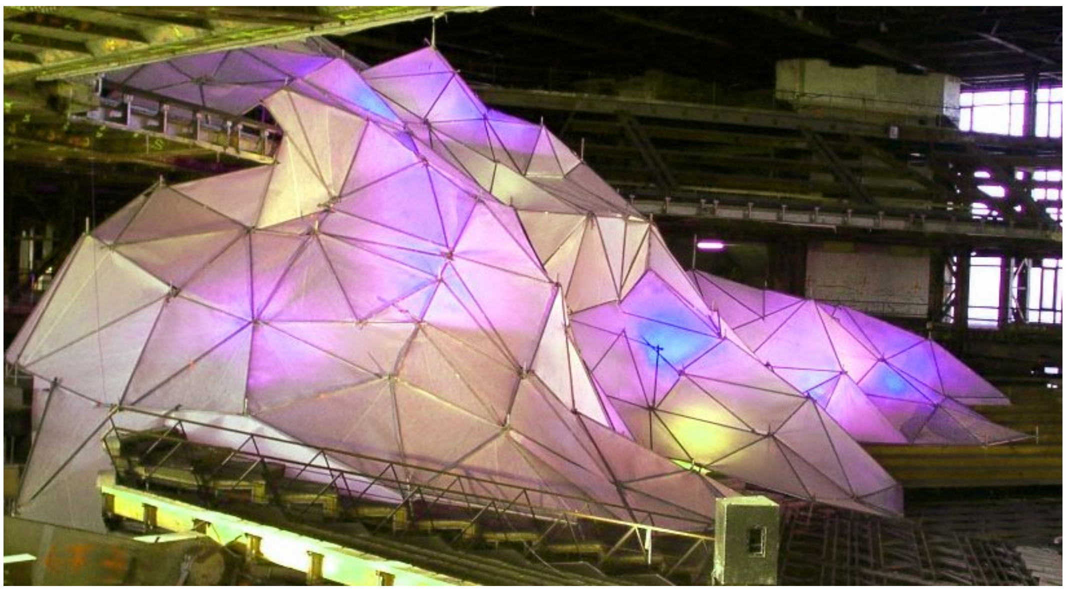
Semigroup methods for evolution equations on networks



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These notes derive from the lecture series "Evolution equations on networks", held in March 2007 at the Mathematical Department at Lousiana State University and are sporadically updated. The lectures were conceived with the aim of presenting known results obtained over the last few years by a group of few researchers with a definite operator theoretical background, most of whom currently are and/or have recently been based at the universities of Tübingen and Ulm.

While elaborating these notes, I have however tried to homogenize the discussion and to provide the reader with several examples and applications, in particular to neuronal modelling. Ideally, the presented method relies upon tools coming from operator semigroups as well as graph theory. Most proofs have only been sketched, and in general I have tried to keep the presentation as fluid and self-contained as possible. Nevertheless, I have tried to mention most relevant recent developments in the theory, including approximation problems, quantum graphs, and non-standard boundary conditions.

Mistakes and typos may well have slipped in the text. I will be glad to receive feedbacks, suggestions, and criticisms: please do not hesitate and send me an e-mail.

Ulm, February 28, 2008

The cover picture has been taken in Berlin during *Der Berg* (The Mountain), an art installation organized in summer 2005 inside the Palast der Republik (Palace of the Republic), the former House of Parliament of the German Democratic Republic. The Palast has been subsequently demolished.

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CHAPTER 1

Introduction

Let me consider some elementary, yet motivating problems.

- **Model 1.** Some people gather at a party. At the beginning, each of them only knows a few other guests. Compute their degrees of separations.
- **Model 2.** Many cars are driving on a highway at constant (but possibly different) speed. Analyze their flow.
- **Model 3.** A passenger wants to fly from an American town to another. Suggest the most efficient way of doing so.
- **Model 4.** A spider has woven its web, which gets stirred by a breath of wind. Study its vibrations.
- **Model 5.** A sexually transmittable disease is spreading. Determine the lowest number of patients to be cured in order to stop the infection.
- **Model 6.** A black-out has occurred. Investigate its propagation through an interconnected electric distribution network.

Are these *models* different?
No: All of them can be represented by means of a network formalism.

Are these *problems* different?

Yes: On one hand, in models 1, 5 (and to some extent in 3) the spatial issue can be neglected. Being in touch with an acquaintance, enjoying the possibility of flying to another town, being infected by somebody: all these are *spatially discrete* phenomena. The system might be evolving in time according to a differential equation, but the state space is finite-dimensional. They are typical problems of **graph theory**.

On the other hand, in models 2, 4, 6 the relevant processes are occurring *in* the links connecting the network's nodes. Such models are best described by partial differential equations: by a conservation law (2), a wave equation (4), or a telegraph equation (6). They are good examples of **evolution equations on networks**.

