in addition v satisfies (1.3.2), then the Cauchy data must satisfy the compatibility condition

$$f'(s) = g(s)\gamma_1'(s) + h(s)\gamma_2'(s).$$

Hence we can only choose *two* of f, g, h, i.e. u, u_x, u_y on γ arbitrarily. Alternatively, we could specify u and its normal (or tangent) derivative on γ:

$$u|_\gamma = f, \qquad \frac{-u_x \cdot g' + u_y \cdot f}{\sqrt{f^2+g^2}} = \tilde{f}.$$

Choosing $v = u_x$ and u_y in (1.3.3), we obtain

$$(g'(s) =)\, \frac{d}{ds} u_x = u_{xx} \cdot \gamma_1' + u_{xy} \cdot \gamma_2'$$

and

$$(h'(s) =)\, \frac{d}{ds} u_y = u_{xy} \cdot \gamma_1' + u_{yy} \cdot \gamma_2',$$

respectively. Suppose now that $u(x, y)$ in solves (1.3.1) and (1.3.2). Then on γ we must have

$$\begin{array}{ccccccc} au_{xx} & + & bu_{xy} & + & cu_{yy} & = & d \\ \gamma_1' u_{xx} & + & \gamma_2' u_{xy} & & & = & g' \\ & & \gamma_1' u_{xy} & + & \gamma_2' u_{yy} & = & h' \end{array} \tag{1.3.4}$$

The given data determine u_{xx}, u_{xy}, u_{yy} uniquely along γ *unless*

$$\triangle := \det \begin{pmatrix} a & b & c \\ \gamma_1' & \gamma_2' & 0 \\ 0 & \gamma_1' & \gamma_2' \end{pmatrix} = a(\gamma_2')^2 - b(\gamma_1'\gamma_2') + c(\gamma_1')^2 = 0. \tag{1.3.5}$$

1.3.1 Definition. The curve γ is *characteristic* (for the given PDE and data, and even the solution u if a, b, c depend on u and its derivatives) if $\triangle = 0$ along γ, and *non-characteristic* otherwise.

1.3.2 Remark. Along a non-characteristic curve we can also successively find all higher order derivatives of u (assuming they exist). For example, supposing a, b, c to be constant, if we differentiate (1.3.1) with respect to x and then repeat the above analysis, we obtain three linear equations, namely for $u_{xxx}, u_{xxy}, u_{xyy}$, which satisfy the same (uniquely soluble) system of equations as (1.3.4). Using (1.3.3) with $v = u_{xx}$ and $v = u_{xy}$ to obtain u_{yyy}. Proceeding iteratively, we can obtain all derivatives of u in a given point $(x_0, y_0) \in \gamma$ and so give a *formal* power series expansion for u in terms of powers of $x - x_0$ and $y - y_0$. That the solution u is in fact *a priori* analytic if $a, b, c, d, \gamma, f, g, h$ are analytic, so that this power series converges, is the statement of the *Theorem of Cauchy–Kovalevskaia*, which however we shall not treat in this course.

Along a characteristic curve the system (1.3.4) is inconsistent unless the data satisfy additional identities; hence the Cauchy problem (1.3.1) and (1.3.2) generally has no solution (i.e. it is ill posed).

When can this happen? Suppose γ is given by a level curve $\phi(x, y) = \textit{const}$. Then (1.3.5) says

$$a(\phi_x)^2 + b\phi_x\phi_y + c(\phi_y)^2 = 0.$$

Solving for x as a function of y, we have

$$a(dx)^2 + b(dx)(dy) + c(dy)^2 = 0$$

$$\frac{dx}{dy} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The existence of characteristic curves depends on the sign of the discriminant $b^2 - 4ac$.

1.3.3 Definition. (a) If $b^2 - 4ac < 0$, the equation is called *elliptic*, since then

$$ax^2 + bxy + cy^2 = d \tag{1.3.6}$$

describes an ellipse. In this case there are no characteristics.

(b) If $b^2 - 4ac = 0$, the equation is *parabolic*, since (1.3.6) describes a parabola. There is one family of characteristics, namely $y = -\frac{2a}{b}x + C$.

(c) If $b^2 - 4ac > 0$, the equation is *hyperbolic*, as (1.3.6) is an hyperbola, and there are two families of characteristics.

1.3.4 Remark. This classification is valid for all two-dimensional quasilinear second order PDEs. In general, since a, b, c, d may be functions of x, y, u etc., the type may depend on the point (x, y) and, in the nonlinear case, even on the solution u.

General dimensions and orders.

Now let $x \in \mathbb{R}^n$ and suppose

$$L := \sum_{|\alpha| \le m} a_\alpha D^\alpha$$

is a general quasilinear operator of order m.

1.3.5 Definition. (a) The *(total) symbol* of L (at a function u) is the polynomial (in ξ)

$$P(x, u, \xi) = \sum_{|\alpha| \le m} a_\alpha(x, u, \ldots)\xi^\alpha, \qquad \xi \in \mathbb{R}^n.$$

Note however that, if L is nonlinear, so that the coefficients a_α themselves may depend on u and its derivatives, then this may not be independent of the particular way of writing L.

(b) The *principal* (or *leading*) *symbol* of L is the (homogeneous in ξ) polynomial

$$p(x, u, \xi) = \sum_{|\alpha| = m} a_\alpha(x, u, \ldots, D^{m-1}u)\xi^\alpha, \qquad \xi \in \mathbb{R}^n.$$

In order to suppress the dependence on u, we introduce dummy variables $z_1 \in \mathbb{R}, \ldots, z_{m-1} \in \mathbb{R}^{n^{m-1}}$ (i.e. z_1 stands for u, z_2 for Du, $\ldots$, z_{m-1} for $D^{m-1}u$) and write

$$p(x, z_1, \ldots, z_{m-1}, \xi) = \sum_{|\alpha| = m} a_\alpha(x, z_1, \ldots, z_{m-1})\xi^\alpha.$$

1.3.6 Remark. Sometimes the symbol is defined as $\sum_{|\alpha| \le m} a_\alpha(\ldots)(i\xi)^\alpha$; this is also known as the Fourier symbol. For, if the a_α are constants, then for the Fourier transform

$$\hat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \int u(x) e^{-i\xi x} \,\mathrm{d}x,$$

we have

$$\widehat{Lu}(\xi) = P(\xi)\hat{u}(\xi).$$

1.3.7 Example. For the three operators from Example 1.2.4, we have:

Operator	Symbol	Principal symbol	
Δ	$\sum_{k=1}^n \xi_k^2 = \lvert\xi\rvert^2$	$\lvert\xi\rvert^2$	$\xi \in \mathbb{R}^n$
$\frac{\partial}{\partial t} - \Delta$	$\tau - \lvert\xi\rvert^2$	$-\lvert\xi\rvert^2$	$(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^{n-1}$
$\frac{\partial^2}{\partial t^2} - \Delta$	$\tau^2 - \lvert\xi\rvert^2$	$\tau^2 - \lvert\xi\rvert^2$	$(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^{n-1}$

Observe again that if $n = 2$, then the equation symbol $= k$ ($k \geq 0$, say) yields, respectively, an ellipse, a parabola, and an hyperbola.

1.3.8 Definition. Suppose L is quasilinear of order $m = 2$, i.e.[3]

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x, u, Du) \frac{\partial^2}{\partial x_i \partial x_j} + a_0(x, u, Du), \tag{1.3.7}$$

with principal symbol

$$p(x, z, w, \xi) = \sum_{i,j=1}^{n} a_{ij}(x, z, w) \xi_i \xi_j, \qquad x \in \Omega,\ z \in \mathbb{R},\ w \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n,$$

where without loss of generality $a_{ij} = a_{ji}$. Then in the set $U \subset \Omega \times \mathbb{R} \times \mathbb{R}^n$ the operator L is

(a) *elliptic* if the coefficient matrix $A = A(x, z, w) := (a_{ij})$ is positive or negative definite for each fixed $(x, z, w) \in U$ (note that if $n = 2$, this means exactly that $4a_{11}a_{22} - a_{12}^2 > 0$, cf. Definition 1.3.3(a));

(b) *parabolic* if A is positive or negative semi-definite with one zero eigenvalue, for each $(x, z, w) \in U$;

(c) *hyperbolic* if A is indefinite but non-degenerate (i.e. 0 is not an eigenvalue) and $n - 1$ eigenvalues have the same sign, for each $(x, z, w) \in U$.

Observe that now even for second order equations this classification is not complete! For equations of higher order there is even less: there seems to be no general definition of parabolic equations, for example, although the terms elliptic and hyperbolic can be meaningfully generalised. We also note that if L is elliptic, then $\frac{\partial}{\partial t} - L$ is (i.e. will be called) parabolic and $\frac{\partial^2}{\partial t^2} - L$ is hyperbolic.

1.3.9 Definition. The quasilinear operator $L = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ is *elliptic* (in a set $U \subset \Omega \times \mathbb{R} \times \ldots \times \mathbb{R}^{n^{m-1}}$) if its principal symbol

$$p(x, z_1, \ldots, z_{m-1}, \xi) = \sum_{|\alpha|=m} a_\alpha(x, z_1, \ldots, z_{m-1}) \xi^\alpha$$

has no real roots, i.e.,

$$p(x, z_1, \ldots, z_{m-1}, \xi) \neq 0 \qquad \forall\, \xi \in \mathbb{R}^n\ \forall\, (x, z_1, \ldots, z_{m-1}) \in U.$$

[3] In the lecture this was originally given for semilinear L. Here and in Definition 1.3.9 etc. we combine this with parts of Definition 3.2.1 from the lecture.

Note that this is genuinely a generalisation of Definitions 1.3.3(a) and 1.3.8(a)!

1.3.10 Lemma. *If L is elliptic, then its order m is even.*

Proof. Fix an arbitrary point $(x, z_1, \ldots, z_{m-1})$; we write $y := (x, z_1, \ldots, z_{m-1})$ for brevity, and let $\xi \in \mathbb{R}^n$. Then $p(y, \xi) \neq 0$. Noting that $p(y, \cdot\,)$ is homogeneous of degree m, if $\eta \in \mathbb{R}^n$ is any vector linearly independent of ξ, then

$$p(y, \eta + t\xi) \quad (\in \mathbb{R})$$

is a polynomial of degree m in t with leading coefficient $p(y, \xi) \neq 0$. If m is odd, then p has at least one real root $t_0 \in \mathbb{R}$, i.e., $p(y, \eta + t_0\xi) = 0$, a contradiction. □

As a consequence of Lemma 1.3.10, we shall instead write the degree of a general elliptic operator as $2m$ for $m \in \mathbb{N}$. Then for each $y = (x, z_1, \ldots, z_{m-1})$ the expression

$$\frac{p(y, \xi)}{|\xi|^{2m}}$$

is homogeneous of degree zero and well defined on the sphere $|\xi| = 1$ (or indeed for all $\xi \neq 0$). Since the sphere is compact and for each fixed y p is just a polynomial in ξ, ellipticity is therefore equivalent to the existence of constants $c = c(y) > 0$ and $C = C(y) > 0$ such that

$$c(y)|\xi|^{2m} \leq p(y, \xi) \leq C(y)|\xi|^{2m} \qquad \forall\, \xi \in \mathbb{R}^n. \tag{1.3.8}$$

(In the literature, this is often taken as the definition of ellipticity.)

1.3.11 Remark. For elliptic operators of second order (1.3.7), this condition is generally written in the form

$$c(y)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(y)\xi_i\xi_j \leq C(y)|\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^n, \tag{1.3.9}$$

as, for example, in the book of Gilbarg–Trudinger [5, Chapter 9].[4] In this case, c and C may be taken as the smallest and largest eigenvalues of the matrix (a_{ij}), respectively.

1.3.12 Definition. Let $c, C > 0$ be as in (1.3.8). We say that the elliptic operator L is

(a) *strongly elliptic* in $U \subset \Omega \times \mathbb{R} \times \ldots \times \mathbb{R}^{n^{m-1}}$ if there exists $c_0 > 0$ such that $c(y) \geq c_0$ for all $y \in U$;

(b) *uniformly elliptic* in U if $C(y)/c(y)$ remains bounded in U.

1.4 Some examples

Here is a list of some of the most common equations found in the literature; we will study many of these in more detail in the sequel. We will omit the proofs of the claimed classifications, except in one exemplary case (the p-Laplacian), which we work through in more detail below: at this juncture we merely observe (i.e. claim) that our very incomplete classification scheme above nevertheless covers a wide variety of important equations.

[4] Be aware that they use the convention of omitting the summation sign in their version of (1.3.8) (the "summation convention").

1.4.1 Example. (a) Nonlinear Poisson equations

$$-\Delta u = f(u)$$

for a suitable $f : \mathbb{R} \to \mathbb{R}$ (semilinear, elliptic; a common choice of f is $f(u) = |u|^p$, $p > 1$).

(b) The p-Laplacian

$$-\Delta_p u := -\operatorname{div}(|Du|^{p-2}Du) = 0 \text{ or } f(u),$$

where $p \in (1, \infty)$ is fixed; see Example 1.4.2 (quasilinear, elliptic).

(c) Scalar reaction-diffusion equations

$$\frac{\partial u}{\partial t} - \Delta u = f(u)$$

(semilinear, parabolic), or the "p-variant"

$$\frac{\partial u}{\partial t} - \Delta u = f(u)$$

(quasilinear, parabolic).

(d) The porous medium equation

$$\frac{\partial u}{\partial t} - \Delta(u^\gamma) = 0,$$

where $\gamma \in [1, \infty)$ is fixed (quasilinear, parabolic).[5]

(e) The minimal surface equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = 0$$

(quasilinear, (non-uniformly) elliptic).

(f) The Monge–Ampère equation

$$\det(D^2 u) = f$$

(fully nonlinear, elliptic).[6]

(g) The m-Laplacian ($m \geq 1$)

$$(-\Delta^m)u := -\Delta(-\Delta(\ldots(-\Delta u))) = f$$

(linear, elliptic).

[5] In fact this equation is considered *degenerate* parabolic, since the term $\Delta(u^\gamma)$ can vanish even if u and its derivatives of highest order are nonzero; see Example 1.4.2 for a similar phenomenon in the case of the p-Laplacian.

[6] Although we have only introduced the notion of ellipticity for quasilinear equations, it turns out to be useful and correct to classify this equation as elliptic.

1.4.2 Example. [7] We look at the p-Laplacian more closely. A fairly long but elementary calculation shows that it is in fact quasilinear elliptic:

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left(\sum_{k=1}^{n}\left(\frac{\partial u}{\partial x_k}\right)^2\right)^{\frac{p-2}{2}} \frac{\partial u}{\partial x_i}\right)$$

$$\vdots$$

$$= \sum_{i=1}^{n} \underbrace{\left(\left(\sum_{k=1}^{n}\left(\frac{\partial u}{\partial x_k}\right)^2\right)^{\frac{p-2}{2}} + (p-2)\left(\frac{\partial u}{\partial x_i}\right)^2 \left(\sum_{k=1}^{n}\left(\frac{\partial u}{\partial x_k}\right)^2\right)^{\frac{p}{2}-2}\right)}_{a_{ii}(x,u,Du)} \frac{\partial^2 u}{\partial x_i^2},$$

where the important thing to note is that the coefficients a_{ii} do not depend on the second-order derivatives of u, i.e.

$$a_{ij}(x,z,w) = \begin{cases} \left(\sum_{k=1}^{n} w_k^2\right)^{\frac{p-2}{2}} + (p-2)\left(\sum_{k=1}^{n} w_k^2\right)^{\frac{p}{2}-2} w_i^2 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{1.4.1}$$

In particular, $a_{ii}(x,z,w) \geq 0$ for all $i = 1,\dots,n$ and $a_{ii} = 0$ only if $p \neq 2$ and $w = 0$, that is, $Du = 0$. In this case we speak of a degeneracy. (Note that whether $p > 2$ or $p < 2$ will have consequences for the behaviour of the a_{ii}: if $p < 2$ then we have a singularity whenever $Du = 0$; if $p > 2$, then we have a genuine degeneracy $a_{ii} = 0$.)

[7] Example 3.2.2 in the lectures.

2 Weak solutions

2.1 Test functions

Throughout, we suppose $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is an arbitrary open set.

2.1.1 Definition. We denote by

$$C_c^\infty(\Omega) = C_c^\infty(\Omega, \mathbb{K})$$

the set of *test functions* on Ω, i.e. $\varphi \in C_c^\infty(\Omega)$ if and only if

(1) $\varphi \in C^\infty(\Omega) := \bigcap_{k \in \mathbb{N}} C^k(\Omega)$, and

(2) $\operatorname{supp} \varphi := \overline{\{x \in \Omega : \varphi(x) \neq 0\}} \subset\subset \Omega$ (φ is "compactly supported" in Ω).

We also write $\varphi \in C_c^k(\Omega)$ $(k \in \mathbb{N}_0;\ C_c^0(\Omega) \equiv C_c(\Omega))$ if $\varphi \in C^k(\Omega)$ and condition (2) holds.

In the literature one sometimes finds the notation $C_0^\infty(\Omega)$, $C_0^k(\Omega)$ instead.

2.1.2 Lemma. *We have*

$$C_c^k(\Omega) \subset C_c^j(\Omega) \qquad \text{if } 0 \leq j \leq k \leq \infty$$

and

$$C_c^k(\Omega) \subset L^p(\Omega) \qquad \text{for all } 0 \leq k \leq \infty \text{ and all } 1 \leq p \leq \infty.$$

Moreover, if $\Omega \subset \widetilde{\Omega}$, *then* $C_c^k(\Omega)$ *may be canonically identified with a subset of* $C_c^k(\widetilde{\Omega})$ *for* $0 \leq k \leq \infty$ *(extension of elements of* $C_c^k(\Omega)$ *by zero outside* Ω*).*

We write $C_c^k(\Omega) \subset C_c^k(\widetilde{\Omega})$ $(0 \leq k \leq \infty)$. The proof is obvious and omitted.

2.1.3 Theorem. $C_c^\infty(\Omega)$ *is dense in* $L^p(\Omega)$ *for any* $1 \leq p < \infty$.

2.1.4 Corollary. $L^p(\Omega) \cap L^q(\Omega)$ *is dense in* $L^p(\Omega)$ *for any* $1 \leq p < \infty$ *and any* $1 \leq q \leq \infty$.

We will not give a full proof of Theorem 2.1.3 here, but merely outline some of the ideas of the proof:

1. Show that $C_c(\Omega)$ is dense in $L^p(\Omega)$ (a deep measure-theoretic result which can be found for example in [10, Theorem 3.14]);
2. Show that any $\varphi \in C_c(\Omega)$ can be approximated by $\varphi_n \in C_c^\infty(\Omega)$ in the L^p-norm, for example using *mollifiers*:

2.1.5 Definition. A function $\eta \in C_c^\infty(\mathbb{R}^n)$ is called a *mollifier* if

(1) $\operatorname{supp}(\eta) \subset \overline{B(0,1)}$, and

(2) $\displaystyle\int_{\mathbb{R}^n} \eta \, dx = 1$.

If, in addition, η satisfies

(3) $\eta(x) \geq 0$ for all $x \in \mathbb{R}^n$, and

(4) $\eta(x) = \zeta(|x|)$ for some function $\zeta : [0, \infty) \to \mathbb{R}$,

then we say η is a *positive symmetric* mollifier.

If η is any mollifier and $\varepsilon > 0$, then we set

$$\eta_\varepsilon := \varepsilon^{-n} \eta \left(\frac{x}{\varepsilon} \right).$$

2.1.6 Example. A typical example of a positive symmetric mollifier is

$$\eta(x) = \begin{cases} c \exp\left(\frac{1}{1-|x|^2}\right), & |x| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $c > 0$ is chosen to ensure $\int_{\mathbb{R}^n} \eta \, dx = 1$.

For a function $f \in L^1_{loc}(\mathbb{R}^n) := \{u : \mathbb{R}^n \to \mathbb{K} \text{ measurable} : u \in L^1(K) \text{ for all } K \subset \mathbb{R}^n \text{ compact}\}$ (we say f is "locally integrable"), we may then define the convolution

$$(f * \eta_\varepsilon)(x) := \varepsilon^{-n} \int_{\mathbb{R}^n} f(t) \eta \left(\frac{x-t}{\varepsilon} \right) dt.$$

2.1.7 Theorem. *Suppose $f \in L^1_{loc}(\mathbb{R}^n)$, η is a positive symmetric mollifier and $\varepsilon > 0$. Then*

(a) $f * \eta_\varepsilon \in C^\infty(\mathbb{R}^n)$;

(b) $f * \eta_\varepsilon \to f$ in $L^p(\mathbb{R}^n)$ as $n \to \infty$, if $f \in L^p(\mathbb{R}^n)$ for some $1 \leq p \leq \infty$;

(c) if $f \in C(\mathbb{R}^n)$ and $\operatorname{supp} f \subset U$, then

$$\operatorname{supp}(f * \eta_\varepsilon) \subset \overline{U}_\varepsilon := \{x \in \mathbb{R}^n : \operatorname{dist}(x, U) \leq \varepsilon\}.$$

Moreover, if $f \in L^p(\mathbb{R}^n)$ and $f = 0$ a.e. outside $U \subset \mathbb{R}^n$, then $\operatorname{supp}(f * \eta_\varepsilon) \subset \overline{U}_\varepsilon$;

(d) if $a \leq f(x) \leq b$ for all $x \in \mathbb{R}^n$, then $a \leq (f * \eta_\varepsilon)(x) \leq b$ for all $x \in \mathbb{R}^n$;

(e) if $f \in C^1(\mathbb{R}^n)$, then

$$\frac{\partial}{\partial x_i}(f * \eta_\varepsilon) = \frac{\partial f}{\partial x_i} * \eta_\varepsilon$$

for all $i = 1, \ldots, n$.

The proof, which uses various tools from measure theory (Fubini's theorem, interchange of derivative and integral, Jensen's inequality, continuity of $\| \cdot \|_p$ with respect to translation,...), is omitted. Statements (a), (b) and (c) hold for general mollifiers.

Step 2 of the proof of Theorem 2.1.3 follows by approximating $\varphi \in C_c^\infty(\Omega)$ by $\varphi * \eta_\varepsilon$:

1. $\varphi * \eta_\varepsilon \in C^\infty(\Omega)$ by Theorem 2.1.7(a);

2. $\operatorname{supp}(\varphi * \eta_\varepsilon) \subset \Omega$ for $\varepsilon > 0$ small enough by Theorem 2.1.7(c), so that $\varphi * \eta_\varepsilon \in C_c^\infty(\Omega)$ for such ε;
3. $\varphi * \eta_\varepsilon \to \varphi$ in $L^p(\Omega)$ by Theorem 2.1.7(b), noting that $\varphi \in L^p(\Omega)$ by Lemma 2.1.2.

2.1.8 Definition. We denote by $\mathcal{D} = \mathcal{D}(\Omega)$ the topological vector space consisting of (the vector space) $C_c^\infty(\Omega)$ equipped with the following notion of convergence: $\varphi_n \overset{\mathcal{D}}{\to} \varphi$ if and only if

1. $\bigcup_{n\in\mathbb{N}} \operatorname{supp} \varphi_n \subset\subset \Omega$, and
2. for all $\alpha \in \mathbb{N}_0^n$ we have $D^\alpha \varphi_n \to D^\alpha \varphi$ uniformly in $x \in \Omega$.

We write $\mathcal{D}(\Omega)$ and $C_c^\infty(\Omega)$ interchangeably, the former usually only when we are explicitly interested in the introduced notion of convergence.

2.2 Distributions

Let Ω and $\mathbb{K}$ be as in Section 2.1.

2.2.1 Definition. A *distribution* (over $\mathcal{D} = \mathcal{D}(\Omega)$) is a continuous linear functional $f : \mathcal{D} \to \mathbb{K}$,

$$f : \varphi \mapsto f[\varphi] \equiv \langle f, \varphi \rangle \in \mathbb{K},$$

that is,

1. for $\varphi, \psi \in \mathcal{D}$ and $\mu, \lambda \in \mathbb{K}$, we have
$$\langle f, \mu\varphi + \lambda\psi \rangle = \mu\langle f, \varphi \rangle + \lambda\langle f, \psi \rangle \qquad \text{(linearity), and}$$
2. for all $\varphi_n, \varphi \in \mathcal{D}$ with $\varphi_n \overset{\mathcal{D}}{\to} \varphi$, we have $\langle f, \varphi_n \rangle \to \langle f, \varphi \rangle$ in $\mathbb{K}$ as $n \to \infty$ (continuity).

2.2.2 Remark. (a) Two distributions f and g are equal if and only if $\langle f, \varphi \rangle = \langle g, \varphi \rangle$ for all $\varphi \in \mathcal{D}$. We say $f = g$ in the distributional sense (or the sense of distributions).

(b) To show a linear functional $f : \mathcal{D} \to \mathbb{K}$ is continuous, due to the linearity it suffices to check continuity at 0, i.e., to check if $\varphi_n \to 0$ in $\mathcal{D}$ always implies $\langle f, \varphi_n \rangle \to 0$.

(c) The set of distributions is a vector space.

2.2.3 Definition. We set $\mathcal{D}' = \mathcal{D}'(\Omega)$ to be the topological vector space of all distributions equipped with the induced convergence

$$f_n \to f \text{ in } \mathcal{D}' :\Longleftrightarrow \langle f_n, \varphi \rangle \to \langle f, \varphi \rangle \quad \text{for all } \varphi \in \mathcal{D}.$$

Obviously, this recalls the notion of the dual space of a normed vector space, except that there is no norm on $C_c^\infty(\Omega)$ compatible with the topology we have introduced (this is a deep statement whose proof we omit). We also omit the proof of the next lemma, which can be found in [9, Theorem 6.17]; we refer more generally to [9, Chapter 6] for more details on the spaces $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ and their topologies.

2.2.4 Lemma ("Completeness" of $\mathcal{D}'$)**.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}'$ and suppose that $\lim_{n\to\infty}\langle f_n,\varphi\rangle =: \ell_\varphi \in \mathbb{K}$ exists for each $\varphi\in\mathcal{D}$. Then there exists $f\in\mathcal{D}'$ such that $\ell_\varphi = \langle f,\varphi\rangle$.*

If $U\subset\Omega$ is open, then we write $f|_U = 0$ (where $f\in\mathcal{D}'(\Omega)$) if $\langle f,\varphi\rangle = 0$ for all $\varphi in \mathcal{D}(U)$, noting that we may identify $\mathcal{D}(U)$ in the obvious way with a subspace of $\mathcal{D}(\Omega)$ (extend functions by zero on $\Omega\setminus U$).

2.2.5 Definition. The *support* of a distribution $f\in\mathcal{D}'(\Omega)$ is defined to be the closed set

$$\operatorname{supp} f := \Omega \setminus \bigcup_{\substack{U\subset\Omega \text{ open}\\ f|_U=0}} U.$$

Suppose now that

$$f\in L^1_{loc}(\Omega) := \{u:\Omega\to\mathbb{K} \text{ measurable} : u|_K \in L^1(K) \text{ for all } K\subset\Omega \text{ compact}\}.$$

Claim: The mapping

$$\mathcal{D}\ni\varphi\mapsto\int_\Omega f\varphi\,dx =: \langle f,\varphi\rangle\in\mathbb{K}$$

defines a distribution.

Proof. Linearity is immediate from linearity of the integral. For continuity: suppose $\varphi_n\to 0$ in $\mathcal{D}$. Then, if U is any bounded open set containing $\bigcup_{n\in\mathbb{N}}\operatorname{supp}\varphi_n$ such that $\overline{U}\subset\Omega$, which exists by definition of $\varphi_n\to\varphi$, we have

$$|\langle f,\varphi\rangle| = \left|\int_\Omega f\varphi_n\,dx\right| = \left|\int_U f\varphi_n\,dx\right| \le \underbrace{\sup_{x\in U}|\varphi_n(x)|}_{\to 0 \text{ by assumption}}\ \underbrace{\int_U |f(x)|\,dx}_{<\infty \text{ fixed}} \to 0 \text{ as } n\to\infty.$$

□

Hence we identify the *function* f with the corresponding distribution.[1] This is in particular well defined, i.e.

$$f=g \text{ in } \mathcal{D}', \text{ i.e. } \int_\Omega f\varphi\,dx = \int_\Omega g\varphi\,dx \text{ for all } \varphi\in C_c^\infty(\Omega), \iff f=g \text{ in } L^1_{loc}(\Omega),$$

due to:

2.2.6 Theorem (du Bois-Reymond/Fundamental Lemma of the Calculus of Variations)**.** *Suppose $\Omega\subset\mathbb{R}^n$ is open and $f\in L^1_{loc}(\Omega)$ is such that*

$$\int_\Omega f\varphi\,dx = 0 \qquad \textit{for all } \varphi\in C_c^\infty(\Omega).$$

Then $f=0$ a.e. in Ω.

[1] Analogous to the way we identify a function in $L^{p'}(\Omega)$ with an element of the dual space of $L^p(\Omega)$, if $\frac{1}{p}+\frac{1}{p'}=1$. The difference is that not every element of $\mathcal{D}'$ can be described this way, as we will see shortly.

Proof. Let η be a positive symmetric mollifier and fix $U \subset\subset \Omega$. If $\varepsilon > 0$ is small enough, then for any $x \in U$, $\eta_\varepsilon(x - \cdot) \in C^\infty(\mathbb{R}^n)$ is compactly supported in $B(x, \varepsilon \subset \Omega$, and so by assumption

$$f * \eta_\varepsilon(x) = \int_\Omega f(t)\eta_\varepsilon(x-t)\,dt = 0,$$

for each fixed $x \in U$. Now by Theorem 2.1.7(b), $f * \eta_\varepsilon \to f$ in $L^1_{loc}(\mathbb{R}^n)$, hence $f = 0$ a.e. in U. Since $U \subset\subset \Omega$ was arbitrary, we conclude $f = 0$ a.e. in Ω. □

In this way, we may identify $L^1_{loc}(\Omega)$ with a subset of $\mathcal{D}'(\Omega)$. In particular, up to this identification, $C_c^\infty(\Omega)$, $L^p(\Omega)$ etc. are subsets of $\mathcal{D}'(\Omega)$.

2.2.7 Definition. A distribution $f \in \mathcal{D}'(\Omega)$ is called *regular* if $f \in L^1_{loc}(\Omega)$ and *singular* otherwise.

2.2.8 Example (The delta distribution[2]). Suppose $0 \in \Omega$. Define $\delta \in \mathcal{D}'(\Omega)$ by $\langle \delta, \varphi\rangle = \varphi(0)$. We first check that this really is a distribution: linearity is as usual immediate, and for continuity, if $\varphi_n \to 0$ in $\mathcal{D}$, then in particular $\varphi_n(x) \to 0$ uniformly in Ω; hence

$$\langle \delta, \varphi_n\rangle = \varphi_n(0) \to 0.$$

One can show that $\operatorname{supp}\delta = \{0\}$ and hence conclude δ must be singular (make a contradiction assumption and use Theorem 2.2.6 to conclude that $\delta = 0$ a.e. in $\Omega \setminus \{0\}$ and hence in Ω); nevertheless, one often writes

$$\varphi(0) = \int_\Omega \delta(x)\varphi(x)\,dx.$$

2.3 Weak derivatives and Sobolev spaces

Suppose now that $\Omega \subset \mathbb{R}^n$ is smooth and bounded, with outer unit normal $\nu = (\nu_1, \dots, \nu_n)$. If $f \in C^1(\overline{\Omega})$ and $\varphi \in C_c^\infty(\Omega)$, then we may apply the integration by parts formula, i.e., the Theorem of Gauß–Green in the form

$$\int_\Omega \frac{\partial u}{\partial x_i}\varphi\,dx = \int_{\partial\Omega} u\nu_i\,d\sigma \qquad (i = 1, \dots, n) \tag{2.3.1}$$

to $u = f\varphi$ to obtain

$$\int_\Omega \frac{\partial f}{\partial x_i}\varphi\,dx = -\int_\Omega f\frac{\partial \varphi}{\partial x_i}\,dx + \underbrace{\int_{\partial\Omega} f\varphi\,\nu_i\,d\sigma}_{=0 \text{ since } \varphi \in C_c^\infty(\Omega)} \qquad \text{for all } i = 1, \dots, n.$$

More generally, if $\Omega \subset \mathbb{R}^n$ is open, $\alpha \in \mathbb{N}_0^n$, $f \in C^{|\alpha|}(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$, then applying Gauß–Green $|\alpha|$ times on any smooth set U such that $\operatorname{supp}\varphi \subset\subset U \subset\subset \Omega$, we have

$$\int_\Omega D^\alpha f\varphi\,dx = (-1)^{|\alpha|}\int_\Omega fD^\alpha\varphi\,dx.$$

This motivates the following definition.

[2] Also called the Dirac delta distribution or, more sloppily, the Dirac delta function. Obviously we could take any $x \in \Omega$ in place of 0 in this definition.

2.3.1 Definition. Suppose $\Omega \subset \mathbb{R}^n$ is open, $f \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^n$. Then the *$\alpha$th distributional (partial) derivative* $D^\alpha f \in \mathcal{D}'(\Omega)$ is given by

$$\langle D^\alpha f, \varphi \rangle := (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

Let us check that this is in fact well defined (we give the argument in some detail explicitly as an example of how to work with these definitions):

1. $\varphi \in \mathcal{D}(\Omega) \implies D^\alpha \varphi \in \mathcal{D}(\Omega)$ (in fact for all $\alpha \in \mathbb{N}_0^n$), so in particular $\langle f, D^\alpha \varphi \rangle$ makes sense.
2. Linearity: using the definition of $D^\alpha f$ and the linearity of f, we have
$$\begin{aligned}\langle D^\alpha f, \mu\varphi + \lambda\psi \rangle &= (-1)^{|\alpha|} \langle f, D^\alpha(\mu\varphi + \lambda\psi) \rangle = (-1)^{|\alpha|} \langle f, \mu D^\alpha \varphi + \lambda D^\alpha \psi \rangle \\ &= (-1)^{|\alpha|} \mu \langle f, D^\alpha \varphi \rangle + (-1)^{|\alpha|} \lambda \langle f, D^\alpha \psi \rangle = \mu \langle D^\alpha f, \varphi \rangle + \lambda \langle D^\alpha f, \psi \rangle.\end{aligned}$$
3. Continuity: it follows from the definition that if $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$, then $D^\alpha \varphi_n \to D^\alpha \varphi$ in $\mathcal{D}(\Omega)$ and hence
$$\langle D^\alpha f, \varphi_n \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi_n \rangle \to (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle = \langle D^\alpha f, \varphi \rangle.$$

2.3.2 Example (Heaviside function)**.** Set

$$\Theta(x) = \begin{cases} 1 & x \geq 0, \\ 0 & x < 0. \end{cases}$$

Then $\Theta \in L^1_{loc}(\mathbb{R})$ may be identified with an element of $\mathcal{D}'(\mathbb{R})$. For $\varphi \in \mathcal{D}(\mathbb{R})$, using compact support,

$$\langle \Theta', \varphi \rangle = -\int_{\mathbb{R}} \Theta(x) \varphi'(x)\, dx = -\int_0^\infty \varphi'(x)\, dx = \varphi(0) = \langle \delta, \varphi \rangle.$$

Hence $\Theta' = \delta$ in the sense of distributions.

Note that if $f \in C^{|\alpha|}(\Omega) \subset L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$, then the distribution $D^\alpha f \in \mathcal{D}'(\Omega)$ agrees with the classical derivative $D^\alpha f \in C(\Omega)$. (This is an exercise using Gauß–Green and Theorem 2.2.6.)

2.3.3 Definition. Suppose $f \in L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^n$. If $D^\alpha f \in \mathcal{D}'(\Omega)$ is itself in $L^1_{loc}(\Omega)$ (i.e. regular), then we call it the *αth weak (partial) derivative* of f. That is, g is the αth weak partial derivative of f if and only if $g \in L^1_{loc}(\Omega)$ and

$$\int_\Omega f D^\alpha \varphi\, dx = (-1)^{|\alpha|} \int_\Omega g \varphi\, dx \qquad \text{for all } \varphi \in C_c^\infty(\Omega).$$

Example 2.3.2 shows that not every $f \in L^1_{loc}(\Omega)$ possesses an αth weak partial derivative, although the corresponding distributional derivative always exists.

An example: in fact $\Theta = f'$ in the weak sense, where

$$f(x) = \begin{cases} x & x \geq 0, \\ 0 & x < 0. \end{cases}$$

Note that weak derivatives, if they exist, are unique by Theorem 2.2.6.

2.3.4 Definition. Suppose $1 \le p \le \infty$ and $k \in \mathbb{N}_0^n$. We set

$$W^{k,p}(\Omega) := \{f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \le k\}.$$

If $p = 2$, we write $H^k(\Omega)$ for $W^{k,2}(\Omega)$.

If $k = 0$, then obviously $W^{0,p}(\Omega) = L^p(\Omega)$ and $H^0(\Omega) = L^2(\Omega)$. In general, functions in $W^{k,p}(\Omega, H^k(\Omega)$ are only defined almost everywhere.[3]

We also write $f \in W^{k,p}_{loc}(\Omega)$ if $f \in W^{k,p}(U)$ for all $U \subset\subset \Omega$. We denote by $\operatorname{supp} f$ the closure of the smallest set U such that $f = 0$ a.e. on $\Omega \setminus U$.

2.3.5 Theorem (Elementary properties of weak derivatives)**.** *Suppose $f, g \in W^{k,p}(\Omega)$ and $|\alpha| \le k$, where $k \ge 0$ and $1 \le p \le \infty$. Then*

(a) $D^\alpha \in W^{k-|\alpha|,p}(\Omega)$ and $D^\beta(D^\alpha f) = D^\alpha(D^\beta f) = D^{\alpha+\beta} f$ for all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| + |\beta| \le k$.

(b) $\lambda f + \mu g \in W^{k,p}(\Omega)$ for all $\lambda, \mu \in \mathbb{K}$ and $D^\alpha(\lambda f + \mu g) = \lambda D^\alpha f + \mu D^\alpha g$ for all $|\alpha| \le k$. In particular, $W^{k,p}(\Omega)$ is a vector space.

(c) If $U \subset \Omega$ is open, then $f|_U \in W^{k,p}(U)$.

(d) If $\varphi \in C_c^\infty(\Omega)$, then $f\varphi \in W^{k,p}(\Omega)$ and

$$D^\alpha(f\varphi) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \varphi D^{\alpha-\beta} f \qquad \textit{(Leibniz' formula).}$$

For (d), we note $\beta \le \alpha :\Longleftrightarrow \beta_i \le \alpha_i$ for all $i = 1, \dots, n$ and

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!} = \prod_{i=1}^n \frac{\alpha_i!}{\beta_i!(\alpha_i - \beta_i)!}.$$

The proof is elementary and omitted (see [3, Section 5.2.3]).

2.3.6 Theorem. *With respect to the norm*

$$\|f\|_{W^{k,p}}(\Omega) := \begin{cases} \left(\sum_{|\alpha| \le k} \int_\Omega |D^\alpha f|^p \, dx\right)^{1/p} & 1 \le p < \infty) \\ \sum_{|\alpha| \le k} \text{ess sup}_\Omega |D^\alpha f| & (p = \infty) \end{cases}$$

the space $W^{k,p}(\Omega)$ is a Banach space; $H^k(\Omega)$ is a Hilbert space with inner product

$$(f, g)_{H^k(\Omega)} := \sum_{|\alpha| \le k} \int_\Omega D^\alpha f \, \overline{D^\alpha g} \, dx.$$

This can be proved by using the definition of weak derivatives to show that $W^{k,p}(\Omega)$ may be identified with a *closed* subspace of the Banach space

$$\underbrace{L^p(\Omega)}_{\ni f} \times \underbrace{L^p(\Omega)^n}_{\ni Df} \times \underbrace{L^p(\Omega)^{n^2}}_{\ni D^2 f} \times \dots \times \underbrace{L^p(\Omega)^{n^k}}_{\ni D^k f}.$$

Oberve that $C_c^\infty(\Omega) \subset W^{k,p}(\Omega)$ for all $1 \le p \le \infty$ and all $k \in \mathbb{N}_0$.

[3] More precisely, as with L^p-functions we really mean that the "functions" are equivalence classes of functions which agree almost everywhere in Ω.

2.3.7 Definition. We denote by $W_0^{k,p}(\Omega)$ the closure of C_c^∞ in $W^{k,p}(\Omega)$ and write $H_0^k(\Omega)$ for $W_0^{k,2}(\Omega)$ ($1 \le p \le \infty$ and $k \in \mathbb{N}_0$).

Roughly speaking, $W_0^{k,p}(\Omega)$ consists of those $f \in W^{k,p}(\Omega)$ for which $D^\alpha f = 0$ on $\partial\Omega$" in some sense for all $0 \le |\alpha| \le k-1$. (This will be made somewhat more precise when we discuss traces below.) It can be shown that $W^{k,p}(\mathbb{R}^n) = W_0^{k,p}(\mathbb{R}^n)$ for all $k \in \mathbb{N}_0$ and all $1 \le p < \infty$, but in general $W^{k,p}(\Omega) \neq W_0^{k,p}(\Omega)$ if $\Omega \subsetneq \mathbb{R}^n$.

Question: how smooth are $W^{k,p}$-functions? Two complementary directions:

1. Sobolev inequalities/embeddings;
2. density of smooth (C^∞) functions in $W^{k,p}(\Omega)$.

In what follows, we will use the notation $\Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$ to denote the part of Ω up to distance $\varepsilon > 0$ to the boundary. If $\varepsilon = 1/i$, we will write Ω_i in place of $\Omega_{1/i}$.

2.3.8 Theorem. *Suppose $\Omega \subset \mathbb{R}^n$ is bounded and open*[4] *and $f \in W^{k,p}(\Omega)$ for some $1 \le p < \infty$ and $k \in \mathbb{N}_0$. Then there exist $f_m \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that $f_m \to f$ in $W^{k,p}(\Omega)$.*[5]

We will use the following lemma without proof.

2.3.9 Lemma. *Suppose $f \in W^{k,p}(\Omega)$ and set $f_\varepsilon := f * \eta_\varepsilon$ in Ω_ε, where η_ε is as in Definition 2.1.5 (or rather just after it). Then $f_\varepsilon \in C^\infty(\Omega_\varepsilon)$ for each $\varepsilon > 0$ and $f_\varepsilon \to f$ in $W^{k,p}_{loc}(\Omega)$ as $\varepsilon \to 0$.*

Proof of Theorem 2.3.8. Fix $f \in W^{k,p}(\Omega)$ and $\delta > 0$. We will prove the existence of $g \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that $\|f - g\|_{W^{k,p}(\Omega)} < \delta$.

1. We have, in the notation introduced above, $\Omega = \cup_{i\in\mathbb{N}}\Omega_i$. Set

$$U_i := \Omega_{i+3} \setminus \overline{\Omega_{i+1}}$$

("overlapping onion rings", which become finer near $\partial\Omega$), and choose any $U_0 \subset\subset \Omega$ open so that

$$\Omega = \bigcup_{i\in\mathbb{N}_0} \Omega_i.$$

Further, let $\{\varphi_i\}_{i\in\mathbb{N}_0}$ be a smooth partition of unity subordinate to the open sets $\{U_i\}_{i\in\mathbb{N}_0}$, i.e.

$$\varphi_i \in C_c^\infty(U_i), \qquad 0 \le \varphi_i \le 1 \qquad \text{and} \quad \sum_{i=0}^\infty \varphi_i(x) = 1 \quad \text{for all } x \in \Omega.$$

By Theorem 2.3.5(d), $f\varphi_i \in W^{k,p}(\Omega)$, and since $\varphi_i \equiv 0$ outside U_i, it follows that $f\varphi_i = 0$ a.e. outside U_i.

[4] Actually, the theorem is valid with essentially the same proof without the assumption of boundedness on Ω. It suffices to "cut off" the sets Ω_i in an appropriate way, e.g., consider $\widetilde{\Omega}_i := \Omega_i \cap B(0,i)$ in place of Ω_i in the above proof.

[5] We could have *defined* $W^{k,p}(\Omega)$ to be the abstract completion of the space of smooth functions with respect to the $W^{k,p}(\Omega)$, i.e., C^∞-functions with finite $W^{k,p}$-norm, as some books do. This theorem (together with Theorem 2.3.6) shows that the two definitions are equivalent.