Problem 1.1. *Find the correct equations that model a system properly, and then discuss them by means of operator theoretical tools.*

CHAPTER 2

Basics on graph theory

A network structure can be described by means of the underlying graph.

Definition 2.1. *An* oriented graph *is a triple* $\mathsf{G} := (\mathsf{V}, \mathsf{E}, \phi)$*, where* V *and* E *are nonempty, disjoint, countable sets and* ϕ *is a mapping from* E *to* $\mathsf{V} \times \mathsf{V}$*. The elements of* V *will be called* nodes *or* vertices*, the elements of* E links *or* edges*, and* ϕ *is called an* orientation*. The* Euler characteristic *of* G *is the number* $|\mathsf{V}| - |\mathsf{E}|$*.*

Let me explicitly observe that in an oriented graph multiple links between two given nodes are allowed in either direction, since ϕ is not assumed to be injective. In fact, an oriented graph G is sometimes called a *directed graph* if for any two nodes $\mathsf{v}, \mathsf{w} \in \mathsf{V}$ there is at most one link $\mathsf{e} \in \mathsf{E}$ such that $\phi(\mathsf{e}) = (\mathsf{v}, \mathsf{w})$.

Definition 2.2. *Let* G *be an oriented graph. If* $\phi(\mathsf{e}) = (\mathsf{v}, \mathsf{w})$*, then* $\mathsf{v}, \mathsf{w} \in V$ *are called the* initial *and* terminal endpoint *of* e*, respectively, and one says that* e connects v to w*. In particular,* v, w *are said to be* adjacent*. One also says that* e *is* incident *in* v *(as well as in* w*): to be more precise,* e goes out of v *and* comes into w*. If additionally* $\mathsf{e}' \in \mathsf{E}$ *such that* $\phi(\mathsf{e}') = (\mathsf{w}, \mathsf{z})$ *for some* $\mathsf{z} \in \mathsf{V}$*, then the links* e, e' *are said to be* adjacent*.*

Although one can guess that a non-oriented graph structure always underlies an oriented graph, several analytical properties depend dramatically on orientations – even if the process taking place on network structures is (or seems to be) physically isotropic. Therefore, in this note oriented graphs only will be considered.

Remark 2.3. Let $\mathsf{G} = (\mathsf{V}, \mathsf{E}, \phi)$ be an oriented graph and take $\mathsf{e}_0 \in \mathsf{E}$ with $\phi(\mathsf{e}_0) = (\mathsf{v}, \mathsf{w}) \in \mathsf{E}$. Then a new oriented graph $\tilde{\mathsf{G}} = (\mathsf{V}, \mathsf{E}, \tilde{\phi})$ can be obtained defining a mapping $\tilde{\phi} : \mathsf{E} \to \mathsf{V} \times \mathsf{V}$ by

$$\tilde{\phi}(\mathsf{e}) = \begin{cases} (\mathsf{w}, \mathsf{v}) & \text{if } \mathsf{e} = \mathsf{e}_0, \\ \phi(\mathsf{e}) & \text{otherwise.} \end{cases}$$

Thus, the graph $\tilde{\mathsf{G}}$ has been constructed by *reorienting* G.

In applications, nodes will be often identified with geometric objects. Any oriented graph can be naturally embedded in a 3-dimensional space. Recall that a *simple arc* is a continuous function $\mathsf{e} : [0, 1] \to \mathbb{R}^3$ whose restriction to $[0, 1)$ is injective, and $\mathsf{e}(0), \mathsf{e}(1)$ are its *endpoints*.

Definition 2.4. *An oriented graph* G *is called a* geometric *graph if its nodes are points of* $\mathbb{R}^3$ *and its links are identified with simple arcs whose endpoints are all nodes of* G*.*

Thus, in the following I will write $\mathsf{v} = \mathsf{e}(0)$ and $\mathsf{w} = \mathsf{e}(1)$ if $\phi(\mathsf{e}) = (\mathsf{v}, \mathsf{w})$. Since the graph is countable, one can write $\mathsf{V} = \{\mathsf{v}_1, \ldots, \mathsf{v}_n, \ldots\}$ as well as $\mathsf{E} := \{\mathsf{e}_1, \ldots, \mathsf{e}_m, \ldots\}$ and introduce the following.