2. For each $i \geq 0$ choose $\varepsilon_i > 0$ small enough that $f_i := (f\varphi_i) * \eta_{\varepsilon_i} \in C_c^\infty(\Omega)$ satisfies

$$\begin{aligned} \|f_i - (f\varphi_i)\|_{W^{k,p}(\Omega)} &\leq \frac{\delta}{2^{i+1}}, \\ \operatorname{supp} f_i &\subset \Omega_{i+1} \setminus \overline{\Omega_i} \ (\supset U_i,\ i \geq 1), \end{aligned} \tag{2.3.2}$$

which we can do by Lemma 2.3.9 and Theorem 2.1.7(d), respectively.

3. Write $g := \sum_{i=0}^\infty f_i$, then $g \in C^\infty(\Omega$, since for each $U \subset\subset \Omega$ there are at most finitely many nonzero terms in the sum. Since $f = \sum_{i=0}^\infty f\varphi_i$ by choice of the φ, we have

$$\|f - g\|_{W^{k,p}(\Omega)} = \left\| \sum_{i=0}^{\infty} (f\varphi_i - f_i) \right\|_{W^{k,p}(\Omega)} \leq \sum_{i=0}^{\infty} \|f\varphi_i - f_i\|_{W^{k,p}(\Omega)} \leq \delta \sum_{i=0}^{\overset{\frac{1}{2^{i+1}}}{}} = \delta,$$

where for the last inequality we have used (2.3.2). □

Note however that the approximating function g is not necessarily continuous up to the boundary. If we wish to find approximating functions in $C^\infty(\overline{\Omega})$, then we need to assume that $\partial\Omega$ is not "too wild".[6]

2.3.10 Theorem. *Assume $\Omega \subset \mathbb{R}^n$ is bounded and open with C^1 boundary $\partial\Omega$. Suppose $f \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$ and $k \in \mathbb{N}_0$. Then there exist $f_m \in C^\infty(\overline{\Omega})$ such that $f_m \to f$ in $W^{k,p}(\Omega)$.*

Proof. See [3, Section 5.3.3] □

2.3.11 Theorem. *Assume $\Omega \subset \mathbb{R}^n$ is bounded and open with C^1 boundary and let $\widetilde{\Omega} \supset\supset \Omega$ be open. Then there exists a bounded linear operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ such that for each $f \in W^{1,p}(\Omega)$*

(a) $Ef = f$ *a.e. in* Ω,

(b) $\operatorname{supp} Ef \subset \widetilde{\Omega}$, *and*

(c) $\|Ef\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|f\|_{W^{1,p}(\Omega)}$, *where $C > 0$ depends only on p, Ω and $\widetilde{\Omega}$.*

We call Ef an *extension* of f to $\mathbb{R}^n$.

Proof. See [3, Section 5.4] □

Since we are interested in Sobolev functions as "weak" solutions of PDEs on Ω (this will be defined later; see Definition 2.6.1 and Remark 2.6.7) we need to be able to assign "boundary values" along $\partial\Omega$. The problem is that $W^{k,p}$-functions are only defined almost everywhere, and $\partial\Omega$ usually has measure zero.

2.3.12 Theorem. *Suppose $\Omega \subset \mathbb{R}^n$ is bounded and open with C^1 boundary and $1 \leq p < \infty$. Then there exists a bounded linear operator*

$$\operatorname{tr} : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that

[6] The next two theorems will not be needed in the sequel; we include their statements for the sake of "completeness" and refer to the book of Evans [3, Chapter 5] for the proofs.

(a) $\operatorname{tr} f = f|_{\partial\Omega}$ *if* $f \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, *and*

(b) $\|\operatorname{tr} f\|_{L^p(\partial\Omega)} \leq C\|f\|_{W^{1,p}(\Omega)}$ *for all* $f \in W^{1,p}(\Omega)$, *where* $C > 0$ *depends only on* p *and* Ω.

We call tr the *trace* (dt: Spur) of f on $\partial\Omega$. The proof, which we again omit (see [3, Section 5.5]), roughly speaking consists in taking the "closure" of the restriction mapping $C(\overline{\Omega}) \to C(\partial\Omega)$, $f \mapsto f|_{\partial\Omega}$, with respect to the $W^{1,p}$- and L^p-norms.

The trace has a natural relation to the spaces $W_0^{1,p}(\Omega)$.

2.3.13 Theorem. *Suppose* $\Omega \subset \mathbb{R}^n$ *is bounded and open with* C^1 *boundary*[7] *and* $f \in W^{1,p}(\Omega)$. *Then*

$$f \in W_0^{1,p}(\Omega) \qquad \Longleftrightarrow \qquad \operatorname{tr} f = 0.$$

The direction " $\Longrightarrow$ " is easy: take $f_m \in C_c^\infty(\Omega)$ such that $f_m \to f$ in $W^{1,p}(\Omega)$. Since $f_m \equiv 0$ on $\partial\Omega$, we have $\operatorname{tr} f_m = 0$ for all $m \in \mathbb{N}$. Since tr is bounded,

$$\|\underbrace{\operatorname{tr} f_m}_{=0} - \operatorname{tr} f\|_{L^p(\partial\Omega)} \leq C\|f_m - f\|_{W^{1,p}(\Omega)} \longrightarrow 0$$

as $m \to \infty$. The other direction is much deeper, and we again refer to Evans [3, Section 5.5] for the proof.

2.4 Sobolev inequalities

We give a short summary of the most important inequalities satisfied by Sobolev functions. The general principle is that a function in $W^{k,p}(\Omega)$ is "smoother" than a general function in $L^p(\Omega)$ and so automatically belongs to other spaces; this is shown by finding inequalities controlling the norm in the target space of a (sufficiently smooth) function in terms of its $W^{k,p}$-norm. In many cases, the resulting embeddings turn out to be of extreme importance to the nonlinear theory. We will prove a couple of the results on which we will rely most heavily in the future, but a full treatment of all the generally very technical proofs would unfortunately take far too long. Omitted proofs can generally be found in Evans' very accessible book [3, Chapter 5], or alternatively in [5, Chapter 7]. The classical books on Sobolev space theory [1, 7] are not recommended for beginners.

Throughout this section, we *always* assume $\Omega \subset \mathbb{R}^n$ is a bounded, open set with C^1 boundary, unless otherwise stated.

2.4.1 Definition. If $1 \leq p < n$, the *Sobolev conjugate* of p is

$$p^* := \frac{np}{n-p}.$$

That is,

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n},$$

and in particular $p^* > p$.

[7] Most statements about Sobolev functions which are true for "C^1 boundary", including this one, continue to hold when the boundary is merely Lipschitz, i.e., locally the graph of a Lipschitz function. However, the proofs become considerably more technical and can, for example, no longer be found in the highly readable – and recommended – book of Evans.

2.4.2 Theorem (Gagliardo–Nirenberg–Sobolev inequality). *Suppose* $1 \le p < n$. *Then*

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \le C(p,n)\|Df\|_{L^p(\mathbb{R}^n)} \qquad \textit{for all } f \in C_c^1(\mathbb{R}^n). \tag{2.4.1}$$

By $C(p,n)$ we mean that the constant $C > 0$ depends only on p and n. Note that the assumption that f have compact support is obviously necessary (consider, e.g., $f \equiv 1$), but the constant does not depend on the *size* of $\operatorname{supp} f$.

Proof. The proof proceeds by applying the (one-dimensional) fundamental theorem of calculus in each coordinate direction and then the *generalised Hölder inequality*, which can be obtained by applying inductively the usual Hölder inequality: if $1 \le p_1, \dots, p_m \le \infty$ with $\frac{1}{p_1} + \ldots + \frac{1}{p_m} = 1$ and $u_k \in L^{p_k}(\Omega)$ for $k = 1, \dots, m$, then

$$\int_\Omega |u_1 \dots u_m|\, dx \le \prod_{k=1}^m \|u_k\|_{L^{p_k}(\Omega)}. \tag{2.4.2}$$

1. Suppose $p = 1$. Then for each $i = 1, \dots, n$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, the fundamental theorem of calculus implies

$$f(x) = \int_{-\infty}^{x_i} \frac{\partial f}{\partial x_i}(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)\, dy_i$$

since f has compact support; since

$$\left|\frac{\partial f}{\partial x_i}\right| \le |Df|$$

pointwise, in particular

$$|f(x)| \le \int_{-\infty}^{\infty} |Df(x_1, \dots, y_i, \dots, x_n)|\, dy_i, \qquad i = 1, \dots, n.$$

Multiplying the n inequalities together and taking $1/(n-1)$-th powers,

$$\begin{aligned}(|f(x)|^n)^{\frac{1}{n-1}} &\le \left(\prod_{i=1}^n \int_{-\infty}^{\infty} |Df(x_1, \dots, y_i, \dots, x_n)|\, dy_i\right)^{\frac{1}{n-1}} \\ &= \prod_{i=1}^n \left(\int_{-\infty}^{\infty} |Df(x_1, \dots, y_i, \dots, x_n)|\, dy_i\right)^{\frac{1}{n-1}}.\end{aligned}$$

We now integrate with respect to x_1:

$$\int_{-\infty}^{\infty} |f(x)|^{\frac{n}{n-1}}\, dx_1 \le \int_{-\infty}^{\infty} \prod_{i=1}^n \left(\int_{-\infty}^{\infty} |Df(x_1, \dots, y_i, \dots, x_n)|\, dy_i\right)^{\frac{1}{n-1}} dx_1.$$

Since the first term ($i = 1$) does not depend on x_1, we may pull it outside the outer integral:

$$\begin{aligned}\int_{-\infty}^{\infty} |f(x)|^{\frac{n}{n-1}}\, dx_1 &\le \left(\int_{-\infty}^{\infty} |Df|\, dy_1\right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^n \left(\int_{-\infty}^{\infty} |Df(x_1, \dots, y_i, \dots, x_n)|\, dy_i\right)^{\frac{1}{n-1}} dx_1 \\ &\le \Bigg(\underbrace{\int_{-\infty}^{\infty} |Df|\, dy_1}_{=:I_1}\Bigg)^{\frac{1}{n-1}} \prod_{i=2}^n \Bigg(\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Df|\, dx_1 dy_i}_{=:I_i,\, i\ge 2}\Bigg)^{\frac{1}{n-1}}\end{aligned}$$

using (2.4.2) with $m = n-1$ and $p_2 = \ldots = p_n = 1/(n-1)$. We now integrate wit respect to x_2

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x)|^{\frac{n}{n-1}}\,dx_1dx_2 \le \Bigg(\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Df|\,dx_1dy_2}_{I_2}\Bigg)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{\substack{i=1\\ i\neq 2}}^{n} I_i^{\frac{1}{n-1}}\,dx_2$$

and again apply (2.4.2) with $m = n-1$ and $p_1 = p_3 = \ldots = p_n = 1/(n-1)$:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x)|^{\frac{n}{n-1}}\,dx_1dx_2 \le \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Df|\,dx_1dy_2\right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Df|\,dy_1dx_2\right)^{\frac{1}{n-1}} \cdot$$
$$\prod_{i=3}^{n}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Df|\,dx_1dx_2dy_i\right)^{\frac{1}{n-1}}.$$

Proceeding inductively, we obtain

$$\int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}}\,dx \le \prod_{i=1}^{n}\left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} |Df|\,dx_1\ldots dy_i\ldots dx_n\right)^{\frac{1}{n-1}} = \left(\int_{\mathbb{R}^n} |Df|\,dx\right)^{\frac{n}{n-1}}.$$

Thus we have shown

$$\|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)}^{\frac{n}{n-1}} \le \|Df\|_{L^1(\mathbb{R}^n)}^{\frac{n}{n-1}}, \tag{2.4.3}$$

which is (2.4.1) when $p = 1$ and so $p^* = n/(n-1)$.

2. Now suppose $1 < p < n$. We apply (2.4.3) to the function $g := |f|^\gamma$ for some $\gamma > 1$ to be chosen later:

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |f|^{\frac{\gamma n}{n-1}}\,dx\right)^{\frac{n-1}{n}} &\le \int_{\mathbb{R}^n} |D|f|^\gamma|\,dx = \gamma\int_{\mathbb{R}^n} |f|^\gamma|Df|\,dx\\ &\le \gamma\left(\int_{\mathbb{R}^n} |f|^{(\gamma-1)\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^n} |Df|^p\,dx\right)^{\frac{1}{p}}\end{aligned}$$

by Hölder's inequality. Here we have also made use of the chain rule to obtain $D|f|^\gamma = |f|^{\gamma-1}Df$ a.e. in $\mathbb{R}^n$: this is certainly true (pointwise) whenever $f \neq 0$; and one may show (for a general differentiable function g) that $Dg = 0$ a.e. on $\{g = 0\}$.

Now we choose γ so that

$$\frac{\gamma n}{n-1} = (\gamma-1)^{\frac{p}{p-1}}, \qquad \text{i.e. } \gamma := \frac{p(n-1)}{n-p}$$

in order to equate the powers of $|f|$ appearing on the left- and right-hand sides; since $p < n$, we have $\gamma > 1$. We also note that with this choice

$$\frac{\gamma n}{n-1} = (\gamma-1)\frac{p}{p-1} = \frac{np}{n-p} = p^*.$$

Thus

$$\frac{1}{\gamma}\left(\int_{\mathbb{R}^n} |f|^{p^*}\,dx\right)^{\frac{n-1}{n}-\frac{p-1}{p}} \le \left(\int_{\mathbb{R}^n} |Df|^p\,dx\right)^{\frac{1}{p}}.$$

Since
$$\frac{n-1}{n}-\frac{p-1}{p}=\frac{n-p}{np}=\frac{1}{p^*},$$
we finally obtain
$$\left(\int_{\mathbb{R}^n}|f|^{p^*}\,dx\right)^{\frac{1}{p^*}}\le\underbrace{\frac{p(n-1)}{n-p}}_{=:C(n,p)}\left(\int_{\mathbb{R}^n}|Df|^p\,dx\right)^{\frac{1}{p}},$$
which is (2.4.1). □

Theorem 2.4.2 together with the Extension Theorem 2.3.11 leads to

2.4.3 Theorem. *If $f\in W^{1,p}(\Omega)$ for some $1\le p<n$, then $f\in L^{p^*}(\Omega)$ and*
$$\|f\|_{L^{p^*}(\Omega)}\le C(p,n,\Omega)\|f\|_{W^{1,p}(\Omega)}.$$

See [3, Section 5.6.1]. We write $W^{1,p}(\Omega)\hookrightarrow L^{p^*}(\Omega$ ("embeds continuously"); by Hölder's inequality, this also implies $W^{1,p}(\Omega)\subset L^q(\Omega)$ for all $q\in[1,p^*]$, since we are assuming Ω is bounded.

A consequence is the assertion that $\|Df\|_{L^p(\Omega)}$ is an equivalent norm on $W_0^{1,p}(\Omega)$; in fact, this holds for all $p\in[1,\infty)$.

2.4.4 Theorem (Poincaré's inequality)**.** *Suppose $f\in W_0^{1,p}(\Omega)$ for some $1\le p<\infty$ and $1\le q\le p^*$.*[8] *Then*
$$\|f\|_{L^q(\Omega)}\le C(p,q,n,\Omega)\|Df\|_{L^p(\Omega)}.$$
In particular,
$$\|f\|_{L^p(\Omega)}\le C(p,n,\Omega)\|Df\|_{L^p(\Omega)}.$$

Proof. Let $f_m\in C_c^\infty(\Omega)$ such that $f_m\to f$ in $W^{1,p}(\Omega)$. We may extend each function f_m by 0 on $\mathbb{R}^n\setminus\overline{\Omega}$ to obtain a function which we still denote by $f_m\in C_c^\infty(\mathbb{R}^n)$. Then by Theorem 2.4.2, for each m,
$$\|f_m\|_{L^{p^*}(\Omega)}=\|f_m\|_{L^{p^*}(\mathbb{R}^n)}\le C\|Df_m\|_{L^p(\mathbb{R}^n)}=C\|Df_m\|_{L^p(\Omega)}.$$
Passing to the limit as $m\to\infty$, we have $\|f_m\|_{L^{p^*}(\Omega)}\to\|f\|_{L^{p^*}(\Omega)}$ by Theorem 2.4.3. Since Ω is bounded, we therefore have
$$\|f\|_{L^q(\Omega)}\le C(p,q,\Omega)\|f\|_{L^{p^*}(\Omega)}\le\widetilde{C}(p,q,n,\Omega)\|Df\|_{L^p(\Omega)}$$
for any $1\le q\le p^*$. □

If $p>n$, then $W^{1,p}(\Omega)$-functions are even (Hölder) continuous. We recall that for $k\in\mathbb{N}_0$ and $\gamma\in[0,1]$ the Hölder space $C^{k,\gamma}(\overline{\Omega})$ is defined by
$$f\in C^{k,\gamma}(\overline{\Omega}):\iff\|f\|_{C^{k,\gamma}(\overline{\Omega})}:=\sum_{|\alpha|\le k}\|D^\alpha f\|_{C(\overline{\Omega})}+\sum_{|\alpha|=k}\underbrace{\sup_{\substack{x,y\in\Omega\\x\neq y}}\frac{|D^\alpha f(x)-D^\alpha f(y)|}{|x-y|^\gamma}}_{\gamma\text{th Hölder seminorm of }D^\alpha f}<\infty.$$

This is a Banach space with respect to the given norm.

[8] In the lecture we assumed $1\le p<n$ and noted afterwards that this is true for all $p<\infty$.

2.4.5 Theorem (Morrey's inequality). *Assume $n < p \leq \infty$.*

(a) $\|f\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C(p,n)\|f\|_{W^{1,p}(\mathbb{R}^n)}$ *for all* $f \in W^{1,p}(\mathbb{R}^n)$.

(b) Suppose $f \in W^{1,p}(\Omega)$. *Then* $f \in C^{0,1-\frac{n}{p}}(\overline{\Omega})$ *(more precisely, there exists a Hölder continuous representative of f as an equivalence class of functions agreeing almost everywhere) with*

$$\|f\|_{C^{0,1-\frac{n}{p}}(\overline{\Omega})} \leq C(p,n,\Omega)\|f\|_{W^{1,p}(\Omega)}$$

The proof can be found in [3, Section 5.6.2], a brief discussion of the subtle borderline case $p = n$ in [3, 5.8.1].

2.4.6 Remark. Applying the estimates in Theorems 2.4.3 and 2.4.5 repeatedly, we can obtain more general Sobolev inequalities such as

1. $f \in W^{k,p}(\Omega)$, $k < \frac{n}{p}$, and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ implies $f \in L^q(\Omega)$ and

$$\|f\|_{L^q(\Omega)} \leq C(k,p,n,\Omega)\|f\|_{W^{k,p}(\Omega)}, \tag{2.4.4}$$

   i.e. $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$, or more generally

2. if $k > l$ and $1 \leq p < q < \infty$ such that $(k-l)p < n$ and

$$\frac{1}{q} = \frac{1}{p} - \frac{k-l}{n}, \qquad \text{then} \qquad W^{k,p}(\Omega) \hookrightarrow W^{l,q}(\Omega), \tag{2.4.5}$$

   as well as

3. $$\frac{k-l-\alpha}{n} = \frac{1}{p} \text{ for some } \alpha \in (0,1) \qquad \Longrightarrow \qquad W^{k,p}(\Omega) \hookrightarrow C^{l,\alpha}(\overline{\Omega}). \tag{2.4.6}$$

It is an important observation that in many cases the embeddings $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ are compact, by which we mean that each bounded sequence in $W^{k,p}(\Omega)$ has a convergent subsequence in $L^q(\Omega)$.

2.4.7 Theorem (Rellich–Kondrachov). *We continue to suppose that* $\Omega \subset \mathbb{R}^n$ *is bounded and open. Then*

(a) the embeddings $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ *and* $W_0^{k,p}(\Omega) \hookrightarrow W_0^{k-1,p}(\Omega)$ *are compact.*

If in addition Ω *has* C^1 *boundary, then also*

(b) $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ $(1 \leq p \leq \infty)$ *and more generally* $W^{k,p}(\Omega) \hookrightarrow W^{k-1,p}(\Omega)$, *as well as*

(c) $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ $(1 \leq p < n$ *and* $1 \leq q < p^*)$

are compact.

The proof relies on the Arzelà–Ascoli theorem. See [3, Section 5.7].

2.5 The dual spaces $W^{-k,p}(\Omega)$ and $H^{-k}(\Omega)$

For $1 \leq p \leq \infty$, denote by p' the conjugate exponent, $\frac{1}{p} + \frac{1}{p'} = 1$, with the usual convention $1/0 = \infty$ and $1/\infty = 0$.

2.5.1 Definition. For $k \in \mathbb{N}$, we define the dual space

$$W^{-k,p'}(\Omega) := \Big(W_0^{k,p}(\Omega)\Big)'.$$

This can be identified with a space of distributions:[9]

$$W^{-k,p'}(\Omega) = \{f \in \mathcal{D}'(\Omega) : f = \sum_{|\alpha|\le k} D^\alpha g_\alpha \text{ for some } g_\alpha \in L^{p'}(\Omega)\}. \tag{2.5.1}$$

This is a Banach space when equipped with the canonical dual space norm

$$\|f\|_{W^{-k,p'}(\Omega)} := \sup_{0\neq g\in W^{k,p}(\Omega)} \frac{|\langle f, g\rangle|}{\|g\|_{W^{k,p}(\Omega)}} = \sup\{\langle f, g\rangle : \|g\|_{W^{k,p}(\Omega)} \le 1\}, \tag{2.5.2}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the dual pairing between $W^{-k,p'}(\Omega)$ and $W^{k,p}(\Omega)$. Also observe that D^α is a bounded operator from $W^{k,p}(\Omega)$ to $W^{k-|\alpha|,p}(\Omega)$ for all $k \in \mathbb{Z}$ (by definition of D^α and (2.5.1)).

If $p = q = 2$, then we set $H^{-k}(\Omega) := W^{-k,2}(\Omega) = (H_0^k(\Omega))'$, a Hilbert space. We do *not* identify H_0^k with its dual; instead, we consider

$$H_0^k(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-k}(\Omega).$$

Let us consider this more carefully in the special case of $H_0^1(\Omega)$.

2.5.2 Proposition. *(a) If $f \in H^{-1}(\Omega)$, then there exist $f_0, f_1, \dots, f_n \in L^2(\Omega)$ such that*

$$\langle f, g\rangle = \int_\Omega f_0 g + \sum_{i=1}^n f_i \frac{\partial g}{\partial x_i}\,dx \qquad \textit{for all } g \in H_0^1(\Omega). \tag{2.5.3}$$

(b) $$\|f\|_{H^{-1}(\Omega)} = \inf\left\{\left(\int_\Omega \sum_{i=0}^n |f_i|^2\,dx\right)^{1/2} : f \textit{ satisfies } (2.5.3) \textit{ for some } f_0,\dots,f_n \in L^2(\Omega)\right\}.$$

(c) For all $f \in L^2(\Omega) \subset H^{-1}(\Omega)$, we have

$$(f, g)_{L^2(\Omega)} = \langle f, g\rangle \qquad \textit{for all } g \in H_0^1(\Omega).$$

Thus if $f \in H^{-1}(\Omega)$ is additionally in $L^2(\Omega)$, then the H^{-1}–H_0^1 duality in this case agrees with the L^2-inner product. We write

$$f = f_0 - \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}$$

if (2.5.3) holds; this is to be interpreted in the distributional sense.

[9] This is the principal reason for taking $W_0^{k,p}(\Omega)$ and not $W^{k,p}(\Omega)$: we wish to consider the space of test functions equipped with the $W^{k,p}$-norm.

Proof. (a) If $u, v \in H_0^1(\Omega)$, then their inner product is given by

$$(u,v)_{H_0^1(\Omega)} = \int_\Omega Du \cdot Dv + uv \, dx.$$

Suppose now $f \in H^{-1}(\Omega)$. By the Riesz Representation Theorem[10], there exists a unique $u \in H_0^1(\Omega)$ such that

$$(u,v)_{H_0^1(\Omega)} = \langle f, v \rangle \qquad \text{for all } v \in H_0^1(\Omega),$$

that is,

$$\int_\Omega Du \cdot Dv + uv \, dx = \langle f, v \rangle \qquad \text{for all } v \in H_0^1(\Omega).$$

(2.5.3) follows immediately with $f_0 = u$, $f_i = \frac{\partial u}{\partial x_i}$, $i = 1, \dots, n$.

(c) follows immediately from (a), since if $f \in L^2(\Omega)$, then $f = f_0$, and (b) is left as an exercise. ☐

2.6 Weak solutions

Consider the linear equation

$$\begin{cases} Lu := \displaystyle\sum_{|\alpha \leq k} a_\alpha(x) D^\alpha u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.6.1}$$

on an open set $\Omega \subset \mathbb{R}^n$, where $a_\alpha \in C^\infty(\Omega)$. Then (2.6.1) may be interpreted classically, that is, $f \in C(\Omega)$ and we seek a solution $u \in C^k(\Omega) \cap C(\overline{\Omega})$.

Or we can note that $Lu \in \mathcal{D}'(\Omega)$ whenever $u \in \mathcal{D}'(\Omega)$, since $a_\alpha D^\alpha u$ acts as a distribution via

$$\langle a_\alpha D^\alpha u, \varphi \rangle = \langle u, \underbrace{D^\alpha a_\alpha \varphi}_{\in \mathcal{D}(\Omega)} \rangle \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

Thus (2.6.1) may be interpreted in the sense of distributions: given $f \in \mathcal{D}'(\Omega)$, a solution u is an element of $\mathcal{D}'(\Omega)$ satisfying

$$\langle Lu, \varphi \rangle = \langle f, \varphi \rangle \qquad \text{for all } \varphi \in \mathcal{D}(\Omega). \tag{2.6.2}$$

Problem: Finding C^2-solutions is difficult (even proving their existence), since the requirement that a function be C^2 is very strong. But (2.6.2) does not reflect the boundary condition; finding "solutions" might be easy, but there might be too many, and they might have too little connection to the original equation.

Now suppose L is second order strictly elliptic and in divergence form:

$$\begin{cases} Lu := -\displaystyle\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j}(x) \right) + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i}(x) + c(x) u(x) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.6.3}$$

[10] Also known as the Fréchet–Riesz theorem, i.e., that a Hilbert space is canonically isometrically isomorphic to its dual.

We write $Lu = -\operatorname{div}(ADu) + b \cdot Du + cu$, where $A = (a_{ij})$ is without loss of generality symmetric, A, $b = (b_1, \ldots, b_n)$ and c are smooth enough, and $f \in L^2(\Omega)$. Multiplying this equation by $\varphi \in C_c^\infty(\Omega)$[11] and applying Gauß–Green as in the definition of weak derivatives to the principal term,

$$a(u, \varphi) := \int_\Omega \sum_{i,j}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} \varphi + cu\varphi \, dx = \int_\Omega f \varphi \, dx \equiv (f, \varphi)_{L^2(\Omega)} \tag{2.6.4}$$

for all $\varphi \in C_c^\infty(\Omega)$. This is analogous to (2.6.2) and makes sense if $u, \varphi \in H_0^1(\Omega)$; but it also encodes the boundary condition $u = 0$ on $\partial\Omega$, cf. Theorem 2.3.13. Moreover, if (2.6.4) holds and u, f and $\partial\Omega$ are smooth enough (C^2 is certainly sufficient), then (2.6.3) holds pointwise (exercise; use Gauß–Green and Theorems 2.2.6 and 2.3.13).

2.6.1 Definition. (a) If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies (2.6.3) pointwise, it is called a *classical solution* (of (2.6.3)).

(b) If $u \in H_0^1(\Omega)$, $\operatorname{div}(ADu) \in L^2(\Omega)$ and (2.6.3) holds pointwise a.e., then u is a *strong solution*.

(c) If $u \in H_0^1(\Omega)$ satisfies (2.6.4), i.e. $a(u, v) = (f, v)_{L^2(\Omega)}$ for all $v \in H_0^1(\Omega)$, it is a *weak* (or *generalised*) *solution*.

(d) If $u \in \mathcal{D}'(\Omega)$ satisfies (2.6.2) (where we assume the coefficients of L are smooth enough that (2.6.2) makes sense), it is a *distributional solution*.

(e) If $u \in H_0^1(\Omega)$ satisfies $a(u, v) \geq (f, v)_{L^2(\Omega)}$ ($\leq$) for all $0 \leq v \in H_0^1(\Omega)$, it is a weak super- (sub-) solution.

Obviously (a) $\implies$ (b) $\implies$ (c) $\implies$ (d). Most of these definitiions can be generalised to most PDEs,[12] although in many cases it can be a difficult question to determine the correct spaces in which to work and hence the "correct" definition of "weak solution".

Much of the theory of PDEs is concerned with two distinct questions arising from these definitions:

Existence theory: search for (or prove existence of) a weak solution of $Lu = f$.

Regularity theory: show that a weak solution is in fact classical (or at least strong) if f, Ω and the coefficients of L are smooth enough, using structural properties of the equation.

In this course we will be concerned with existence theory; as a first example we will give a proof of the existence of a unique weak solution of (2.6.3) (for any given $f \in L^2(\Omega)$ under suitable assumptions on L. This will also help to motivate our entry into the general nonlinear theory.

2.6.2 Lemma. *Suppose $a_{ij}, b_i, c \in L^\infty(\Omega)$. Then $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is a bounded, bilinear*[13] *form, i.e. for all $u, v, w \in H_0^1(\Omega)$ and all $\lambda, \mu \in \mathbb{R}$, we have*

$$a(\lambda u + \mu v, w) = \lambda a(u, w) + \mu a(v, w)$$
$$a(u, \lambda v + \mu w) = \lambda a(u, v) + \mu a(u, w)$$

[11] We emphasise again that $C_c^\infty(\Omega) \equiv \mathcal{D}(\Omega)$; here we are no longer interested in the topology on $C_c^\infty(\Omega)$.

[12] Note however that (d) in this form requires both the linearity of the equation and the C^∞-smoothness of the coefficientsof L.

[13] Here we will always be interested in *real* problems and solutions, i.e., all our functions are real-valued. If one allows complex values, then a will become sesquilinear.

and there exists $C > 0$ such that

$$|a(u,v)| \leq C\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)} \qquad \textit{for all } u,v \in H_0^1(\Omega).$$

Proof. Linearity is obvious. For $u, v \in H_0^1(\Omega)$,

$$\begin{aligned}|a(u,v)| &\leq \sum_{i,j=1}^{n} \|a_{ij}\|_{L^\infty(\Omega)} \int_\Omega |Du||Dv|dx + \sum_{i=1}^{n} \|b_i\|_{L^\infty(\Omega)} \int_\Omega |Du||v|\,dx + \|c\|_{L^\infty(\Omega)} \int_\Omega |u||v|dx \\ &\leq C\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}\end{aligned}$$

using Cauchy–Schwarz on each integral and taking $C > 0$ large enough. □

The key theoretical tool in the existence proof will be the following theorem from functional analysis.

2.6.3 Theorem (Lax–Milgram lemma)**.** *Suppose H is a real Hilbert space and $a : H \times H \to \mathbb{R}$ is a bounded, bilinear form satisfying the coercivity condition*

$$a(u,u) \geq c\|u\|_H^2 \qquad \textit{for all } u \in H,$$

for some $c > 0$. Then for any $f \in H'$ there exists a unique $u \in H$ such that

$$a(u,v) = \langle f, v\rangle_{H',H} \qquad \textit{for all } v \in H.$$

If $a(\,\cdot\,,\,\cdot\,)$ is symmetric (i.e. $a(u,v) = a(v,u)$ for all $u, v \in H$), then $a(\,\cdot\,,\,\cdot\,)$ is an equivalent inner product on H hand Theorem 2.6.3 is the Riesz Representation Theorem.

The next theorem uses the ellipticity assumption on L to show that the bilinear form given by (2.6.4) is "almost" coercive; more precisely, it becomes coercive on $H_0^1(\Omega)$ if we add enough L^2-norm.

2.6.4 Theorem (Energy estimates)**.** *Suppose $a(\,\cdot\,,\,\cdot\,)$ is given by (2.6.4), where $a_{ij}, b_i, c \in L^\infty(\Omega)$. Then there exist constants $\omega, \alpha > 0$ such that*

$$a(u,u) + \omega\|u\|_{L^2(\Omega)}^2 \geq \alpha\|u\|_{H_0^1(\Omega)}^2 \qquad \textit{for all } H_0^1(\Omega).$$

Proof. Let $u \in H_0^1(\Omega)$. Strict ellipticity of L (see Definitions 1.3.8 and 1.3.12) implies

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx \geq c_0 \int_\Omega |Du|^2\,dx,$$

where $c_0 > 0$ may be chosen as the constant $c > 0$ appearing in Definition 1.3.12. Now

$$\begin{aligned}\int_\Omega \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx &= a(u,u) - \int_\Omega \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} u + cu^2\,dx \\ &\leq a(u,u) + \sum_{i=1}^{n} \|b_i\|_{L^\infty(\Omega)} \int_\Omega |Du||u|\,dx + \|c\|_{L^\infty(\Omega)} \int_\Omega u^2\,dx.\end{aligned}$$

We need to control the integral terms on the right-hand side of the above inequality. To this end, we use the inequality

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon} \qquad (a, b > 0, \varepsilon > 0) \tag{2.6.5}$$

with $a = |Du|$, $b = |u|$ at every point in Ω and integrate:

$$\int_\Omega |Du||u|\,dx \leq \varepsilon \int_\Omega |Du|^2\,dx + \frac{1}{4\varepsilon}\int_\Omega u^2\,dx$$

and choose $\varepsilon > 0$ small enough that

$$\varepsilon \sum_{i=1}^n \|b_i\|_{L^\infty(\Omega)} \leq \frac{c_0}{2}$$

to obtain

$$\frac{c_0}{2}\int_\Omega |Du|^2\,dx \leq a(u,u) + \underbrace{\left(\|c\|_{L^\infty(\Omega)} + \frac{1}{4\varepsilon}\right)}_{=:\omega}\int_\Omega u^2\,dx.$$

Poincaré's inequality, Theorem 2.4.4, guarantees that the left-hand side of the above inequality is an equivalent norm on $H_0^1(\Omega)$; in particular, there exists $\widetilde{C} > 0$ such that

$$\|u\|^2_{H_0^1(\Omega)} \leq \widetilde{C}\|Du\|_{L^2(\Omega)} \qquad \text{for all } u \in H_0^1(\Omega).$$

Hence

$$\underbrace{\frac{c_0}{2\widetilde{C}}}_{=:\alpha}\|u\|^2_{H_0^1(\Omega)} \leq a(u,u) + \omega\|u\|^2_{L^2(\Omega)}.$$

□

Now we can combine the two theorems above to prove existence.

2.6.5 Theorem. *There exists $\omega \geq 0$ such that for all $\lambda \geq \omega$ and all $f \in L^2(\Omega)$ there exists a unique weak solution $u \in H_0^1(\Omega)$ of the problem*

$$\begin{cases} Lu + \lambda u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.6.6}$$

with L as in (2.6.3), where $a_{ij}, b_i, c \in L^\infty(\Omega)$. The constant $\omega > 0$ may be taken as in Theorem 2.6.4.

Note that the form

$$a_\lambda(u,v) := a(u,v) + \lambda(u,v)_{L^2(\Omega)}$$

corresponds to the shifted operator $Lu + \lambda u$ in the sense that weak solutions of (2.6.6) should satisfy

$$a_\lambda(u,v) \equiv a(u,v) + \lambda(u,v)_{L^2(\Omega)} = (f,v)_{L^2(\Omega)} \qquad \text{for all } v \in H_0^1(\Omega).$$

Proof. Let $\omega \geq 0$ be as in Theorem 2.6.4 and suppose $\lambda \geq \omega$. Then $a_\lambda(\,\cdot\,,\,\cdot\,)$ satisfies the hypotheses of the Lax–Milgram lemma, Theorem 2.6.3. Now fix $f \in L^2(\Omega)$ and consider f as a bounded linear functional on $H_0^1(\Omega)$ which acts via

$$\langle f, v\rangle := (f, v)_{L^2(\Omega)} \qquad \text{for all } v \in H_0^1(\Omega),$$

as in Proposition 2.5.2. Then by Lax–Milgram there exists a unique $u \in H_0^1(\Omega)$ satisfying

$$a_\lambda(u, v) = \langle f, v\rangle = (f, v)_{L^2(\Omega)} \qquad \text{for all } v \in H_0^1(\Omega).$$

Hence u is the unique weak solution of (2.6.6). □

2.6.6 Remark. The proof of Theorem 2.6.5 also shows that (2.6.6) has a unique weak solution $u \in H_0^1(\Omega)$ if $f \in H^{-1}(\Omega)$; in this case we mean

$$a_\lambda(u, v) = \langle f, v\rangle \qquad \text{for all } v \in H_0^1(\Omega).$$

2.6.7 Remark (Other boundary conditions)**.** Suppose we wish to solve the same equation with an *inhomogeneous Dirichlet condition*,

$$\begin{cases} Lu = f & \text{in } \Omega, \\ \ \ u = g & \text{on } \partial\Omega \end{cases} \tag{2.6.7}$$

weakly (as in the sense of (2.6.4)), where we now search for $u \in H^1(\Omega)$ and interpret $u = g$ on $\partial\Omega$ as meaning $\operatorname{tr} u = g$ in $L^2(\partial\Omega)$ (cf. Theorem 2.3.12). We need to assume $g = \operatorname{tr} w$ for *some* $w \in H^1(\Omega)$. Then $\tilde{u} := u - w \in H_0^1(\Omega)$ by Theorem 2.3.13 and u is a weak solution of (2.6.7) if and only if $\tilde{u}$ is a weak solution of

$$\begin{cases} L\tilde{u} = \tilde{f} & \text{in } \Omega, \\ \ \ \tilde{u} = 0 & \text{on } \partial\Omega \end{cases}$$

with $\tilde{f} := f - Lw \in H^{-1}(\Omega)$.

The "natural" Neumann condition (see Section 1.2) corresponding to the operator $Lu = -\operatorname{div}(ADu) + b \cdot Du + cu$ is *not* $\frac{\partial u}{\partial \nu} \equiv Du \cdot \nu = 0$, but rather

$$ADu \cdot \nu = 0 \qquad \text{on } \partial\Omega.$$

To see this, multiplying the equation $Lu = f$ by a test function $\varphi \in H^1(\Omega) \cap C^\infty(\Omega)$ and integrating,

$$\int_\Omega (-\operatorname{div}(ADu))\varphi + b \cdot Du\varphi + cu\varphi\,dx = \int_\Omega f\varphi\,dx.$$

If we now apply Gauß–Green to the principal term, we obtain

$$\int_\Omega ADu \cdot D\varphi\,dx + \int_{\partial\Omega} \varphi ADu \cdot \nu\,d\sigma + \int_\Omega b \cdot Du\varphi + cu\varphi\,dx = \int_\Omega f\varphi\,dx$$

for all φ. Since we want the boundary term to vanish (as in the case where $A = I$ if we assume $Du \cdot \nu = 0$), our condition in the "strong" or "classical" form now reads $ADu \cdot \nu = 0$. A weak solution should now satisfy

$$\int_\Omega ADu \cdot D\varphi + (b \cdot Du + cu)\varphi\,dx = \int_\Omega f\varphi\,dx.$$

But note that the left-hand side is exactly $a(u, \varphi)$ from (2.6.4). In order to "see" that $ADu \cdot \nu = 0$ for a weak solution $u \in H^1(\Omega)$, we therefore require that all $\varphi \in H^1(\Omega)$ be allowed as test functions. Thus the *only* difference between the Dirichlet and the Neumann problems is that we replace the space $H_0^1(\Omega)$ with $H^1(\Omega)$.

The expression $ADu \cdot \nu$ is sometimes called the *conormal derivative* of u (with respect to the operator L).

Obviously this method cannot be generalised directly to nonlinear equations. But we still wish to exploit the structural condition of ellipticity without requiring the linearity.

3 Variational methods

3.1 Energy functionals

Suppose V is a Banach space (or a closed subset of one), $U \subset V$ is an open set and $\mathcal{E} : U \to \mathbb{R}$ is a (not necessarily linear) functional, i.e. a mapping. We often refer to $\mathcal{E}$ as an "energy functional".

3.1.1 Definition. We say $\mathcal{E}$ is *Fréchet differentiable* at a point $u \in U$ if a bounded linear map $A : V \to \mathbb{R}$ (i.e. $A \in V'$) exists such that

$$\lim_{\|\varphi\|_V \to 0} \frac{|\mathcal{E}(u+\varphi) - \mathcal{E}(u) - A\varphi|}{\|\varphi\|_V} = 0. \tag{3.1.1}$$

The operator $A =: \mathcal{E}'(u)$ is called the *Fréchet derivative* of $\mathcal{E}$ in u.

3.1.2 Example. If $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}$ and $x, y \in \mathbb{R}^n$, then $\mathcal{E}'(x)y = D\mathcal{E}(x) \cdot y$, since obviously

$$\lim_{|h| \to 0} \frac{|\mathcal{E}(x+h) - \mathcal{E}(x) - D\mathcal{E}(x) \cdot h|}{|h|} = 0, \quad \text{that is,} \quad \frac{\mathcal{E}(x+h) - \mathcal{E}(x)}{|h|} \overset{h \to 0}{\sim} D\mathcal{E}(x) \cdot \frac{h}{|h|}.$$

3.1.3 Definition. Suppose $u \in U$ and $\varphi \in V$ is fixed, so that the map $t \mapsto \mathcal{E}(u + t\varphi)$ is well defined for $t \in \mathbb{R}$ small enough (since U is open). If this map is differentiable in $t = 0$, then

$$\delta\mathcal{E}(u)(\varphi) := \frac{d}{dt}\mathcal{E}(u + t\varphi)\Big|_{t=0} \tag{3.1.2}$$

is called the *first variation* of $\mathcal{E}$ in u, in the direction φ.

This generalises the notion of a classical derivative. In particular, if $\mathcal{E}'(u)$ exists, then

$$\delta\mathcal{E}(u)(\varphi) = \mathcal{E}'(u)\varphi \qquad \text{for all } \varphi \in V. \tag{3.1.3}$$

If $\mathcal{E}$ reaches a maximum or a minimum in $u^* \in U$, i.e. $\mathcal{E}(u^*) \geq \mathcal{E}(u)$ or $\mathcal{E}(u^*) \leq \mathcal{E}(u)$ for all $u \in U$, respectively, and is Fréchet differentiable in U, then $\mathcal{E}'(u^*) = 0$. (Exercise. Note that by (3.1.3), it suffices to show $\delta\mathcal{E}(u^*)(\varphi) = 0$ for all $\varphi \in V$.)

3.1.4 Example. Suppose $U = V = H_0^1(\Omega)$ and

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + cu2 - 2fu\,dx, \qquad u \in H_0^1(\Omega),$$

where $a_{ij}, c \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$ are given and $a_{ij} = a_{ji}$. Suppose also that $u \in H_0^1(\Omega)$ is a critical point $\mathcal{E}$. Let $\varphi \in H_0^1(\Omega)$ and $t \in \mathbb{R}$, then

$$\mathcal{E}(u+t\varphi) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}\left(\frac{\partial u}{\partial x_i} + t\frac{\partial \varphi}{\partial x_i}\right)\left(\frac{\partial u}{\partial x_j} + t\frac{\partial \varphi}{\partial x_j}\right) + c(u^2 + 2tu\varphi + t^2\varphi^2) + 2f(u + t\varphi)\,dx.$$

For $\delta\mathcal{E}(u)(\varphi) = \lim_{t\to 0} \frac{1}{t}(\mathcal{E}(u+t\varphi) - \mathcal{E}(u))$ we need the coefficients of t:

$$\delta\mathcal{E}(u)(\varphi) = \frac{1}{2}\int_\Omega \underbrace{\sum_{i,j=1}^n a_{ij}\left(\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j} + \frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i}\right)}_{=2\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j} \text{ since } a_{ij}=a_{ji}} + 2cu\varphi - 2f\varphi\,dx.$$

That is,

$$0 = \delta\mathcal{E}(u)(\varphi) = \int_\Omega \sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j} cu\varphi\,dx - \int_\Omega f\varphi\,dx \qquad \text{for all } \varphi \in H_0^1(\Omega).$$

Comparing this with (2.6.4), we see $u \in H_0^1(\Omega)$ is a weak solution of

$$\begin{cases} Lu := -\displaystyle\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + cu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Special case: $a_{ij} = \delta_{ij}$, $c = f = 0$: a minimiser[1] of the *Dirichlet integral* (occasionally referred to as the *Dirichlet energy*)

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega |Du|^2\,dx$$

is a (weak) solution of $\Delta u = 0$, i.e. a (weakly) harmonic function. This is called *Dirichlet's principle.*

General principle: interpret the weak form of a PDE as

$$\delta\mathcal{E}(u)(\varphi) \equiv \mathcal{E}'(u)\varphi = 0 \qquad \text{for all } \varphi \in V \tag{3.1.4}$$

for an appropriate Banach space V and energy functional $\mathcal{E}$, so that solution(s) of the PDE are critical points of $\mathcal{E}(\,\cdot\,)$.

A (differential) equation in the form (3.1.4) is said to be in *variational form*; it is also called the (weak form of the) *Euler–Lagrange equation* associated with $\mathcal{E}$.

Now consider

$$\mathcal{E}(u) = \int_\Omega F(x, u, Du)\,dx \tag{3.1.5}$$

for $\Omega \subset \mathbb{R}^n$ bounded and open with sufficiently smooth boundary and $u \in W_0^{1,p}(\Omega)$ ($1 \le p < \infty$), where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, often called the *Lagrangian*, is also sufficiently smooth. We also implicitly assume that F is of the right form so that (3.1.5) is actually well defined for all $u \in W_0^{1,p}(\Omega)$ (we will give sufficient conditions – which also influence the choice of p – for this later; see Section 3.4 and in particular Theorem 3.4.11).

We write[2]

$$F(x, z, w) \qquad \text{for } x \in \Omega,\ z \in \mathbb{R},\ w \in \mathbb{R}^n,$$

[1] Actually any critical point has this property, but it turns out this functional only has the one, which is a global minimum. Also note that one usually minimises this integral over a subset of $H^1(\Omega)$ such as of the form $\{u \in H^1(\Omega) : \operatorname{tr} u = g\}$ for some given $g \in L^2(\partial\Omega)$. But the principle is the same.

[2] The notation (x, z, p) is more common.

as well as

$$D_x F = \left(\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}\right)$$
$$D_z F = \frac{\partial F}{\partial z}$$
$$D_w F = \left(\frac{\partial F}{\partial w_1}, \dots, \frac{\partial F}{\partial w_n}\right).$$

3.1.5 Proposition. *The (formal) Euler–Lagrange equation associated with the functional $\mathcal{E}$ given by* (3.1.5) *is given by*[3]

$$\int_\Omega \sum_{i=1}^n \frac{\partial F}{\partial w_i}(x,u,Du)\frac{\partial \varphi}{\partial x_i} + \frac{\partial F}{\partial z}(x,u,Du)\varphi\, dx = 0 \qquad \textit{for all } \varphi \in W_0^{1,p}(\Omega) \tag{3.1.6}$$

(weak form), or

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial w_i}(x,u,Du)\right) + \frac{\partial F}{\partial z}(x,u,Du) = 0 \tag{3.1.7}$$

(strong/classical form).

We will see later that this covers a wide variety of (second order elliptic) quasilinear equations.

Proof. We first show that (3.1.6) corresponds to $\delta\mathcal{E}(u)(\varphi) = 0$, $\varphi \in W_0^{1,p}(\Omega)$:

$$\mathcal{E}(u+t\varphi) = \int_\Omega F(x, u+t\varphi, Du+tD\varphi)\, dx$$
$$\frac{d}{dt}\mathcal{E}(u+t\varphi) = \int_\Omega \frac{\partial F}{\partial z}(x,u+t\varphi, Du+t\varphi)\varphi + \sum_{i=1}^n \frac{\partial F}{\partial w_i}(x,u+t\varphi,Du+tD\varphi)\frac{\partial\varphi}{\partial x_i}\, dx.$$

Hence

$$\delta\mathcal{E}(u)(\varphi) = \frac{d}{dt}\mathcal{E}(u+t\varphi)\Big|_{t=0} = \int_\Omega \frac{\partial F}{\partial z}(x,u,Du)\varphi + \sum_{i=1}^n \frac{\partial F}{\partial w_i}(x,u,Du)\frac{\partial \varphi}{\partial x_i}\, dx,$$

which yields (3.1.6) if u is a critical point. If u is sufficiently smooth, then applying Gauß–Green to (3.1.6) with $\varphi \in C_c^\infty(\Omega) \subset W_0^{1,p}(\Omega)$ arbitrary and using Theorem 2.2.6 yields (3.1.7). □

3.1.6 Definition. A function $u : \Omega \to \mathbb{R}$ is a *weak solution* of the (formal Euler–Lagrange) equation

$$\begin{cases} -\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial w_i}(x,u,Du)\right) + \frac{\partial F}{\partial z}(x,u,Du) = 0 & \text{in } \Omega, u \;= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.1.8}$$

if $u \in W_0^{1,p}(\Omega)$ and[4]

$$\int_\Omega \sum_{i=1}^n \frac{\partial F}{\partial w_i}(x,u,Du)\frac{\partial \varphi}{\partial x_i} + \frac{\partial F}{\partial z}(x,u,Du)\varphi\, dx = 0 \qquad \text{for all } \varphi \in W_0^{1,p}(\Omega). \tag{3.1.9}$$

[3] At this stage we are merely writing down formal identities; we assume that F is "nice" enough to justify finiteness of the integrals, exchange of derivative and integral etc.; we will give examples of functions F for which this holds later (see Theorem 3.4.11). Alternatively, one may for now assume $\varphi \in C_c^\infty(\Omega)$ and F is smooth and even bounded, say.

[4] Again, we are implicitly assuming here that this integral is well defined and finite for all $\varphi \in W_0^{1,p}(\Omega)$.

3.2 Important examples

We recall the second-order quasilinear equation

$$Lu := \sum_{|\alpha|=2} a_\alpha(x,u,Du)D^\alpha u + a_0(x,u,Du) = f$$

is in divergence form if

$$Lu = -\operatorname{div} A(x,u,Du) + \tilde{a}_0(x,u,Du)$$

for some $A = (A^1, \dots, A^n)$, that is, with the a_{ij} as in (1.3.7),

$$a_{ij}(x,z,w) = \frac{1}{2}\left(\frac{\partial}{\partial w_i}A^j(x,z,w) + \frac{\partial}{\partial w_j}A^i(x,z,w)\right).$$

(Any quasilinear equation in divergence form satisfies this relation; on the other hand, a general *semilinear* equation can be written in divergence form provided the leading-order coefficients are smooth enough; we leave the proof as an exercise.)

To be in variational form, i.e., to be the Euler–Lagrange equation

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial w_i}(x,u,Du)\right) - \frac{\partial F}{\partial z}(x,u,Du) = 0$$

of the functional

$$\mathcal{E}(u) = \int_\Omega F(x,u,Du)\,dx$$

(with $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ sufficiently smooth), we need to have

$$A^i(x,z,w) = \frac{\partial F}{\partial w_i}(x,z,w),$$

that is,

$$a_{ij}(x,z,w) = \frac{\partial^2 F}{\partial w_i \partial w_j}(x,z,w), \qquad i,j = 1,\dots,n. \tag{3.2.1}$$

3.2.1 Example. [5]

(a) The p-Laplacian:

$$\mathcal{E}(u) = \frac{1}{p}\int_\Omega |Du|^p - fu\,dx, \qquad u \in W_0^{1,p}(\Omega)$$

with Euler–Lagrange equation

$$\int_\Omega |Du|^{p-2}Du \cdot D\varphi\,dx = \int_\Omega f\varphi\,dx \qquad \text{for all } \varphi \in W_0^{1,p}(\Omega).$$

[5] Example 3.2.3 from the lectures.

(b) The nonlinear Poisson equation $-\Delta u = f(u)$: let

$$F(t) := \int_0^t f(s)\,ds \qquad , t \in \mathbb{R}.$$

Then

$$\mathcal{E}(u) = \int_\Omega \frac{1}{2}|Du|^2 - F(u(x))\,dx, \qquad u \in H_0^1(\Omega),$$
$$\int_\Omega Du \cdot D\varphi\,dx = \int_\Omega f(u)\varphi\,dx \qquad \text{for all } \varphi \in H_0^1(\Omega).$$

(c) The minimal surface equation

$$\mathcal{E}(u) = \int_\Omega (1+|Du|^2)^{1/2}\,dx, \qquad u \in H_0^1(\Omega),$$
$$\int_\Omega (1+|Du|^2)^{-1/2} Du \cdot D\varphi\,dx = 0 \qquad \text{for all } \varphi \in H_0^1(\Omega).$$

We recall the strong form

$$\operatorname{div}\left(\frac{Du}{(1+|Du|^2)^{1/2}}\right) = 0.$$

Minimal surfaces, which by definition are minimisers of $\mathcal{E}$, thus correspond to solutions of the minimal surface equation (hence the name). Since the mean curvature of the graph of u is given by

$$\frac{1}{n}\operatorname{div}\left(\frac{Du}{(1+|Du|^2)^{1/2}}\right),$$

this yields the fundamental observation that *minimal surfaces have zero mean curvature.*

3.3 The second variation

Let V be a Banach space, $U \subset V$ be open and $\mathcal{E} : U \to \mathbb{R}$ be a functional.[6]

3.3.1 Definition. The *second variation* of $u \in U$ in the direction $\varphi \in V$ is defined to be

$$\delta^2\mathcal{E}(u)(\varphi) := \frac{d^2}{dt^2}\mathcal{E}(u+t\varphi)\Big|_{t=0},$$

assuming it exists.

3.3.2 Lemma. *Suppose $u^* \in U$ is a local minimum (maximum) of $\mathcal{E}$. Then*

$$\delta^2\mathcal{E}(u)(\varphi) \geq 0 \ (\leq 0) \qquad \textit{for all } \varphi \in V.$$

whenever it exists.

[6]In the lectures Definition 3.3.1 and Lemma 3.3.2 were only given for the case $V = W_0^{1,p}(\Omega)$ and for minima, but obviously they hold in general.

Suppose now that we have, as in (3.1.5), a functional

$$\mathcal{E}(u) = \int_\Omega F(x,u,Du)\,dx$$

defined on a subset of $W_0^{1,p}(\Omega)$ $(1 \leq p < \infty)$, where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is sufficiently smooth. What does the assertion of Lemma 3.3.2 say in this case?[7]

Recall

$$\frac{d}{dt}\mathcal{E}(u+t\varphi) = \int_\Omega \frac{\partial F}{\partial z}(x,u+t\varphi,Du+tD\varphi)\varphi + \sum_{i=1}^n \frac{\partial F}{\partial w_i}(x,u+t\varphi,Du+tD\varphi)\frac{\partial \varphi}{\partial x_i}\,dx$$

so

$$\frac{d^2}{dt^2}\mathcal{E}(u+t\varphi) = \int_\Omega \frac{\partial^2 F}{\partial z^2}(x,u+t\varphi,Du+tD\varphi)\varphi^2 + 2\sum_{i=1}^n \frac{\partial^2 F}{\partial z \partial w_i}(x,u+t\varphi,Du+tD\varphi)\varphi\frac{\partial \varphi}{\partial x_i}$$
$$+ \sum_{i,j=1}^n \frac{\partial^2 F}{\partial w_i \partial w_j}(x,u+t\varphi,Du+tD\varphi)\frac{\partial \varphi}{\partial x_i}\frac{\partial \varphi}{\partial x_i}\,dx,$$

that is,

$$0 \overset{!}{\leq} \delta^2\mathcal{E}(u)(\varphi) = \int_\Omega \frac{\partial^2 F}{\partial z^2}(x,u,Du)\varphi^2 + 2\sum_{i=1}^n \frac{\partial^2 F}{\partial z \partial w_i}(x,u,Du)\varphi\frac{\partial \varphi}{\partial x_i}$$
$$+ \sum_{i,j=1}^n \frac{\partial^2 F}{\partial w_i \partial w_j}(x,u,Du)\frac{\partial \varphi}{\partial x_i}\frac{\partial \varphi}{\partial x_i}\,dx \qquad (3.3.1)$$

for all $\varphi \in W_0^{1,p}(\Omega)$. We now make a specific choice of φ: suppose $\psi \in C_c^\infty(\Omega)$ is arbitrary and $\rho : \mathbb{R} \to \mathbb{R}$ is the "sawtooth" (or "tent") function given by

$$\rho(x) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 1-x & \text{if } \frac{1}{2} < x \leq 1, \\ x+1 & \text{for all } x \in \mathbb{R}, \end{cases}$$

so that $|\rho'(x)| = 1$ a.e.. Choosing, for fixed $\xi \in \mathbb{R}^n$,

$$\varphi(x) := \varepsilon\rho\left(\frac{x\cdot\xi}{\varepsilon}\right)\psi(x), \qquad x \in \Omega,$$

we claim (without proof) that $\varphi \in W_0^{1,p}(\Omega)$ and

$$\frac{\partial \varphi}{\partial x_i} = \rho'\left(\frac{x\cdot\xi}{\varepsilon}\right)\xi_i\psi + \underbrace{\varepsilon\rho\left(\frac{x\cdot\xi}{\varepsilon}\right)\frac{\partial \psi}{\partial x_i}}_{=\mathcal{O}(\varepsilon)\text{ as }\varepsilon\to 0}$$

[7] As this is again a kind of motivation for the following sections, the calculations will be *formal*: we will not give precise conditions on F or justify the interchange of integral and derivative, etc..

in $L^p(\Omega)$. Since $\varphi = \mathcal{O}(\varepsilon)$ as well, substituting this into (3.3.1) yields

$$0 \le \int_\Omega \sum_{i,j=1}^n \frac{\partial^2 F}{\partial w_i \partial w_j}(x,u,Du)\,\xi_i\xi_j\,\psi^2 \cdot \underbrace{\left(\rho'\left(\frac{x\cdot\xi}{\varepsilon}\right)\right)^2}_{=1 \text{ a.e.}} dx + \mathcal{O}(\varepsilon).$$

Letting $\varepsilon \to 0$, we see

$$\int_\Omega \sum_{i,j}^n \frac{\partial^2 F}{\partial w_i \partial w_j}(x,u,Du)\,\xi_i\xi_j\,\psi^2\,dx \ge 0$$

for all $\psi \in C_c^\infty(\Omega)$, which, similar to the Fundamental Lemma of the Calculus of Variations implies, assuming F is smooth enough,

$$\frac{\partial^2 F}{\partial w_i \partial w_j}(x,u,Du)\,\xi_i\xi_j \ge 0 \qquad \text{in } \Omega, \text{ for all } \xi \in \mathbb{R}^n. \tag{3.3.2}$$

This structural condition on F needs to be satisfied in order for a minimum to exist. But this is guaranteed whenever F is associated with an elliptic operator (in fact this characterises second-order elliptic operators in variational form, assuming everything is smooth enough); cf. (3.2.1).

3.4 Minimisation of energy functionals

Suppose V is a reflexive Banach space and $\mathcal{E} : U \subset V \to \mathbb{R}$ is a functional which is bounded from below, i.e. there exists $c \in \mathbb{R}$ such that

$$\mathcal{E}(u) \ge c \qquad \text{for all } u \in U. \tag{3.4.1}$$

Without loss of generality, we may assume $\mathcal{E}$ is defined on the whole of V by setting

$$\widetilde{\mathcal{E}}(u) := \begin{cases} \mathcal{E}(u) & \text{if } u \in U, \\ \infty & \text{if } u \in V \setminus U; \end{cases}$$

we will identify $\mathcal{E}$ und $\widetilde{\mathcal{E}}$. Obviously there exists a *minimising sequence* $u_k \in V$ such that $\mathcal{E}(u_k) \to \inf\{\mathcal{E}(u) : u \in V\}$ as $k \to \infty$.

Simple one-dimensional examples show that we need additional assumptions on $\mathcal{E}$ to ensure the existence of a minimiser: take, e.g., $\mathcal{E}(x) = e^x$ or $\mathcal{E}(x) = 1/(1+x)$ ($x \in \mathbb{R}$). We need a property which ensures $\mathcal{E}(u) \to \infty$ as $\|u\|_V \to \infty$:

3.4.1 Definition. The functional $\mathcal{E}$ is *coercive* if there exist constants $c_1, c_2, p > 0$ such that

$$c_1\mathcal{E}(u) + c_2 \ge \|u\|_V^p \qquad \text{for all } u \in V. \tag{3.4.2}$$

This implies in particular that the sublevel sets

$$\{u \in V : \mathcal{E}(u) \le \alpha\} \tag{3.4.3}$$

are bounded for all $\alpha \in \mathbb{R}$. (Sometimes this is taken as the definition of coercivity.) Note also that coercive functionals are automatically bounded from below.

We also need a form of continuity. Requiring actual continuity (i.e. $u_k \to u$ in V implies $\mathcal{E}(u_k) \to \mathcal{E}(u)$) is however too strong.[8] Recall that $u_k \rightharpoonup u$ weakly in V if

$$\langle \varphi, u_k \rangle \to \langle \varphi, u \rangle$$

in $\mathbb{R}$ for every fixed $\varphi \in V'$.

3.4.2 Definition. $\mathcal{E}$ is (sequentially) *weakly lower semicontinuous* (in V) if

$$u_k \rightharpoonup u \text{ in } V \quad \Longrightarrow \quad \mathcal{E}(u) \le \liminf_{k\to\infty} \mathcal{E}(u_k). \tag{3.4.4}$$

3.4.3 Theorem. *Suppose V is a reflexive Banach space and $\mathcal{E} : V \to \mathbb{R} \cup \{\infty\}$ is a functional which is not identically ∞. If $\mathcal{E}$ is coercive and weakly lower semicontinuous, then there exists a global minimiser $u \in V$ of $\mathcal{E}$, i.e.*

$$\mathcal{E}(u) = \inf\{\mathcal{E}(v) : v \in V\}.$$

Proof. Let $a \in \mathbb{R}$ be this infimum; this is indeed a real number by coercivity (and $\mathcal{E} \not\equiv \infty$). Suppose $(u_k) \subset V$ is a minimising sequence, i.e. $\mathcal{E}(u_k) \to a$ as $k \to \infty$. Since the sequence $(\mathcal{E}(u_k))$ is bounded, the same is true of (u_k) in V, cf. (3.4.3). Since V is reflexive, (u_k) has a subsequence, which we shall again denote by (u_k), which converges weakly to some $u \in V$. Weak lower semicontinuity implies

$$\mathcal{E}(u) \le \liminf_{k\to\infty} \mathcal{E}(u_k) = \lim_{k\to\infty} \mathcal{E}(u_k) = a.$$

□

Two natural questions:

1. When is the minimiser unique?
2. Can we replace *weak* lower semicontinuity with an easier condition?

In both cases the notion of *convexity* provides an answer.

3.4.4 Definition. The functional $\mathcal{E} : V \to \mathbb{R}$ is *convex* if

$$\mathcal{E}(tu + (1-t)v) \le t\mathcal{E}(u) + (1-t)\mathcal{E}(v)$$

for all $t \in [0,1]$ and all $u, v \in V$, and *strictly convex* if for $u \neq v$ and $t \in (0,1)$

$$\mathcal{E}(tu + (1-t)v) < t\mathcal{E}(u) + (1-t)\mathcal{E}(v).$$

Obviously, a strictly convex functional can have at most one minimiser, while we can show for convex functionals in general:

3.4.5 Proposition. *Suppose V is a Banach space and $\mathcal{E} : V \to \mathbb{R} \cup \{\infty\}$ is convex and lower semicontinuous. Then $\mathcal{E}$ is also weakly lower semicontinuous.*

[8] Or else in many practical examples useless since we usually only have weak convergence of a minimising sequence.

Convexity is in practice often too strong, so we omit the proof here; it can be found in [11, Theorem 13.8]. This idea will however motivate our approach: we now return to the question of finding minimiser(s) of

$$\mathcal{E}(u) = \int_\Omega F(x, u, Du)\, dx, \qquad u \in W_0^{1,p}(\Omega)$$

under our previous assumptions on F, Ω, p, etc..

3.4.6 Remark. $W_0^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$ are reflexive Banach spaces if $1 < p < \infty$, since they may be identified with closed subspaces of the reflexive space $L^p(\Omega)^{n+1}$.

We make the following additional coercivity assumption on F: there exist constants $\alpha > 0$ and $\beta \geq 0$ such that

$$F(x, z, w) \geq \alpha |x|^p - \beta \qquad \text{for all } (x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^n. \tag{3.4.5}$$

This implies the coercivity condition (3.4.2) for $\mathcal{E}$ on $W_0^{1,p}(\Omega)$, since then

$$\mathcal{E}(u) \geq \alpha \|Du\|_{L^p(\Omega)}^p - \beta |\Omega|,$$

and by Poincaré's inequality (Theorem 2.4.4), $\|Du\|_{L^p(\Omega)}$ defines an equivalent norm on $W_0^{1,p}(\Omega)$.

3.4.7 Theorem. *Suppose $\Omega \subset \mathbb{R}^n$ is bounded and open, $F \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ is bounded from below as in* (3.4.1)*, and the mapping*

$$w \mapsto F(x, z, w)$$

is convex, that is, for all $t \in [0, 1]$ and all $v, w \in \mathbb{R}^n$,

$$F(x, z, tw + (1-t)v) \leq tF(x, z, w) + (1-t)F(x, z, v)$$

for each fixed $x \in \Omega$ and $z \in \mathbb{R}$. Then $\mathcal{E}$ given by (3.1.5) *is weakly lower semicontinuous on $W_0^{1,p}(\Omega)$ (or also $W^{1,p}(\Omega)$ if $\partial\Omega$ is of class C^1), where $1 < p < \infty$ is fixed.*

Now $w \mapsto F(x, z, w)$ being convex means that for all $w \in \mathbb{R}^n$ there exists $\xi \in \mathbb{R}^n$ such that

$$F(x, z, w) \geq F(x, z, w) + \xi \cdot (v - w) \qquad \text{for all } v \in \mathbb{R}^n, \tag{3.4.6}$$

where

$$\xi = D_w F \equiv \left(\frac{\partial F}{\partial w_1}, \ldots, \frac{\partial F}{\partial w_n} \right)(x, z, w)$$

if F is C^1. If F is C^2, then convexity means exactly that the Hessian matrix

$$D_w^2 F = \left(\frac{\partial^2 F}{\partial w_i \partial w_j} \right) \geq 0, \qquad \text{i.e.} \quad \sum_{i,j=1}^n \frac{\partial^2 F}{\partial w_i \partial w_j} \xi_i \xi_j \geq 0 \qquad \text{for all } \xi \in \mathbb{R}^n$$

(cf. (3.3.2)). We also say that $w \mapsto F(x, z, w)$ is *uniformly convex* if there exists $\alpha > 0$ such that

$$D_w^2 F \geq \alpha I, \qquad \text{i.e.} \quad \sum_{i,j=1}^n \frac{\partial^2 F}{\partial w_i \partial w_j} \xi_i \xi_j \geq \alpha |\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^n,\ x \in \Omega \text{ and } z \in \mathbb{R}.$$

This corresponds to the associated Euler–Lagrange equation (3.1.7) being (stongly) elliptic, cf. Definition 1.3.12 and (3.2.1).

The proof of Theorem 3.4.7 uses the following measure-theoretic result, which we state for the case of Lebesgue measure in $\mathbb{R}^n$.

3.4.8 Theorem (Egorov). *Suppose $X \subset \mathbb{R}^n$ is measurable with $|X| < \infty$ and suppose (f_k) is a sequence of measurable functions such that $f_k \to f$ a.e. in X. Then for any $\varepsilon > 0$ there exists a measurable set $E \subset X$ with $|X \setminus E| < \varepsilon$, such that $f_k \to f$ uniformly in E.*

Proof of Theorem 3.4.7. 1. Let $(u_k) \subset W_0^{1,p}(\Omega)$, $u_k \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$. This implies in particular that $u_k \rightharpoonup u$ in $L^p(\Omega)$ and $Du_k \rightharpoonup Du$ in $L^p(\Omega, \mathbb{R}^n)$ (exercise). Let $a := \liminf_{k\to\infty} \mathcal{E}(u_k)$. Passing to a subsequence if necessary, we may assume $a = \lim_{k\to\infty} \mathcal{E}(u_k)$. We wish to show $a \geq \mathcal{E}(u)$.

Now since (u_k) is weakly convergent, it is bounded, and since the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact (Theorem 2.4.7(a)), there exists a subsequence, which we will still denote by (u_k), such that $u_k \to u$ (strongly) in $L^p(\Omega)$. In particular, passing to another subsequence, $u_k \to u$ a.e. in Ω.

2. Now fix $\varepsilon > 0$. Theorem 3.4.8 implies that there exists a measurable $E_\varepsilon \subset \Omega$, $|\Omega \setminus E_\varepsilon| < \varepsilon$, such that $u_k \to u$ uniformly in E_ε. WLOG we may also assume

$$0 < \varepsilon' < \varepsilon \quad \Longrightarrow \quad E_\varepsilon \subset E_{\varepsilon'}.$$

Now write

$$F_\varepsilon := \{x \in \Omega : |u(x)| + |Du(x)| \leq \varepsilon^{-1}\}$$

measurable (where "$\leq$" is to be understood a.e.), so that $|\Omega \setminus F_\varepsilon| \to 0$ as well. If we set

$$G_\varepsilon := E_\varepsilon \cap F_\varepsilon$$

measurable, it follows that $|\Omega \setminus G_\varepsilon| \to 0$ as $\varepsilon \to 0$.

3. WLOG me may assume $F(x, z, w) \geq 0$ (as F is bounded from below, since otherwise we may consider $\widetilde{F} := F + c$ for some $c > 0$ large enough). Then

$$\mathcal{E}(u_k) = \int_\Omega F(x, u_k, Du_k)\,dx \geq \int_{G_\varepsilon} F(x, u_k, Du_k)\,dx.$$

Using the convexity of F in the form (3.4.6), we may write

$$F(x, u_k, Du_k) \geq F(x, u_k, Du) + D_w F(x, u_k, Du) \cdot (Du_k - Du)$$

for each $x \in \Omega$ and each $k \in \mathbb{N}$; thus

$$\mathcal{E}(u_k) \geq \int_{G_\varepsilon} F(x, u_k, Du)\,dx + \underbrace{\int_{G_\varepsilon} D_w F(x, u_k, Du) \cdot (Du_k - Du)\,dx}_{\stackrel{!}{\longrightarrow} 0} \tag{3.4.7}$$

4. Now since $u_k \to u$ uniformly on $E_\varepsilon \supset G_\varepsilon$, since u is bounded on $F_\varepsilon \supset G_\varepsilon$ (so in particular u_k is bounded on G_ε for k large enough) and since F is Lipschitz on the compact interval containing all values of u and u_k (k large), it follows that

$$\lim_{k\to\infty} \int_{G_\varepsilon} F(x, u_k, Du)\,dx = \int_{G_\varepsilon} F(x, u, Du)\,dx.$$

Moreover, we have $D_w F(x, u_k, Du) \to D_w F(x, u, Du)$ uniformly in G_ε and hence strongly in $L^{p'}(G_\varepsilon)$ and $Du_k \rightharpoonup Du$ weakly in $L^p(\Omega; \mathbb{R}^n)$ and so in $L^p(G_\varepsilon; \mathbb{R}^n)$; thus

$$\int_{G_\varepsilon} D_w F(x, u_k, Du) \cdot (Du_k - Du)\, dx = \langle D_w F(x, u_k, Du), Du_k - Du\rangle_{L^{p'}, L^p} \longrightarrow 0.$$

Hence

$$\lim_{k\to\infty} \mathcal{E}(u_k) \geq \int_{G_\varepsilon} F(x, u, Du)\, dx,$$

for all $\varepsilon > 0$. Letting $\varepsilon \to 0$, we have

$$\int_{G_\varepsilon} F(x, u, Du)\, dx \to \int_\Omega F(x, u, Du) = \mathcal{E}(u)$$

by the monotone convergence theorem by our assumptions on G_ε from Step 2 and $F \geq 0$. □

3.4.9 Remark. The convexity of $w \mapsto F(x, z, w)$ is used to offset the fact that we only have $Du_k \rightharpoonup Du$ weakly (note the convexity does *not* imply, e.g., $Du_k \to Du$ a.e. up to a subsequence). We do not need $z \mapsto F(x, z, w)$ to be convex since $u_k \rightharpoonup u$ strongly in $L^p(\Omega)$.

3.4.10 Corollary. *Suppose $F \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ is coercive in the sense of* (3.4.5) *and $w \mapsto F(x, z, w)$ is convex in the sense of Theorem 3.4.7, where $\Omega \subset \mathbb{R}^n$ is bounded and open. Then the energy functional $\mathcal{E} : W_0^{1,p}(\Omega) \to \mathbb{R} \cup \{\infty\}$ (for given $1 < p < \infty$),*

$$\mathcal{E}(u) = \int_\Omega F(x, u, Du)\, dx,$$

admits a global minimiser $u^* \in W_0^{1,p}(\Omega)$.

Proof. Since $1 < p < \infty$, $W_0^{1,p}(\Omega)$ is reflexive; by (3.4.5), $\mathcal{E}$ is coercive in the sense of Definition 3.4.1, and by Theorem 3.4.7, $\mathcal{E}$ is weakly lower semicontinuous. Hence Theorem 3.4.3 is applicable. □

We now wish to show that any minimiser is in fact a (weak) solution of the corresponding Euler–Lagrange equation *formally* derived in Proposition 3.1.5. To do so wee need more assumptions on F.

3.4.11 Theorem. *Suppose that $F \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ satisfies the growth assumptions*

$$\begin{cases} |F(x, z, w)| \leq C(|z|^p + |w|^p + 1) \\ |D_z(x, z, w)| \leq C(|z|^{p-1} + |w|^{p-1} + 1) \\ |D_w(x, z, w)| \leq C(|z|^{p-1} + |w|^{p-1} + 1) \end{cases} \quad \text{for all } (x, z, w) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n, \tag{3.4.8}$$

for some $C > 0$, where $\Omega \subset \mathbb{R}^n$ is bounded and open and $1 < p < \infty$. If $u \in W_0^{1,p}(\Omega)$ satisfies

$$\mathcal{E}(u) = \min_{v \in W_0^{1,p}(\Omega)} \mathcal{E}(v), \qquad \text{where} \quad \mathcal{E}(v) = \int_\Omega F(x, u, Du)\, dx,$$

then u is a weak solution of (3.1.8) *in the sense of Definition 3.1.6.*

Proof. Fix $\varphi \in W_0^{1,p}(\Omega)$. For any $t \neq 0$ we have

$$\frac{\mathcal{E}(u+t\varphi)-\mathcal{E}(u)}{t} = \int_\Omega \underbrace{\frac{1}{t}\left(F(x,u+t\varphi,Du+tD\varphi)-F(x,u,Du)\right)}_{=:F^t(x)}\,dx.$$

Since F is C^1,

$$F^t(x) \to \sum_{i=1}^n \frac{\partial F}{\partial w_i}(x,u,Du)\frac{\partial \varphi}{\partial x_i} + \frac{\partial F}{\partial z}(x,u,Du)\varphi \tag{3.4.9}$$

pointwise a.e. in Ω as $t \to \infty$[9] and for a.e. $x \in \Omega$

$$\begin{aligned} F^t(x) &= \frac{1}{t}\int_0^t \frac{d}{ds}F(x,u+s\varphi,Du+sD\varphi)\,ds \\ &= \frac{1}{t}\int_0^t \sum_{i=1}^n \frac{\partial F}{\partial w_i}(x,u+s\varphi,Du+sD\varphi)\frac{\partial\varphi}{\partial x_i} + \frac{\partial F}{\partial z}(x,u+s\varphi,Du+sD\varphi)\varphi\,ds \end{aligned}$$

using the chain rule for Sobolev functions.[10] An elementary (but long) calculation using the bounds 3.4.8 and the inequality

$$ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'} \qquad \text{where } a,b \ge 0 \text{ and } \frac{1}{p}+\frac{1}{p'}=1,$$

shows that there exists $\widetilde{C} > 0$ such that for a.e. $x \in \Omega$

$$|F^t(x)| \le \widetilde{C}\left(|Du(x)|^p + |u(x)|^p + |D\varphi(x)|^p + |\varphi(x)|^p + 1\right),$$

where the right-hand side, which is independent of $t \neq 0$, is in $L^1(\Omega)$ since $u, \varphi \in W_0^{1,p}(\Omega)$. Hence the dominated convergence theorem may be applied to (3.4.9) to obtain

$$0 = \frac{d}{dt}\mathcal{E}(u+t\varphi) = \lim_{t\to 0}\int_\Omega F^t(x)\,dx = \int_\Omega \sum_{i=1}^n \frac{\partial F}{\partial w_i}(x,u,Du)\frac{\partial\varphi}{\partial x_i} + \frac{\partial F}{\partial z}(x,u,Du)\varphi\,dx$$

for all $\varphi \in W_0^{1,p}(\Omega)$, using that u is a minimiser for the first equality. □

3.4.12 Remark. In general it is possible for (3.1.9) to have solutions not corresponding to minima of $\mathcal{E}$. If, however, the mapping

$$(z,w) \mapsto F(x,z,w)$$

is convex for each $x \in \Omega$, i.e.

$$F(x,y,v) \ge F(x,z,w) + D_zF(x,z,w)\cdot(y-z) + D_wF(x,z,w)\cdot(v-w) \tag{3.4.10}$$

for all $y,z \in \mathbb{R}$ and all $v,w \in \mathbb{R}^n$, then each weak solution of the Euler–Lagrange equation is a (global) minimiser of $\mathcal{E}$. Indeed, suppose $u \in W_0^{1,p}(\Omega)$ solves (3.1.9) and take any other

[9] As in Proposition 3.1.5.
[10] See [5, Lemma 7.5].

$\tilde{u} \in W_0^{1,p}(\Omega)$. Letting $z = u(x)$, $y = \tilde{u}(x)$, $w = Du(x)$ and $v = D\tilde{u}(x)$ in (3.4.10) and integrating over Ω, we have

$$\mathcal{E}(\tilde{u}) = \int_\Omega F(x, \tilde{u}, D\tilde{u})\, dx \geq \underbrace{\int_\Omega F(x, u, Du)\, dx}_{=\mathcal{E}(u)} + \int_\Omega \frac{\partial F}{\partial z}(x, u, Du)(u - \tilde{u}) + \sum_{i=1}^n \frac{\partial F}{\partial w_i}(x, u, Du)\frac{\partial}{\partial x_i}(u - \tilde{u})\, dx.$$

Setting $\varphi := u - \tilde{u} \in W_0^{1,p}(\Omega)$, by virtue of (3.1.9) the second integral on the right is zero; hence $\mathcal{E}(u) \geq \mathcal{E}(\tilde{u})$ for each $\tilde{u} \in W_0^{1,p}(\Omega)$.

Finally for this section, we address the question of the uniqueness of the minimiser. Here we need additional assumptions on F, without which it is possible that multiple (global) minima could exist.

3.4.13 Theorem. *Suppose $F \in C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ (where $\Omega \subset \mathbb{R}^n$ is bounded and open) satisfies*

$$F(x, z, w) = F(x, 0, w) =: F(x, w) \qquad \text{for all } (x, z, w) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$$

(i.e. F does not depend on z) and there exists $\alpha > 0$ such that

$$\sum_{i,j=1}^n \frac{\partial^2 F}{\partial w_i \partial w_j}(x, w)\xi_i\xi_j \geq \alpha|\xi|^2 \qquad \text{for all } (x, w) \in \overline{\Omega} \times \mathbb{R}^n \text{ and all } \xi \in \mathbb{R}^n,$$

i.e. the mapping $w \mapsto F(x, w)$ is uniformly convex (and thus the Euler–Lagrange equation is strongly elliptic). Then there exists at most one minimiser in $W_0^{1,p}(\Omega)$ of

$$\mathcal{E}(u) = \int_\Omega F(x, u, Du)\, dx.$$

Proof. Assume $u, \tilde{u} \in W_0^{1,p}(\Omega)$ are two minimisers of $\mathcal{E}$ in $W_0^{1,p}(\Omega)$. We claim

$$\mathcal{E}\left(\frac{u + \tilde{u}}{2}\right) \leq \frac{\mathcal{E}(u) + \mathcal{E}(\tilde{u})}{2}, \tag{3.4.11}$$

with strict inequality unless $u = \tilde{u}$ a.e. in Ω. To see this, the uniform convexity implies

$$F(x, v) \geq F(x, w) + D_w F(x, w) \cdot (v - w) + \frac{\alpha}{2}|v - w|^2 \qquad \text{for all } x \in \Omega \text{ and all } v, w \in \mathbb{R}^n.$$

Set $w = (Du + D\tilde{u})/2$, $v = Du$, and integrate over Ω:

$$\mathcal{E}(u) \geq \mathcal{E}\left(\frac{u + \tilde{u}}{2}\right) + \int_\Omega D_w F\left(x, \frac{u + \tilde{u}}{2}\right) \cdot \left(\frac{Du - D\tilde{u}}{2}\right) dx + \frac{\alpha}{2}\int_\Omega \frac{|Du - D\tilde{u}|^2}{4}\, dx.$$

Interchanging the roles of u and $\tilde{u}$,

$$\mathcal{E}(\tilde{u}) \geq \mathcal{E}\left(\frac{u + \tilde{u}}{2}\right) + \int_\Omega D_w F\left(x, \frac{u + \tilde{u}}{2}\right) \cdot \left(\frac{D\tilde{u} - Du}{2}\right) dx + \frac{\alpha}{2}\int_\Omega \frac{|Du - D\tilde{u}|^2}{4}\, dx.$$

Summing these two identities, we have

$$\mathcal{E}(u) + \mathcal{E}(\tilde{u}) \geq 2\mathcal{E}\left(\frac{u+\tilde{u}}{2}\right) + \frac{\alpha}{2}\int_\Omega \frac{|Du - D\tilde{u}|^2}{4}\,dx,$$

i.e.

$$\mathcal{E}\left(\frac{u+\tilde{u}}{2}\right) + \frac{\alpha}{8}\int_\Omega \frac{|Du - D\tilde{u}|^2}{4}\,dx \leq \frac{\mathcal{E}(u) + \mathcal{E}(\tilde{u})}{2},$$

proving (3.4.11). Since

$$\mathcal{E}(u) = \mathcal{E}(\tilde{u}) \leq \mathcal{E}\left(\frac{u+\tilde{u}}{2}\right)$$

as u and $\tilde{u}$ are minimisers, this is only possible if $Du = D\tilde{u}$ a.e. in Ω, and so $\|Du - D\tilde{u}\|_{L^p(\Omega)} = 0$. Since $\|Dv\|_{L^p(\Omega)}$ gives an equivalent norm on $W_0^{1,p}(\Omega)$, this means $u = \tilde{u}$ in $W_0^{1,p}(\Omega)$, i.e. $u = \tilde{u}$ a.e. in Ω. □

3.4.14 Remark. Suppose $\partial\Omega$ is C^1 and $g \in L^p(\partial\Omega)$ is in $\operatorname{tr}(W^{1,p}(\Omega))$, say $g = \operatorname{tr} w$. The results of this section apply in the same way if we seek solutions $u \in W^{1,p}(\Omega)$ in the affine space

$$\{u \in W^{1,p}(\Omega) : \operatorname{tr} u = g\} = W_0^{1,p} + w.$$

In this way we may replace the boundary condition $u = 0$ on $\partial\Omega$ with $u = g$ on $\partial\Omega$.

3.5 Constraints

Often we wish to search for minimisers within a special class of functions satisfying additional constraints. Sometimes the constraints come from the (physical) problem itself; these typically lead to variational inequalities. Sometimes we impose "artificial" constraints on the admissible set of functions to obtain a solution with additional properties. We will illustrate this with a couple of prototypical examples.

Variational inequalities and a free boundary problem.

Let $\Omega \subset \mathbb{R}^n$ be bounded and open and suppose $h : \overline{\Omega} \to \mathbb{R}$ is a smooth (say C^2) function and $f \in L^2(\Omega)$. We minimise

$$\mathcal{E}(u) := \int_\Omega \frac{1}{2}|Du|^2 - fu\,dx$$

on the (convex) set

$$\mathcal{A} := \{u \in H_0^1(\Omega) : u \geq h \text{ a.e. in } \Omega\}.$$

Note that the minimiser in $H_0^1(\Omega)$ is a (weak) solution of $-\Delta u = f$ in Ω (if it exists). The function h is called an *obstacle*; the constraint $u \geq h$ is often referred to as *unilateral* (i.e. one-sided).

3.5.1 Theorem. *If $\mathcal{A} \neq \emptyset$, then there exists a unique function $u \in \mathcal{A}$ such that*

$$\mathcal{E}(u) = \inf_{v \in \mathcal{A}} \mathcal{E}(v).$$

Proof. 1. Existence follows as in Section 3.4: if $u_k \in \mathcal{A}$ is a minimising sequence with $u_k \rightharpoonup u$ weakly in $H_0^1(\Omega)$, then

$$\mathcal{E}(u) \leq \liminf_{k\to\infty} \mathcal{E}(u_k)$$

since $\|Du\|_{L^2(\Omega)} \leq \liminf_{k\to\infty} \|Du_k\|_{L^2(\Omega)}$ and $u_k \to u$ strongly in $L^2(\Omega)$ by Theorem 2.4.7. In particular, since $u_k \to u$ pointwise a.e. (up to a subsequence) and $u_k \geq h$ pointwise a.e., we also have $u \in \mathcal{A}$. (Thus $\mathcal{A}$ is weakly closed.)

2. Uniqueness: suppose $u, \tilde{u} \in \mathcal{A}$ are distinct, i.e. $u \neq \tilde{u}$ on a set of positive measure; then

$$v := \frac{u + \tilde{u}}{2} \in \mathcal{A}$$

and we claim

$$\mathcal{E}(v) < \frac{\mathcal{E}(u) + \mathcal{E}(\tilde{u})}{2}.$$

Indeed

$$\begin{aligned}
\mathcal{E}(v) &= \int_\Omega \frac{1}{2}\left|\frac{Du + D\tilde{u}}{2}\right|^2 - f\,\frac{u+\tilde{u}}{2}\,dx \\
&= \int_\Omega \frac{1}{8}(|Du|^2 + 2Du\cdot D\tilde{u} + |D\tilde{u}|^2) - f\,\frac{u+\tilde{u}}{2}\,dx \\
&= \int_\Omega \frac{1}{8}(2|Du|^2 + 2|D\tilde{u}|^2 - |Du - D\tilde{u}|^2) - f\,\frac{u+\tilde{u}}{2}\,dx \\
&< \frac{\mathcal{E}(u) + \mathcal{E}(\tilde{u})}{2},
\end{aligned}$$

where we have used $2Du \cdot D\tilde{u} = |Du|^2 + |D\tilde{u}|^2 - |Du - D\tilde{u}|^2$ and $|Du - D\tilde{u}|^2 \neq 0$ on a set of positive measure since $u \neq \tilde{u}$ by assumption. □

The analogue of the Euler–Lagrange equation is now an inequality; we speak of a variational characterisation of the minimum.

3.5.2 Theorem. *Let u be the unique minimiser of $\mathcal{E}$ on $\mathcal{A}$. Then*

$$\int_\Omega Du \cdot D(v-u)\,dx \geq \int_\Omega f(v-u)\,dx \qquad \textit{for all } v \in \mathcal{A}. \tag{3.5.1}$$

Proof. 1. Since $\mathcal{A}$ is convex, for any fixed $v \in \mathcal{A}$ and all $t \in [0,1]$,

$$u + t(v-u) = tv + (1-t)u \in \mathcal{A}.$$

Thus

$$\mathcal{E}(u) \leq \mathcal{E}(u + t(v-u)) \qquad \text{for all } t \in [0,1].$$

Hence[11]

$$\frac{d}{dt}\mathcal{E}(u + t(v-u))\Big|_{t=0} \geq 0.$$

[11] Note that this derivative exists, $\mathcal{E}$ being defined (and C^1) on the whole of $H_0^1(\Omega)$. The minimising property of u on $\mathcal{A}$ delivers the inequality ≥ 0.

2. If $0 < t \leq 1$, then

$$\frac{\mathcal{E}(u+t(v-u))-\mathcal{E}(u)}{t} = \frac{1}{t}\int_\Omega \frac{|Du+tD(v-u)|^2-|Du|^2}{2} - f\,(u+t(v-u)-u)\,dx$$
$$= \int_\Omega Du\cdot D(v-u) + \underbrace{t\,\frac{|D(v-u)|^2}{2}}_{\to 0} - f(v-u)\,dx.$$

Letting $t \to 0$, we conclude

$$\int_\Omega Du\cdot D(v-u) - f(v-u)\,dx \geq 0.$$

□

Although $\delta E(u)(v-u)$ exists for any $v \in \mathcal{A}$, we only have a one-sided estimate on its sign since we can only vary in one direction.

(Formal) interpretation: a free boundary problem

It can be shown that if $\partial\Omega$ is smooth enough, then the minimiser $u \in W^{2,\infty}(\Omega) \hookrightarrow C(\overline{\Omega})$ (cf. Theorem 2.4.5 for the embedding statement). Hence

$$\mathcal{O} := \{x \in \Omega : u(x) > h(x)\}$$
$$\mathcal{C} := \{x \in \Omega : u(x) = h(x)\}$$

are open and (relatively) closed (in Ω), respectively. We claim that

$$-\Delta u = f \quad \text{(weakly) in } \mathcal{O};$$

to see this, fix any $\varphi \in C_c^\infty(\mathcal{O})$; then for $|t|$ sufficiently small $u + t\varphi \geq h$, so $u + t\varphi \in \mathcal{A}$. For such φ, we may vary in both directions to obtain

$$\int_\Omega Du\cdot D\varphi - f\varphi\,dx = 0,$$

proving the claim (since $C_c^\infty(\mathcal{O})$ is dense in $H_0^1(\mathcal{O})$).

Now suppose $\varphi \in C_c^\infty(\Omega)$ satisfies $\varphi \geq 0$, so $u + t\varphi \in \mathcal{A}$ if $t > 0$. Thus

$$\int_\Omega Du\cdot D\varphi - f\varphi\,dx \geq 0.$$

Since $u \in W^{2,\infty}(\Omega)$, we may integrate by parts to obtain

$$\int_\Omega (-\Delta u - f)\varphi\,dx \geq 0 \qquad \text{for all } \varphi \in C_c^\infty(\Omega).$$

This can be shown to imply $-\Delta u = f$ a.e. in Ω.

Conclusion: u solves

$$\begin{cases} u \geq h, \;\; -\Delta u \geq f & \text{a.e. in } \Omega, \\ \qquad\quad\; -\Delta u = f & \text{in } \Omega \cap \{u > h\} = \mathcal{O}. \end{cases}$$

The set $F := \partial\mathcal{O} \cap \Omega$ is called the *free boundary*.

3.5.3 Remark. More generally, if V is a reflexive Banach space, $\mathcal{A}$ is a weakly closed set in V and $\mathcal{E} : \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ is coercive and weakly lower semicontinuous, then there exists $u \in \mathcal{A}$ with $\mathcal{E}(u) = \inf_{v \in \mathcal{A}} \mathcal{E}(v)$. If $\mathcal{A}$ is in addition convex and $\delta\mathcal{E}(u)(\varphi)$ exists for any $\varphi \in V$,[12] then u satisfies the variational inequality

$$\delta E(u)(v-u) \geq 0 \qquad \text{for all } v \in \mathcal{A}.$$

3.5.4 Example (Weak sub- and supersolutions: a variational version of Perron's method)**.** Suppose $\Omega \subset \mathbb{R}^n$ is smooth, bounded, and $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is smooth with a growth constraint: there exists a smooth $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$|g(x, u(x))| \leq \psi(\|u\|_{L^\infty(\Omega)}), \qquad u \in L^\infty(\Omega).$$

Denote by

$$G(x,u) = \int_0^u g(x,t)\,dt$$

the antiderivative of g with $G(x,0) = 0$. We consider

$$\begin{cases} -\Delta u = g(\,\cdot\,, u) & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega, \end{cases} \tag{3.5.2}$$

where $h \in \operatorname{tr} H^1(\Omega)$.

3.5.5 Definition. $u \in H^1(\Omega)$ is called a *(weak) subsolution* of (3.5.2) if $u \leq h$ a.e. on $\partial\Omega$ and

$$\int_\Omega Du \cdot D\varphi - g(\,\cdot\,, u)\varphi\,dx \leq 0 \qquad \text{for all } \varphi \in C_c^\infty(\Omega) \text{ with } \varphi \geq 0;$$

u is a *(weak) supersolution* if $-u$ is a weak subsolution.

3.5.6 Theorem. *Suppose $\underline{u}, \overline{u} \in H^1(\Omega)$ are weak sub- and supersolutions of* (3.5.2)*, respectively, such that*

$$-\infty < \underline{c} \leq \underline{u} \leq \overline{u} \leq \overline{c} < \infty \qquad a.e. \text{ in } \Omega$$

for some constants $\underline{c}, \overline{c} \in \mathbb{R}$. Then there exists a weak solution $u \in H^1(\Omega)$ of (3.5.2) *satisfying*

$$\underline{u} \leq u \leq \overline{u} \qquad a.e. \text{ in } \Omega.$$

Sketch of proof. [13] WLOG $h = 0$. The associated functional is

$$\mathcal{E}(u) = \int_\Omega \frac{1}{2}|Du|^2 - G(x,u)\,dx, \qquad u \in H_0^1(\Omega).$$

Note that $\mathcal{E}$ does not have to be bounded from below (or differentiable) on all of $H_0^1(\Omega)$; instead we consider

$$\mathcal{A} := \{u \in H_0^1(\Omega) : \underline{u} \leq u \leq \overline{u} \text{ a.e. in } \Omega\}.$$

Then by assumption $\mathcal{A} \subset L^\infty(\Omega)$ and there exists $c > 0$ such that

$$|G(x,u)| \leq c \qquad \text{for all } u \in \mathcal{A},$$

[12] We are also assuming implicitly that $\mathcal{E}(u) < \infty$.
[13] Full details can be found in [12, Section 1.2.3].

so that $\mathcal{E}$ is coercive on $\mathcal{A}$. The set $\mathcal{A}$ is weakly closed and convex and $\mathcal{E}$ is weakly lower semicontinuous on $\mathcal{A}$ since

$$\int_\Omega G(x,u_k)\,dx \to \int_\Omega G(x,u)\,dx \qquad \text{if } u_k \rightharpoonup u \text{ weakly in } \mathcal{A} \subset H_0^1(\Omega),$$

which follows from the dominated convergence theorem using $|G(x,u_k)| \le c$ for all $k \in \mathbb{N}$.

By Remark 3.5.3, there exists a minimiser of $\mathcal{E}$ in $\mathcal{A}$. To show it is a weak solution of (3.5.2), we fix $\varphi \in C_c^\infty(\Omega)$ and $\varepsilon > 0$ and set

$$v_\varepsilon := \min\{\overline{u}, \max\{\underline{u}, u+\varepsilon\varphi\}\} \in \mathcal{A}$$

(i.e. we cut it off at $\underline{u}$ and $\overline{u}$; this is then again in $H_0^1(\Omega)$ by an argument similar to Exercise 6). Then $\partial E(u)(v_\varepsilon - u)$ exists. A rather long calculation using $\delta\mathcal{E}(\overline{u})(\varphi) \ge 0$ and a passage to the limit shows $\delta\mathcal{E}(u)(\varphi) \ge 0$. Replacing φ with $-\varphi$ and repeating yields $\delta\mathcal{E}(u)(\varphi) = 0$ for all $\varphi \in C_c^\infty(\Omega)$; since $C_c^\infty(\Omega)$ it follows that u is genuinely a weak solution of (3.5.2). □

Lagrange multipliers and (nonlinear) eigenvalue problems.

Here we are interested in the special case in Remark 3.5.3 where $\mathcal{A}$ is the zero set of a mapping $J : V \to \mathbb{R}$ (V being a reflexive Banach space). More precisely, we assume $J \in C^1(V,\mathbb{R})$ (or $C^1(V,\mathbb{R}^n)$) and set

$$\mathcal{N} := \{u \in V : J(u) = 0\}.$$

We wish to find $u \in \mathcal{N}$ such that

$$\mathcal{E}(u) = \inf_{v\in\mathcal{N}} \mathcal{E}(v). \tag{3.5.3}$$

3.5.7 Theorem. *Suppose V is a reflexive Banach space and the set $\mathcal{N} = \{J = 0\}$ is weakly closed. If $\mathcal{E} : \mathcal{N} \to \mathbb{R} \cup \{\infty\}$, $\mathcal{E}(u) \not\equiv \infty$, is coercive and weakly lower semicontinuous, then there exists $u \in \mathcal{N}$ satisfying* (3.5.3).

The proof is exactly the same as for Remark 3.5.3. The question is which equation the minimiser satisfies: obviously $\partial\mathcal{E}(u)(\varphi) = 0$ only for some φ in general: more precisely, this holds for those φ which are tangent vectors to $\mathcal{N}$ at u.

3.5.8 Theorem (Lagrange multipliers)**.** *Suppose $V = H$ is a Hilbert space, $J \in C^1(H,\mathbb{R})$ is a mapping and $u \in \mathcal{N} = \{J = 0\}$ solves* (3.5.3) *under the assumptions of Theorem 3.5.7. If the linear mapping $J'(u) : H \to \mathbb{R}$ is surjective, then there exists $\lambda \in \mathbb{R}$ such that*

$$\delta\mathcal{E}(u)(\varphi) = \lambda J'(u)\varphi \qquad \textit{for all } \varphi \in H. \tag{3.5.4}$$

Proof. 1. We decompose H into the orthogonal direct sum

$$H = \underbrace{\ker J'(u)}_{=:H_1} \oplus \underbrace{(\ker J'(u))^\perp}_{=:H_0}.$$

Since $J'(u)$ is surjective, we have $\dim H_0 = 1$ and $J'(u)|_{H_0}$ is bijective with continuous inverse (as the inverse of a one-dimensional bounded linear map). We consider the mapping

$$\Lambda : \mathbb{R} \to \mathbb{R}, \qquad y \mapsto \delta\mathcal{E}(u)\left(\left(J'(u)|_{H_0}\right)^{-1}(y)\right).$$

This is linear, so there exists $\lambda \in \mathbb{R}$ such that $\Lambda(y) = \lambda y$ for all $y \in \mathbb{R}$. We claim that λ satisfies (3.5.4).

2. We first suppose that $\varphi = \varphi_0 \in H_0$ and check (3.5.4): indeed, setting $y := J'(u_0)\varphi_0$, we have

$$\lambda J'(u_0) = \lambda y = \delta\mathcal{E}(u)\left((J'(u_0)|_{H_0})^{-1}J'(u)\varphi\right) = \delta\mathcal{E}(u)(\varphi_0).$$

3. Now suppose $\varphi = \varphi_1 \in H_1$; since $H_0 \oplus H_1$ is an orthogonal decomposition, to finish the proof, it suffices to prove (3.5.4) for φ_1. In this case we have $J'(u)\varphi_1 = 0$ since $\varphi_1 \in \ker J'(u)$; we need to show

$$\delta\mathcal{E}(u)\varphi_1 = 0.$$

To that end we use that φ_1 is a variation which is tangential to $\mathcal{N}$ and u is a minimum. We use the implicit function theorem, which is valid in general Banach spaces:[14]

In a neighbourhood of $u \in \mathcal{N}$, the set $\mathcal{N}$ can be represented as the graph of a (C^1-) mapping $N : H_1 \to H_0$, i.e. for $v \in N$ near u, $v = v_0 + v_1 \in H_0 \oplus H_1$, we have $N(v_1) = v_0$.[15] In particular, $u = u_0 + u_1$ for some $u_0 \in H_0$ and $u_1 \in H_1$. We consider the path

$$\gamma : (-\varepsilon, \varepsilon) \to \mathcal{N}, \qquad t \mapsto (u_1 + t\varphi_1) + N(u_1 + t\varphi_1)$$

and the corresponding energies

$$\mathcal{F} := \mathcal{E} \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}.$$

We claim that $N'(u_1)\varphi_1 = 0$. This together with the fact that $\mathcal{E}$ and hence $\mathcal{F}$ are minimal for $t = 0$ yield

$$0 = \frac{d}{dt}\mathcal{F}(u_1 + t\varphi_1)\Big|_{t=0} = \delta\mathcal{E}(u)(\varphi_1 + N'(u_1)\varphi_1) = \delta\mathcal{E}(u)(\varphi_1),$$

as desired. The claim follows since $J(\gamma(t)) = 0$ for all $t \in (-\varepsilon, \varepsilon)$, which implies

$$0 = \frac{d}{dt}J(\gamma(0)) = J'(u)\gamma'(0) = J'(u)(\varphi_1 + N'(u_1)\varphi) = J'(u)N'(u_1)\varphi_1,$$

where the last equality follows since $J'(u)\varphi_1 = 0$ as $\varphi_1 \in \ker J'(u)$. Since $N'(u_1)\varphi_1 \in H_0 = (\ker J'(u))^\perp$, it follows that $N'(u_1)\varphi_1 = 0$, as claimed. $\square$

3.5.9 Example (Nonlinear eigenvalue problems)**.** We minimise

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega |Du|^2\, dx, \qquad u \in H_0^1(\Omega),$$

where $\Omega \subset \mathbb{R}^n$ is bounded, open and *connected*, subject to the following constraint: for a given C^1-function $G : \mathbb{R} \to \mathbb{R}$, $g = G'$, we assume

$$J(u) := \int_\Omega G(u)\, dx$$

Claim: If

$$|g(z)| \leq C(|z| + 1) \qquad \text{for all } z \in \mathbb{R},$$

so that also

$$|G(z)| \leq \widetilde{C}(|z|^2 + 1) \qquad \text{for all } z \in \mathbb{R},$$

then $\{J = 0\}$ is weakly closed.

[14] See [6, Theorem I.5.9].

[15] Note that here we have used the surjectivity assumption on $J'(u)$.

Proof. Suppose $u_k \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Then as usual $u_k \to u$ strongly in $L^2(\Omega)$ and so, since

$$|G(u(x)) - G(u_k(x))| \leq \sup_{z \in u(x), u_k(x)} |g(z)||u(x) - u_k(x)|$$

pointwise,

$$\begin{aligned} |J(u) - J(u_k)| \leq \int_\Omega |G(u) - G(u_k)|\, dx &\leq C \int_\Omega |u - u_k|(1 + |u| + |u_k|)\, dx \\ &\leq C\|u - u_k\|_{L^2(\Omega)} \underbrace{\|1 + |u| + |u_k|\|_{L^2(\Omega)}}_{\text{bounded}} \to 0. \end{aligned}$$

□

Theorem 3.5.7 yields a solution u; if $u \not\equiv 0$, we have in particular

$$J'(u)\varphi = \int_\Omega g(u)\varphi\, dx \qquad \text{for all } \varphi \in H_0^1(\Omega).$$

We claim $g(u) \not\equiv 0$ in Ω unless $u = 0$ a.e., from which follows the surjectivity of $J'(u)$ (just take $\varphi \in H_0^1(\Omega)$ such that

$$\int_\Omega g(u)\varphi\, dx \neq 0$$

and consider $t \mapsto t\varphi$, $t \in \mathbb{R}$). Suppose that in fact $g(u) = 0$ a.e. in Ω. By assumption,

$$D_x G(u(x)) = g(u(x))Du(x) = 0 \quad \text{a.e.}.$$

Since Ω is connected, $G(u)$ is constant a.e. in Ω.[16] Since

$$J(u) = \int_\Omega G(u)\, dx = 0,$$

in fact $G(u) = 0$ a.e.. Since $u = 0$ in the trace sense, for any $\varepsilon > 0$ the set $\{x \in \Omega : |u(x)| < \varepsilon\}$ has positive measure; as G is continuous, it follows that $G(0) = 0$. But then for u to be a minimiser, we need $u = 0$ a.e.; otherwise

$$\mathcal{E}(u) > \mathcal{E}(0) = 0,$$

and we have shown 0 to be admissible. (Obviously if $u = 0$ is a minimiser, then it will still trivially solve (3.5.4).)

So we assume $u \not\equiv 0$ and may then apply Theorem 3.5.8: there exists $\lambda \in \mathbb{R}$ such that

$$\int_\Omega Du \cdot D\varphi\, dx = \delta\mathcal{E}(u)(\varphi) = \lambda J'(u)\varphi = \lambda \int_\Omega g(u)\varphi\, dx \qquad \text{for all } \varphi \in H_0^1(\Omega).$$

This is the weak form of $-\Delta u = \lambda g(u)$. The value λ is given by

$$\frac{\int_\Omega |Du|^2\, dx}{\int_\Omega u g(u)\, dx}.$$

[16] Note that $G \circ u$ is weakly differentiable in Ω and by our growth assumptions and Cauchy–Schwarz $G \circ u \in W^{1,1}(\Omega)$ at least. Hence a standard variant of the Fundamental Lemma of the Calculus of Variations implies that having zero derivative a.e. is sufficient to ensure $G \circ u$ is genuinely constant a.e..

Special case:[17]

$$J(u) = \frac{1}{2}\left(\int_\Omega u^2\,dx - 1\right), \quad \text{i.e.} \quad G(z) = \frac{1}{2}\left(z^2 - \frac{1}{|\Omega|}\right),\ g(z) = z.$$

Then there exists $\psi_1 \in H_0^1(\Omega)$ such that

$$\int_\Omega \psi_1^2\,dx = 1 \quad \text{and} \quad -\Delta\psi_1 = \lambda\psi_1 \text{ weakly,}$$

where

$$\int_\Omega |D\psi_1|^2\,dx = \inf\left\{\int_\Omega |Dv|^2\,dx : \int_\Omega v^2 = 1\right\},$$

that is,[18]

$$\lambda = \inf_{0\neq v\in H_0^1(\Omega)} \frac{\int_\Omega |Dv|^2\,dx}{\int_\Omega v^2\,dx}, \tag{3.5.5}$$

which is > 0 by Poincaré's inequality, Theorem 2.4.4. This is the *variational characterisation* of the first (smallest) eigenvalue $\lambda = \lambda_1$ of the Dirichlet Laplacian; the quotient on the right is called the *Rayleigh quotient* (of v).

Note $-\Delta v = \mu v$ for $v \in H_0^1(\Omega)$ holds if and only if, by definition,

$$\int_\Omega Dv \cdot D\varphi\,dx = \mu\int_\Omega v\varphi\,dx \qquad \text{for all } \varphi \in H_0^1(\Omega),$$

so choosing $\varphi = v$, any such eigenvalue μ is given by

$$\mu = \frac{\int_\Omega |Dv|^2\,dx}{\int_\Omega v^2\,dx}.$$

In particular, λ_1 is the *smallest* eigenvalue; this also gives the optimal constant in Poincaré's inequality when $p = 2$.

Higher eigenvalues may be found inductively by replacing $H_0^1(\Omega)$ with the smaller Hilbert space $(\operatorname{span}\psi_1)^\perp$ to find a new minimiser ψ_2, then $(\operatorname{span}\{\psi_1,\psi_2\})^\perp$ for ψ_3, etc.. This yields a sequence of eigenpairs $(\lambda_k,\psi_k)_{k\in\mathbb{N}}$,

$$0 < \lambda_1 \le \lambda_2 \le \ldots.$$

The spectral theorem guarantees that this sequence is discrete and $\{\psi_k\}$ can be chosen to be an orthonormal basis of $L^2(\Omega)$.

3.5.10 Example. Suppose $\Omega \subset \mathbb{R}^n$ is bounded and open with C^1-boundary and let $p > 2$; if $n \ge 3$, we also assume $p < 2^* = 2n/(n-2)$ (cf. Definition 2.4.1). For $\lambda \in \mathbb{R}$ we consider

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5.6}$$

Claim: Denote by λ_1 the first eigenvalue from Example 3.5.9. Then for any $\lambda > -\lambda_1$ there exists a nonzero, positive (a.e.) weak solution $u \in H_0^1(\Omega)$ of (3.5.6).

[17] Obviously $u = 0$ cannot be a solution here.
[18] Since both $\int_\Omega |Dv|^2\,dx$ and $\int_\Omega v^2\,dx$ scale the same way if v is multiplied by a constant.

Proof. First note that under our assumptions on p, $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ and this embedding is compact (Theorems 2.4.3 and 2.4.7). The functional corresponding to (3.5.6) is

$$\widetilde{\mathcal{E}}u = \frac{1}{2}\int_\Omega |Du|^2 + \lambda u^2\,dx - \frac{1}{p}\int_\Omega |u|^p\,dx,$$

but since $\|u\|^2_{L^2(\Omega)}$ and $\|u\|^p_{L^p(\Omega)}$ scale differently, this is not bounded from below (or above) on $H_0^1(\Omega)$. Instead we apply Theorem 3.5.8 to

$$\mathcal{E}(u) := \frac{1}{2}\int_\Omega |Du|^2 + \lambda u^2\,dx \qquad\qquad \text{on } H_0^1(\Omega),$$

with[19]

$$J(u) := \int_\Omega |u|^p\,dx - 1;$$

as before, $\{J = 0\}$ is weakly closed, since $u_k \rightharpoonup u$ weaky in $H_0^1(\Omega)$ implies $u_k \to u$ strongly in $L^p(\Omega)$, and if $J(u) = 0$, then

$$\int_\Omega |u|^p\,dx = 1,$$

so $J'(u)$ is surjective, since

$$J'(u)\varphi = \int_\Omega |u|^{p-2}u\varphi\,dx$$

(and now take, e.g., $\varphi = tu$, $t \in \mathbb{R}$). As usual, $\mathcal{E}$ is weakly lower semicontinuous, cf. Theorem 3.4.7. We check coercivity: we may assume $\lambda \in (-\lambda_1, 0)$, since otherwise this is immediate. Then for $u \in H_0^1(\Omega)$, since

$$\int_\Omega |Du|^2\,dx \geq \lambda_1 \int_\Omega u^2\,dx,$$

we have

$$\begin{aligned}\mathcal{E}(u) &= \frac{1}{2}\int_\Omega \underbrace{-\frac{\lambda}{\lambda_1}}_{\in(0,1)}|Du|^2 + \underbrace{\left(1+\frac{\lambda}{\lambda_1}\right)}_{\in(0,1)}|Du|^2 + \lambda|u|^2\,dx \\ &\geq \int_\Omega -\lambda u^2 + \left(1+\frac{\lambda}{\lambda_1}\right)|Du|^2 + \lambda u^2\,dx = \frac{1}{2}\left(1+\frac{\lambda}{\lambda_1}\right)\int_\Omega |Du|^2\,dx \geq C\|u\|^2_{H_0^1(\Omega)}.\end{aligned}$$

Theorem 3.5.8 yields the existence of $u \in H_0^1(\Omega)$ and $\mu \in \mathbb{R}$ such that

$$\int_\Omega Du\cdot D\varphi + \lambda u\varphi\,dx = \delta\mathcal{E}(u)(\varphi) = \mu J'(u)\varphi = \mu\int_\Omega |u|^{p-2}u\varphi\,dx \tag{3.5.7}$$

for all $\varphi \in H_0^1(\Omega)$.

Observe that $\mathcal{E}(u) = \mathcal{E}(|u|)$ and $J(u) = J(|u|)$;[20] thus if u is a minimiser, then so is $|u|$, and hence we may assume that u in (3.5.7) is positive. Setting $\varphi = u$ in (3.5.7),

$$2\mathcal{E}(u) = \mu\underbrace{\int_\Omega |u|^p\,dx}_{=1\,\Rightarrow\,u\neq 0} = \mu > 0.$$

[19] Alternatively, we could apply Remark 3.5.3 to $\mathcal{E}$ on $\mathcal{A} := \{u \in H_0^1(\Omega) : \int_\Omega |u|^p\,dx = 1\}$.

[20] Cf. Exercise 6: if $u \in H_0^1(\Omega)$ then $|u| \in H_0^1(\Omega)$ and $|D|u|| = |Du|$ as L^2-functions.

We now rescale u: set

$$u^* := \mu^{\frac{1}{p-2}} u \in H_0^1(\Omega);$$

then (3.5.7) implies

$$\int_\Omega Du^* \cdot D\varphi + \lambda u^* \varphi \, dx = \int_\Omega |u^*|^{p-2} u^* \varphi \, dx \qquad \text{for all } \varphi \in H_0^1(\Omega).$$

□

4 Critical points and minimax methods

We will now search for saddle points of $\mathcal{E}$; i.e. $u \in V$ (Banach) such that $\mathcal{E}'(u) = 0$ but $\mathcal{E}$ does not necessarily reach a maximum or a minimum at u. This requires studying the topology of the sublevel sets

$$S_\beta = \{u \in V : \mathcal{E}(u) \le \beta\}, \qquad \beta \in \mathbb{R}.$$

To that end, we will always assume that $\mathcal{E} \in C^1(V,\mathbb{R})$ (at least), that is, $\mathcal{E}'$ exists as a Fréchet derivative (and is continuous).

Notation: For $\beta \in \mathbb{R}$ we will always write

(a) $S_\beta := \{u \in V : \mathcal{E}(u) \le \beta\}$ for the sublevel set of $\mathcal{E}$ (at the level β);

(b) $K_\beta := \{u \in V : \mathcal{E}(u) = \beta \text{ and } \mathcal{E}'(u) = 0\}$ for the critical points of $\mathcal{E}$ at the level β.

4.1 Mountain pass theorems

Suppose $\mathcal{E} \in C^1(\mathbb{R}^n,\mathbb{R})$ is coercive in the sense that S_β is bounded for each $\beta \in \mathbb{R}$. Then one way to find saddle points is as follows.

4.1.1 Theorem. *Suppose $x_1 \neq x_2$ are local minima of the coercive function $\mathcal{E} \in C^1(\mathbb{R}^n,\mathbb{R})$, i.e. there exist open neighbourhoods $U_1 \ni x_1$ and $U_2 \ni x_2$ such that*

$$\mathcal{E}(x_1) \le \mathcal{E}(y) \quad \text{for all } y \in U_1, \qquad \mathcal{E}(x_2) \le \mathcal{E}(z) \quad \text{for all } z \in U_2.$$

Then $\mathcal{E}$ has a critical point $x_3 \neq x_1, x_2$ given by

$$\mathcal{E}(x_3) = \inf_{\gamma\in\Gamma} \max\{\mathcal{E}(\gamma(t)) : t \in [0,1]\},$$

where

$$\Gamma = \{\gamma \in C([0,1],\mathbb{R}^n) : \gamma(0) = x_1, \gamma(1) = x_2\}.$$

We think of the sets γ as being "paths" between the low points ("valleys") at x_1 and x_2. If we take the lowest possible path connecting x_1 to x_2, then at its highest point we must cross a "mountain ridge" (at a "mountain pass", a low point along this ridge).

We will not give the proof of Theorem 4.1.1 in the finite-dimensional case,[1] but instead wish to generalise it to the infinite-dimensional case. Having two local minima is however too restrictive. What do we need?

Principle 1: The set $\{x : \mathcal{E}(x) \le \beta\}$ should be disconnected for some (energy level) $\beta \in \mathbb{R}$. However, even in the finite-dimensional case this is not enough: consider

$$\mathcal{E}(x,y) = e^{-y} - x^2, \qquad (x,y) \in \mathbb{R}^2.$$

[1] This can be found in [12, Section II.1].

Since $\mathcal{E}(0,y) > 0$ for all $y \in \mathbb{R}$, the set

$$\operatorname{int} S_0 = \{(x,y) \in \mathbb{R}^2 : \mathcal{E}(x,y) < 0\}$$

is disconnected (and consists of two connected components). Clearly,

$$\inf\{\max\{\mathcal{E}(x,y) : (x,y) \in \gamma\} : \gamma \text{connects these two connected components}\} = 0,$$

but $0 \neq \max_\gamma \mathcal{E}$ for any admissible γ.

If we take a minimising sequence of paths γ_k, then WLOG $\mathcal{E}$ reaches its maximum on γ_k at a point $(0, y_k)$, $y_k \to \infty$. Then $\mathcal{E}((0,y_k)) \to 0$, $\nabla\mathcal{E}((0,y_k)) \to 0$, but the sequence $((0,y_k))_{k\in\mathbb{N}}$ does not have an accumulation point.

Principle 2: We need a "compactness" assumption on $\mathcal{E}$.

Now suppose $\mathcal{E} \in C^1(V, \mathbb{R})$, where V is a (reflexive) Banach space.

4.1.2 Definition. (a) A sequence (u_k) in V is a *Palais–Smale (P.-S.) sequence* (for $\mathcal{E}$) if

(i) $(\mathcal{E}(u_k))$ is bounded (in $\mathbb{R}$), and

(ii) $\|E'(u_k)\|_{V'} \to 0$ as $k \to \infty$.

(b) The functional $\mathcal{E}$ satisfies the *Palais–Smale condition* if every Palais–Smale sequence is precompact in V, i.e., it contains a convergent subsequence.

There are several variants of this condition in the literature, although this is probably the most common.

4.1.3 Example. (a) Suppose $\mathcal{E} \in C^1(\mathbb{R}^n, \mathbb{R})$ is such that the function $|\nabla\mathcal{E}| + |\mathcal{E}| : \mathbb{R}^n \to \mathbb{R}$ is coercive (in the sense that its sublevel sets are bounded). Then every P.-S. sequence is bounded and so has a convergent subsequence; hence $\mathcal{E}$ satisfies the Palais–Smale condition.

(b) If $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}$ is a quadratic polynomial:

$$\mathcal{E}(x) = \sum_{i,j=1}^n a_{ij}x_ix_j + \sum_{i=1}^n b_ix_i + c, \qquad x = (x_1, \ldots, x_n) \in \mathbb{R}^n,\ a_{ij}, b_i, c \in \mathbb{R}$$

such that $D^2\mathcal{E}(x) = (a_{ij})$ is invertible, then $\mathcal{E}$ satisfies the Palais–Smale condition. This seems to be unknown if $\mathcal{E}$ is instead a general polynomial $\mathcal{E}(x) = \sum_{|\alpha|\leq m} a_\alpha x^\alpha$ such that $D^2\mathcal{E}(x)$ is non-degenerate for each $x \in \mathbb{R}^n$.[2]

For the rest of this section, we will assume $V = H$ is a Hilbert space.

4.1.4 Lemma. *Suppose $\mathcal{E} \in C^1(H, \mathbb{R})$. Then for each $u \in H$ there exists a unique vector $v \in H$ such that*

$$\mathcal{E}'(u)\varphi = (v, \varphi)_H \qquad \text{for all } \varphi \in H.$$

In this case we write $v = \nabla\mathcal{E}(u)$ or, if there is no danger of confusion, $v = \mathcal{E}'(u)$. Note also that

$$\|\mathcal{E}'(u)\|_{\mathcal{L}(H,\mathbb{R})} = \|\nabla\mathcal{E}(u)\|_H.$$

[2] See [12, Section II.2].

Proof. Since $\mathcal{E}'(u)$ is, for any fixed u, a bounded linear map from H to $\mathbb{R}$, this follows immediately from the Riesz Representation Theorem. □

We will also assume for the rest of the section that $\mathcal{E}' : H \to H$ is Lipschitz continuous on bounded sets; we write $\mathcal{E} \in C^{1,1}_{loc}(H, \mathbb{R})$.

We recall that $u \in H$ is a critcal point of $\mathcal{E}$ if $\mathcal{E}'(u) = 0$ (in H) and say that β is a *critical value* (of $\mathcal{E}$) if the set of critical points $K_\beta \neq \emptyset$.

Principle: if $\beta \in \mathbb{R}$ is *not* a critical value of $\mathcal{E}$ (and $\mathcal{E}$ is "nice enough"), then we can smoothly deform the set $S_{\beta+\varepsilon}$ into the set $S_{\beta-\varepsilon}$ for $\varepsilon > 0$ small enough

4.1.5 Theorem (Deformation theorem)**.** *Suppose $\mathcal{E} \in C^{1,1}_{loc}(H, \mathbb{R})$ satisfies the Palais–Smale condition and suppose $K_\beta = \emptyset$ for some β. Then for any sufficiently small $\varepsilon > 0$, there exists $\delta > 0$ and a function (family of deformations)*

$$\Phi \in C([0,1] \times H, H)$$

such that the mappings

$$\Phi_t(u) \equiv \Phi(t, u) \qquad (0 \leq t \leq 1, u \in H)$$

satisfy

(i) $\Phi_0 = Id$*, i.e.* $\Phi_0(u) = u$ *for all* $u \in H$*;*

(ii) $\Phi_1(u) = u$ *if* $\mathcal{E}(u) \notin [\beta - \varepsilon, \beta + \varepsilon]$*;*

(iii) $\mathcal{E}(\Phi_t(u)) \leq \mathcal{E}(u)$ *for all* $u \in H$ *and all* $t \in [0,1]$*;*

(iv) $\Phi_1(S_{\beta+\delta}) \subset S_{\beta-\delta}$.

The idea here is to solve an appropriate ordinary differential equation (ODE) in H, modelled on $\frac{d\Phi}{dt} = \mathcal{E}'(\Phi(u))$, and following the resulting flow "downhill"; this can be done if no critical points are in the way. (We will see similar ideas later, in Chapter 5, when we come to gradient flows.)

We will need the following lemma. For $A, B \subset H$ and $u \in H$ we denote by

$$\begin{aligned} \operatorname{dist}(u, A) &= \inf\{\|u - v\|_H : v \in A\}, \\ \operatorname{dist}(A, B) &= \inf\{\|u - v\|_H : u \in A, v \in B\} = \inf\{\operatorname{dist}(u, B) : u \in A\}. \end{aligned}$$

4.1.6 Lemma. *For any nonempty $A \subset H$, the function $u \mapsto \operatorname{dist}(u, A)$ is Lipschitz continuous (with Lipschitz constant 1).*

Proof. By the triangle inequality applied to sequences approaching the corresponding infimum, if $u, v \in H$, then

$$\begin{aligned} \operatorname{dist}(u, A) &\leq \|u - v\|_H + \operatorname{dist}(v, A) \\ \operatorname{dist}(v, A) &\leq \|u - v\|_H + \operatorname{dist}(u, A). \end{aligned}$$

□

Proof of Theorem 4.1.5. 1. The Palais–Smale condition implies the existence of constants $0 < \gamma, \varepsilon (< 1)$ such that

$$\|\mathcal{E}'(u)\|_H \geq \gamma \qquad \text{for all } u \in S_{\beta+\varepsilon} \setminus S_{\beta-\varepsilon} \equiv \{v \in H : \beta - \varepsilon < \mathcal{E}(v) \leq \beta + \varepsilon\}. \tag{4.1.1}$$

Indeed, if not, we could find sequences $\gamma_k \to 0$, $\varepsilon_k \to 0$ and $u_k \in H$ such that

$$u_k \in S_{\beta+\varepsilon_k} \setminus S_{\beta-\varepsilon_k}, \qquad \|\mathcal{E}'(u_k)\|_H \leq \gamma_k.$$

This says exactly that (u_k) is a P.-S. sequence, so up to a subsequence $u_k \to u$ in H. Since $\mathcal{E} \in C^1$, it follows that $\mathcal{E}(u) = \beta$ and $\mathcal{E}'(u) = 0$, contradicting $K_\beta = \emptyset$.

2. Now fix $\delta > 0$ such that

$$0 < \delta < \min\left\{\varepsilon, \frac{\gamma^2}{2}\right\}$$

and set

$$A := \{u \in H : \mathcal{E}(u) \leq \beta - \varepsilon \text{ or } \mathcal{E}(u) \geq \beta + \varepsilon\}$$
$$B := \{u \in H : \beta - \delta \leq \mathcal{E}(u) \leq \beta + \delta\}.$$

Claim: On any *bounded* set $M \subset H$, there exists a constant $c = c_M > 0$ such that

$$\operatorname{dist}(u, A) + \operatorname{dist}(u, B) \geq c_M \qquad \text{for all } u \in M.$$

Indeed, if $\operatorname{dist}(M, A)$ or $\operatorname{dist}(M, B) > 0$, then there is nothing to prove; otherwise, by enlarging M if necessary, WLOG we have $A \cap M, B \cap M \neq \emptyset$ and hence

$$\operatorname{dist}(u, A) + \operatorname{dist}(u, B) \geq \operatorname{dist}(M \cap A, M \cap B)$$

by the triangle inequality. Now since

$$|\mathcal{E}(u) - \mathcal{E}(v)| \leq \sup_{w \in M} |\mathcal{E}'(w)| \|u - v\|_H,$$

where $\sup_{w \in M} |\mathcal{E}'(w)| < \infty$ by assumption, and

$$|\mathcal{E}(u) - \mathcal{E}(v)| \geq \varepsilon - \delta \qquad \text{if } u \in A,\ v \in B,$$

we have

$$\|u - v\|_H \geq \frac{\varepsilon - \delta}{\sup_{w \in M} |\mathcal{E}'(w)|} \qquad \text{for all } u \in M \cap A \text{ and all } v \in M \cap B.$$

This proves the claim.

Combined with Lemma 4.1.6, it follows that the function

$$g(u) := \frac{\operatorname{dist}(u, A)}{\operatorname{dist}(u, A) + \operatorname{dist}(u, B)}, \qquad u \in H,$$

is (well defined and) Lipschitz continuous on bounded sets, i.e., locally Lipschitz, with

$$0 \leq g \leq 1, \quad g|_A = 0, \quad g|_B = 1.$$

Finally, we let $\xi : H \to H$ be defined by

$$\xi(u) := -g(u)\frac{\mathcal{E}'(u)}{\max\{1, \|\mathcal{E}'(u)\|_H\}}, \qquad u \in H, \tag{4.1.2}$$

which is once again bounded and locally Lipschitz.

3. We now consider, for any given $u \in H$, the ODE[3]

$$\begin{aligned}\frac{d\Phi}{dt}(t) &= \xi(\Phi(t))\\ \Phi(0) &= u.\end{aligned}$$

Since ξ is bounded and locally Lipschitz continuous, the Picard–Lindelöf theorem[4] yields the existence of a solution for all $t \in [0,1]$, which we denote by

$$\Phi(t,u) \equiv \Phi_t(u);$$

then obviously $\Phi \in C([0,1]\times H, H)$ satisfies (i), and if $\mathcal{E}(u) \notin [\beta-\varepsilon, \beta+\varepsilon]$, then $u \in A$, so that $g(u) = 0$ and $\xi(u) = 0$. It follows that $\Phi_t(u) = u$ for all $t \geq 0$, so (ii) is satisfied.

4. To show (iii), we compute, using the chain rule, the ODE property, the definition of ξ and the fact that $g \geq 0$, respectively,

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(\Phi_t(u)) &= \left(\mathcal{E}'(\Phi_t(u)), \frac{d}{dt}\Phi_t(u)\right)_H\\ &= (\mathcal{E}'(\Phi_t(u)), \xi(\Phi_t(u))_H\\ &= -g(\Phi_t(u))\,\frac{1}{\max\{1, \|\mathcal{E}'(u)\|_H\}}\,\|\mathcal{E}'(\Phi_t(u))\|_H^2 \leq 0\end{aligned}$$

for all $u \in H$ and all $t \in [0,1]$.

5. Finally, to show (iv), fix any $u \in S_{\beta+\delta}$. If $\Phi_{t_0}(u) \notin B$ for some $t_0 \in [0,1]$, then by (iii) it follows that $\mathcal{E}(\Phi_t(u)) < \beta - \delta$ for all $t \geq t_0$, so there is nothing to prove. Hence suppose $\Phi_t \in B$ for all $t \in [0,1]$. Then $g(\Phi_t(u)) = 1$ for all $t \in [0,1]$ and

$$\frac{d}{dt}\mathcal{E}(\Phi_t(u)) = -\frac{\|\mathcal{E}'(\Phi_t(u))\|_H^2}{\max\{1, \|\mathcal{E}'(\Phi_t(u))\|_H\}}.$$

If $\|\mathcal{E}'(\Phi_t(u))\|_H \geq 1$, then

$$\frac{d}{dt}\mathcal{E}(\Phi_t(u)) = -\|\mathcal{E}'(\Phi_t(u))\|_H \leq -\gamma \leq -\gamma^2$$

by (4.1.1), as $\Phi_t(u) \in B \subset S_{\beta+\varepsilon} \setminus S_{\beta-\varepsilon}$, and $\gamma \in (0,1)$. If $\|\mathcal{E}'(\Phi_t(u))\|_H \geq 1$, then also

$$\frac{d}{dt}\mathcal{E}(\Phi_t(u)) \leq -\|\mathcal{E}'(\Phi_t(u))\|_H^2 \leq -\gamma^2$$

by (4.1.1). It follows from the fundamental theorem of calculus that[5]

$$\mathcal{E}(\Phi_1(u)) = \int_0^1 \frac{d}{dt}\mathcal{E}(\Phi_t(u))\,dt + \mathcal{E}(\Phi_0(u)) \leq \mathcal{E}(u) - \gamma^2 \leq \beta + \delta - \gamma^2 \leq \beta - \delta$$

by choice of δ. Thus $\Phi_1(u) \in S_{\beta-\delta}$, as desired. □

[3] Observe that, up to our cut-off argument, we are essentially solving $\frac{d\Phi}{dt} = -\mathcal{E}'(\Phi(t))$.

[4] Obviously valid for a Banach space-valued ODE, since Banach's fixed point theorem is equally valid in a general Banach space.

[5] Note $\frac{d}{dt}\mathcal{E}(\Phi_t(u))$ is continuous as the composition of continuous functions.

4.1.7 Remark. (a) We actually only needed a weaker, local form of the Palais–Smale condition: if for any sequence (u_k) such that there exists $\varepsilon > 0$ with $|\mathcal{E}(u_k) - \beta| \leq \varepsilon$ and $\mathcal{E}'(u_k) \to 0$ as $k \to \infty$, then the sequence is precompact. We say $\mathcal{E}$ satisfies a *local Palais–Smale condition* at $\beta \in \mathbb{R}$.

(b) If $\mathcal{E}$ is even, i.e. $\mathcal{E}(-u) = \mathcal{E}(u)$ for all $u \in H$, then it is possible to choose Φ_t to be odd. To see this, note that then $\mathcal{E}' : H \to H$ is odd, and since $u \in A$ or B if and only if $-u \in A$ or B, g is even in this case. Hence ξ in (4.1.2) is odd, and so the solution Φ as well.

(c) Suppose $K_\beta \neq \emptyset$ is compact for some $\beta \in \mathbb{R}$. Then if $\mathcal{O} \supset K_\beta$ is any open neighbourhood of K_β, we may obtain, with essentially the same method of proof, a deformation Φ_t satisfying (i)-(iii) of Theorem 4.1.5, the conclusion of Remark 4.1.7(b) and

(iv') $\Phi_1(S_{\beta+\delta} \setminus \mathcal{O}) \subset S_{\beta-\delta}$.

(d) The theorem (and remarks) continue to hold if H is replaced either with a (smooth) *Hilbert manifold* M, or more generally a Banach space/manifold V. (The proofs are far more technical and require more sophisticated tools.) A Hilbert manifold M is a separable Hausdorff space in which each point $u \in M$ has a neighbourhood which is homeomorphic to a Hilbert space, e.g. $M = \{u \in H : \|u\|_H = r\}$.

More details on (c) and (d) can be found in [8, Appendix A].

Now we can give a model mountain pass theorem for functionals.

4.1.8 Theorem (Mountain pass theorem)**.** *Suppose that* $\mathcal{E} \in C^{1,1}_{loc}(H, \mathbb{R})$ *satisfies the Palais–Smale condition of Definition 4.1.2 and*

(i) $\mathcal{E}(0) = 0$,

(ii) there exist $r, \alpha > 0$ *such that* $\mathcal{E}(u) \geq \alpha$ *whenever* $\|u\|_H = r$, *and*

(iii) there exists $v \in H$ *such that* $\|v\|_H > r$ *and* $\mathcal{E}(v) \leq 0$.

Set

$$\Gamma := \{\gamma \in C([0,1], H) : \gamma(0) = 0, \gamma(1) = v\}.$$

Then

$$\beta := \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \mathcal{E}(\gamma(t)) \tag{4.1.3}$$

is a critical value of $\mathcal{E}$, *i.e.* $K_\beta \neq \emptyset$.

Obviously, by considering $\widetilde{\mathcal{E}}(x) := \mathcal{E}(x - y)$ for fixed $y \in H$, we may shift the location of the points 0 and v arbitrarily. We also only need to assume the local Palais–Smale condition at β given by (4.1.3).

Proof. By construction, $\beta \geq \alpha$. Suppose β is not a critical value, so that $K_\beta = \emptyset$. Fix any $\varepsilon \in (0, \alpha/2)$ sufficiently small. Then by Theorem 4.1.5, there exist $\delta \in (0, \varepsilon)$ and $\Phi = \Phi_1 : H \to H$ such that $\Phi(S_{\beta+\delta}) \subset S_{\beta-\delta}$ and $\Phi(u) = u$ if $\mathcal{E}(u) \notin [\beta - \varepsilon, \beta + \varepsilon]$. Now suppose $\gamma \in \Gamma$ satisfies

$$\max_{0\leq t\leq 1} \mathcal{E}(\gamma(t)) \leq \beta + \delta.$$

Then $\Phi \circ \gamma \in \Gamma$ as well, since $\Phi(\gamma(0)) = \Phi(0) = 0$ and $\Phi(\gamma(1)) = \Phi(v) = v$ since $\mathcal{E}(v) \leq 0 \notin [\beta - \varepsilon, \beta + \varepsilon]$. But then, by construction of Φ,

$$\max_{0 \leq t \leq 1} \mathcal{E}(\Phi(\gamma(t))) \leq \beta - \delta,$$

contradicting (4.1.3). □

4.2 An application

Consider the semilinear problem (nonlinear Poisson equation)

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \Omega, \end{cases} \tag{4.2.1}$$

where $\Omega \subset \mathbb{R}^n$ is bounded and open, $n \geq 2$, and f is smooth, say $f \in C^1(\mathbb{R}, \mathbb{R})$, and satisfies the growth conditions

$$\begin{matrix} |f(z)| \leq C(1 + |z|^p) \\ |f'(z)| \leq C(1 + |z|^{p-1}) \end{matrix}, \qquad z \in \mathbb{R}, \tag{4.2.2}$$

where $C > 0$ is a constant and $1 < p < \frac{n+2}{n-2} = 2^* - 1$ (the "critical exponent"; see Section 7.3). Setting

$$F(z) := \int_0^z f(s)\, ds, \qquad z \in \mathbb{R},$$

we also assume there exist constants $\widetilde{C}, c_1, c_2 > 0$, with $\widetilde{C} < \frac{1}{2}$, such that

$$\begin{matrix} 0 \leq F(z) & \leq \widetilde{C} f(z) z \\ c_1 |z|^{p+1} \leq |F(z)| & \leq c_2 |z|^{p+1} \end{matrix}, \qquad z \in \mathbb{R}. \tag{4.2.3}$$

In particular, $F(0) = 0$. A model f is $f(z) = |z|^{p-1} z$, i.e. $-\Delta u = |u|^{p-1} u$.[6]

4.2.1 Theorem. *Under the above assumptions,* (4.2.1) *has at least one weak solution* $u \not\equiv 0$.

Such solutions, which are found via application of a mountain pass theorem, are often referred to as being of mountain pass type.

Proof. 1. We wish to apply Theorem 4.1.8 to

$$\mathcal{E}(u) = \int_\Omega \frac{1}{2} |Du|^2 - F(u)\, dx, \qquad u \in H_0^1(\Omega),$$

since solutions of $\mathcal{E}'(u)\varphi = 0$ for all $\varphi \in H_0^1(\Omega)$ are (by definition) weak solutions of (4.2.1). We work in $H = H_0^1(\Omega)$ with norm

$$\|u\|_H = \|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |Du|^2\, dx \right)^{1/2}$$

[6] This particular f was already treated in Example 3.5.10 via completely different methods – note that $\lambda = 0$ is always allowed in (3.5.6) and that the p of Example 3.5.10 corresponds to $p + 1$ here.

and corresponding inner product

$$(u, v)_H = \int_\Omega Du \cdot Dv \, dx.$$

Then

$$\mathcal{E}(u) = \frac{1}{2}\|u\|_H^2 - \int_\Omega F(u)\, dx =: \mathcal{E}_1(u) - \mathcal{E}_2(u).$$

2. *Claim:* $\mathcal{E} \in C^{1,1}_{loc}(H, \mathbb{R})$.

Proof. Here we need the assumptions on f and F. For $u, v \in H$, since $\mathcal{E}_1(u) = \frac{1}{2}\|u\|_H^2$,

$$(\mathcal{E}_1'(u), v)_H = \mathcal{E}_1'(u)v = \int_\Omega Du \cdot Dv \, dx = (u, v)_H.$$

Thus $\mathcal{E}_1'(u)$ exists and equals u for all $u \in H$, cf. Lemma 4.1.4. Hence $\mathcal{E}_1 \in C^{1,1}_{loc}$.
For $\mathcal{E}_2$, we first recall that for each $g \in H^{-1}(\Omega) = H'$ there is a unique weak solution $v \in H_0^1(\Omega)$ of

$$\begin{cases} -\Delta v = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega; \end{cases} \tag{4.2.4}$$

see Theorem 2.6.5 and Remark 2.6.6 (and note that we may choose $\omega = 0$ for the Laplacian, cf. Theorem 2.6.4). It follows that the mapping

$$K : H^{-1}(\Omega) \to H_0^1(\Omega), \qquad g \mapsto v,$$

is a bijection. Then in fact K is an isometry by the Riesz Representation Theorem, since (4.2.4) says that

$$(v, u)_H = \langle g, u \rangle \qquad \text{for all } u \in H_0^1(\Omega).$$

Now if $w \in L^{\frac{2n}{n+2}}(\Omega)$, then, since

$$\langle w^*, u \rangle := \int_\Omega wu \, dx \qquad u \in H_0^1(\Omega)$$

defines a (bounded) linear functional, $w^* \in H^{-1}(\Omega)$; in a slight abuse of notation, we identify w and w^* and write $L^{\frac{2n}{n+1}} \hookrightarrow H^{-1}$. Observe next that

$$p \cdot \frac{2n}{n+2} < \frac{n+2}{n-2} \cdot \frac{2n}{n+2} = 2^*.$$

The growth assumption (4.2.2) implies that

$$\int_\Omega |f(u)|^{\frac{2n}{n+2}} dx \leq \int_\Omega |u|^{p\frac{2n}{n+2}} dx + C|\Omega| \leq \int_\Omega |u|^{2^*} dx + C|\Omega|;$$

hence, using Theorem 2.4.3, we have shown that

$$f(u) \in L^{\frac{2n}{n+2}}(\Omega) \hookrightarrow H^{-1}(\Omega) \qquad \text{whenever } u \in H_0^1(\Omega).$$

In particular, this means there exists $k = k_f > 0$ such that

$$\|f(u)\|_{H^{-1}(\Omega)} \leq k \|f(u)\|_{L^{\frac{2n}{n+1}}(\Omega)}.$$

Let $u \in H_0^1(\Omega)$ be fixed. We now show that

$$\mathcal{E}_2'(u) = K(f(u)) \tag{4.2.5}$$

by proving

$$\mathcal{E}_2(v) = \mathcal{E}_2(u) + (K(f(u)), v-u)_H + o(\|u-v\|_H) \qquad \text{as } \|u-v\|_H \to 0. \tag{4.2.6}$$

Since $F' = f$, integrating by parts gives

$$F(a+b) = F(a) + f(a)b + \int_0^1 (1-s)f'(a+sb)\,ds\,b^2, \qquad a, b \in \mathbb{R}.$$

Hence, for $v \in H_0^1(\Omega)$,

$$\begin{aligned}
\mathcal{E}_2(v) = \int_\Omega F(v)\,dx &= \int_\Omega F(\underbrace{u}_{a} + \underbrace{v-u}_{b})\,dx \\
&= \int_\Omega F(u) + \underbrace{f(u)(v-u)}_{f(u)\in H^{-1}(\Omega)}\,dx + R \\
&= \mathcal{E}_2(u) + \int_\Omega D(K(f(u)))\cdot D(v-u)\,dx + R,
\end{aligned}$$

where, using the growth assumption on f',

$$\begin{aligned}
|R| &\le \int_\Omega |v-u|^2 \int_0^1 (1-s)|f'(u+s(v-u))|\,ds\,dx \\
&\le \int_\Omega |v-u|^2 \cdot C \int_0^1 (1-s)(1+|u+s(v-u)|^{p-1})\,ds\,dx \\
&\le \widehat{C} \int_\Omega |v-u|^2(1+|u|^{p-1}+|v-u|^{p-1})\,dx
\end{aligned}$$

since

$$\int_0^1 (1-s)|a+s(b-a)|^{p-1}\,ds \le c(p)(|a|^{p-1}+|b|^{p-1}).$$

Hence, by Hölder's inequality with exponents $(p+1)/2$ and $(p+1)/(p-1)$,

$$|R| \le \widehat{C}\int_\Omega |v-u|^2 + |v-u|^{p+1}\,dx + \widehat{C}\underbrace{\left(\int_\Omega |u|^{p+1}\,dx\right)^{\frac{p-1}{p+1}}}_{\le \text{const.}\,\|u\|_H^{p+1}}\left(\int_\Omega |v-u|^{p+1}\,dx\right)^{\frac{2}{p+1}}.$$

Since $2, p+1 < 2^*$, this is $o(\|v-u\|_H)$ since (using Poincaré's inequality, Theorem 2.4.4)

$$\begin{aligned}
\int_\Omega |v-u|^2\,dx &\le c\|v-u\|_H^2 \\
\int_\Omega |v-u|^{p+1}\,dx &\le c\|v-u\|_H^{p+1} \\
\left(\int_\Omega |v-u|^{p+1}\,dx\right)^{\frac{2}{p+1}} &\le c\|v-u\|_H^2.
\end{aligned}$$

This proves (4.2.6) and hence $\mathcal{E}_2'(u) = K(f(u))$. We still have to show that $\mathcal{E}_2'$ is Lipschitz on bounded sets: if $u, v \in H_0^1(\Omega)$ and $\|u\|_H, \|v\|_H \leq m$ for some given $m > 0$, then

$$\begin{aligned}\|\mathcal{E}_2'(u) - \mathcal{E}_2'(v)\|_H &= \|K(f(u)) - H(f(v))\|_H \\ &= \|f(u) - f(v)\|_{H^{-1}(\Omega)} \\ &\leq C_1 \|f(u) - f(v)\|_{L^{\frac{2n}{n+2}}(\Omega)}.\end{aligned}$$

Using the growth condition,

$$|f(z) - f(w)| \leq \sup_{y \in (z,w) \text{ or } (w,z)} |f'(y)||z - w| \leq C(1 + |z|^{p-1} + |w|^{p-1})|z - w|;$$

hence

$$\|\mathcal{E}_2'(u) - \mathcal{E}_2'(v)\|_H \leq C_1 \|f(u) - f(v)\|_{L^{\frac{2n}{n+2}}(\Omega)} \leq C_2 \left(\int_\Omega \left((1 + |u|^{p-1} + |v|^{p-1})|u - v|\right)^{\frac{2n}{n+2}} dx \right)^{\frac{n+2}{2n}}.$$

Applying Hölder's inequality with exponents $(n-2)/(n+2)$ and $(n+2)/4$, the right-hand side is no larger than

$$C_2 \underbrace{\left(\int_\Omega |u - v|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{2n}}}_{=\|u-v\|_{L^{2^*}(\Omega)}} \underbrace{\left(\int_\Omega (1 + |u|^{p-1} + |v|^{p-1})^{\frac{2n}{n+2} \cdot \frac{n+2}{4}} dx \right)^{\frac{n+2}{2n} \cdot \frac{4}{n+2}}}_{\leq C_3(m) \text{ since } (p-1)\frac{n}{2} < \frac{2n}{n-2} = 2^*}$$

Putting this together,

$$\|\mathcal{E}_2'(u) - \mathcal{E}_2'(v)\|_H \leq C_4(m) \|u - v\|_H. \tag{4.2.7}$$

This shows that $\mathcal{E}_2$ and hence also $\mathcal{E}$ are in $C^{1,1}_{loc}(H, \mathbb{R})$, proving the claim. □

3. Now we show that $\mathcal{E}$ satisfies the Palais–Smale condition: suppose (u_k) is in $H_0^1(\Omega)$ such that $(\mathcal{E}(u_k))$ is bounded and $\mathcal{E}'(u_k) \to 0$ in $H_0^1(\Omega)$. We have shown that

$$\mathcal{E}'(u_k) = u_k - K(f(u_k)).$$

Hence, for any $\varepsilon > 0$,

$$\left| \int_\Omega Du_k \cdot Dv - f(u_k) v \, dx \right| = |(\mathcal{E}'(u_k), v)_{H_0^1(\Omega)}| \leq \varepsilon \|v\|_{H_0^1(\Omega)} \qquad \text{for all } v \in H_0^1(\Omega), \tag{4.2.8}$$

for all k large enough, where we recall $f(u_k) \in H^{-1}(\Omega)$ may be written in this way (as an L^2-function) since $f(u_k) \in L^{2^*}(\Omega)$, cf. Proposition 2.5.2.

Choosing $v = u_k$,

$$\left| \int_\Omega |Du_k|^2 - f(u_k) u_k \, dx \right| \leq \varepsilon \|u_k\|_{H_0^1(\Omega)}.$$

In particular, taking $\varepsilon = 1$,

$$\int_\Omega f(u_k) u_k \, dx \leq \|u_k\|^2_{H_0^1(\Omega)} + \|u_k\|_{H_0^1(\Omega)}$$

for k large enough. Now since $(\mathcal{E}(u_k))$ is bounded,

$$\mathcal{E}(u_k)\frac{1}{2}\|u_k\|^2_{H^1_0(\Omega)} - \int_\Omega F(u_k)\,dx \le M < \infty$$

for all k, for some $M > 0$. Thus, by (4.2.3),

$$\|u_k\|^2_{H^1_0(\Omega)} \le 2M+2\int_\Omega F(u_k)\,dx \le 2M+2\widetilde{C}\int_\Omega f(u_k)u_k\,dx \le 2M+2\widetilde{C}(\|u_k\|^2_{H^1_0(\Omega)}+\|u_k\|_{H^1_0(\Omega)}).$$

Since $2\widetilde{C} < 1$ by assumption, it follows that (u_k) is bounded in $H^1_0(\Omega)$. Thus up to a subsequence, $u_k \rightharpoonup u$ weakly in $H^1_0(\Omega)$, so that $u_k \to u$ in $L^{p+1}(\Omega)$ since $p+1 < 2^*$ (Theorem 2.4.7). An argument similar to the one used to obtain (4.2.7), using $L^{\frac{p+1}{p}}(\Omega) \hookrightarrow H^{-1}(\Omega)$, shows that $f(u_k) \to f(u)$ in $H^1_0(\Omega)$.

(That is:

$$\begin{aligned}\|f(u)-f(u_k)\|_{H^{-1}(\Omega)} &\le C_1\|f(u)-f(u_k)\|_{L^{\frac{p+1}{p}}(\Omega)}\\ &\le C_2\left(\int_\Omega \left((1+|u|^{p-1}+|u_k|^{p-1})|u-u_k|\right)^{\frac{p+1}{p}}\,dx\right)^{\frac{p}{p+1}}\\ &\le C_2\left(\int_\Omega |u-u_k|^{p+1}\,dx\right)^{\frac{1}{p+1}}\underbrace{\left(\int_\Omega (1+|u|^{p-1}+|u_k|^{p-1})^{\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p+1}}}_{\text{bounded}}\end{aligned}$$

again using Hölder's inequality, this time with p and $p/(p-1)$, for the last inequality.)

Hence also $K(f(u_k)) \to K(f(u))$ in $H^1_0(\Omega)$ since K is an isometry. Since $u_k - K(f(u_k)) \to 0$, it follows that $u_k \to u$ in $H^1_0(\Omega)$.

4. We finally check the other assumptions of Theorem 4.1.8, using the growth conditions (4.2.3) on F: obviously $\mathcal{E}(0) = 0$ since $F(0) = 0$. If $u \in H^1_0(\Omega)$ and $\|u\|_{H^1_0(\Omega)} = r$ (to be chosen later), then

$$\mathcal{E}(u) = \mathcal{E}_1(u) - \mathcal{E}_2(u) = \frac{r^2}{2} - \mathcal{E}_2(u).$$

By (4.2.3)

$$|\mathcal{E}_2(u)| = \left|\int_\Omega F(u)\,dx\right| \le c_2\int_\Omega |u|^{p+1}\,dx = c_2\|u\|^{p+1}_{L^{p+1}(\Omega)} \le \tilde{c}\|u\|^{p+1}_{H^1_0(\Omega)} \le \tilde{c}r^{p+1}$$

using $H^1_0(\Omega) \hookrightarrow L^{p+1}(\Omega)$ since $p+1 < 2^*$. Thus

$$\mathcal{E}(u) \ge \frac{r^2}{2} - \tilde{c}r^{p+1} > 0$$

if $r > 0$ is small enough, since $p+1 > 2$.

Now fix $0 \not\equiv u \in H^1_0(\Omega)$ and set $v := tu$ for $t > 0$ to be chosen later. Then

$$\begin{aligned}\mathcal{E}(v) &= \mathcal{E}_1(tu) - \mathcal{E}_2(tu)\\ &= t^2E_1(u) - \int_\Omega F(tu)\,dx\\ &\le t^2\underbrace{\mathcal{E}_1(u)}_{\text{fixed}} - t^{p+1}c_2\underbrace{\int_\Omega |u|^{p+1}\,dx}_{\text{fixed}} \qquad \text{since } -F(tu) \ge -c_2\int_\Omega |u|^{p+1}t^{p+1}\,dx\\ &< 0\end{aligned}$$

if $t > 0$ is large enough.

5. By Theorem 4.1.8, there exists $0 \not\equiv u \in H_0^1(\Omega)$ such that

$$\mathcal{E}'(u) = u - K(f(u)) = 0.$$

This means (cf. (4.2.8))

$$\int_\Omega Du \cdot D\varphi \, dx = \int_\Omega f(u)\varphi \, dx \qquad \text{for all } \varphi \in H_0^1(\Omega).$$

□

4.3 Multiple critical points via symmetry and index theory

Goal: Prove the existence of infinitely many eigenpairs (λ, u) of problems such as

$$\{ -\Delta_p u = \lambda |u|^{p-2} u \qquad \text{in } \Omega, u \; = 0 \quad \text{on } \partial\Omega,$$

$1 < p < \infty$, or (in this lecture) its semilinear equivalent.

Idea: If $\mathcal{G}$ is a group of mappings of a Banach space to itself and $\mathcal{E} \in C^1(V, \mathbb{R})$ is such that

$$\mathcal{E}(gu) = \mathcal{E}(u) \qquad \text{for all } g \in \mathcal{G} \text{ and all } u \in V,$$

then we say $\mathcal{E}$ is *invariant* under $\mathcal{G}$, e.g. $\mathcal{E}$ is even: $\mathcal{E}(-u) = \mathcal{E}(u)$, i.e. $\mathcal{G} = \{\mathrm{id}, -\mathrm{id}\} \simeq \mathcal{Z}_2$. We exploit this structure to find multiple critical points which have a "minimax characterisation". The original, finite-dimensional result:

4.3.1 Theorem (Ljusternik–Schnirelman,[7] 1930s)**.** *Suppose $\mathcal{E} \in C^1(\mathbb{R}^n, \mathbb{R})$ is even. Then $\mathcal{E}|_{\mathbb{S}^{n-1}}$ possesses at least n pairs of critical points (considered as a functional on $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$).*

When referring to minimax results in this direction we often speak of Ljusternik–Schnirelman theory. Here we will only work with the special case of even functionals.

We need a measure of the "size" of sets invariant under the group action (here "symmetric sets").

4.3.2 Definition. Suppose V is a Banach space and denote by $\mathcal{A}$ the family of closed sets $A \subset V \setminus \{0\}$ such that $x \in A$ if and only if $x \in -A$. Then the *genus* $\gamma(A)$ of $A \in \mathcal{A}$ is the smallest $k \in \mathbb{N}$ such that there exists an odd map $\varphi \in C(A, \mathbb{R}^k \setminus \{0\})$, or ∞ if no such k exists. We also set $\gamma(\emptyset) = 0$.

4.3.3 Example. (a) If $B \subset V$ is closed and $B \cap (-B) = \emptyset$, then $A := B \cup (-B) \in V$ and $\gamma(A) = 1$ since we may set

$$\varphi = \begin{cases} 1, & x \in B, \\ -1, & x \in -B, \end{cases}$$

then $\varphi \in C(A, \mathbb{R} \setminus \{0\})$.

[7] As with Chebyshev, there are many different spellings of these names.

(b) If $V = \mathbb{R}^n$ and $A = \mathbb{S}^{n-1}$, then $\gamma(A) = n$.[8]

(c) If $A \in \mathcal{A}$ and $\gamma(A) > 1$, then A contains infinitely many points, since if A were finite, then we could write $A = B \cup (-B)$ as in (a).

This is well adapted to even functionals $\mathcal{E}$ since the sublevel sets $S_\alpha = \{u \in V : \mathcal{E}(u) \leq \alpha\}$ are in $\mathcal{A}$.[9]

4.3.4 Proposition. *Let $A, B \in \mathcal{A}$.*

(a) Mapping property: if there exists an odd $f \in C(A, B)$, then $\gamma(A) \leq \gamma(B)$.

(b) Monotonicity: if $A \subset B$, then $\gamma(A) \leq \gamma(B)$.

(c) Subadditivity: $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.

(d) "Continuity": if A is compact, then $\gamma(A) < \infty$ and there is a $\delta > 0$ such that

$$A_\delta := \{u \in V : \operatorname{dist}(u, A) < \delta\} \in \mathcal{A} \quad \text{and} \quad \gamma(A_\delta) = \gamma(A).$$

Proof. We may assume WLOG in (a), (b) and (c) that $\gamma(A), \gamma(B) < \infty$, since otherwise the statements are trivial.

(a) If $\gamma(B) = n$, then there exists an odd $\varphi \in C(B, \mathbb{R}^n \setminus \{0\})$, so $\varphi \circ f \in C(A, \mathbb{R}^n \setminus \{0\})$ is also odd; hence $\gamma(A) \leq n$.

(b) Choose $f = \mathrm{id}$ in (a).

(c) Suppose $\gamma(A) = m$ and $\gamma(B) = n$ with corresponding odd mappings $\varphi \in C(A, \mathbb{R}^n \setminus \{0\})$ and $\psi \in C(B, \mathbb{R}^m \setminus \{0\})$. By Tietze's Extension Theorem (see [10, Theorem 20.4]) these can be extended to $\tilde{\varphi} \in C(V, \mathbb{R}^n \setminus \{0\})$, $\tilde{\psi} \in C(V, \mathbb{R}^m \setminus \{0\})$ (such that $\tilde{\varphi}|_A = \varphi$, $\tilde{\psi}|_B = \psi$). Replacing $\tilde{\varphi}$ and $\tilde{\psi}$ with their odd parts, we may assume they are odd. Setting $f : V \to \mathbb{R}^{n+m}$, $f(x) := (\tilde{\varphi}(x), \tilde{\psi}(x))$, it follows that $f|_{A \cup B} \in C(A \cup B, \mathbb{R}^{n+m} \setminus \{0\})$ is odd, and so $\gamma(A \cup B) \leq n + m$.

(d) For each $x \in A$, set $r(x) := \frac{1}{2}\|x\|_V =: r(-x)$ and $T_x := B(x, r(x)) \cup B(-x, r(x))$. Then $\overline{T}_x \in \mathcal{A}$ and $\gamma(\overline{T}_x) = 1$ by Example 4.3.3(a). Then since A is compact, there exist $x_1, \ldots, x_k \in A$ such that $A \subset \bigcup_{x \in A} \overline{T}_x$. Hence

$$\gamma(A) \overset{(b)}{\leq} \gamma\left(\bigcup_{i=1}^k \overline{T}_x\right) \overset{(c)}{\leq} \sum_{i=1}^k \gamma(\overline{T}_x) = k$$

(noting that (c) obviously holds for any finite union of sets by induction). Hence $\gamma(A) < \infty$.

If $\gamma(A) = n$, with $\varphi \in C(A, \mathbb{R}^n \setminus \{0\})$ odd, then extend φ to an odd $\tilde{\varphi}$ as in (c). Since $\varphi \neq 0$ on the compact set A, there exists $\delta > 0$ such that $\tilde{\varphi} \neq 0$ on A_δ. Hence $\gamma(A_\delta) \leq n = \gamma(A)$. But $\gamma(A) \leq \gamma(A_\delta)$ by (b). □

4.3.5 Proposition. *If $A \subset V$, $\Omega \subset \mathbb{R}^k$ is a bounded neighbourhood of 0 and there exists an odd homeomorphism $h \in C(A, \partial\Omega)$, then $\gamma(A) = k$.*[10]

Proof. Obviously $\gamma(A) \leq k$. If $\gamma(A) = j < k$, then there exists an odd $\varphi \in C(A, \mathbb{R}^j \setminus \{0\})$, so $\varphi \circ h^{-1}$ is odd and in $C(\partial\Omega, \mathbb{R}^j\{0\})$. This contradicts the Borsuk–Ulam theorem.[11] □

[8] See Proposition 4.3.5.

[9] As long as $0 \notin S_\alpha$, of course: typically we will restrict $\mathcal{E}$ and hence S_α to spheres of the form $\{\|u\|_V = r\}$.

[10] This proves Example 4.3.3(b).

[11] There is no continuous odd map $h : \mathbb{S}^n \to \mathbb{S}^{n-1}$; equivalently, if $f : \mathbb{S}^n \to \mathbb{R}^n$ is continuous, then there exists $x \in \mathbb{S}^n$ such that $f(-x) = f(x)$.

Now suppose for simplicity that $V = H$ is a Hilbert space.

4.3.6 Proposition. *Suppose $X \subset H$ is a subspace of codimension $k \in \mathbb{N}$ and $A \in \mathcal{A}$ with $\gamma(A) > k$. Then $A \cap X \neq \emptyset$.*

Proof. Write $H = X \oplus Y$, where Y is k-dimensional, and denote by $P : H \to Y$ the corresponding projection. If $A \cap X = \emptyset$, then $P \in C(A, Y \setminus \{0\})$, and $P|_A$ is odd (as the identity on A). Hence $\gamma(A) \leq \gamma(PA)$ by Proposition 4.3.4(a). The radial projection of PA into $\{\|x\|_H = 1\} \cap Y$ (via $PA \ni x \mapsto \frac{x}{\|x\|_H}$) is also continuous and odd, so

$$\gamma(A) \leq \gamma(\{\|x\|_H = 1\} \cap Y) = k$$

by Proposition 4.3.5. □

Now we show how genus allows us to obtain critical points of functionals with a norm constraint.

4.3.7 Theorem. *Suppose H is an (infinite-dimensional) separable Hilbert space and $\mathcal{E} \in C^1(H, \mathbb{R})$ is even. Suppose that for some $r > 0$, $\mathcal{E}|_{\{\|u\|_H = r\}}$ satisfies the Palais–Smale condition and is bounded from below. Then $\mathcal{E}|_{\{\|u\|_H = r\}}$ possesses infinitely many distinct pairs of critical points (considered as a functional on the Hilbert manifold $M_r := \{\|u\|_H = r\} = \partial B(0, r)$).*

Proof. For all $j \in \mathbb{N}$ set

$$\Gamma_j := \{A \in \mathcal{A} : A \subset M_r \text{ and } \gamma(A) \geq j\}$$

(with $\mathcal{A}$ as in Definition 4.3.2).

Claim 1:

(a) $\Gamma_j \neq \emptyset$, for all $j \in \mathbb{N}$;

(b) $\Gamma_1 \supset \Gamma_2 \supset \ldots \supset \Gamma_j \supset \ldots$;

(c) If $\varphi \in C(M_r, M_r)$ is odd, then $\varphi(A) \in \Gamma_j$ whenever $A \in \Gamma_j$;

(d) If $A \in \Gamma_j$ and $B \in \mathcal{A}$ with $\gamma(B) \leq s < j$, then $\overline{A \setminus B} \in \Gamma_{j-s}$.

Proof of Claim 1: (a) Let $\{u_n\}_{n \in \mathbb{N}}$ be an arbitrary (orthonormal) basis of H and fix $j \in \mathbb{N}$. Then $\operatorname{span}\{u_1, \ldots, u_j\} \cap M_r \in \mathcal{A}$ can be identified with $\{x \in \mathbb{R}^j : |x| = r\}$, so it has genus j by Proposition 4.3.5.

(b) is immediate from the definition.

(c) follows from Proposition 4.3.4(a).

(d) follows from the general observation that

$$\gamma(B) < \infty \implies \gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$$

if $A, B \in \mathcal{A}$, which follows from Proposition 4.3.4(b) and (c) since $A \subset \overline{A \setminus B} \cup B$. This proves the claim. □

Now define

$$\beta_j := \inf_{A\in\Gamma_j}\sup_{u\in A}\mathcal{E}(u), \qquad j\in\mathbb{N}, \tag{4.3.1}$$

so that $\beta_1 \le \beta_2 \le \beta_3 \le \ldots$. We claim that $\{\beta_j\}$ is an infinite set of critical levels: we first use the Deformation Theorem 4.1.5 (on the Hilbert manifold M_r as in Remark 4.1.7(d)) to show that every level β_j is in fact critical. Note that the Palais–Smale condition implies that

$$K_\beta = \{u\in M_r : \mathcal{E}(u)=\beta \text{ and } \mathcal{E}|'_{M_r}(u)=0\} \tag{4.3.2}$$

is compact for any $\beta\in\mathbb{R}$, since any sequence in K_β satisfies the Palais–Smale condition.

Claim 2: If $\beta_j = \ldots \beta_{j+p} = \beta$ for some $p\ge 0$ (with β_j as in (4.3.1)), then

$$\gamma(K_\beta)\ge p+1.$$

(In particular, $K_\beta\neq\emptyset$, i.e., this *is* a critical level.)

The theorem now follows from Claim 2 and Example 4.3.3(c), since if $\beta_j=\beta_{j+1}$ for some j, then K_{β_j} is an infinite set, and if not, then $K_{\beta_j}\cap K_{\beta_i}=\emptyset$ $(i\neq j)$, so $\bigcup_{j\in\mathbb{N}} K_{\beta_j}$ is an infinite set.

Proof of Claim 2: Suppose $\gamma(K_\beta)\le p$. Then since K_β is compact, by Proposition 4.3.4(d) (and (b)), there exists $\eta>0$ such that

$$\gamma(\overline{\{u\in M_r : \operatorname{dist}(u,K_\beta)<\eta\}})\le p.$$

We now use Theorem 4.1.5 in the following modified form (with Remark 4.1.7(b)–(d)): setting

$$\mathcal{O} := \{u\in M_r : \operatorname{dist}(u,K_\beta)<\eta\}$$

open and taking $\varepsilon\le 1$ small enough, there exist $\Phi\in C([0,1]\times M_r, M_r)$ *odd* and $0<\delta<\varepsilon$ such that

$$\Phi(S_{\beta+\delta}\setminus\mathcal{O})\subset S_{\beta-\delta}$$

(where $S_\alpha=\{u\in M_r : \mathcal{E}(u)=\alpha\}$).

Now choose $A\in\Gamma_{j+p}$ such that $\sup_{u\in A}\mathcal{E}(u)\le\beta+\delta$. Since $\overline{\mathcal{O}}$ satisfies $\gamma(\overline{\mathcal{O}})\le p$, by Claim 1(d), we have

$$S_{\beta+\delta}\setminus\mathcal{O}\supset\overline{A\setminus\overline{\mathcal{O}}}\in\Gamma_j.$$

By Claim 1(c) applied to Φ, we have $\Phi(\overline{A\setminus\overline{\mathcal{O}}})\in\Gamma_j$ and hence $\Phi(\overline{A\setminus\overline{\mathcal{O}}})\subset S_{\beta-\delta}$. Thus

$$\beta=\beta_j=\inf_{A\in\Gamma_j}\sup_{u\in A}\mathcal{E}(u)\le\sup_{u\in\Phi(\overline{A\setminus\overline{\mathcal{O}}})}\mathcal{E}(u)\le\beta-\delta,$$

a contradiction. □

4.3.8 Remark. (a) The minimax values β_j may be characterised geometrically as

$$\beta_j=\inf\{\alpha\in\mathbb{R} : \gamma(S_\alpha)\ge j\}, \tag{4.3.3}$$

i.e., the β_j are the numbers at which $S_\alpha=\{u\in M_r : \mathcal{E}(u)=\alpha\}$ changes genus. (We emphasise $S_\alpha\in\mathcal{A}$ since $\mathcal{E}$ is even.) Indeed, denote the right-hand side of (4.3.3) by $\bar\beta_j$. If $\alpha>\bar\beta_j$, then $\gamma(S_\alpha)\ge j$, so $S_\alpha\in\Gamma_j$ and thus

$$\beta_j\le\sup_{u\in S_\alpha}\mathcal{E}(u)=\alpha.$$

Taking the infimum over $\alpha > \bar\beta_j$ yields $\beta_j \le \bar\beta_j$. If $\beta_j < \bar\beta_j$, then set

$$\beta := \begin{cases} \beta_j + 1 & \text{if } \bar\beta_j = \infty, \\ \frac{1}{2}(\beta_j + \bar\beta_j) & \text{if } \bar\beta_j < \infty; \end{cases}$$

then there is an $A \in \Gamma_j$ such that $\sup_{u\in A} \mathcal{E}(u) \le \beta$. Hence $\gamma(S_\beta) \ge \gamma(A) \ge j$ since $A \subset S_\beta$ (see Proposition 4.3.4(d)). But $\beta < \bar\beta_j$, contradicting the definition of $\bar\beta_j$.

(b) The requirement in Theorem 4.3.7 that $\mathcal{E}|_{M_r}$ satisfy the Palais–Smale condition can be significantly weakened (and is too strong for many applications). We needed it in two places in the proof:

1. compactness of the K_{β_j}, and
2. the application of the Deformation Theorem 4.1.5 at the levels β_j.

In both cases it suffices to assume $\mathcal{E}|_{M_r}$ satisfies the local Palais–Smale condition at the levels β_j in the sense of Remark 4.1.7(a).

4.3.9 Example. Consider

$$\begin{cases} -\Delta u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3.4}$$

where $1 < p < 2^* = \frac{2n}{n-2}$. We set

$$\mathcal{E}(u) := \frac{1}{p} \int_\Omega |u|^p \, dx, \qquad u \in H_0^1(\Omega),$$

which is even and in $C^1(H_0^1(\Omega), \mathbb{R})$ by our assumption on p. At a critical point u of $\mathcal{E}|_{M_1 = \{\|u\|_{H_0^1(\Omega)} = 1\}}$, since M_1 is the zero set of

$$J(u) = \|u\|^2_{H_0^1(\Omega)} - 1,$$

by Theorem 3.5.8 there exists $\mu \in \mathbb{R}$ such that[12]

$$\mathcal{E}'(u)\varphi = \mu(u, \varphi)_{H_0^1(\Omega)} \qquad \text{for all } \varphi \in H_0^1(\Omega), \tag{4.3.6}$$

that is,

$$\int_\Omega |u|^{p-2} u\varphi \, dx = \mu \int_\Omega Du \cdot D\varphi \, dx \qquad \text{for all } \varphi \in H_0^1(\Omega).$$

Thus critical points of $\mathcal{E}|_{M_1}$ correspond to weak solutions of (4.3.4) with $\mu = \lambda^{-1}$.

We check the conditions of Theorem 4.3.7. First note that $\mathcal{E}$ is weakly continuous (and bounded from below on M_1) due to the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ (Theorem 2.4.7).

[12] Actually, Theorem 3.5.8 was only stated for minima. But an inspection of the proof shows that the same conclusion holds for *any* critical point of $\mathcal{E}$ on $\{J = 0\}$. Alternatively, following through the reasoning of Theorem 3.5.8, for any $u \in M_1$, we have

$$\mathcal{E}|'_{M_1}(u) = \mathcal{E}'(u) - (\mathcal{E}'(u), u)_H u \tag{4.3.5}$$

(where we regard $\mathcal{E}|'_{M_1}(u)$ as an element of $H = H_0^1(\Omega)$). If u is a critical point, then $u = \varphi \in M_1$, we have $\mu = \mathcal{E}'(u)u$ and thus (4.3.5) equals 0 for such u. This yields (4.3.6).

We now merely need to check the Palais–Smale condition, which we do in the local sense for any $\beta \neq 0$. That is, given $\beta \neq 0$, suppose $u_k \in H_0^1(\Omega)$, $\|u_k\| = 1$, such that $|\mathcal{E}(u_k) - \beta| < \varepsilon$ for some $\varepsilon < |\beta|$ independent of k, and $\mathcal{E}|'_{M_1}(u_k) \to 0$ as $k \to \infty$. Then

$$\mathcal{E}|'_{M_1}(u_k) = \mathcal{E}'(u_k) - (\mathcal{E}'(u_k), u_k)_{H_0^1(\Omega)} u_k \to 0. \tag{4.3.7}$$

We next claim that $\mathcal{E}'$ is compact as a mapping from H to H: indeed, $\mathcal{E}'$ is given by

$$(\mathcal{E}'(u), \varphi)_{H_0^1(\Omega)} \equiv \langle \mathcal{E}'(u), \varphi \rangle_{H^{-1}(\Omega)} = \int_\Omega |u|^{p-2} u \varphi \, dx, \qquad \varphi \in H_0^1(\Omega)$$

If $(v_k) \subset H_0^1(\Omega)$ is bounded, then WLOG $v_k \rightharpoonup v$ in $H_0^1(\Omega)$. We claim $\mathcal{E}'(v_k) \to \mathcal{E}(v)$ in $H_0^1(\Omega)$: by the Riesz Representation Theorem,

$$\|\mathcal{E}'(v_k) - \mathcal{E}'(v)\|_{H_0^1(\Omega)} = \|\mathcal{E}'(v_k) - \mathcal{E}'(v)\|_{H^{-1}(\Omega)},$$

so it suffices to show $\mathcal{E}(v_k) \to \mathcal{E}(v)$ in $H^{-1}(\Omega)$. Now, for $\varphi \in H_0^1(\Omega)$, using the definition of $\mathcal{E}'$, Hölder's inequality (with exponents $2n/(n-2)$ and $2n/(n+2)$) and the fact that $H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}$, respectively,

$$\begin{aligned}
|\langle \mathcal{E}'(v_k) - \mathcal{E}'(v), \varphi \rangle| &\leq \int_\Omega \big||v_k|^{p-2} v_k - |v|^{p-2} v\big| |\varphi| \, dx \\
&\leq \left(\int_\Omega \big||v_k|^{p-2} v_k - |v|^{p-2} v\big|^{\frac{2n}{n+2}} dx \right)^{\frac{n+2}{2n}} \|\varphi\|_{L^{\frac{2n}{n-2}}(\Omega)} \\
&\leq C \left(\int_\Omega \big||v_k|^{p-2} v_k - |v|^{p-2} v\big|^{\frac{2n}{n+2}} dx \right)^{\frac{n+2}{2n}} \|\varphi\|_{H_0^1(\Omega)}.
\end{aligned}$$

Since $v_k \to v$ in $L^{\frac{2n}{n-2}}(\Omega)$ and $(p-1)\frac{2n}{n+2} < \frac{2n}{n-2}$, in particular $|v_k|^{p-2} v_k \to |v|^{p-2} v$ in $L^{\frac{2n}{n+2}}$. It follows that $\|\mathcal{E}'(v_k) - \mathcal{E}'(v)\|_{H^{-1}(\Omega)} \to 0$, and hence $\mathcal{E}'$ is compact, as claimed.

We return to considering our Palais–Smale sequence (u_k) in M_1. Since it is bounded and $\mathcal{E}'$ is compact, there exists $u \in H_0^1(\Omega)$ such that (up to a subsequence) $u_k \rightharpoonup u$ in $H_0^1(\Omega)$ and

$$0 \leftarrow \mathcal{E}|'_{M_1}(u_k) \to \mathcal{E}|'_{M_1}(u) = \mathcal{E}'(u) - (\mathcal{E}'(u), u) u.$$

Since $\mathcal{E}$ is weakly continuous, $\mathcal{E}(u_k) \to \mathcal{E}(u)$, so $\mathcal{E}(u) \in [\beta - \varepsilon, \beta + \varepsilon]$ and in particular, by choice of ε, $\mathcal{E}(u) \neq 0$. In this case $u \neq 0$ as well (by choice of $\mathcal{E}$); hence

$$(\mathcal{E}'(u), u) = \int_\Omega |u|^p \, dx = p\mathcal{E}(u) \neq 0$$

and so $(\mathcal{E}'(u_k), u_k) \neq 0$ as well, for $k \in \mathbb{N}$ large enough. It follows from (4.3.7) that we may write

$$u_k = \frac{1}{(\mathcal{E}'(u_k), u_k)} (\mathcal{E}|'_{M_1}(u_k) - \mathcal{E}'(u_k)),$$

which contains a convergent subsequence as $\mathcal{E}'$ is compact.

(Note that this does *not* hold if $\beta = 0$; *any* sequence $u_k \rightharpoonup 0$ satisfies $\mathcal{E}(u_k) \to 0$ and $\mathcal{E}|'_{M_1}(u_k) \to 0$.)

Finally, we observe that 0 cannot be a critical level of $\mathcal{E}$. Indeed, any critical point u of $\mathcal{E}$ satisfies

$$\int_\Omega Du \cdot D\varphi \, dx = \lambda \int_\Omega |u|^{p-2} u\varphi \, dx$$

for some $\lambda \in \mathbb{R}$; choosing $u = \varphi \in M_1$, we obtain the Rayleigh quotient expression

$$\lambda = \frac{\int_\Omega |Du|^2 \, dx}{\int_\Omega |u|^p \, dx} \left(\equiv \frac{1}{(\mathcal{E}'(u), u)_{H_0^1(\Omega)}} \right).$$

But Theorem 2.4.4 ensures that

$$\inf \left\{ \frac{1}{\int_\Omega |u|^p \, dx} : u \in H_0^1(\Omega) \text{ with } \int_\Omega |Du|^2 \, dx = 1 \right\} > 0.$$

Hence the minimax levels β_j defined in (4.3.1) must be > 0 for all $j \in \mathbb{N}$. We conclude that there exists an infinite sequence of eigenpairs $(\lambda_k, \pm u_k)$ solving (4.3.4) weakly, given by

$$\lambda_k = \inf_{A \in \Gamma_j} \sup_{u \in A} \frac{1}{p} \int_\Omega |u|^p \, dx$$

for some $j \in \mathbb{N}$.

Observe:

(a) Although

$$\lambda_1 = \inf \left\{ \frac{1}{\int_\Omega |u|^p \, dx} : u \in H_0^1(\Omega) \text{ with } \int_\Omega |Du|^2 \, dx = 1 \right\},$$

there is no orthogonality result for the eigenfunctions in general as there was in the linear case considered in Example 3.5.9.

(b) Equally, we cannot talk of linear subspaces of eigenfunctions: u is an eigenfunction if and only if $-u$ is an eigenfunction. But for $u \mapsto tu$, if $-\Delta u = \lambda |u|^{p-2} u$ (weakly or strongly), where $p \neq 2$, then

$$-\Delta(tu) = t(-\Delta u) = \lambda t |u|^{p-2} u = \frac{\lambda}{t^{p-2}} |tu|^{p-2} (tu),$$

reflecting the fact that we obtain an infinite set of critical points whenever we fix the scaling

$$\|u\|_{H_0^1(\Omega)} = r > 0.$$

4.3.10 Remark. The same conclusion as in Example 4.3.9 holds for the p-Laplacian:[13] the equation

$$\left\{ -\Delta u_p = \lambda |u|^{p-2} u \qquad \text{in } \Omega, u \;= 0 \quad \text{on } \partial\Omega \right.$$

$(1 < p < \infty)$ possesses an infinite sequence of eigenpairs (λ_k, u_k) satisfying[14]

$$\lambda_k = \frac{\int_\Omega |Du|^p \, dx}{\int_\Omega |u|^p \, dx};$$

[13] Although we will not prove this: here we need the corresponding Ljusternik–Schnirelman theory in general reflexive Banach spaces.

[14] However, it still seems to be an open problem whether this minimax method yields *all* eigenvalues of the p-Laplacian, or whether there can exist others.

these are now invariant under rescaling $u_k \mapsto tu_k$. In particular

$$\lambda_1 = \inf\left\{\frac{\int_\Omega |Du|^p\,dx}{\int_\Omega |u|^p\,dx} : 0 \neq u \in W_0^{1,p}(\Omega)\right\},$$

again giving the optimal constant in the corresponding Poincaré inequality.

5 Parabolic problems

Goal: Study parabolic problems of the form[1]

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au = f \\ \qquad u(0) = u_0, \end{cases}$$

where A is a possibly nonlinear, usually elliptic operator and $u_0, f \in H$ (here a Hilbert space). For given data u_0, f, the solution should be a function $t \mapsto u(t) \in H$, $u(0) = u_0$. We therefore need to start by studying Hilbert space-valued functions of a real variable.[2]

5.1 Bochner integrals and Bochner–Sobolev spaces

Let V be a real, separable Banach space and $\Omega \subset \mathbb{R}^n$ open. We denote by $\mathcal{A}$ the Lebesgue σ-algebra on Ω, μ Lebesgue measure and χ_M the characteristic function of a set $M \subset \Omega$. We wish to define the so-called *Bochner integral*

$$\int_\Omega f \, d\mu \in V$$

for $f : \Omega \to V$. Our treatment will be brief; more details on the construction of Bochner integrals can be found in [13, Section V.4–5]; the classical reference is [2].

5.1.1 Definition. (a) $f : \Omega \to V$ is a *step function* if there exist $M_k \in \mathcal{A}$, pairwise disjoint, and $v_k \in V$ such that

$$f = \sum_{k \in \mathbb{N}} \chi_{M_k} v_k.$$

(b) $f : \Omega \to V$ is *measurable* if there exist step functions $f_k : \Omega \to V$ such that $f_k \to f$ in V pointwise almost everywhere.[3]

5.1.2 Theorem. *The function $f : \Omega \to V$ is measurable if and only if f is weakly measurable, i.e. the mapping*

$$\Omega \ni x \mapsto \langle \varphi, f(x) \rangle$$

is measurable for each $\varphi \in V'$.

The statement is more complicated if V is not separable: we also need the existence of a null set $N \in \mathcal{A}$ such that $f(\Omega \setminus N)$ is separable. For the proof we refer to [13, Section V.4].

[1] Since we did not end up considering the corresponding hyperbolic problems $\frac{\partial^2 u}{\partial t^2} + Au = f$, this chapter has also been renamed accordingly.

[2] Or more generally Banach space-valued, since for much of the theory the difference is minimal.

[3] If $V = \mathbb{R}$, this is equivalent to the preimage of measurable sets being measurable.

5.1.3 Lemma. *Suppose V and W are real, separable Banach spaces.*

(a) Every continuous function $f : \Omega \to V$ is measurable.

(b) If $f : \Omega \to V$ is measurable, then $\|f\|_V : \Omega \to \mathbb{R}$ is measurable.

(c) If $f : \Omega \to V$ is measurable and $g : V \to W$ is continuous, then $g \circ f : \Omega \to W$ is measurable.

(d) If $f : \Omega \to V$ and $g : \Omega \to \mathbb{R}$ are measurable, then so is $fg : \Omega \to V$.

(e) If $f : \Omega \to V$ and $g : \Omega \to V'$ are measurable, then so is $\langle g, f\rangle : \Omega \to \mathbb{R}$.

(f) If $f_k : \Omega \to V$ are measurable and $f_k \to f$ pointwise a.e., then f is measurable.

5.1.4 Definition. (a) $f : \Omega \to V$ is *integrable* if f is measurable and

$$\int_\Omega \|f\|_V \, d\mu < \infty.$$

(We emphasise $\|f\|_V : \Omega \to \mathbb{R}$ is a real-valued function.)

(b) If $f : \Omega \to V$ is an integrable step function, $f = \sum_{k\in\mathbb{N}} \chi_{M_k} v_k$, we define its *Bochner integral* to be

$$\int_\Omega f \, d\mu := \sum_{k\in\mathbb{N}} \mu(M_k) v_k,$$

the series converging absolutely due to the integrability assumption.

(c) If $f : \Omega \to V$ is a general integrable function, we set

$$\int_\Omega f \, d\mu := \lim_{k\to\infty} \int_\Omega f_k \, d\mu,$$

where $f_k : \Omega \to V$ are any step functions such that $f_k \to f$ pointwise a.e. and $\|f_k\|_V \le \|f\|_V$.

We claim without proof that (b) is independent of the representation of f and (c) of the approximating sequence, whose existence we also assume without proof.

5.1.5 Theorem. *(a) If f is integrable, then*

$$\left\| \int_\Omega f \, d\mu \right\|_V \le \int_\Omega \|f\|_V \, d\mu.$$

(b) If $f, g : \Omega \to V$ are integrable, then so is $f + g$, and

$$\int_\Omega \alpha f + \beta g \, d\mu = \alpha \int_\Omega f \, d\mu + \beta \int_\Omega g \, d\mu \qquad (\alpha, \beta \in \mathbb{R}).$$

(c) If f is integrable and $T : V \to W$ is continuous and linear, then $Tf : \Omega \to W$ is integrable and

$$\int_\Omega Tf \, d\mu = T \int_\Omega f \, d\mu.$$

(d) *(Dominated convergence theorem) Suppose $f_k : \Omega \to V$ are measurable, $f_k \to f$ pointwise a.e., and there exists an integrable $g : \Omega \to \mathbb{R}$ such that $\|f\|_k \le g$ for all $k \in \mathbb{N}$. Then f is integrable and*

$$\int_\Omega f\,d\mu = \lim_{k\to\infty} \int_\Omega f_k\,d\mu.$$

If $\Omega = (a, b) \subset \mathbb{R}$, we will also write $\displaystyle\int_a^b f(x)\,dx$ for $\displaystyle\int_{(a,b)} f\,d\mu$.

As in the scalar case, we identify functions which agree for almost all $x \in \Omega$. With this convention:

5.1.6 Definition. For $p \in [1, \infty]$ we define the *Bochner–Lebesgue space*

$$L^p(\Omega, V) := \{f : \Omega \to V \text{ measurable} : \|f\|_{L^p(\Omega,V)} < \infty\},$$

where

$$\|f\|_{L^p(\Omega,V)} := \left(\int_\Omega \|f\|^p\,d\mu\right)^{1/p}.$$

if $p \in [1, \infty)$, and

$$\|f\|_{L^\infty(\Omega,V)} := \inf\{c \ge 0 : \mu(\{\|f\|_V \ge c\}) = 0\}.$$

These are Banach spaces; moreover, if $\Omega \subset \mathbb{R}^n$ is bounded[4] and $1 \le p \le q \le \infty$, then by Hölder's inequality

$$C(\overline{\Omega}, V) \subset L^\infty(\Omega, V) \subset L^q(\Omega, V) \subset L^p(\Omega, V) \subset L^1(\Omega, V).$$

We also define $L^p_{loc}(\Omega, V)$ in the usual way.

5.1.7 Theorem. *Let V be a real, separable Banach space.*

(a) *If $p \in [1, \infty)$, then $L^p(\Omega, V)$ is separable.*

(b) *If $p \in (1, \infty)$ and V is reflexive, then $L^p(\Omega, V)$ is reflexive and*

$$L^p(\Omega, V)' \simeq L^{p'}(\Omega, V'), \qquad \frac{1}{p} + \frac{1}{p'} = 1. \tag{5.1.1}$$

The identification (5.1.1) *still holds when $p = 1$.*

(c) *If $V = H$ is a Hilbert space, then so is $L^2(\Omega, H)$ with inner product*

$$(f, g)_{L^2(\Omega,H)} := \int_\Omega (f(x), g(x))_H\,d\mu, \qquad f, g \in L^2(\Omega, H(.$$

The proof of (a) proceeds by showing that if $(f_k) \subset L^p(\Omega) = L^p(\Omega, \mathbb{R})$ and $(v_j) \subset V$ are countable dense sets, then for

$$M := \{f_k v_j : \Omega \to V : k, j \in \mathbb{N}\}$$

we have that $\operatorname{span}_{\mathbb{Q}} M$ is countable and dense in $L^p(\Omega, V)$.

[4]Or more generally has finite Lebesgue measure.

To understand (5.1.1), suppose $f \in L^p(\Omega, V)$ and $g \in L^{p'}(\Omega, V)$. By using that $|\langle g(x), f(x)\rangle_{V',V}| \leq \|g(x)\|_{V'}\|f(x)\|_V$ pointwise a.e. in Ω) and then applying Hölder's inequality to $\|g\|_{V'} \in L^{p'}(\Omega, \mathbb{R})$ and $\|f\|_V \in L^p(\Omega, \mathbb{R})$,

$$\left|\int_\Omega \langle g, f\rangle_{V',V}\, d\mu\right| \leq \int_\Omega \|g\|_{V'}\|f\|_V\, d\mu \leq \|f\|_{L^p(\Omega,V)}\|g\|_{L^{p'}(\Omega,V')}.$$

Moreover, as usual,

$$\|g\|_{L^{p'}(\Omega,V')} = \sup_{\|f\|_{L^p(\Omega,V)}=1} \left|\int_\Omega \langle g, f\rangle_{V',V}\, d\mu\right|.$$

Thus $L^{p'}(\Omega, V')$ may be identified with a (closed) subspace of $L^p(\Omega, V)'$; as usual, they are equal in the reflexive case.

If $\Omega = (a, b)$, $a, b \in \mathbb{R}$, then we will write $L^p(a, b; V)$ in place of $L^p(\Omega, V)$. We can define the weak derivative of a $L^p(a, b; V)$-function in the following way.

5.1.8 Definition. (a) $v \in L^1_{loc}(a, b; V)$ is the weak derivative of $u \in L^1_{loc}(a, b; V)$ if[5]

$$\int_a^b u\varphi'\, dx = -\int_a^b v\varphi\, dx \qquad \text{for all } \varphi \in C_c^\infty(a, b) \equiv C_c^\infty(a, b; \mathbb{R}).$$

(This is uniquely determined if it exists by an easy variant of the Fundamental Lemma of the Calculus of Variations, Theorem 2.2.6.)

(b) For $p \in [1, \infty]$ and $k \in \mathbb{N}_0$ we set

$$W^{k,p}(a, b; V) := \{u \in L^p(a, b; V) : u', \ldots, u^{(k)} \text{ exist and } \in L^p(a, b; V)\},$$

with norm

$$\|u\|_{W^{k,p}(a,b;V)} = \left(\sum_{j=0}^k \|u^{(j)}\|^p_{L^p(a,b;V)}\right)^{1/p}$$

if $p \in [1, \infty)$ and

$$\|u\|_{W^{k,\infty}(a,b;V)} = \sup\left\{\|u^{(j)}\|_{L^\infty(a,b;V)} : j = 0, \ldots, k\right\}.$$

We also set

$$W_0^{k,p}(a, b; V) := \overline{C_c^\infty(a, b; V)}^{W^{k,p}}$$

and, if $V = H$ is a Hilbert space,

$$H^k(a, b; H) := W^{k,2}(a, b; H), \qquad H_0^k(a, b; H) := W_0^{k,2}(a, b; H).$$

5.1.9 Theorem. *Let V be a real, separable Banach space and H a real, separable Hilbert space.*

(a) $W^{k,p}(a, b; V)$ and $W_0^{k,p}(a, b; V)$ are Banach spaces when equipped with the $W^{k,p}$-norm, for all $p \in [1, \infty]$ and all $k \in \mathbb{N}_0$. Moreover, $H^k(a, b; H)$ and $H_0^k(a, b; H)$ are Hilbert spaces with respect to the canonical inner product.

(b) If $p \in [1, \infty)$, then $W^{k,p}(a, b; V)$ is separable.

[5] Higher order derivatives are defined accordingly.

(c) If $p \in (1,\infty)$ and V is reflexive, then $W^{k,p}(a,b;V)$ is reflexive.

Proof. As in the real-valued case, the linear mapping $W^{k,p}(a,b;V) \to L^p(a,b;V)^{k+1}$, $u \mapsto (u, u', \dots, u^{(k)})$, allows us to identify $W^{k,p}(a,b;V)$ with a closed subspace of $L^p(a,b;V)^{k+1}$ (closed using the definition of weak derivatives). Hence the assertions follow from the corresponding assertions for $L^p(a,b;V)$. □

5.1.10 Theorem. *Let $(a,b) \subset \mathbb{R}$ be bounded, let $p \in [1,\infty]$ and let $u \in W^{1,p}(a,b;V)$, $v \in W^{1,p}(a,b;\mathbb{R})$.*

(a) For almost every $x, y \in [a,b]$,

$$u(y) - u(x) = \int_x^y u'(t)\,dt.$$

In particular, if $u' = 0$ a.e., then u is constant a.e..

(b) (Sobolev embedding theorem) $W^{1,p}(a,b;V) \hookrightarrow C([a,b],V)$. In particular, there exists a constant $c = c(a,b,p) > 0$ such that

$$\|u\|_{L^\infty(a,b;V)} \le c\|u\|_{W^{1,p}(a,b;V)} \qquad \text{for all } u \in W^{1,p}(a,b;V).$$

(c) (Product rule) $uv \in W^{1,p}(a,b;V)$ and $(uv)' = u'v + uv'$.

(d) (Integration by parts)

$$\int_a^b u'v\,dx = u(b)v(b) - u(a)v(a) - \int_a^b uv'\,dx.$$

(e) $u \in W_0^{1,p}(a,b;V)$ if and only if $u \in W^{1,p}(a,b;V)$ and $u(a) = u(b) = 0$.

(f) (Poincaré's inequality) If $p \in [1,\infty)$, then there exists $c = c(a,b,p) > 0$ such that

$$\int_a^b \|u\|_V^p\,dx \le c\int_a^b \|u'\|_V^p\,dx \qquad \text{for all } u \in W_0^{1,p}(a,b;V).$$

In particular, $\|u'\|_{L^p(a,b;V)}$ defines an equivalent norm on $W_0^{1,p}(a,b;V)$.

The proof of (a) is analogous to the real-valued case; (b) follows from (a) and the Closed Graph Theorem (the identity mapping $W^{1,p} \to C$ is closed). (c), (d) and (e) make sense, i.e., the formulae are well defined, due to (b). The direction "only if" in (e) is immediate from the definition and (b); the "if" direction is the harder part. Most of the proofs can be found in [4, Chapter 5].

We will write $\dot{u}$ for $u' = \frac{du}{dt}$ if $u \in W^{1,p}(a,b;V)$.

5.2 Gradient flows

Suppose V is a Banach space and $\mathcal{E} \in C^1(V,\mathbb{R})$. Suppose also that H is a Hilbert space such that V is continuously and densely embedded into H, i.e. there exists a bounded, injective, linear mapping $i : V \to H$ with dense range.

(Prototype: $V = W^{1,p}(\Omega)$ or $W_0^{1,p}(\Omega)$, $H = L^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is bounded and open and $p \ge 1$ is large enough.)

5.2.1 Definition. The *gradient* $\nabla_H\mathcal{E} : D(\nabla_H\mathcal{E}) \subset V \to H$ of $\mathcal{E} \in C^1(V,\mathbb{R})$ *with respect to the inner product* $(\,\cdot\,,\,\cdot\,)_H$ is given by

$$\begin{cases} D(\nabla_H\mathcal{E}) := \{u \in V : \text{there exists } v \in H \text{ such that } \mathcal{E}'(u)\varphi = (v,\varphi)_H \text{ for all } \varphi \in V \hookrightarrow H\} \\ \nabla_H\mathcal{E}(u) := v. \end{cases}$$

That is,

$$\mathcal{E}'(u)\varphi = (\nabla_H\mathcal{E}(u),\varphi)_H \qquad \text{for all } \varphi \in V,$$

if $\mathcal{E}'(u)$ may be identified in this way as an element of H. In the special case that $i = \mathrm{id}$, i.e. $V = H$ up to isometric isomorphism, then this coincides with the statement of Lemma 4.1.4. In the general case, this depends on the space H. The gradient of a vector $u \in V$ is however always unique, if it exists, since V is dense in H.

5.2.2 Definition. (a) A (non-autonomous)[6] *gradient flow* (sometimes called *gradient system*) is a differential equation of the form

$$\begin{cases} \dot{u} + \nabla_H\mathcal{E}(u) = f, \\ \qquad\quad u(0) = u_0, \end{cases} \tag{5.2.1}$$

where $t \in I := (0,T) \subset \mathbb{R}$, $T > 0$, and $f \in L^2(I,H)$, $u_0 \in V$.

(b) We *define* a solution of (5.2.1) to be a function

$$u \in H^1(I,H) \cap L^\infty(I,V) \overset{\text{Thm 5.1.10}(b)}{\hookrightarrow} C(\overline{I},H)$$

such that, for almost every $t \in I$, $u(t) \in D(\nabla_H\mathcal{E})$ and (5.2.1) holds, and $\lim_{t\to 0} u(t) = u_0$ in H.

With this definition, both terms on the left-hand side of (5.2.1) have the same degree of regularity, i.e., they are both in $L^2(I,H)$, exactly the same as f. In this case we speak of *maximal regularity* (obviously we cannot expect *more* regularity in general without stronger assumptions on f).

Since V is dense in H, by definition of $\nabla_H\mathcal{E}$, u is a solution of (5.2.1) if and only if $u \in H^1(I,H) \cap L^\infty(I,V)$, $u(t) \to u_0$ in H as $t \to 0$, and u is a solution of the *variational form* of the gradient flow

$$(\dot{u},\varphi)_H + \mathcal{E}'(u)\varphi = (f,\varphi)_H \qquad \text{for all } \varphi \in V, \tag{5.2.2}$$

for almost every $t \in I$.

5.2.3 Theorem. *Suppose in addition to the above assumptions on V and H that V is a real, reflexive, separable Banach space and $\mathcal{E} \in C^1(V,\mathbb{R})$ satisfies*

(i) $\mathcal{E}$ is convex (in the sense of Definition 3.4.4);

(ii) $\mathcal{E}$ is coercive in the sense that $S_\alpha = \{u \in V : \mathcal{E}(u) \le \alpha\}$ is bounded in V for all $\alpha \in \mathbb{R}$ (cf. (3.4.3)*);*

[6] "Non-autonomous" refers to the fact that the right-hand side, i.e. f, may depend on t.

(iii) $\mathcal{E}'$ *maps bounded sets in* V *into bounded sets in* V'*, i.e.* $\sup\{\|\mathcal{E}'(u)\|_V : u \in M\} < \infty$ *whenever* M *is bounded in* V*.*

Then for any $T \in (0,\infty)$*, any* $f \in L^2(0,T;H)$ *and any* $u_0 \in V$ *the gradient flow* (5.2.1) *has a unique solution* $u \in H^1(0,T;H) \cap L^\infty(0,T;V)$ *in the sense of Definition 5.2.2(b). For this solution and for almost every* $t \in (0,T)$*, the energy inequality*

$$\int_0^t \|\dot{u}(s)\|_H^2 \, ds + \mathcal{E}(u(t)) - \mathcal{E}(u_0) \le \int_0^t (f(s), \dot{u}(s))_H \, ds \tag{5.2.3}$$

holds.

This is a *global* existence result, since we obtain solutions on the whole interval where f is defined.

5.2.4 Example. Suppose $V = W_0^{1,p}(\Omega)$ and $H = L^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is bounded and open and $p \in [\frac{2n}{n+2}, \infty)$ (or $p \in (1,\infty)$ if $n = 2$), so that the embedding $V \hookrightarrow H$ is continuous and dense (see Theorems 2.4.3 and 2.1.3). We take

$$\mathcal{E}(u) = \frac{1}{p}\int_\Omega |Du|^p \, dx, \qquad u \in W_0^{1,p}(\Omega),$$

so that

$$\mathcal{E}'(u)\varphi = \int_\Omega |Du|^{p-2} Du \cdot D\varphi \, dx, \qquad u, \varphi \in W_0^{1,p}(\Omega).$$

We claim (without proof) that $\mathcal{E}$ satisfies (i)–(iii). The equation

$$\mathcal{E}'(u)\varphi = (\nabla_{L^2}\mathcal{E}(u), \varphi)_{L^2(\Omega)}$$

reads

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, dx = \int_\Omega \nabla_{L^2}\mathcal{E}(u)\varphi \, dx \qquad \text{for all } \varphi \in W_0^{1,p}(\Omega),$$

that is, $g := \nabla_{L^2}\mathcal{E}(u)$ is the weak solution of

$$\left\{ -\Delta_p u = g \qquad \text{in } \Omega, u \;= 0 \quad \text{on } \partial\Omega, \right.$$

and $u \in D(\nabla_{L^2}\mathcal{E})$ if and only if this equation has a weak solution, i.e., its (distributional) p-Laplacian $\Delta_p u$ is in $L^p(\Omega)$.

Thus Theorem 5.2.3 yields a unique (weak) solution of the evolution equation

$$\begin{cases} \dot{u} - \Delta_p u = f, & t \in [0,T], \\ u = 0 & \text{on } \partial\Omega \times [0,T], \\ u(0) = u_0, & \end{cases}$$

where $T > 0$, $u_0 \in W_0^{1,p}(\Omega)$ and $f \in L^2(0,T;L^2(\Omega))$.[7]

[7] The boundary condition $u = 0$ on $\partial\Omega \times [0,T]$ is satisfied in the sense that $u(t) \in W_0^{1,p}(\Omega)$ (and hence $\operatorname{tr} u = 0$) for almost every $t \in [0,T]$, including $u(0) = u_0 \in W_0^{1,p}(\Omega)$.

5.2.5 Remark. Instead of taking the gradient with respect to $(\,\cdot\,,\,\cdot\,)_H$, we may allow the inner product and gradient to depend on $u \in V$: we suppose g is a *metric*, i.e. a function which maps each $u \in V$ to an *inner product* $(\,\cdot\,,\,\cdot\,)_{g(u)}$ on H, such that $(v,w)_{g(u_n)} \to (v,w)_{g(u)}$ for all $v, w \in H$ whenever $u_n \to u$ in V (we say g is *strongly convergent*. Note that by "inner product" we mean "bounded, coercive, bilinear form"). Then the *gradient* $\nabla_g \mathcal{E}(u) \in H$ (with respect to g) is given by

$$\mathcal{E}'(u)\varphi = (\nabla_g \mathcal{E}(u), \varphi)_{g(u)} \qquad \text{for all } \varphi \in V,$$

if there exists a vector with this property. The corresponding gradient flow is given by

$$\dot{u} + \nabla_g \mathcal{E}(u) = f,$$

with variational form

$$(\dot{u}, \varphi)_{g(u)} + \mathcal{E}'(u)\varphi = (f, \varphi)_{g(u)} \qquad \text{for all } \varphi \in V.$$

Such problems often arise in geoemtric flows, for example. Under slightly stronger convergence conditions on g, we can obtain an analogue of Theorem 5.2.3.[8]

Proof of Theorem 5.2.3: Uniqueness. The uniqueness statement is an easy consequence of the convexity and differentiability of $\mathcal{E}$, and we deal with it first.

We first observe that convexity implies

$$\langle \mathcal{E}'(u) - \mathcal{E}'(v), u - v \rangle \geq 0 \qquad \text{for all } u, v \in V \tag{5.2.4}$$

(exercise!). Hence, if u_1, u_2 are any two solutions of (5.2.1), then for almost every $t \in [0,T]$, using the chain rule and the solution property, respectively,

$$\begin{aligned} \frac{d}{dt}\|u_1(t) - u_2(t)\|_H^2 &= 2(u_1(t) - u_2(t), \dot{u}_1(t) - \dot{u}_2(t))_H \\ &= -2\langle \mathcal{E}'(u_1(t) - \mathcal{E}'u_2(t), u_1(t) - u_2(t)\rangle \leq 0 \end{aligned}$$

by (5.2.4). Hence, by the fundamental theorem of calculus (noting that $\|u_1 - u_2\|_H^2 \in H^1(0,T;\mathbb{R})$ by definition of solutions and properties of H^1-functions),

$$\|u_1(t) - u_2(t)\|_H^2 - \underbrace{\|u_1(0) - u_2(0)\|_H^2}_{=0} \leq 0$$

for all $t > 0$, so $u_1 = u_2$ for almost every $x \in \Omega$ and all $t > 0$. □

Existence: We prove existence by constructing a sequence of solutions of approximating finite-dimensional gradient flow problems. These will be shown to be global solutions via norm and energy bounds; these bounds also allow us to obtain a weakly convergent subsequence whose weak limit solves (5.2.1). This method is sometimes known as the Ritz–Galerkin approximation method; one also speaks of Galerkin approximation.

Step 1: A finite-dimensional existence result.

[8] See [4, Chapter 8].

5.2.6 Theorem (Carathéodory)**.** *Let* $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ *be an open set,* $n \geq 1$*, and let* $F : \Omega \to \mathbb{R}^n$*,* $F = F(t, z)$*, satisfy the* Carathéodory conditions*:*

(i) $F(\,\cdot\,, z)$ *is measurable for each fixed* z*,*

(ii) $F(t, \,\cdot\,)$ *is continuous for each fixed* t*, and*

(iii) for every $(t_0, z_0) \in \Omega$ *there exist* $\alpha > 0$*,* $r > 0$ *and* $g \in L^1(t_0, t_0 + \alpha)$ *such that*

$$|F(t, z)| \leq g(t) \qquad \text{for all } t \in (t_0, t_0 + \alpha) \text{ and all } z \in B(z_0, r)$$

("locally uniform integrability").

Then for every $(t_0, u_0) \in \Omega$ *the ODE*

$$\begin{cases} \dot{u} + F(t, u) = 0 \\ \qquad u(t_0) = u_0 \end{cases} \tag{5.2.5}$$

admits a local solution*, that is, there exist* $I = [t_0, t_0 + \beta] \subset \mathbb{R}$*,* $\beta > 0$*, and* $u \in C(I, \mathbb{R}^n)$ *such that*

$$u(t) = u_0 - \int_{t_0}^{t} F(s, u(s))\, ds \qquad \text{for all } t \in I.$$

Proof. We construct approximate solutions and use Arzelà–Ascoli: let $(t_0, u_0) \in \Omega$ and $\alpha, r > 0$, g be as in (iii), and define

$$G(t) := \int_{t_0}^{t} g(s)\, ds, \qquad t \in [t_0, t_0 + \alpha].$$

Then G is continuous with $G(t_0) = 0$. By taking $\alpha > 0$ smaller if necessary, we may assume

$$|G(t)| \leq r \qquad \text{for all } t \in [t_0, t_0 + \alpha].$$

For $k \geq 2$ we define $u_k \in C([t_0, t_0 + \alpha], \mathbb{R}^n)$ by

$$u_k(t) := \begin{cases} u_0 & \text{if } t \in [t_0, t_0 + \frac{\alpha}{k}], \\ u_0 - \int_{t_0}^{t - \frac{\alpha}{k}} F(s, u_0(s))\, ds & \text{if } t \in (t_0 + \frac{\alpha}{k}, t_0 + \alpha], \end{cases}$$

where this definition is to be understood iteratively, i.e., the definition up to $t_0 + \frac{\alpha}{k}$ allows us to define the integral expression for $t \in (t_0 + \frac{\alpha}{k}, t_0 + \frac{2\alpha}{t}]$, and thus for $t \in (t_0 + \frac{2\alpha}{k}, t_0 + \frac{3\alpha}{t}]$, etc.. The Carathéodory conditions ensure that the integral is well defined and

$$|u_k(t) - u_0| \leq \int_{t_0}^{t - \frac{\alpha}{k}} |F(s, u_k(s))|\, ds.$$

Since $|F(s, u_k(s))| \leq g(s)$ if $2\alpha/k$ because then $u_k(s) \in B(u_0, r)$, this means

$$|u_k(t) - u_0| \leq \int_{t_0}^{t} g(s) = G(t) \leq r$$

for all t up to $2\alpha/k$; thus we may repeat this argument to obtain $|u_k(t) - u_0| \leq r$ for all t up to $3\alpha/k$, and so on, so that

$$|u_k(t) - u_0| \leq r \qquad \text{for all } t \in [t_0, t_0 + \alpha]. \tag{5.2.6}$$

In particular, since this holds independently of k, the sequence (u_k) is uniformly bounded.

Equicontinuity: fix $\varepsilon > 0$. Since G is uniformly continuous on $[t_0, t_0 + \alpha]$, there exists $\delta > 0$ such that

$$|t - t'| \leq \delta \qquad \text{implies} \qquad |G(t) - G(t')| \leq \varepsilon. \tag{5.2.7}$$

Moreover, for all $k \geq 2$ and all $t, t' \in [t_0, t_0 + \alpha]$ such that $t' \geq t$,

$$|u_k(t') - u_k(t)| \leq \begin{cases} \int_{t-\frac{\alpha}{k}}^{t'-\frac{\alpha}{k}} |F(s, u_k(s))|\, ds \leq |G(t' - \frac{\alpha}{k}) - G(t - \frac{\alpha}{k})| & \text{if } t' \geq t \geq t_0 + \frac{\alpha}{k}, \\ \int_{t_0}^{t-\frac{\alpha}{k}} |F(s, u_k(s))|\, dx \leq |G(t' - \frac{\alpha}{k}) - \underbrace{G(t_0)}_{=0}| & \text{if } t' \geq t_0 + \frac{\alpha}{k} \geq t, \\ 0 & \text{if } t_0 + \frac{\alpha}{k} \geq t' \geq t \end{cases}$$
$$\leq \varepsilon \text{ if } |t' - t| \leq \text{ by (5.2.7).}$$

By the Arzelá–Ascoli theorem there exist a subsequence (u_{k_j}) and $u \in C([t_0, t_0 + \alpha], \mathbb{R}^n)$ such that $u_{k_j} \to u$ uniformly in $[t_0, t_0 + \alpha]$. By (ii),

$$\lim_{j\to\infty} F(t, u_{k_j}(t)) = F(t, u(t)) \qquad \text{for each } t \in [t_0, t_0 + \alpha].$$

Now by (iii) (using (5.2.6),

$$|F(t, u(t))| \leq g(t) \qquad \text{for all } k \geq 2 \text{ and all } t \in [t_0, t_0 + \alpha].$$

Hence, by the dominated convergence theorem, for any $t \in [t_0, t_0 + \alpha]$,

$$\begin{aligned} u(t) = \lim_{j\to\infty} u_{k_j}(t) = u_0 - \lim_{j\to\infty} \int_{t_0}^{t-\frac{\alpha}{k_j}} F(s, u_{k_j}(s))\, ds \\ = u_0 - \lim_{j\to\infty} \int_{t_0}^{t} \chi_{[t_0, t-\frac{\alpha}{k_j}]} F(s, u_{k_j}(s))\, ds = u_0 - \int_{t_0}^{t} F(s, u_{k_j}(s))\, ds. \end{aligned}$$

□

A solution $u : [t_0, t_0 + \alpha) \to \mathbb{R}^n$, $\alpha \in (0, \infty]$, of the ODE (5.2.5) is called *maximal* if it cannot be extended to a solution on a larger interval $[t_0, t_0 + \beta)$, $\beta > \alpha$. (We mean that v is an extension of u if it is a solution on $[t_0, t_0 + \beta)$, $\beta > \alpha$, and $v|_{[t_0, t_0+\alpha)} = u$.)

5.2.7 Corollary. *Under the assumptions of Theorem 5.2.6, for every* (t_0, u_0) *there exists a maximal solution of* (5.2.5).

Proof. This uses Zorn's Lemma: denote by $\mathcal{S}$ the set of all pairs (α, u), $\alpha \in (0, \infty]$, $u \in C([t_0, t_0 + \alpha), \mathbb{R}^n)$ solving (5.2.5). Theorem 5.2.6 shows $\mathcal{S} \neq \emptyset$. Define a partial ordering $\prec$ on $\mathcal{S}$ as follows: $(\alpha, u) \prec (\beta, v)$ if $\alpha \leq \beta$ (in $(0, \infty]$) and $v|_{[t_0, t_0+\alpha)} = u$. Then every totally ordered subset has a maximal element (exercise). □

Step 2: Construction of approximating finite-dimensional problems.

Since V is separable, there exists a sequence $(w_n) \subset V$ such that, setting

$$V_n := \operatorname{span}\{w_k : 1 \leq k \leq n\}, \qquad n \in \mathbb{N},$$

we have

$$V = \overline{\bigcup_{n\in\mathbb{N}} V_n}.$$

Given $u_0 \in V$ and $f \in L^2(0,T;H)$ (the data from (5.2.1)), and noting $V_n \subset V_{n+1} \subset \ldots$, we choose $u_0^n \in V_n$ such that $u_0^n \to u_0$ in V.

We now wish to find $u_n \in W^{1,2}_{loc}(0,T_n;V_n)$ $(T_n \le T)$ solving

$$\begin{cases} (\dot{u}_n, \varphi)_H + \mathcal{E}'(u_n)\varphi = (f,\varphi)_H & \text{for all } \varphi \in V_n \text{ and a.e. } t \in (0,T_n), \\ u_n(0) = u_0^n. \end{cases} \tag{5.2.8}$$

Since $\varphi \in V_n$ and V_n is finite dimensional, setting $H_n = V_n$ equipped with the inner product induced by H, solving (5.2.8) is equivalent to finding a variational solution u_n of

$$\begin{cases} \dot{u}_n + \nabla_{H_n}\mathcal{E}_n(u_n) = P_n f \\ u_n(0) = u_0^n, \end{cases} \tag{5.2.9}$$

where $\mathcal{E}_n := \mathcal{E}|_{V_n}$, $\nabla_{H_n}\mathcal{E}_n$ is the gradient of $\mathcal{E}_n$ in V_n with respect to $(\,\cdot\,,\,\cdot\,)_H = (\,\cdot\,,\,\cdot\,)_{H_n}$, and $P_n : H \to H$ is the orthogonal projection of H onto H_n with respect to $(\,\cdot\,,\,\cdot\,)_H$.

It is immediate that $\nabla_{H_n}\mathcal{E}_n(u)$ exists and belongs to V_n, since $\mathcal{E}_n$ is C^1 and V_n is finite dimensional. Moreover, the mapping $(t,z) \mapsto \nabla_{H_n}\mathcal{E}_n(z)$ obviously satisfies the Carathéodory conditions of Theorem 5.2.6; hence (5.2.9) admits a maximal solution $u_n \in C(0,T_n;V_n)$ by Corollary 5.2.7 (that is, either $T_n = T$, or $T_n < T$ and u_n cannot be extended. Also note that the proof of the uniqueness statement in Theorem 5.2.3 given above also applies here, to show that u_n is unique).

Now u_n is by construction weakly differentiable (in t) with derivative $\nabla_{H_n}\mathcal{E}_n(u_n(t))$; we will see in the next step that in fact $\dot{u}_n$ is in $L^2_{loc}(0,T_n;H_n) \simeq L^2_{loc}(0,T_n;V_n)$.

Step 3: Bounds on the u_n.

We claim that the u_n are global solutions (i.e. $T_n = T$) and that

$$(u_n) \text{ is a bounded sequence in } H^1(0,T;H) \cap L^\infty(0,T;V). \tag{5.2.10}$$

Taking $\varphi = \dot{u}_n$ in (5.2.8) and integrating over $[0,t]$ for $t \in (0,T_n)$,

$$\int_0^t \|\dot{u}(s)\|_H^2\,ds + \int_0^t \underbrace{\mathcal{E}'(u_n(s))\dot{u}_n(s)}_{=\frac{d}{dt}\mathcal{E}(u_n(s))}\,ds = \int_0^t (f(s),\dot{u}_n(s))_H\,ds.$$

Since

$$|(f(s),\dot{u}_n(s))_H| \le \|f(s)\|_H\|\dot{u}_n\|_H \le \frac{1}{2}(\|f(s)\|_H^2 + \|\dot{u}_n(s)\|_H^2)$$

for almost every $s > 0$ and since $\mathcal{E}(u_n(\,\cdot\,))$ is weakly differentiable as the composition of a C^1 with a weakly differentiable function (which in particular justifies the calculation of $\frac{d}{dt}\mathcal{E}(u_n(t))$), meaning that we may apply the fundamental theorem of calculus to it, i.e.,

$$\int_0^t \mathcal{E}'(u_n(s))\dot{u}_n(s)\,ds = \int_0^t (\nabla_{H_n}\mathcal{E}_n(u_n),\dot{u}_n)_H\,ds = \int_0^t \frac{d}{dt}\mathcal{E}(u_n(s))\,ds = \mathcal{E}(u_n(t)) - \mathcal{E}(u_0^n),$$

we obtain

$$\int_0^t \|\dot u_n(s)\|_H^2\,ds + \mathcal{E}(u_n(t)) - \mathcal{E}(u_0^n) \le \frac{1}{2}\int_0^t \|f(s)\|_H^2\,ds + \frac{1}{2}\int_0^t \|\dot u_n(s)\|_H^2\,ds \tag{5.2.11}$$

for all $t > 0$ (noting $\mathcal{E}(u_n(\,\cdot\,))$ is continuous).

Now since $u_0^n \to u$ in V and $\mathcal{E}$ is continuous, we have $\mathcal{E}(u_0^n) \to \mathcal{E}(u_0)$ and so in particular $(\mathcal{E}(u_0^n))$ is bounded. Hence there exists a constant $C > 0$ independent of n such that

$$\underbrace{\frac{1}{2}\int_0^t \|\dot u_n\|_H^2\,ds}_{\ge 0} + \mathcal{E}(u_n(t)) \le \underbrace{C + \frac{1}{2}\int_0^T \|f(s)\|_H^2\,ds}_{\text{independent of } n \text{ and } t\in(0,T_n)} \qquad \text{for all } t \in (0, T_n).$$

It follows that

$$\{u_n(t) : n \in \mathbb{N}, t \in (0,T_n)\} \subset S_\alpha = \{v \in V : \mathcal{E}(v) \le \alpha\},$$

where

$$\alpha := C + \frac{1}{2}\int_0^T \|f(s)\|_H^2\,ds < \infty.$$

Since $\mathcal{E}$ is coercive, S_α is bounded, and so

$$\sup_{n\in\mathbb{N}} \sup_{t\in(0,T_n)} \|u_n(t)\|_V < \infty.$$

Moreover, $\mathcal{E}$ is bounded from below on V as a continuous, convex, coercive functional on V (exercise). Hence we also obtain

$$\sup_{n\in\mathbb{N}} \|\dot u_n\|_{L^2(0,T_n;H)} < \infty.$$

In particular, $\dot u_n$ is integrable (L^1) on $[0, T_n)$ as $T_n \le T < \infty$, meaning u_n may be extended to a continuous function on the closed interval $[0, T_n]$. If $T_n < T$, then Theorem 5.2.6 yields an extension of u_n to a larger interval $[0, T_n + \varepsilon)$, contradicting the assumption of maximality. Hence $T_n = T$ and u_n is a global solution on $[0, T]$.

Thus (u_n) is bounded in $L^\infty(0,T;V)$ and $(\dot u_n)$ is bounded in $L^2(0,T;H)$; by continuity of the embedding $V \to H$, (5.2.10) follows. Moreover, since $\mathcal{E}' : V \to V'$ maps bounded sets into bounded sets, the boundedness of (u_n) in $L^\infty(0,T;V)$ implies

$$(\mathcal{E}'(u_n)) \text{ is bounded in } L^\infty(0,T;V').$$

Step 4: (u_n) has a convergent subsequence.

$H^1(0,T;H)$ is a Hilbert space and, since V is reflexive,

$$L^\infty(0,T;V) \simeq L^1(0,T;V')'$$
$$L^\infty(0,T;V') \simeq L^1(0,T;V)';$$

see Theorem 5.1.7(b). Moreover, $L^1(0,T;V)$ is separable by separability of V and Theorem 5.1.7(a). Hence there exist a subsequence of (u_n), which we shall again denote by (u_n), and $u \in H^1(0,T;H)$, $v \in L^\infty(0,T;V)$, $w \in L^\infty(0,T;V')$ such that

$$\begin{aligned} u_n &\rightharpoonup u && \text{in } H^1(0,T;H),\\ u_n &\overset{*}{\rightharpoonup} v && \text{in } L^\infty(0,T;V),\\ \mathcal{E}'(u_n) &\overset{*}{\rightharpoonup} w && \text{in } L^\infty(0,T;V'). \end{aligned}$$

It is an exercise in the uniqueness of weak limits to show that $u = v$, so in particular

$$u \in H^1(0,T;H) \cap L^\infty(0,T;V),$$

and moreover

$$\begin{aligned} \dot u_n &\rightharpoonup \dot u && \text{in } L^2(0,T;H),\\ u_n(0) &\rightharpoonup u(0) && \text{in } H,\\ u_n(T) &\rightharpoonup u(T) && \text{in } H \end{aligned}$$

(the last two following from continuity and linearity of the embeddings $H^1(0,T;H) \hookrightarrow C([0,T],H) \hookrightarrow H$, cf. Theorem 5.1.10). This means that

$$\begin{aligned} \int_0^T \langle \varphi, u_n\rangle\, dt &\to \int_0^T \langle \varphi, u\rangle\, dt && \text{for all } \varphi \in L^1(0,T;V'),\\ \int_0^T (\dot u_n, \varphi)_H\, dt &\to \int_0^T (\dot u_n, \varphi)_H\, dt && \text{for all } \varphi \in L^2(0,T;H)\\ \int_0^T \langle \mathcal{E}'(u_n), \varphi\rangle\, dt &\to \int_0^T \langle w, \varphi\rangle\, dt && \text{for all } \varphi \in L^1(0,T;V). \end{aligned}$$

Step 5: u is a solution.

We have $u_n(0) \rightharpoonup u(0)$ in H and $u_0^n \to u_0$ in V by choice of u_0^n. Since $V \hookrightarrow H$, $u(0) = u_0$, i.e. u satisfies the initial condition in (5.2.1). We need to show

$$\int_0^T (\dot u, \varphi)_H\, dt + \int_0^T \langle E'(u), \varphi\rangle\, dt = \int_0^T (f, \varphi)_H\, dt \qquad \text{for all } \varphi \in L^2(0,T;V) \tag{5.2.12}$$

using (5.2.8)/(5.2.9). Taking the corresponding weak form of (5.2.8) and inserting $\varphi(\,\cdot\,)v$ as a test function, where $v \in V_n$ and $\varphi \in L^2(0,T;\mathbb{R})$,

$$\int_0^T (\dot u_m(t), \varphi(t)v)_H\, dt + \int_0^T \langle \mathcal{E}'(u_m(t)), \varphi(t)v\rangle\, dt = \int_0^T (f(t), \varphi(t)v)_H\, dt \qquad \text{for all } m \geq n.$$

Letting $m \to \infty$ and using the convergence results from Step 4, we obtain

$$\int_0^T (\dot u(t), \varphi(t)v)_H\, dt + \int_0^T \langle w(t), \varphi(t)v\rangle\, dt = \int_0^T (f(t), \varphi(t)v)_H\, dt.$$

(Here we have used that $\varphi v \in L^1(0,T;V') \cap L^2(0,T;H) \cap L^1(0,T;V)$, since V_n is finite dimensional and $\varphi \in L^2(0,T;\mathbb{R}) \hookrightarrow L^1(0,T;\mathbb{R})$.)

Now

$$\operatorname{span}\left\{\varphi(\,\cdot\,)v : v \in \bigcup_{n\in\mathbb{N}} V_n, \varphi \in L^2(0,T;\mathbb{R})\right\}$$

is dense in $L^2(0,T;V)$ (and $L^2(0,T;H)$ etc.), cf. the sketch of proof of Theorem 5.1.7(a). It follows that

$$\int_0^T (\dot u, \psi)_H\, dt + \int_0^T \langle w, \psi\rangle\, dt = \int_0^T (f, \psi)_H\, dt \qquad \text{for all } \psi \in L^2(0,T;V); \tag{5.2.13}$$

comparing this with (5.2.12), it remains to show $w = \mathcal{E}'(u)$.

Again using the weak form of (5.2.8) with u_n as a test function (for any $n \in \mathbb{N}$),

$$\begin{aligned}\int_0^T \mathcal{E}'(u_n)u_n\,dt &= \int_0^T (f,u_n)_H\,dt - \int_0^T (\dot{u}_n, u_n)_H\,dt\\ &= \int_0^T (f,u_n)_H\,dt - \int_0^T \frac{1}{2}\frac{d}{dt}\|u_n\|_H^2\,dt\\ &= \int_0^T (f,u_n)_H\,dt - \frac{1}{2}(\|u_n(T)\|_H^1 - \|u_0^n\|_H^2).\end{aligned}$$

Now since $u_n \rightharpoonup u$ in $H^1(0,T;H)$, the same is true in $C([0,T],H)$, so in particular

$$\|u(T)\|_H^2 \leq \liminf_{n\to\infty} \|u_n(T)\|_H^2.$$

Moreover, since $u_0^n \to u_0$ in $V \hookrightarrow H$,

$$\|u_0^n\|_H^2 \to \|u_0\|_H^2,$$

and since $u \rightharpoonup u$ in $L^2(0,T;H)$,

$$\int_0^T (f,u)_H\,dt = \lim_{n\to\infty} \int_0^T (f,u_n)_H\,dt.$$

Hence

$$\begin{aligned}\limsup_{n\to\infty} \int_0^T \mathcal{E}'(u_n)u_n\,dt &\leq \int_0^T (f,u)_H\,dt - \frac{1}{2}\|u(T)\|_H^2 + \frac{1}{2}\|u_0\|_H^2\\ &= \int_0^T (f,u)_H\,dt - \int_0^T \underbrace{\frac{1}{2}\frac{d}{dt}\|u\|_H^2}_{=(\dot{u},u)_H}\,dt\\ &= \int_0^T \langle w,u\rangle\,dt\end{aligned}$$

by (5.2.13). On the other hand, using the convexity property (5.2.4),

$$\int_0^T \langle \mathcal{E}'(u_n), u_n - u\rangle\,dt \geq \int_0^T \langle \mathcal{E}'(u), u_n - u\rangle\,dt.$$

Since $u_n \overset{*}{\rightharpoonup} u$ and $\mathcal{E}'(u_n) \overset{*}{\rightharpoonup} w$ in $L^\infty(0,T;V)$, it follows that

$$\liminf_{n\to\infty} \int_0^T \langle \mathcal{E}'(u_n), u_n\rangle\,dt \geq \int_0^T \langle w,u\rangle\,dt,$$

and hence

$$\lim_{n\to\infty} \int_0^T \langle \mathcal{E}'(u_n), u_n\rangle\,dt = \int_0^T \langle w,u\rangle\,dt. \tag{5.2.14}$$

Now let $\varphi \in L^\infty(0,T;V)$ and $\lambda \in \mathbb{R}$ be arbitrary. Then (5.2.4) implies

$$\int_0^T \langle \mathcal{E}'(u_n) - \mathcal{E}'(u+\lambda\varphi), u_n - u - \lambda\varphi\rangle\,dt \geq 0 \qquad \text{for all } n \in \mathbb{N}.$$

Letting $n \to \infty$ and using the weak convergences $u_n \overset{*}{\rightharpoonup} u$, $\mathcal{E}'(u_n) \overset{*}{\rightharpoonup} w$ and (5.2.14),

$$-\int_0^T \langle w, \lambda\varphi\rangle + \langle \mathcal{E}'(u+\lambda\varphi), \lambda\varphi\rangle \, dt \geq 0,$$

that is,

$$-\lambda \int_0^T \langle w, \varphi\rangle \, dt \geq -\lambda \int_0^T \langle \mathcal{E}'(u+\lambda\varphi), \varphi\rangle \, dt \qquad \text{for all } \lambda \in \mathbb{R}.$$

If $\lambda > 0$, dividing by λ and letting $\lambda \to 0^+$ yields

$$\int_0^T \langle w, \varphi\rangle \, dt \leq \int_0^T \langle \mathcal{E}'(u), \varphi\rangle \, dt;$$

if $\lambda < 0$ and $\lambda \to 0^-$, we obtain "$\geq$".

Since $\varphi \in L^\infty(0,T;V)$ was arbitrary, it finally follows that

$$w = \mathcal{E}'(u).$$

Recalling (5.2.12), this completes the existence proof.

Step 6: Energy inequality.

Again denote by (u_n) the (sub-) sequence of solutions of the approximating problems obtained in Step 4. We recall the identity (5.2.12):

$$\int_0^t \|\dot{u}_n\|_H^2 \, ds + \mathcal{E}(u_n(t)) - \mathcal{E}(u_0^n) = \int_0^t (f, \dot{u}_n)_H \, ds.$$

Now $\dot{u}_n \rightharpoonup \dot{u}$ in $L^2(0,T;H)$ implies $\dot{u}_n \rightharpoonup \dot{u}$ in $L^2(0,t;H)$ for all $t \in [0,T]$ and hence

$$\lim_{n\to\infty} \int_0^t (f, \dot{u}_n)_H \, ds = \int_0^t (f, \dot{u})_H \, ds \qquad \text{for all } t \in [0,T],$$

as well as

$$\int_0^t \|\dot{u}\|_H^2 \, ds = \|\dot{u}\|_{L^2(0,t;H)}^2 \leq \liminf_{n\to\infty} \|\dot{u}_n\|_{L^2(0,t;H)}^2 = \int_0^t \|\dot{u}_n\|_H^2 \, ds.$$

Also recall that $\mathcal{E}(u_0^n) \to \mathcal{E}(u_0)$. Hence for the energy inequality (5.2.3) we still need

$$\mathcal{E}(u(t)) \leq \liminf_{n\to\infty} \mathcal{E}(u_n(t)).$$

Now $\mathcal{E}$ is convex and continuous and hence weakly lower semicontinuous (cf. Proposition 3.4.5), so we need $u_n(t) \rightharpoonup u(t)$ in V for a.e. t. Since $u_n \rightharpoonup u$ in $H^1(0,T;H)$ implies $u_n(t) \rightharpoonup u(t)$ in H for each $t \in [0,T]$ (via weak convergence in $C([0,T],H)$). Since (u_n) is bounded in $L^\infty(0,T;V)$, it follows (exercise) that indeed $u_n(t) \rightharpoonup u(t)$ in V t-a.e..

5.3 Nonlinear semigroups

5.3.1 Definition. Let H be a Hilbert space. A family $(T(t))_{t\geq 0}$ of (nonlinear) operators is called a *nonlinear semigroup* if it satisfies the following conditions:

(i) $T(0) = \mathrm{Id}$, i.e. $T(0)u = u$ for all $u \in H$;

(ii) $T(t+s)u = T(t)T(s)u$ for all $t, s \geq 0$ and all $u \in H$;

(iii) the mapping $t \mapsto T(t)u$ is continuous from $[0, \infty)$ into H for each $u \in H$.

If in addition

(iv) $\|T(t)u - T(t)v\|_H \leq \|u - v\|_H$ for all $u, v \in H$,

then T is called a *contraction semigroup.*

Solutions of $\dot{u} + Au = 0$ (with initial and boundary condition) always satisfy (i) and (ii), as long as they exist and are unique.

5.3.2 Theorem. *Under the assumptions of Theorem 5.2.3, the solution operator mapping $u_0 \in V$ to the solution of the autonomous problem*

$$\begin{cases} \dot{u}(t) + \nabla_H \mathcal{E}(u(t)) = 0 \\ \qquad\qquad u(0) = u_0 \end{cases} \tag{5.3.1}$$

extends to a contraction semigroup on H, which we denote by $(T(t))_{t \geq 0}$.

One can in fact show that for every $u_0 \in H$ there exists $u(t) = T(t)u_0$ such that $u \in D(\nabla_H \mathcal{E})$ for a.e. $t \in [0, T]$ and u solves (5.3.1) for a.e. $t \in [0, T]$.

Sketch of proof. First note that Theorem 5.2.3 immediately yields the semigroup properties (i)–(iii) of solutions if $u_0 \in V$, (i) since $\lim_{t \to 0} u(t) = u_0$, (ii) for almost every $s \geq 0$ by uniqueness of solutions: if $u_0 \in V$, then, denoting by $T(t)u_0$ the corresponding solution $u(t)$ at time t,

$$T(t+s)u_0 = T(t)T(s)u_0 \tag{5.3.2}$$

whenever $T(s)u_0 \in V$, and (iii) since $H^1(0, T; H) \hookrightarrow C([0, T], H)$. This embedding also shows that (5.3.2) holds as an identity in H for all $t, s \geq 0$. We next show that (iv) holds for all initial conditions $u_0, v_0 \in V$: suppose $u(t), v(t)$ are the corresponding solutions. Then, by the usual convexity argument,

$$\frac{d}{dt}\|u(t) - v(t)\|_H^2 = 2(\dot{u}(t) - \dot{v}(t), u(t) - v(t))_H = -2\langle \mathcal{E}'(u(t)) - \mathcal{E}'(v(t)), u(t) - v(t) \rangle \leq 0$$

for a.e. t. Since $u, v \in H^1(0, T; H)$, the fundamental theorem of calculus holds and hence

$$\|T(t)u_0 - T(t)v_0\|_H = \|u(t) - v(t)\|_H \leq \|u_0 - v_0\|_H \qquad \text{for all } t \geq 0,$$

so (iv) holds on V.

We have shown that $(T(t))_{t \geq 0}$ is densely defined and contractive on H. Hence there exists a unique extension to the whole of H: if $u_0 \in H$ and $u_n \in V$ with $u_n \to u_0$ in H, then $(u_n(t))$ is Cauchy in H for each $t \in [0, T]$; the limit will once again satisfy the properties (i)–(iv). □

6 Non-variational methods

6.1 Fixed point theorems

We recall the following fixed point theorems.

6.1.1 Theorem (Banach's fixed point theorem = contraction mapping theorem)**.** *Suppose V is a Banach space and $A : V \to V$ is a mapping such that*

$$\|Ax - Ay\|_V \le c\|x - y\|_V \qquad \textit{for all } x, y \in V,$$

for some $c < 1$. Then A has a unique point x, i.e., such that $Ax = x$.

6.1.2 Theorem (Brouwer's fixed point theorem)**.** *Denote by $\overline{B(0,1)}$ the closed unit ball in $\mathbb{R}^n$. If $u : \overline{B(0,1)} \to \overline{B(0,1)}$ is continuous, then u has (at least) one fixed point in $\overline{B(0,1)}$.*

The theorem continues to hold if $\overline{B(0,1)}$ is replaced with a more general compact set $K \subset \mathbb{R}^n$, say, which is homeomorphic to the unit ball. In Banach spaces the compactness is necessary: Theorem 6.1.2 does *not* hold in the closed unit ball of a general Banach space. Here is one of the most common generalisations.

6.1.3 Theorem (Schauder's fixed point theorem)**.** *Suppose V is a Banach space and $\emptyset \neq K \subset V$ is compact and convex. If $A : K \to K$ is continuous, then A has a fixed point.*

Proof. 1. Fix $\varepsilon > 0$ and choose $x_1, \ldots, x_{N_\varepsilon} \in K$ such that $\{B(x_i, \varepsilon)\}_{i=1}^{N_\varepsilon}$ is an open cover of K. Denote by K_ε the *closed convex hull* of $\{x_1, \ldots, x_{N_\varepsilon}\}$, i.e. the smallest closed convex set containing $\{x_1, \ldots, x_{N_\varepsilon}\}$,

$$K_\varepsilon := \left\{ \sum_{i=1}^{N_\varepsilon} \lambda_i x_i : 0 \le \lambda_i \le 1, \sum_{i=1}^{N_\varepsilon} \lambda_i = 1 \right\}.$$

Then $K_\varepsilon \subset K$ since K is convex. We define a mapping $P_\varepsilon : K \to K_\varepsilon$ by

$$P_\varepsilon := \frac{\sum_{i=1}^{N_\varepsilon} \operatorname{dist}(x, K \setminus B(x_i, \varepsilon)) x_i}{\sum_{i=1}^{N_\varepsilon} \operatorname{dist}(x, K \setminus B(x_i, \varepsilon))}, \qquad x \in K.$$

(Thus $P_\varepsilon x$ is a weighted combination of the x_i, the weighting only being nonzero for given i if $x \in B(x_i, \varepsilon)$.) This is well defined since

$$\sum_{i=1}^{N_\varepsilon} \operatorname{dist}(x, K \setminus B(x, \varepsilon)) \neq 0,$$

which in turn follows since $K \subset \bigcup_i B(x_i, \varepsilon)$. Moreover, P_ε is continuous (as the same is true of dist) and

$$\|P_\varepsilon x - x\|_V \leq \frac{\sum_{i=1}^{N_\varepsilon} \operatorname{dist}(x, K \setminus B(x_i, \varepsilon))\|x_i - x\|_V}{\sum_{i=1}^{N_\varepsilon} \operatorname{dist}(x, K \setminus B(x_i, \varepsilon))} < \varepsilon \qquad \text{for all } x \in K \tag{6.1.1}$$

since the coefficient of $\|x_i - x\|_V$ in the sum in the numerator is only nonzero if $x \in B(x_i, \varepsilon$, and then $\|x_i - x\|_V < \varepsilon$.

2. Observe that K_ε is homeomorphic to the closed unit ball in $\mathbb{R}^{M_\varepsilon}$ for some $M_\varepsilon \leq N_\varepsilon$. If we define an operator $A_\varepsilon : K_\varepsilon \to K_\varepsilon$ by

$$A_\varepsilon = P_\varepsilon A x, \qquad x \in K_\varepsilon,$$

then Brouwer's fixed point theorem implies there exists $x_\varepsilon \in K_\varepsilon$ such that $A_\varepsilon x_\varepsilon = x_\varepsilon$.

3. Since K is compact there exist $\varepsilon_j \to 0$ and $x \in K$ such that $x_{\varepsilon_j} \to x$ in V. Then x is a fixed point of A since

$$\|x - Ax\|_V \leftarrow \|x_{\varepsilon_j} - Ax_{\varepsilon_j}\|_V = \|A_{\varepsilon_j} x_{\varepsilon_j} - Ax_{\varepsilon_j}\|_V = \|P_{\varepsilon_j}(Ax_{\varepsilon_j}) - Ax_{\varepsilon_j}\|_V \overset{(6.1.1)}{\leq} \varepsilon_j \to 0.$$

□

6.1.4 Definition. A mapping $A : V \to V$ (V a Banach space) is called *compact* if for any bounded sequence (x_k) in V, (Ax_k) is precompact in V.

There are several alternate forms of Theorem 6.1.3 for compact operators; for example, if A is compact, then K may be closed and bounded in place of compact. The following variant eliminates the need to identify a *convex* set K:

6.1.5 Theorem. *Suppose V is a Banach space and $A : V \to V$ is a continuous and compact mapping such that the set*

$$\{x \in V : x = \lambda A x \text{ for some } 0 \leq \lambda \leq 1\}$$

is bounded. Then A has a fixed point.

Proof. 1. Choose $M > 0$ large enough that

$$\{x \in V : x = \lambda A x \text{ for some } 0 \leq \lambda \leq 1\} \subset B(0, M). \tag{6.1.2}$$

Now define

$$\widetilde{A}x := \begin{cases} Ax & \text{if } \|Ax\|_V \leq M, \\ M \cdot \frac{Ax}{\|Ax\|_V} & \text{if } \|Ax\|_V > M. \end{cases}$$

Then $\widetilde{A} : \overline{B(0,M)} \to \overline{B(0,M)}$. Set K to be the closed convex hull of $\widetilde{A}(\overline{B(0,M)})$. Since A is compact, so is $\widetilde{A}$; hence K is compact and convex, and obviously $\widetilde{A} : K \to K$ since $K \subset \overline{B(0,M)}$.

2. By Theorem 6.1.3, there exists $x \in K$ such that $\widetilde{A}x = x$; we claim that $Ax = x$ as well. Indeed, if not, then by construction of $\widetilde{A}$, $\|Ax\|_V \geq M$ (since otherwise $Ax = \widetilde{A}x$), and

$$x = \underbrace{\frac{M}{\|Ax\|_V}}_{=:\lambda < 1} \cdot Ax.$$

But $\|x\|_V = \|\widetilde{A}x\|_V = M$, contradicting the choice of M in (6.1.2). □

6.2 Applications of fixed point theorems

Typical strategy/metaprinciples:

1. Banach: perturbation, e.g. a "small" nonlinear perturbation of a linear equation where existence is known.
2. Schauder/Schaefer: need some sort of compactness, usually works if dealing with inverses of linear elliptic PDEs, which are smoothing (if $u \in L^2(\Omega)$ and $\Delta u \in L^2(\Omega)$, then $u \in H^1(\Omega)$ and often even $H^2(\Omega)$).
3. Schaefer: if we can bound any possible fixed points of λA, $0 \leq \lambda 1$, then a fixed point of A *exists*. That means: if we can prove appropriate estimates for solutions of a PDE under the assumption that they exist, then they do in fact exist! This is the method of *a priori solutions*.

6.2.1 Example. We will use Banach's fixed point theorem and the results of Section 5.2 to prove existence of solutions to the *reaction-diffusion equation* (cf. Example 1.4.1(c))[1]

$$\begin{cases} \dot{u} - \Delta u = f(u) & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times [0,T], \\ u(0) = u_0 & \text{on } \Omega \times \{t=0\}, \end{cases} \tag{6.2.1}$$

where $\Omega \subset \mathbb{R}^n$ is bounded with sufficiently smooth boundary, $u_0 \in H_0^1(\Omega)$, $T > 0$, and $f : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz, which in particular implies the existence of $C > 0$ such that

$$|f(z)| \leq C(|z|+1) \qquad \text{for all } z \in \mathbb{R}. \tag{6.2.2}$$

If f is allowed to grow more quickly (a "stronger" nonlinearity), then it is possible that solutions could "blow up" in finite time (see Section 7.2 below).

As in Section 5.2, we understand a solution of (6.2.1) to be a function

$$u \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)) \hookrightarrow C([0,T],L^2(\Omega))$$

such that $\dot{u}, D_x u \in L^2(\Omega)$ for almost all $t \in [0,T]$ and

$$(\dot{u}(t),\varphi)_{L^2} + (D_x u(t), D_x\varphi)_{L^2} = (f(u(t)),\varphi)_{L^2} \tag{6.2.3}$$

for all $\varphi \in H_0^1(\Omega)$ and almost all $t \in [0,T]$. (Here $D_x u$ denotes the gradient of $u = u(x,t)$ with respect to the x-variables.)

6.2.2 Theorem. *There exists a solution of* (6.2.1) *in the sense of* (6.2.3).

Proof. We will apply Theorem 6.1.1 in the space $V := C([0,T],L^2(\Omega))$; we note explicitly that

$$\|u\|_V = \max_{t\in[0,T]} \|u(t)\|_{L^2(\Omega)}.$$

1. We define the operator $A : V \to V$ as follows. Given $u \in V$, set $g(t)(= g_u(t)) := f(u(t))$, $t \in [0,T]$. Then certainly $g \in L^2(0,T;L^2(\Omega))$ by (6.2.2), so the (linearised) problem

$$\begin{cases} \dot{v} - \Delta v = f(v) & \text{in } \Omega \times (0,T), \\ v = 0 & \text{on } \partial\Omega \times [0,T], \\ v(0) = v_0 & \text{on } \Omega \times \{t=0\}, \end{cases}$$

[1] The same statement holds, with the same arguments, for systems.

has a (unique weak) solution

$$v \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)) \hookrightarrow V$$

satisfying $\dot{v}, D_x v \in L^2(\Omega)$ and

$$(\dot{v}(t),\varphi)_{L^2} + (D_x v(t), D_x\varphi)_{L^2} = (g(t),\varphi) \qquad \text{for all } \varphi \in H_0^1(\Omega) \tag{6.2.4}$$

for almost every $t \in [0,T]$, by Theorem 5.2.3 (see also Example 5.2.4). We now define $A : V \to V$ by setting $Au =: v \, (= v(g(u)))$.

2. *Claim:* If $T > 0$ is small enough, then A is a contraction on V.

Proof. Let $u, \tilde{u} \in V$, $g := f(u)$, $\tilde{g} := f(\tilde{u})$, and $v := Au$, $\tilde{v} := A\tilde{u}$, as above. Then

$$\begin{aligned}
\frac{d}{dt}\|v-\tilde{v}\|_{L^2}^2 &= 2(v-\tilde{v},\dot{v}-\dot{\tilde{v}})_{L^2} \\
&\overset{(6.2.4)}{=} 2(v-v\tilde{v}, g-\tilde{g})_{L^2} - 2(Dv-D\tilde{v}, Dv-D\tilde{v})_{L^2} \\
&\le \varepsilon\|v-\tilde{v}\|_{L^2} + C(\varepsilon)\|g-\tilde{g}\|_{L^2}^2 - 2\|Dv-D\tilde{v}\|_{L^2}^2 \qquad (\varepsilon > 0) \\
&\le (\widetilde{C}\varepsilon - 2)\|Dv-D\tilde{v}\|_{L^2}^2 + C(\varepsilon)\|f(u)-f(\tilde{u})\|_{L^2}^2,
\end{aligned}$$

the last inequality following from Poincaré's inequality. Hence, fixing any $\varepsilon < 2/\widetilde{C}$, setting $C := C(\varepsilon) > 0$, we have

$$\frac{d}{dt}\|v-\tilde{v}\|_{L^2}^2 \le C\|f(u)-f(\tilde{u})\|_{L^2}^2 \le C_1\|u-\tilde{u}\|_{L^2}$$

since f is Lipschitz continuous. Integrating this inequality,

$$\|v(t)-\tilde{v}(t)\|_{L^2}^2 \le C_1 \int_0^t \|u(s)-\tilde{u}(s)\|_{L^2}^2\, ds \le C_1 T \underbrace{\max_{t\in[0,T]} \|u(t)-\tilde{u}(t)\|_{L^2}^2}_{=\|u-\tilde{u}\|_V^2}. \tag{6.2.5}$$

Maximising over $t \in [0,T]$,

$$\|Au - A\tilde{u}\|_V^2 = \|v-\tilde{v}\|_V^2 \le C_1 T\|u-\tilde{u}\|_V^2.$$

Hence A is a (strict) contraction as long as $\sqrt{C_1 T} < 1$. This proves the claim. □

Note that C_1 depends only on Ω (via Poincaré's inequality) and the Lipschitz constant of f.[2]

3. Now assume $T > 0$ is arbitrary. Then there exists some $T_1 \in (0,T]$ depending nly on Ω and f such that A (as above) is a contraction on $V = C([0,T], L^2(\Omega)$. By Banach's fixed point theorem, there exists a unique $u \in C([0,T], L^2(\Omega))$ such that $u \in \text{Range}\,(A)$, meaning $u \in H^1(0,T_1;L^1(\Omega)) \cap L^\infty(0,T_1;H_0^1(\Omega))$ and $u(0) = u_0$ etc., and $u = Au$, so that

$$(\dot{u},\varphi)_{L^2} + (D_x u, D_x\varphi)_{L^2} = (f(u),\varphi)_{L^2} \qquad \text{for all } \varphi \in H_0^1(\Omega),$$

for almost every $t \in [0,T]$. Since $u(t) \in H_0^1(\Omega)$ for almost every $t \in [0,T]$, we may assume $u(T_1) \in H_0^1(\Omega)$ by taking T_1 slightly smaller if necessary. Hence we may repeat the above argument to extend u to a solution on $[T_1, 2T_1]$, since T_1 depends only on f and Ω. Repeating inductively, we obtain a solution on the whole of $[0,T]$. □

[2]This will allow us to obtain a global existence result despite the contraction argument in Step 2 only being local, i.e., proving existence for sufficiently small $T > 0$.

6.2.3 Remark. The solution in Theorem 6.2.2 is unique; this follows from *Gronwall's inequality*: if $h : [0,T] \to [0,\infty)$ is integrable with

$$h(t) \leq a \int_0^t h(s)\,ds + b \qquad \text{for a.e. } t \in [0,T],$$

where $a, b \geq 0$, then

$$h(t) \leq b(1 + ae^{at}) \qquad \text{for a.e. } t \in [0,T];$$

in particular, if b can be chosen to be zero, then $h = 0$ a.e.. Indeed, if u and $\tilde{u}$ are two weak solutions, then by (6.2.5) and the fixed point property $u = v$, $\tilde{u} = \tilde{v}$,

$$\underbrace{\|u(t) - \tilde{u}(t)\|_{L^2}^2}_{=:h(t)} \leq C_1 \int_0^t \|u(s) - \tilde{u}(s)\|_{L^2}^2\,ds \qquad \text{for all } t \in [0,T].$$

6.2.4 Example. We apply Schaefer's fixed point theorem to find a solution of the semilinear PDE

$$\begin{cases} -\Delta u + b(Du) + \mu u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.2.6}$$

where $\Omega \subset \mathbb{R}^n$ is bounded and open and $\partial\Omega$ and $b : \mathbb{R}^n \to \mathbb{R}$ are sufficiently smooth; we assume in addition that b is globally Lipschitz, so that there exists $C > 0$ such that

$$|b(z)| \leq C(|z| + 1) \qquad \text{for all } z \in \mathbb{R}^n.$$

6.2.5 Theorem. *If $\mu > 0$ is sufficiently large, then there exists a (weak) solution $H^2(\Omega) \cap H_0^1(\Omega)$ of (6.2.6).*

Proof. 1. Given $u \in H_0^1(\Omega)$, set

$$f\,(= f_u) := -b(Du);$$

then $f \in L^2(\Omega)$ since b is globally Lipschitz and $Du \in L^2(\Omega)$. Hence there exists a (unique) weak solution $v \in H_0^1(\Omega)$ of the linear problem

$$\begin{cases} -\Delta v + \mu v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

if $\mu > 0$ is large enough (depending only on Ω), see Theorem 2.6.5. It can be shown (linear regularity theory; see [3, Section 6.3.2]) that $v \in H^2(\Omega)$ and

$$\|v\|_{H^2(\Omega)} \leq C(\Omega)\|f\|_{L^2(\Omega)}$$

(here we need that $\partial\Omega$ is smooth enough, say C^2), for some $C = C(\Omega) > 0$. Write $Au := v$. Then, recalling the definition of f,

$$\|Au\|_{H^2(\Omega)} \leq C(\Omega)\|b(Du)\|_{L^2(\Omega)} \leq C_1(\|Du\|_{L^2(\Omega)} + 1) \leq C_1(\|u\|_{H_0^1(\Omega)} + 1). \tag{6.2.7}$$

2. *Claim:* $A : H_0^1(\Omega) \to H_0^1(\Omega)$ is continuous and compact.

Continuous: if $u_k \to u$ in $H_0^1(\Omega)$, then $\sup_k \|Au_k\|_{H^2(\Omega)} < \infty$ by (6.2.7). Since $H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow H_0^1(\Omega)$ is compact (Theorem 2.4.7(b)), there exist $v \in H_0^1(\Omega)$ and a subsequence (u_{k_j}) such that $Au_k \to v$ in $H_0^1(\Omega)$. By definition of A and weak solutions,

$$\int_\Omega DAu_{k_j} \cdot D\varphi + \mu Au_{k_j}\varphi \, dx = -\int_\Omega b(Du_{k_j})\varphi \, dx \qquad \text{for all } \varphi \in H_0^1(\Omega).$$

But convergence in $H_0^1(\Omega)$ implies

$$\int_\Omega DAu_{k_j} \cdot D\varphi + \mu Au_{k_j}\varphi \, dx \to \int_\Omega Dv \cdot D\varphi + \mu v\varphi \, dx,$$

while the convergence $Du_{k_j} \to Du$ in $L^2(\Omega)$ and the assumption that b is Lipschitz continuous imply

$$\int_\Omega b(Du_{k_j})\varphi \, dx \to \int_\Omega b(Du)\varphi \, dx.$$

Thus $v = Au$, that is, if $u_k \to u$ in $H_0^1(\Omega)$, then for a subsequence $Au_{k_j} \to Au$ in $H_0^1(\Omega)$. The hair-splitting lemma implies $Au_k \to Au$ for the whole sequence.

Compact: again, if (u_k) is bounded in $H_0^1(\Omega)$, then (6.2.7) shows (Au_k) is bounded in $H^2(\Omega) \cap H_0^1(\Omega)$, so it has a convergent subsequence in $H_0^1(\Omega)$.

3. We now show that if $\mu > 0$ is large enough, then

$$\{u \in H_0^1(\Omega) : u = \lambda Au \text{ for some } \lambda \in [0,1]\}$$

is bounded in $H_0^1(\Omega)$. So assume $u \in H_0^1(\Omega)$ and $u = \lambda Au$ for $\lambda \in (0,1]$. Then, since $u/\lambda = Au$, we have $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and

$$-\Delta u + \mu u = -\lambda b(Du)$$

(both weakly and strongly, i.e. a.e. in Ω, since $u \in H^2$). In particular, taking u as a test function in the corresponding weak form and using the growth assumption on b,

$$\begin{aligned}
\int_\Omega |\nabla u|^2 + \mu u^2 \, dx &= -\lambda \int_\Omega b(Du) u \, dx \\
&\le \underbrace{|\lambda|}_{\le 1} C_1 \int_\Omega (|Du| + 1)|u| \, dx \\
&\le \frac{1}{2} \int_\Omega |Du|^2 \, dx + C_2 \int_\Omega |u|^2 + 1 \, dx,
\end{aligned}$$

where $C_2 > 0$ is independent of $\lambda \in (0,1]$. If $\mu > C_2$, this implies

$$\|u\|_{H_0^1(\Omega)} \le C_3(b, \Omega, \mu),$$

independent of λ.

4. Hence we may apply Theorem 6.1.5 to obtain a fixed point $u \in H^2(\Omega) \cap H_0^1(\Omega)$ of A, which by construction satisfies (6.2.6) pointwise almost everywhere (and weakly). $\square$

6.3 Monotone operators

Suppose V is a reflexive Banach space, $f \in V'$, and we wish to solve

$$Au = f, \tag{6.3.1}$$

where we think of $A : V \to V'$ as being a PDO. Recall (Chapter 3/Calculus of Variations) that if $\mathcal{E} : V \to \mathbb{R}$ is a C^1 energy functional and $A = \mathcal{E}'$ in $\mathcal{L}(V, V')$, then we interpret (6.3.1) as

$$.\langle \mathcal{E}'(u), \varphi\rangle_{V',V} = \langle f, \varphi\rangle_{V',V} \qquad \text{for all } \varphi \in V$$

If $\mathcal{E}$ is convex, i.e.

$$\langle \mathcal{E}'(u) - \mathcal{E}'(v), u - v\rangle_{V',V} \geq 0 \qquad \text{for all } u, v \in V$$

and coercive, then (6.3.1) has a solution (see Theorem 3.4.3 and Proposition 3.4.5). This motivates:

6.3.1 Definition. Let V be a Banach space and $K \subset V$. An operator $A : K \to V'$ is called *monotone* if

$$\langle Au - Av, u - v\rangle_{V',V} \geq 0 \qquad \text{for all } u, v \in K, \tag{6.3.2}$$

and *strictly monotone* if equality implies $u = v$.

(This is another, non-variational, way in which we can interpret and exploit the structural condition of ellipticity.)

If $K = V = \mathbb{R}$, then “monotone” is equivalent to “monotonically increasing”.

Goal: A (strictly) monotone + technical/auxiliary conditions $\implies$ (6.3.1) has a (unique) solution.

6.3.2 Remark. We remark explicitly: if $\mathcal{E} \in C^1(V, \mathbb{R})$ is convex, then $\mathcal{E}' : V \to V'$ is monotone.

6.3.3 Example. (a) The p-Laplacian with Dirichlet boundary conditions $-\Delta_p = -\operatorname{div}(|D\cdot|^{p-2}D\cdot) : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)'$ ($p \in (1,\infty)$) given by

$$\langle -\Delta_p u, \varphi\rangle = \int_\Omega |Du|^{p-2} Du \cdot D\varphi \, dx, \qquad u, \varphi \in W_0^{1,p}(\Omega),$$

is monotone, being associated with the convex functional

$$\mathcal{E}(u) = \frac{1}{p}\int_\Omega |Du|^p \, dx \qquad \text{on } W_0^{1,p}(\Omega)$$

(see Examples 1.4.2 and 3.2.1(a); we will return to this in Example 6.3.12).

(b) More generally, suppose $a : \mathbb{R}^n \to \mathbb{R}^n$ is a sufficiently smooth vector field such that

$$(a(z) - a(w)) \cdot (z - w) \geq 0 \qquad \text{for all } z, w \in \mathbb{R}^n$$

(a is *monotone*). Then $A := -\operatorname{div}(a(D\cdot))$ is monotone. *If* $a(z) = DF(z)$ for some $F : \mathbb{R}^n \to \mathbb{R}$, then

$$\mathcal{E}(u) = \int_\Omega F(Du) \, dx$$

is the associated energy functional.

6.3.4 Lemma. *Suppose V is a Banach space and $A : V \to V'$ is monotone. Then A is* locally bounded*: for all $u \in V$ there exist $r, \varepsilon > 0$ such that $\|Av\|_{V'} \le r$ for all $v \in B(u, \varepsilon)$.*

Proof. Suppose for a contradiction that there exist $u \in V$ and $(u_k) \subset V$ such that $u_k \to u$ and $\|Au_k\|_{V'} \to \infty$. Since A is monotone, for any $v \in V$,

$$0 \le \langle Au_k - Av, u_k - v\rangle = \langle Au_k - Av, u - v\rangle + \langle Au_k - Au, u_k - u\rangle.$$

Setting $w := v - u$,

$$\langle Au_k, w\rangle \le \langle Av, w\rangle + \langle Au_k - Av, u_k - u\rangle.$$

Since $\langle Av, w\rangle$ is independent of k and WLOG $|\langle Av, u_k - u\rangle| \le \|Av\|_{V'}\|u_k - u\|_V \le \|Av\|_{V'}$, we may write

$$\langle Au_k, w\rangle \le C(w) + \|Au_k\|_{V'}\|u_k - u\|_V,$$

where we assume WLOG that $C(w) \ge 1$. Set

$$a_k := 1 + \|Au_k\|_{V'}\|u_k - u\|_V$$

and consider the renormalisation $\varphi_k \in V'$ given by

$$\langle \varphi_k, w\rangle := \langle a_k^{-1} Au_k, w\rangle \le a_k^{-1} C(w) a_k = C(w).$$

In particular,

$$\sup_{k\in\mathbb{N}} \langle \varphi, w\rangle \le C(w) \qquad \text{for all } w \in V.$$

The Uniform Boundedness Principle implies

$$c_0 := \sup_{k\in\mathbb{N}} \|\varphi_k\|_{V'} < \infty.$$

Hence for the Au_k,

$$\|Au_k\|_{V'} \le c_0 a_k = c_0(1 + \|Au_k\|_{V'}\|u_k - u\|_V).$$

Since $\|u_k - u\|_{V'} \to 0$, this implies $\|Au_k\|_{V'}$ remains bounded as $k \to \infty$, a contradiction. □

6.3.5 Definition. A mapping $A : K \to V'$, $K \subset V$, is called *continuous on finite-dimensional subspaces* if for every finite-dimensional subspace $M \subset V$ of V the restriction $A|_M : M \cap K \to V'$ is continuous in the sense that

$$M \cap K \ni u \mapsto \langle Au, v\rangle \in \mathbb{R}$$

is continuous for each fixed $v \in V$.

6.3.6 Theorem (Minty's Lemma)**.** *Suppose V is a Banach space, $K \subset V$, $A : K \to V'$ is monotone and continuous on finite-dimensional subspaces, and $u \in \operatorname{int} K$. Then the following are equivalent:*

(i) $Au = f$ for some $f \in V'$;

(ii) $\langle Av - f, v - u\rangle \ge 0$ for all $v \in K$.

The expression in (ii) is linear in u. (ii) is sometimes called a *variational inequality* (cf. Section 3.5 and in particular Theorem 3.5.2).

Proof. (ii) $\implies$ (i) is exactly (6.3.2).

(ii) $\implies$ (i): Let $w \in V$, $t > 0$, and consider the "variation" $v := u + tw$. Then $v \in K$ if t is small enough, since $u \in \operatorname{int} K$, and hence

$$\langle Av - f, v - u \rangle = \langle A(u + tw) - f, tw \rangle \geq 0.$$

We now divide by $t > 0$ and pass to the limit:

$$\liminf_{t \to 0} \langle A(u + tw) - f, w \rangle \geq 0.$$

Now A is continuous restricted to the one-dimensional space $\{u + tw : t \in \mathbb{R}\}$; hence $A(u + tw) \to Au$ as $t \to 0$ and therefore

$$\langle Au - f, w \rangle \geq 0.$$

Replacing w with $-w$, we obtain the reverse inequality and conclude

$$\langle Au - f, w \rangle = 0 \qquad \text{for all } w \in V.$$

Hence $Au = f$ in V'. □

6.3.7 Corollary. *Suppose $A : V \to V'$ is monotone and continuous on finite-dimensional subspaces. If for $u_k, u \in V$*

(i) $u_k \rightharpoonup u$ *in* V,

(ii) $Au_k \overset{*}{\rightharpoonup} f$ *in* V', *and*

(iii) $\langle Au_k, u_k \rangle \to \langle f, u \rangle$,

then $Au = f$.

Condition (iii) is automatically satisfied if the convergence in (i) or (ii) is strong.

Proof. By Minty's Lemma, it suffices to show

$$\langle Av - f, v - u \rangle \geq 0 \qquad \text{for all } v \in V.$$

So suppose $v \in V$. Then, using that A is monotone,

$$\begin{aligned} 0 &\leq \langle Au_k - Av, u_k, v \rangle \\ &= \langle Au_k - f, u_k - v \rangle + \langle Av - f, v - u_k \rangle \\ &\to \langle f, u - v \rangle - \langle f, u - v \rangle + \langle Av - f, v - u \rangle \\ &= \langle Av - f, v - u \rangle. \end{aligned}$$

□

6.3.8 Theorem (Browder and Minty)**.** *Let V be a separable, reflexive Banach space, let $A : V \to V'$ be monotone and continuous on finite-dimensional subspaces, and suppose A is coercive in the sense that*

$$\frac{\langle Au, u \rangle}{\|u\|_V} \to \infty \qquad \text{as } \|u\|_V \to \infty. \tag{6.3.3}$$

Then for all $f \in V'$ there exists $u \in V$ such that $Au = f$.

6.3.9 Remark. If $A : K \subset V \to V'$ is *strictly* monotone and $f \in V'$, then the equations $Au = f$ and $\langle Av - f, v - u\rangle \geq 0$ for all $v \in K$ have at most one solution. (Exercise!)

For the proof of Theorem 6.3.8 we first need a finite-dimensional existence result, which relies on Brouwer's fixed point theorem.

6.3.10 Lemma. *Suppose $A : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and coercive in the sense of* (6.3.3)*. Then for all $v \in \mathbb{R}^n$ there exists $u \in \mathbb{R}^n$ such that $Au = v$.*

Proof. 1. We may assume WLOG that $v = 0$; indeed, if $v \neq 0$, then consider $\widetilde{A}u := Au - v$, so that $\widetilde{A}u = 0$ if and only if $Au = v$. Then $\widetilde{A}$ is obviously continuous, and coercivity follows from

$$\frac{\widetilde{A}u \cdot u}{|u|} \geq \frac{Au \cdot u}{|u|} - \frac{|v||u|}{|u|} \to \infty \qquad \text{as } |u| \to \infty.$$

2. Coercivity (6.3.3) implies in particular the existence of a constant $R > 0$ such that

$$Au \cdot u \geq 0 \qquad \text{for all } |u| \geq R. \tag{6.3.4}$$

Then obviously $Au = 0$ does not have a solution outside $\overline{B(0,R)} \subset \mathbb{R}^n$, so we shall restrict our attention to this ball.

3. We wish to apply a fixed point argument to $u \mapsto u - Au$, but this does not necessarily map $\overline{B(0,R)}$ to itself. Hence we consider the mapping $T : \overline{B(0,R)} \to \overline{B(0,R)}$,

$$Tu := \begin{cases} u - Au & \text{if } u - Au \in \overline{B(0,R)}, \\ R \cdot \frac{u-Au}{|u-Au|} & \text{otherwise,} \end{cases}$$

which is in particular continuous. By Theorem 6.1.2, T has a fixed point $u_0 \in \overline{B(0,R)}$.

4. If $u_0 - Au_0 \in \overline{B(0,R)}$, then we have $u_0 = Tu_0 = u_0 - Au_0$ and hence $Au_0 = 0$. We claim that indeed $u_0 - Au_0 \in \overline{B(0,R)}$. If not, then $Tu_0 = u_0 - Au_0 \in \partial B(0,R)$, i.e., $|u_0| = |Tu_0| = R$. Now

$$u_0 = tu_0 = \underbrace{\frac{R}{|u_0 - Au_0|}}_{=:\lambda\in(0,1)}(u_0 - Au_0).$$

Rearranging,

$$(1-\lambda)u_0 = -\lambda Au_0.$$

Taking the scalar product of both sides with u_0,

$$0 < (1-\lambda)|u_0|^2 = -\lambda Au_0 \cdot u_0,$$

but $Au_0 \cdot u_0 > 0$ by (6.3.4) since $|u_0| \geq R$. □

Proof of Theorem 6.3.8. As in the proof of Theorem 5.2.3, we start by constructing a sequence of finite-dimensional approximations and then use "*a priori* estimates" to pass to the limit.

Step 1: Finite-dimensional (Galerkin) approximation.

Since V is separable, there exists a sequence $(w_k) \subset V$ such that for $V_n := \operatorname{span}\{w_k : 1 \le k \le n\}$, we have

$$V = \overline{\bigcup_{n\in\mathbb{N}} V_n}.$$

We write $A_k u$ for $(Au)|_{V_k} : V_k \to \mathbb{R}$ and $f_k := f|_{V_k' \simeq V_k}$ etc.. Then $A_k : V_k \to V_k' \simeq V_k$ is coercive and continuous; hence by Lemma 6.3.10 there exists, for each $k \in \mathbb{N}$, a solution $u_k \in V_k$ of $A_k u_k = f_k$.

Step 2: A priori estimates.

Set $R := \|f\|_{V'}$. Coercivity of A implies the existence of $C > 0$ such that

$$\langle Au, u\rangle > R\|u\|_V \qquad\qquad \text{whenever } \|u\|_V \ge C.$$

In particular, $\|u_k\|_V \le C$ for all $k \in \mathbb{N}$, since

$$\langle Au_k, u_k\rangle \overset{u_k\in V_k}{=} \langle f_k, u_k\rangle = \langle f, u_k\rangle \le R\|u_k\|_V.$$

Since V is reflexive, there exists a subsequence (u_k) and $u \in V$ such that $u_k \rightharpoonup u$.[3]

Now since A is monotone,

$$\langle Au_k, v\rangle \le \langle Au_k, u_k\rangle - \langle Av, u_k - v\rangle = \langle f, u_k\rangle - \langle Av, u_k - v\rangle \qquad\qquad \text{for all } v \in V.$$

Fix $k \in \mathbb{N}$ and $\varepsilon > 0$. Taking the supremum over all $v \in \overline{B(0,\varepsilon)}$, we obtain

$$\varepsilon\|Au_k\|_{V'} = \sup_{v\in\overline{B(0,\varepsilon)}} \langle Au_k, v\rangle \le C\|f\|_{V'} + C \sup_{v\in\overline{B(0,\varepsilon)}} \|Av\|_{V'} + \sup_{v\in\overline{B(0,\varepsilon)}} \|Av\|_{V'}\|v\|_V$$

(with $C > 0$ as in Step 2). We apply Lemma 6.3.4 with $u = 0$ to obtain, for $\varepsilon > 0$ small enough,

$$\|Au_k\|_{V'} \le \frac{C}{\varepsilon}\|f\|_{V'} + \frac{C}{\varepsilon}r + r$$

for some $r > 0$. Hence (Au_k) is bounded in the reflexive space V'. Up to another subsequence, $Au_k \overset{*}{\rightharpoonup} g$ for some $g \in V'$.

Step 3: Limit.

We claim $f = g$ and $Au = f$. To that end fix $v \in V$ arbitrary and choose $v_k \in V_k$ such that $v_k \to v$ in V. Then $\langle Au_k, v_k\rangle \to \langle g, v\rangle$ since $Au_k \overset{*}{\rightharpoonup} g$ and $v_k \to v$ strongly, hence

$$\langle g, v\rangle \leftarrow \langle Au_k, v_k\rangle = \langle f, v_k\rangle \to \langle f, v\rangle,$$

and so $f = g$ in V'. We now pass to the limit using Corollary 6.3.7: we have shown (i) $u_k \hookrightarrow u$ in V and (ii) $Au_k \overset{*}{\rightharpoonup} f$ in V'; by construction, (iii)

$$\langle Au_k, u_k\rangle = \langle f, u_k\rangle \to \langle f, u\rangle.$$

We conclude that $Au = f$. □

[3] At this point we know $\|A_k u_k\|_{V'} = \|f_k\|_{V'} \le \|f\|_{V'}$ is bounded, but we do not yet know anything about Au_k: this is where we need that A is monotone.

6.3.11 Remark. We emphasise that in general $u_k \rightharpoonup u$ does not imply $Au_k \rightharpoonup Au$ if A is nonlinear (and, say, continuous); we overcame this problem in the above proof using the assumption that A was monotone.

6.3.12 Example (p-Laplacian)**.** We now interpret the operator $-\Delta_p = -\operatorname{div}(|D\,\cdot\,|^{p-2}D\,\cdot\,)$ ($1 < p < \infty$) on $\Omega \subset \mathbb{R}^n$ as a mapping from $W_0^{1,p}(\Omega)$ to $W_0^{1,p}(\Omega)' \simeq W^{-1,p'}(\Omega)$.[4] As before we interpret

$$\begin{cases} -\Delta_p u = f \in W^{-1,p'}(\Omega) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

via

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, dx = \langle f, \varphi \rangle \qquad \text{for all } \varphi \in W_0^{1,p}(\Omega); \tag{6.3.5}$$

this is well defined since by Hölder's inequality

$$\begin{aligned} \langle -\Delta_p u, \varphi \rangle &\le \| |Du|^{p-2} Du \|_{L^{p'}(\Omega)} \|D\varphi\|_{L^p(\Omega)} \\ &= \left(\int_\Omega |Du|^{(p-1)p'} \, dx \right)^{p'} \|D\varphi\|_{L^p(\Omega)} \\ &= \|Du\|_{L^p(\Omega)}^{p-1} \|\varphi\|_{W_0^{1,p}(\Omega)} \end{aligned}$$

using $p' = p/(p-1)$, i.e., $(p-1)p' = p$.

6.3.13 Theorem. *For each $f \in W^{-1,p'}(\Omega)$ there exists a unique solution $u \in W_0^{1,p}(\Omega)$ of* (6.3.5).

Note $L^{p'}(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$ in the obvious way (e.g., use the representation from Exercise 4), so this holds in particular for all $f \in L^{p'}(\Omega)$.

Proof. It suffices to check the assumptions of Theorem 6.3.8 and Remark 6.3.9, where $V = W_0^{1,p}(\Omega)$ is separable and reflexive and $A = -\Delta_p$ is as above:

1. $-\Delta_p$ is coercive:

$$\langle -\Delta_p u, u \rangle = \int_\Omega |Du|^p \, dx = \|u\|_{W_0^{1,p}(\Omega)}^p,$$

and so

$$\frac{\langle -\Delta_p u, u \rangle}{\|u\|_{W_0^{1,p}(\Omega)}} = \|u\|_{W_0^{1,p}(\Omega)}^{p-1} \to \infty$$

as $\|u\| \to \infty$ since $p > 1$.

2. $-\Delta_p$ is strictly monotone:[5] The functional $\Phi : \mathbb{R}^n \to \mathbb{R}$, $\Phi(\xi) = \frac{1}{p}|\xi|^p$ is (strictly) convex and differentiable everywhere (including 0 since $p > 1$) with gradient

$$\langle D\Phi(\xi), \eta \rangle = |\xi|^{p-2} \xi \cdot \eta;$$

by Exercise 9(i), it follows that

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) = \langle D\Phi(\xi) - D\Phi(\eta), \xi - \eta \rangle \ge 0 \qquad \text{for all } \xi, \eta \in \mathbb{R}^n. \tag{6.3.6}$$

[4] Cf. Exercise 4 for this identification.

[5] Here we essentially use the convexity of the associated functional $\mathcal{E}(u) = \frac{1}{p}\int_\Omega |Du|^p \, dx$, or rather, this argument also shows that the functional is convex.

Choosing $\xi = Du(x)$ and $\eta = Dv(x)$ for given functions $u, v \in W_0^{1,p}(\Omega)$ and integrating over $x \in \Omega$,

$$\langle(-\Delta_p u) - (-\Delta_p v), u - v\rangle = \int_\Omega (|Du|^{p-2}Du - |Dv|^{p-2}Dv) \cdot (Du - Dv)\,dx \geq 0, \qquad (6.3.7)$$

which says exactly that $-\Delta_p$ is monotone. Since (6.3.6) implies the integrand is non-negative a.e., equality in (6.3.7) implies the integrand is in fact 0 a.e., that is, there is equality in (6.3.6) for a.e. $x \in \Omega$ (with $\xi = Du(x)$ etc.). Since Φ is strictly convex, it follows that $\xi = \eta$, i.e. $Du(x) = Dv(x)$ a.e. in Ω. It follows that $\|u - v\|_{W_0^{1,p}(\Omega)} = 0$ and hence $u = v$ a.e.. Hence $-\Delta_p$ is in fact strictly convex.

3. $-\Delta_p$ is *continuous*: Suppose $u_k \to u$ in $W_0^{1,p}(\Omega)$; up to a subsequence $Du_k \to Du$ pointwise a.e. in Ω, so the same is true of $|Du_k|^{p-2}Du_k \to |Du|^{p-2}Du$.

Claim: this convergence is strong in $L^{p'}(\Omega)$.

Assuming the claim, we have

$$\langle -\Delta_p u_k + \Delta_p u, \varphi\rangle \leq \underbrace{\||Du_k|^{p-2}Du_k - |Du|^{p-2}Du\|_{L^{p'}(\Omega)}}_{\to 0} \|D\varphi\|_{L^p(\Omega)} \qquad \text{for all } \varphi \in W_0^{1,p}(\Omega),$$

implying (since $\|D\varphi\|_{L^p(\Omega)} = \|u\|_{W_0^{1,p}(\Omega)}$) that $\| - \Delta u_p u_k + \Delta u_p\|_{W_0^{1,p}(\Omega)'} \to 0$ for this subsequence; the hair-splitting lemma yields convergence of the whole sequence.

Hence it remains to prove the claim; we shall attempt to apply the dominated convergence theorem by dominating

$$\big||Du_k|^{p-2}Du_k - |Du|^{p-2}Du\big|^{p'}$$

in $L^1(\Omega)$. Since for any $a, b \geq 0$ and $q \geq 1$,

$$(a + b)^q \leq 2^q(a^q + b^q)$$

(exercise; WLOG $a \leq b$) we have, choosing $a = |Du_k|^{p-1}$, $b = |Du|^{p-1}$, $q = p'$,

$$\big||Du_k|^{p-2}Du_k - |Du|^{p-2}Du\big|^{p'} \leq 2^{p'}(|Du_k|^p + |Du|^p)$$

pointwise. The right-hand side converges strongly in $L^1(\Omega)$; hence a straightforward generalisation of the dominated convergence theorem implies

$$\big||Du_k|^{p-2}Du_k - |Du|^{p-2}Du\big|^{p'} \to 0$$

in $L^1(\Omega)$, i.e. $|Du_k|^{p-2}Du_k \to |Du|^{p-2}Du$ in $L^{p'}(\Omega)$, as claimed.

4. We may now apply Theorem 6.3.8 and Remark 6.3.9 to obtain a unique solution of (6.3.5) for each $f \in W^{-1,p'}(\Omega)$. □

7 Long-time behaviour and (non-) existence of solutions

7.1 Long-time behaviour

Consider again the equation from Example 6.2.1:

$$\begin{cases} \dot{u} - \Delta u = f(u) & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times [0,\infty), \\ u(0) = u_0 \in H_0^1(\Omega) & \text{on } \Omega \times \{t = 0\}, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is bounded and sufficiently smooth and f is lobally Lipschitz (so there exists $C > 0$ such that $|f(z)| \leq C(|z| + 1)$ for all $z \in \mathbb{R}$, as usual).

Question: What happens to u as $t \to \infty$?[1]

Suppose for the above problem that

$$F(z) := \int_0^z f(s)\, ds$$

is *bounded from above* on $\mathbb{R}$.[2] Let $u(t) = T(t)u_0$ be the corresponding solution from Theorem 6.2.2 satisfying, for all $T > 0$,

$$u \in H^1(0,T; L^2(\Omega)) \cap L^\infty(0,T; H_0^1(\Omega))$$

and

$$(\dot{u}, \varphi)_{L^2(\Omega)} + (D_x u, D_x \varphi)_{L^2(\Omega)} = (f(u), \varphi)_{L^2(\Omega)}$$

for all $\varphi \in H_0^1(\Omega)$ and a.e. $t \in [0,T]$ and denote by

$$\mathcal{E}(u) := \int_\Omega \underbrace{\frac{1}{2}|Du|^2 - F(u)}_{=:G(x,u,Du)} dx, \qquad u \in H_0^1(\Omega). \tag{7.1.1}$$

the energy functional.

7.1.1 Theorem. *With the above assumptions and notation, if we have the additional regularity $\mathcal{E}(u(\,\cdot\,))$ is absolutely continuous[3] and $\nabla_{L^2(\Omega)}\mathcal{E}(u) \in L^2(\Omega)$ for almost all $t > 0$, then there exist*

[1] We observe that since Theorem 6.2.2 guarantees the existence of a global solution on $(0,T]$ for any $T > 0$, we may extend this to obtain a solution on $(0,\infty)$.

[2] $f(z) = -z$ is admissible, for example.

[3] Or rather $\mathcal{E}(u(\,\cdot\,))$ can be extended on the null set where $u(t) \notin H_0^1(\Omega)$ to an absolutely continuous function on $[0,T]$

a sequence $t_k \to \infty$ *and* $u_\infty \in H^1_0(\Omega)$ *such that* $u(t_k) \rightharpoonup u_\infty$ *in* $H^1_0(\Omega)$ *as* $k \to \infty$*. Moreover,* u_∞ *is a weak solution of the* stationary equation

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.1.2}$$

7.1.2 Remark. (a) Problem (7.1.2) always has (at least) one solution under our assumptions: in the notation of (7.1.1), F is bounded from above, G is coercive in the sense of (3.4.5). Since $w \mapsto G(x, z, w)$ is convex, $\mathcal{E}$ admits a global minimiser in $H^1_0(\Omega)$ by Corollary 3.4.10; by Theorem 3.4.11, this is also a weak solution of (7.1.2) since G satisfies the necessary growth estimates:

$$\begin{aligned} |G(x, z, w)| &\le |w|^2 + \int_0^z C(|s| + 1)\, ds \le \widetilde{C}(|w|^2 + |z|^2 + 1), \\ |D_z G(x, z, w)| &\le |f(z)| \le \widetilde{C}(|w| + |z| + 1), \\ |D_w G(x, z, w)| &= |w| \le \widetilde{C}(|w| + |z| + 1). \end{aligned}$$

(b) Theorem 7.1.1 holds for more general elliptic equations than (7.1.2) (although not all, even when solutions exist for all $t > 0$); for this reason, elliptic equations are sometimes called stationary equations: if u_0 solves the stationary equation, then $u(t) = u_0$ solves the time-dependent (i.e. non-stationary) equation for all $t > 0$.

Proof of Theorem 7.1.1. 1. Since u is a solution, $u(t) \in H^1_0(\Omega)$ for a.e. $t > 0$ and, using our additional regularity assumptions,

$$(\nabla_{L^2(\Omega)} u(t), \varphi)_{L^2(\Omega)} = \langle E'(u(t)), \varphi\rangle = (Du(t), D\varphi)_{L^2(\Omega)} - (f(u(t)), \varphi)_{L^2(\Omega)} = -(\dot{u}(t), \varphi)_{L^2(\Omega)}$$

for all $\varphi \in H^1_0(\Omega)$. Since $H^1_0(\Omega)$ is dense in $L^2(\Omega)$, we obtain the relation

$$(\nabla_{L^2(\Omega)} \mathcal{E}(u(t)), \varphi)_{L^2(\Omega)} = -(\dot{u}(t), \varphi)_{L^2(\Omega)}$$

for all $\varphi \in L^2(\Omega)$, and hence, choosing $\varphi = \dot{u}$,

$$\frac{d}{dt}\mathcal{E}(u(t)) = (\nabla_{L^2(\Omega)} \mathcal{E}(u(t)), \dot{u})_{L^2(\Omega)} = -(\dot{u}(t), \dot{u}(t))_{L^2(\Omega)} \le 0 \tag{7.1.3}$$

for a.e. $t \ge 0$. Since $\mathcal{E}(u(\,\cdot\,))$ is absolutely continuous by assumption, it is in particular monotonically decreasing.

2. By our assumptions on F, $\mathcal{E}$ is coercive in the sense that there exist $\alpha > 0$ and $\beta \in R$ such that

$$\mathcal{E}(v) \ge \alpha \|v\|^2_{H^1_0(\Omega)} - \beta \qquad \text{for all } v \in H^1_0(\Omega)$$

(cf. Remark 7.1.2(a)). Since $\mathcal{E}(u(\,\cdot\,))$ is monotonically decreasing, it is therefore bounded in $t \ge 0$. Integrating (7.1.3) with respect to t,

$$0 \ge -\int_0^T \|\dot{u}\|^2_{L^2(\Omega)}\, dt = \mathcal{E}(u(T)) - \mathcal{E}(u(0)) \qquad \text{for all } T > 0.$$

Thus $\int_0^T \|\dot{u}\|^2_{L^2(\Omega)}\, dt$ is bounded from above uniformly in T; in particular, $\int_0^\infty \|\dot{u}\|^2_{L^2(\Omega)}\, dt$ exists and

$$\int_T^\infty \|\dot{u}\|^2_{L^2(\Omega)}\, dt \to 0 \qquad \text{as } T \to \infty. \tag{7.1.4}$$

3. Now since $\{\mathcal{E}(u(t))\}_{t\geq 0}$ is bounded and $\mathcal{E}$ is coercive, there exists $C > 0$ such that

$$\|u(t)\|_{H_0^1(\Omega)} \leq C \qquad \text{for (almost) all} t > 0.$$

Choose $t_k \to \infty$ such that $u(t_k) \in H_0^1(\Omega)$, $\dot{u}(t_k) \in L^2(\Omega)$ (both holding at least almost everywhere) and

$$\|\dot{u}(t_k)\|_{L^2(\Omega)} \to 0 \qquad \text{as } k \to \infty.$$

Then there exist a subsequence, which we shall also denote by (t_k), and $u_\infty \in H_0^1(\Omega)$ such that $u_{t_k} \rightharpoonup u_\infty$ in $H_0^1(\Omega)$ as $k \to \infty$. Then

$$(Du(t_k), D\varphi)_{L^2(\Omega)} \to (Du_\infty, D\varphi)_{L^2(\Omega)} \qquad \text{for all } \varphi \in H_0^1(\Omega),$$

and since f is Lipschitz continuous,

$$\begin{aligned}(f(u(t_k)) - f(u_\infty), \varphi)_{L^2(\Omega)} &\leq \|f(u(t_k)) - f(u_\infty)\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)} \\ &\leq \mathrm{Lip}(f)\|u(t_k) - u_\infty\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)} \to 0.\end{aligned}$$

Hence

$$(\dot{u}(t_k), \varphi)_{L^2(\Omega)} + (Du(t_k), D\varphi)_{L^2(\Omega)} = (f(u(t_k)), \varphi)_{L^2(\Omega)} \qquad \text{for all } \varphi \in H_0^1(\Omega).$$

Passing to the limit as $k \to \infty$, since $\|\dot{u}(t_k)\| \to 0$, we obtain

$$(Du_\infty, D\varphi)_{L^2(\Omega)}(f(u_\infty), \varphi)_{L^2(\Omega)} \qquad \text{for all } \varphi \in H_0^1(\Omega).$$

□

7.1.3 Remark. (a) Modulo the additional regularity assumptions, this gives another method of proof of the *existence* of a solution u_∞ of (7.1.2) (which we did not need to assume): take any initial condition and "evolve" it to obtain a solution of the elliptic equation in the (weak) limit.

(b) If (7.1.2) only has one solution, then, for any $u_0 \in H_0^1(\Omega)$, by the hair-splitting lemma $u(t_k) \rightharpoonup u_\infty$ for *any* sequence $t_k \to \infty$ such that $u(t_k) \in H_0^1(\Omega)$, $\dot{u}(t_k) \in L^2(\Omega)$ and $\|\dot{u}(t_k)\|_{L^2(\Omega)} \to 0$ as $k \to \infty$.

7.2 Blow-up

7.2.1 Example (ODEs)**.** Consider

(a) $\dot{u} = u$;

(b) $\dot{u} = u^2$;

(c) $\dot{u} + u = u^2$.

Then (a) has the solution $u(t) = ce^t$ and thus global existence for any initial condition $u(0) = c$, with at most exponential growth. For (b) all solutions have the form

$$\dot{u}(t) = \frac{1}{c - t}$$

and thus $u(t) \to \infty$ as $t \to c^-$: for any initial condition, the solution "blows up" in finite time and cannot be continuously extended. Solutions of (c) are given by

$$u(t) = \frac{1}{1 - ce^t}.$$

If $0 < u(0) < 1$, then $c < 0$ and hence we have global existence and convergence to 0 as $t \to \infty$; if $u(0) = 1$, then $u \equiv 1$ for all t (these are of course the two solutions of $u = u^2$). If $u(0) > 1$, then $0 < c < 1$ and the solution blows up at $t = -\ln c$.

Heuristic idea: A strong enough nonlinearity implies blow-up in finite time. A "damping term" means such a blow-up might only occur for large enough initial data.

For PDEs, "blow-up" can mean the solution itself becomes unbounded (e.g. in some L^p-norm), or also that the solution cannot be extended *smoothly* beyond a finite point of time (e.g. the formation of singularities, unboundedness of (spatial) derivatives).[4] Here we will consider two prototypical (and common) examples.

7.2.2 Example. Consider

$$\begin{cases} \dot{u} - \Delta u = u^2 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times [0,T], \\ u(0) = u_0 \in H_0^1(\Omega), \end{cases} \tag{7.2.1}$$

where $\Omega \subset \mathbb{R}^n$ is bounded and sufficiently smooth, say C^2. (Compare with Example 6.2.1.)

Claim: if $u_0 \geq 0$ and $T > 0$ are "large enough", then (7.2.1) does not have a smooth solution.

Denote by $\lambda_1 > 0$ the smallest eigenvalue and $\psi_1 \in H^2(\Omega) \cap H_0^1(\Omega)$ $(\cap C(\overline{\Omega}))$ the corresponding eigenfunction of

$$\begin{cases} -\Delta\psi_1 = \lambda_1\psi_1 & \text{in } \Omega, \\ \psi_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

where WLOG $\psi_1 > 0$ in Ω and $\int_\Omega \psi_1 \, dx = 1$.[5]

7.2.3 Theorem. *Suppose $0 \leq u_0 \in H_0^1(\Omega)$ satisfies*

$$(u_0, \psi_1)_{L^2(\Omega)} > \lambda_1.$$

Then if $T > 0$ is large enough, there does not exist a classical solution

$$u \in C^2(\Omega \times (0,T)) \cap C(\overline{\Omega} \times [0,T])$$

[4] There does not seem to be a precise, universally accepted definition of "blow-up" available, although in practice, for any particular equation it is usually clear when one observes it.

[5] See Example 3.5.9 for existence. That ψ_1 can be chosen ≥ 0 everywhere in Ω follows since if ψ_1 minimises the Rayleigh quotient (3.5.5), then so too does $|\psi_1| \in H_0^1(\Omega)$. It then follows from the strong maximum principle that ψ_1 cannot be zero in the interior of Ω; see [5, Theorem 8.19] for a version valid for weak solutions. Then $\int_\Omega \psi_1 \, dx = \|\psi\|_{L^1(\Omega)} < \infty$ since obviously $H_0^1(\Omega) \hookrightarrow L^1(\Omega)$, so the assumed normalisation is also possible. The H^2-regularity of ψ_1 follows from Theorem 4 of [3, Section 6.3.2], by taking $f = \lambda u$ (which is certainly in $L^2(\Omega)$). In fact, if we assume $\partial\Omega$ is of class C^∞, then since $u \in H^2(\Omega)$, by Theorem 5 of the same section, we obtain $u \in H^4(\Omega)$, and then $u \in H^6(\Omega)$, and so on. By using the embedding theorem stated in Remark 2.4.6(3), we eventually obtain $u \in C^l(\overline{\Omega})$ for all $l \in \mathbb{N}$, i.e., $u \in C^\infty(\overline{\Omega})$. This argument is known as a *bootstrapping argument*.

of (7.2.1).[6]

Proof. Suppose $u = u(t)$ is a smooth solution. We show that

$$\eta(t) := (u(t), \psi_1)_{L^2(\Omega)}$$

satisfies a differential inequality whose solutions blow up in finite time. Observe that $\eta(0) > \lambda_1$ by assumption.

Now η is well defined and is in C^1 as long as u exists (and is sufficiently smooth), with

$$\begin{aligned}\frac{d}{dt}\eta(t) = (\dot{u}, \psi_1)_{L^2(\Omega)} &= -(Du, D\psi_1)_{L^2(\Omega)} + (u^2, \psi_1)_{L^2(\Omega)}\\ &= -\lambda_1 \underbrace{(u, \psi_1)_{L^2(\Omega)}}_{=\eta} + (u^2, \psi_1)_{L^2(\Omega)},\end{aligned}$$

where we have used that u is a (weak) solution of (7.2.1) and ψ_1 is a (weak) solution of the eigenvalue equation $(D\psi_1, D\varphi)_{L^2(\Omega)} = \lambda_1(\psi_1, \varphi)_{L^2(\Omega)}$ for all $\varphi \in H_0^1(\Omega)$. Also, a clever application of Cauchy–Schwarz yields

$$\eta = \int_\Omega u\psi_1\, dx = \int_\Omega u\psi_1^{1/2}\psi_1^{1/2}\, dx \le \left(\int_\Omega u^2\psi_1\, dx\right)^{1/2} \underbrace{\left(\int_\Omega \psi_1\, dx\right)^{1/2}}_{=1},$$

that is, $\eta^2 \le (u^2, \psi_1)_{L^2(\Omega)}$, and hence

$$\dot{\eta} \ge -\lambda_1\eta + \eta^2 \tag{7.2.2}$$

as long as u exists and is sufficiently smooth. We now show η blows up in finite time: since $\eta(0) > \lambda_1$, (7.2.2) implies $\dot{\eta}$ remains positive and $\eta > \lambda_1$ as long as it exists;[7] for

$$\xi(t) := e^{\lambda_1 t}\eta(t),$$

we have

$$\dot{\xi}(t) = e^{\lambda_1 t}\dot{\eta}(t) + \lambda_1 e^{\lambda_1 t}\eta(t) \overset{(7.2.2)}{\ge} e^{\lambda_1 t}\eta^2(t) = e^{-\lambda_1 t}\xi^2(t).$$

Since $\eta > 0$, also $\xi > 0$, and hence we may write

$$\frac{d}{dt}\left(-\frac{1}{\xi}\right) = \frac{\dot{\xi}}{\xi^2} \ge e^{-\lambda_1 t}.$$

Integrating from 0 to t,

$$-\frac{1}{\xi(t)} + \frac{1}{\xi(0)} \ge -\frac{e^{\lambda_1 s}}{\lambda_1}\Big|_0^t,$$

[6] With somewhat more work, one could weaken the smoothness assumption on u somewhat: in fact it suffices that $\dot{u} \in L^2(\Omega)$ – which holds whenever u is a weak solution in our sense – and $\frac{d}{dt}(u(t), \psi_1)_{L^2(\Omega)}$ exists, is smooth enough, say L^2 in t, so that the fundamental theorem of calculus holds (cf. Theorem 5.1.10(a)), and equals $(\dot{u}(t), \psi_1)_{L^2(\Omega)}$.

[7] Alternatively, assuming $u \in C^2$, one could use the strong parabolic maximum principle to show $u(t) > 0$ in Ω for all $t > 0$, see Theorem 12 in [3, Section 7.1.4], and then conclude $\eta = (u, \psi_1)_{L^2(\Omega)} > 0$ since $\psi_1 > 0$ as well.

i.e.

$$-\frac{1}{\xi(t)} \geq -\frac{1}{\xi(0)} + \frac{1 - e^{-\lambda_1 t}}{\lambda_1} = \frac{-\lambda_1 + (1 - e^{-\lambda_1 t})\xi(0)}{\xi(0)\lambda_1}.$$

Since $\xi(0) = \eta(0)$, we conclude

$$\xi(t) \geq \frac{\eta(0)\lambda_1}{\lambda_1 - \eta(0)(1 - e^{-\lambda_1 t})},$$

as long as the denominator is nonzero. By assumption, $\eta(0) > \lambda_1$; hence the right-hand side, which is finite (and equal to $\eta(0)$) for $t = 0$, diverges to ∞ as

$$t \to t^* := -\frac{1}{\lambda_1} \ln\left(\frac{\eta(0) - \lambda_1}{\eta(0)}\right).$$

It follows that $\|u(t)\|_{L^2(\Omega)} \geq (u(t), \psi_1)_{L^2(\Omega)} = \eta(t) \to \infty$ as $t \to t^*$, if u remains smooth enough to justify the above calculations (e.g., if u is a classical solution). □

7.2.4 Remark. (a) Technically speaking, it is not clear from the above proof whether $\|u\|_{L^2(\Omega)} \to \infty$ in finite time, or whether the solution is not smooth enough/stops being smooth enough for some $t \in (0, t^*)$.

(b) If $u_0 \geq 0$ is small enough, there are several methods available to prove global existence of solutions, although to do so would go outside the scope of this course. One possible idea is as follows: there exists a stationary solution of $-\Delta\psi_0 = \psi_0^2$ by Theorem 4.2.1 (if $n \leq 3$). Then $\psi(t) = \psi_0$ is a stationary solution for all $t > 0$. One can show ("comparison principles") that if $0 \leq u_0 \leq \psi_0$ in Ω, then $0 \leq u(t) \leq \psi_0$ for all $t > 0$ exists for all times.

(c) The positive solution of the ODE $\dot{u} = f(u)$ blows up in finite time for any positive $u(0)$ if $f : [0, \infty) \to \mathbb{R}$ satisfies $f(z) > 0$ for all $z > 0$ and

$$\int_w^\infty \frac{dz}{f(z)} < \infty \qquad \text{for some } w > 0.$$

Under these assumptions on f, if in addition f is smooth and $f'' > 0$ (e.g. $f(u) = |u|^p$), then one can show that any C^2 solution of

$$\begin{cases} \dot{u} - \Delta u = f(u) & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times [0, T], \\ u(0) = u_0 \geq 0 \text{ smooth}, \end{cases}$$

blows up in finite time if $(u_0, \psi_1)_{L^2(\Omega)}$ is large enough, for the same reason. In this case the differential inequality (7.2.2) becomes $\dot{\eta} \geq -\lambda_1\eta + f(\eta)$.

Another example exploiting "convexity":

7.2.5 Theorem. *Consider*

$$\begin{cases} \dot{u} - \Delta u = |u|^{p-1}u & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times [0, T] \\ u(0) = u_0 \in H_0^1(\Omega) \cap L^p(\Omega), \end{cases} \tag{7.2.3}$$

where $\Omega \subset \mathbb{R}^n$ is open with finite volume and $p \in (1, \infty)$. If u_0 satisfies

$$\mathcal{E}(u_0) := \frac{1}{2}\int_\Omega |\nabla u_0|^2\,dx - \frac{1}{p+1}\int_\Omega |u_0|^{p+1}\,dx < 0, \tag{7.2.4}$$

then any corresponding (weak) solution u of (7.2.3) such that $u \in D(\nabla_H \mathcal{E})$ t-a.e. and $\mathcal{E}(u(\,\cdot\,))$ is absolutely continuous must blow up in finite time.

For simplicity, assume $p > 1$ is such that $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, so that a solution is a function

$$u \in H^1(0,T; L^2(\Omega)) \cap L^\infty(0,T; H_0^1(\Omega))$$

satisfying

$$(\dot{u}, \varphi)_{L^2(\Omega)} + \underbrace{(Du, D\varphi)_{L^2(\Omega)} - (|u|^{p-2}u, \varphi)_{L^2(\Omega)}}_{=\mathcal{E}'(u)\varphi = (\nabla_H \mathcal{E}(u), \varphi)_{L^2(\Omega)}} = 0 \qquad \text{for all } \varphi \in H_0^1(\Omega) \text{ and a.e. } t > 0. \tag{7.2.5}$$

Existence of solutions for $T > 0$ small enough could be proved, e.g., using fixed point methods, although we are not actually asserting that they exist here.

Proof. 1. Since $u \in D(\nabla_H \mathcal{E})$,

$$(\dot{u}, \varphi)_{L^2(\Omega)} + (\nabla_H \mathcal{E}(u), \varphi)_{L^2(\Omega)} = 0 \qquad \text{for all } \varphi \in L^2(\Omega),\ \text{for a.e. } t > 0.$$

Choosing $\varphi = \dot{u} \in L^2(\Omega)$ t-a.e.,

$$(0 \le)\ \|\dot{u}(t)\|_{L^2(\Omega)} = -\frac{d}{dt}\mathcal{E}(u(t)) \tag{7.2.6}$$

for almost every $t > 0$, so in particular, since $\mathcal{E}(u(\,\cdot\,)$ is absolutely continuous,

$$\mathcal{E}(u(t)) = \mathcal{E}(u_0) - \int_0^t \|\dot{u}\|_{L^2(\Omega)}^2\,ds.$$

Choosing $\varphi = u$ in (7.2.5), since $\frac{d}{dt}\|u\|_{L^2(\Omega)} = 2(\dot{u}, u)_{L^2(\Omega)}$, we also have

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega u^2\,dx\right) + \int_\Omega |Du|^2\,dx = \int_\Omega |u|^{p+1}\,dx. \tag{7.2.7}$$

2. Now define

$$I(t) := \int_0^t \|u(s)\|_{L^2(\Omega)}^2\,dx + C,$$

for some $C > 0$ to be determined later. We will show that I satisfies the differential inequality

$$I''(t)I(t) - (1+\alpha)I'(t)^2 > 0$$

for a.e. $t > 0$. To that end, we compute

$$I'(t) = \|u(t)\|_{L^2(\Omega)}^2 = \int_0^t \frac{d}{ds}\|u(s)\|_{L^2(\Omega)}^2\,ds + \|u_0\|_{L^2(\Omega)}^2 = 2\int_0^t (\dot{u}, u)_{L^2(\Omega)}\,ds + \|u_0\|_{L^2(\Omega)}^2$$

for all $t > 0$ (noting that this is in $H^1(0,T;\mathbb{R}) \cap C([0,T];\mathbb{R})$ since $u \in H^1(0,T;L^2(\Omega))$) and so

$$\begin{aligned} I'(t)^2 &= 4\left(\int_0^t (\dot{u},u)_{L^2(\Omega)}\,ds\right)^2 + 4\int_0^t (\dot{u},u)_{L^2(\Omega)}\,ds \int_\Omega u_0^2\,dx + \left(\int_\Omega u_0^2\,dx\right)^2 \\ &\le 4(1+\varepsilon)\left(\int_0^t\int_\Omega u^2\,dx\,dt\right)\left(\int_0^t\int_\Omega \dot{u}^2\,dx\,dt\right) + \left(1+\frac{1}{\varepsilon}\right)\left(\int_\Omega u_0^2\,dx\right)^2 \end{aligned}$$

for any $\varepsilon > 0$, where we used Cauchy–Schwarz on the first term and (2.6.5) on the second. We also have

$$\begin{aligned} I''(t) = \frac{d}{dt}\|u(t)\|_{L^2(\Omega)}^2 &\overset{(7.2.7)}{=} -2\int_\Omega |Du|^2\,dx + 2\int_\Omega |u|^{p+1}\,dx \\ &\overset{(7.2.4)}{\ge} (2-2p)\mathcal{E}(u(t)) \\ &=: 4(1+\delta)\left(-\mathcal{E}(0) + \int_0^t \|\dot{u}(s)\|_{L^2(\Omega)}^2\,ds\right), \end{aligned}$$

where $\delta := (p-1)/2 > 0$. Combining the two estimates, and introducing another constant $\alpha > 0$ to be chosen later,

$$\begin{aligned} I''(t)I(t) + (1+\alpha)I'(t)^2 \ge 4(1+\delta)&\left(-\mathcal{E}(u_0) + \int_0^t\int_\Omega \dot{u}^2\,dx\right)\left(\int_0^t\int_\Omega u^2\,dx\,dt + C\right) \\ &- (1+\alpha)\left(4(1+\varepsilon)\int_0^t\int_\Omega \dot{u}^2\,dx\,dt\int_0^t\int_\Omega u^2\,dx\,dt + \left(1+\frac{1}{\varepsilon}\right)\left(\int_\Omega u_0^2\,dx\right)^2\right) \end{aligned}$$

for almost every $t > 0$. We want this expression to be > 0; to that end, we now choose $\varepsilon, \alpha > 0$ small enough that

$$1+\delta \ge (1+\alpha)(1+\varepsilon).$$

Since $\mathcal{E}(u_0) < 0$ by assumption, if $C > 0$ is large enough, e.g.

$$-\mathcal{E}(u_0)C > \left(1+\frac{1}{\varepsilon}\right)\left(\int_\Omega u_0^2\,dx\right)^2,$$

then indeed $I''(t)I(t) - (1+\alpha)I'(t)^2 > 0$ for almost every $t > 0$. Noting that $I'' \in L^2(0,T)$ (so that the fundamental theorem of calculus holds) and $I(t) > 0$ for all $t > 0$, we may write

$$\frac{d}{dt}\left(\frac{I'(t)}{I^{\alpha+1}(t)}\right) = \frac{I''(t)I^{\alpha+1}(t) - (1+\alpha)I^\alpha(t)I'(t)}{I^{\alpha+1}(t)^2} > 0$$

for a.e. $t > 0$, and hence

$$\frac{I'(t)}{I^{\alpha+1}(t)} \ge \frac{I'(0)}{I^{\alpha+1}(0)} > 0$$

for all $t > 0$ (noting $I' \in H^1(0,T) \hookrightarrow C([0,T])$), that is,

$$I'(t) \ge cI^{\alpha+1}(t), \qquad c,\ I(0) > 0.$$

It follows that there exists $t^* > 0$ such that

$$I(t) = \int_0^t\int_\Omega u^2\,dx\,ds + C = \|u\|_{L^2(0,t;L^2(\Omega))} + C \to \infty$$

as $t \to t^*$, cf. Remark 7.2.4(c). □

7.3 Non-existence and critical exponents for elliptic problems: Pohozaev's identity

Prototype problem:

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.3.1}$$

where $\Omega \subset \mathbb{R}^n$ is bounded and sufficiently smooth, as usual, and $p > 1$.

If $1 < p < \frac{n+2}{n-2} = 2^* - 1$, then there exists a weak solution $u \not\equiv 0$; see Theorem 4.2.1 (or Example 3.5.10).

What happens if $p > \frac{n+2}{n-2}$?

Claim: under a certain geometrical condition on Ω, if $p > \frac{n+2}{n-2}$, then the only smooth solution of (7.3.1) is $u \equiv 0$.[8]

We therefore say that $p = \frac{n+2}{n-2}$ is a *critical exponent* (for the problem (7.3.1)).

Idea: Any (sufficiently smooth) solution of (7.3.1) must satisfy a Sobolev-type inequality which can only hold if $p \le \frac{n+2}{n-2}$.[9]

7.3.1 Theorem (Pohozaev identity)**.** *Assume $u \in C^2(\overline{\Omega})$ is a classical solution of* (7.3.1)*, where $\Omega \subset \mathbb{R}^n$ is bounded and open with C^1-boundary. Then*

$$\left(\frac{n-2}{2}\right)\int_\Omega |Du|^2\,dx + \frac{1}{2}\int_{\partial\Omega} |Du|^2 x\cdot\nu\,d\sigma = \frac{n}{p+1}\int_\Omega |u|^{p+1}\,dx, \tag{7.3.2}$$

where $\nu = (\nu^1, \dots, \nu^n)$ is the outward-pointing unit normal to $\partial\Omega$.

(7.3.2) is also known as the Derrick–Pohozaev identity.

Proof. 1. Multiply (7.3.1) by $x \cdot Du(x)$ and integrate over Ω:

$$\int_\Omega (-\Delta u)(x\cdot Du)\,dx = \int_\Omega |u|^{p-1}u(x\cdot Du)\,dx. \tag{7.3.3}$$

The left-hand side may be written as

$$-\sum_{i,j=1}^n \int_\Omega \frac{\partial^2 u}{\partial x_i^2} x_j \frac{\partial u}{\partial x_j}\,dx.$$

We now manipulate the integrands in order to apply Gauß–Green:

$$\begin{aligned}
\int_\Omega \frac{\partial^2 u}{\partial x_i^2} x_j \frac{\partial u}{\partial x_j}\,dx &= \int_\Omega \frac{\partial^2 u}{\partial x_i} x_j \frac{\partial u}{\partial x_j} + \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}\left(x_j \frac{\partial u}{\partial x_j}\right)dx - \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}\left(x_j\frac{\partial u}{\partial x_j}\right)dx \\
&= \int_\Omega \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i} x_j \frac{\partial u}{\partial x_j}\right)dx - \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial}{\partial i}\left(x_j \frac{\partial u}{\partial x_j}\right)dx \\
&= \int_{\partial\Omega} \frac{\partial u}{\partial x_i}\nu^i x_j \frac{\partial u}{\partial x_j}\,dx - \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}\left(x_j\frac{\partial u}{\partial x_j}\right)dx,
\end{aligned}$$

[8]The same assertion holds if we consider $-\Delta u = f(u)$, where f satisfies a corresponding growth condition, such as $f(z)$ grows at least as fast as $|z|^{p-1}$ as $z \to \pm\infty$. The method of proof is also essentially the same; we replace the term $\int_\Omega |u|^{p+1}\,dx$ in (7.3.2) with $\int_\Omega F(u)\,dx$, F being the antiderivative of f with $F(0) = 0$.

[9]Again, we will assume our solutions are smooth enough to justify all necessary calculations without trying to find the minimal possible assumptions under which everything works.

where in the last line we applied Gauß–Green in the form (2.3.1). Hence the left-hand side of (7.3.3) is equal to

$$\sum_{i,j=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_i} \left(x_j \frac{\partial u}{\partial x_j} \right) dx - \sum_{i,j=1}^{n} \int_{\partial\Omega} \frac{\partial u}{\partial x_i} \nu^i x_j \frac{\partial u}{\partial x_j} \, d\sigma =: A_1 + A_2.$$

2. Now

$$\begin{aligned} A_1 &= \sum_{i,j=1}^{n} \int_\Omega \delta_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \frac{\partial u}{\partial x_i} x_j \frac{\partial^2 u}{\partial x_i \partial x_j} \, dx \\ &= \int_\Omega |Du|^2 + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\underbrace{\frac{1}{2} \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2}_{=|Du|^2/2} \right) x_j \, dx \\ &= \int_\Omega |Du|^2 + \operatorname{div} \left(\frac{|Du|^2}{2} x \right) - n \frac{|Du|^2}{2} \, dx \\ &= \int_\Omega \left(1 - \frac{n}{2}\right) |Du|^2 \, dx + \int_{\partial\Omega} \frac{|Du|^2}{2} x \cdot \nu \, d\sigma, \end{aligned}$$

by another application of Gauß–Green. For A_2, since $u = 0$ on $\partial\Omega$, $Du(x)$ is parallel to $\nu(x)$ at every $x \in \partial\Omega$, so

$$Du(x) = \pm |Du(x)| \nu(x),$$

that is,

$$\frac{\partial u}{\partial x_i} = \pm \sqrt{\sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2} \, \nu^i \qquad \text{for all } i = 1, \dots, n,$$

and so

$$\sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \nu^i x_j \frac{\partial u}{\partial x_j} = \sum_{i,j=1}^{n} \underbrace{\sum_{k=1}^{n} \left(\frac{\partial u}{\partial x_k} \right)^2}_{=|Du|^2} (\nu^i)^2 \nu^j x_j = |Du|^2 x \cdot \nu.$$

Thus

$$A_2 = - \int_{\partial\Omega} |Du|^2 x \cdot \nu \, d\sigma.$$

3. The right-hand side of (7.3.3) is

$$\begin{aligned} \sum_{j=1}^{n} \int_\Omega |u|^{p-1} u x_j \frac{\partial u}{\partial x_j} \, dx &= \sum_{j=1}^{n} \int_\Omega \frac{\partial}{\partial x_j} \left(\frac{|u|^{p+1}}{p+1} \right) x_j \, dx \\ &= - \sum_{j=1}^{n} \int_\Omega \frac{|u|^{p+1}}{p+1} \frac{\partial}{\partial x_j} x_j \, dx = - \frac{n}{p+1} \int_\Omega |u|^{p+1} \, dx, \end{aligned}$$

again using Gauß–Green; the boundary integral vanishes since $u = 0$ on $\partial\Omega$.

4. Putting this together with A_1, A_2 and (7.3.3),

$$\frac{2-n}{2}\int_\Omega |Du|^2\,dx - \frac{1}{2}\int_{\partial\Omega} |Du|^2 x\cdot\nu\,d\sigma = -\frac{n}{p+1}\int_\Omega |u|^{p+1}\,dx.$$

□

Now we will give a geometric condition on Ω which allows us to control the sign of the integral over $\partial\Omega$.

7.3.2 Definition. An open set Ω is called *star-shaped* with respect to a point $x_0 \in \Omega$ if for all $x \in \overline{\Omega}$, the line segment joining x_0 and x,

$$\{\lambda x + (1-\lambda)x_0 : \lambda \in [0,1]\}$$

lies in $\overline{\Omega}$.

Obviously, convex sets are star-shaped but the converse is not true.

7.3.3 Lemma. *Suppose $\partial\Omega$ is C^1 and Ω is star-shaped with respect to $0 \in \Omega$. Then*

$$x\cdot\nu(x) \geq 0 \qquad \textit{for all } x \in \partial\Omega.$$

Proof. Fix $x \in \partial\Omega$. Since $\partial\Omega$ and hence ν are C^1, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $y \in \overline{\Omega}$,

$$|y-x| < \delta \qquad \text{implies} \qquad \frac{y-x}{|y-x|}\cdot\nu(x) \leq \varepsilon.$$

In particular,

$$\limsup_{\substack{y\to x\\ y\in\overline{\Omega}}} \frac{y-x}{|y-x|}\cdot\nu(x) \leq 0.$$

Let $y = \lambda x$ for $\lambda \in (0,1)$, then $y \in \overline{\Omega}$ since Ω is star-shaped with respect to 0, and

$$\frac{x}{|x|} = -\frac{\lambda x - x}{|\lambda x - x|}.$$

Hence

$$\frac{x}{|x|}\cdot\nu(x) = -\lim_{\lambda\to 1^-} \frac{\lambda x - x}{|\lambda x - x|}\cdot\nu(x) \geq 0.$$

□

7.3.4 Theorem. *Suppose Ω is bounded, open and star-shaped with respect to 0 and $\partial\Omega$ is C^1. Suppose also that $u \in C^2(\overline{\Omega})$ solves (7.3.1) for some $p > \frac{n+2}{n-2}$. Then $u \equiv 0$ in Ω.*

Proof. Since $x\cdot\nu \geq 0$,

$$\frac{1}{2}\int_{\partial\Omega} |Du|^2 x\cdot\nu\,d\sigma \geq 0,$$

and so Theorem 7.3.1 implies

$$\left(\frac{n-2}{2}\right)\int_\Omega |Du|^2\,dx \leq \frac{n}{p+1}\int_\Omega |u|^{p+1}\,dx.$$

On the other hand, choosing u as a test function in the weak form of (7.3.1) yields

$$\int_\Omega |Du|^2\,dx = \int_\Omega |u|^{p+1}\,dx.$$

It follows that

$$\left(\frac{n-2}{2} - \frac{n}{p+1}\right)\int_\Omega |u|^{p+1}\,dx \le 0.$$

If $u \not\equiv 0$, then necessarily

$$\frac{n-2}{2} - \frac{n}{p+1} \le 0,$$

that is,

$$p \le \frac{n+2}{n-2}.$$

□

Bibliography

[1] R. A. Adams, *Sobolev spaces.* Academic Press, New York-London, 1975.

[2] N. Dunford and J. Schwartz, *Linear operators*, Vol. 1, Interscience, New York, 1958.

[3] L. C. Evans, *Partial differential equations*, 2nd edition, American Mathematical Society, Providence, R. I., 2010.

[4] R. Chill and E. Fašangová, *Gradient systems*, 13th International Internet Seminar on Evolution Equations, 2010.

[5] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order.* Grundlehren der mathematischen Wissenschaften Vol. 224, Springer, Berlin-New York, 1977.

[6] S. Lang, *Fundamentals of differential geometry*, Graduate texts in mathematics, Vol. 191, Springer, New York-Heidelberg, 1999.

[7] V. G. Maz'ja, *Sobolev spaces*, Springer, Berlin, 1985.

[8] P. Rabinowitz, *Minimax methods in critical point theory with applications to partial differential equations*, CBMS regional conference series in mathematics, 65. AMS, Providence, R. I., 1986.

[9] W. Rudin, *Functional analysis*, 2nd edition, McGraw-Hill, New York, 1991.

[10] W. Rudin, *Real and complex analysis*, 3rd international edition, McGraw-Hill, New York, 1987.

[11] B. Schweizer, *Partielle Differentialgleichungen: Eine anwendungsorientierte Einführung*, Springer, Berlin-Heidelberg, 2013.

[12] M. Struwe, *Variational methods: Applications to nonlinear partial differential equations and Hamiltonian systems*, 4th edition, Springer, Berlin-Heidelberg, 2008.

[13] K. Yosida, *Functional analysis.* Grundlehren der mathematischen Wissenschaften Vol. 123, 6th edition, Springer, Berlin, 1980.