Definition 2.5. *Let* G *be a geometric graph. For* $i \in \mathbb{N}$ *I will denote by* $\Gamma^+(\mathsf{v}_i)$ *(resp.,* $\Gamma^-(\mathsf{v}_i)$*) the index set of all links that go out of (resp., come into)* v_i*, i.e.,* $\{j \in \mathbb{N} : \mathsf{v}_i = \mathsf{e}_j(0)\}$ *(resp.* $\{j \in \mathbb{N} : \mathsf{v}_i = \mathsf{e}_j(1)\}$*) and by* $\Gamma(\mathsf{v}_i)$ *their union. Its cardinality* $|\Gamma^+(\mathsf{v}_i)|$ *(resp.,* $|\Gamma^-(\mathsf{v}_i)|$*) is called* outdegree *(resp.,* indegree*) of the node* v_i*, while the* degree *of* v_i *is* $|\Gamma(\mathsf{v}_i)| = |\Gamma^+(\mathsf{v}_i)| + |\Gamma^-(\mathsf{v}_i)|$*.*

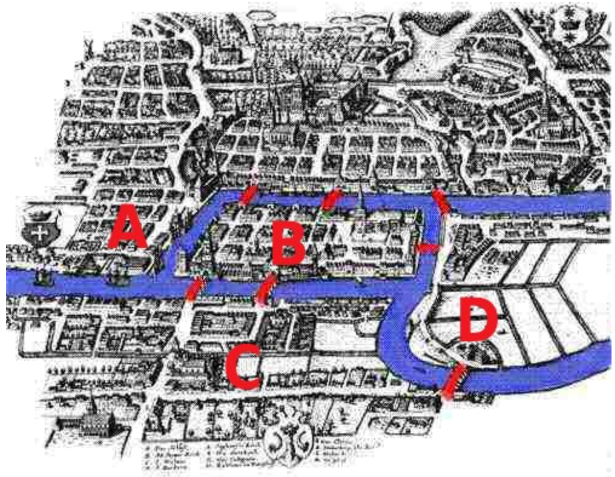

FIGURE 1. The seven bridges of Königsberg...

EXAMPLE 2.6. Let me revise the models considered in the introduction. It can be seen that all of them can be represented by graphs.

- If there exists $\mathsf{e} \in \mathsf{E}$ such that $\phi(\mathsf{e}) = (\mathsf{v}, \mathsf{w})$, this means that v *knows* w, then **model 1** can be represented by means of an oriented graph. Multiple links connecting v to w cannot arise. Though, even if v knows w, it may or may not be that w knows v, too: this is particularly clear if social networking www-sites are consider instead, and let e.g. $(\mathsf{v}, \mathsf{w}) \in \mathsf{E}$ if v *has offered a Gmail account to* w, or if v *has added* w *as a flickr contact*, or...
- If there exists $\mathsf{e} \in \mathsf{E}$ such that $\phi(\mathsf{e}) = (\mathsf{v}, \mathsf{w})$, this means that v *is the highway junction preceding* w, then **model 2** can be represented by an oriented graph.
- Define $\phi(\mathsf{e}) = (\mathsf{v}, \mathsf{w})$ as "*the flight* e *connects* v *to* w": then **model 3** takes the form of an oriented graph.
- Let $\phi(\mathsf{e}) = (\mathsf{v}, \mathsf{w})$ mean that *a spider's thread* e *links* v *and* w. It seems that no preferential direction can be naturally assigned to the network considered in **model 4**, so that it is legitimate to arbitrarily assume that $(\mathsf{v}, \mathsf{w}) \in \mathsf{E}$ if and only if $(\mathsf{w}, \mathsf{w}) \in \mathsf{E}$.
- If the existence of $\mathsf{e} \in \mathsf{E}$ such that $\phi(\mathsf{e}) = (\mathsf{v}, \mathsf{w}) \in \mathsf{E}$ means that v *has infected* w, then **model 5** has the same features of an oriented invitation network from model 1.
- Setting $\phi(e) = (\mathsf{v}, \mathsf{w})$ for $\mathsf{e} \in \mathsf{E}$ whenever v and w are adjacent nodes of the power grid, there is no natural way of directing the graph arising from **model 6**. A direction can be assigned arbitrarily.

Observe that the network topology of models 1, 3, 5 might change in time. For instance, talking with an acquaintance at a party will likely let me know his/her friends, too, thus allowing for new links from me to them. Such variable, time-evolving networks are the topic of the now mature theory of random graphs, and of the still tumultuously growing theory of scale-free networks, see [**1**] for a classical monograph about the first and [**2**] for a casual survey about the second one. I will not deal with these topic in the present notes. □

Graph theory is unanimously given a precise birthday: the solution to a then-famous problem concerning the traversability of seven bridges in the town of Königsberg in Eastern Prussia (now Kaliningrad, Russia).

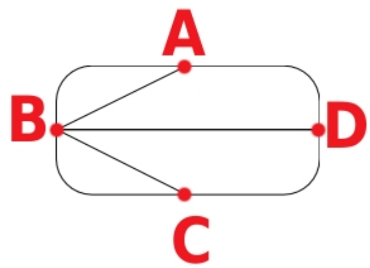

FIGURE 2. ...and its reduction to a graph theoretical setting

PROBLEM 2.7. *Is there a way to traverse all bridges of Königsberg in a single trip, without doubling back, in such a way that the trip ends in the same place it began?*

Such a solution, which is in fact the first proof of a graph theoretical theorem, has been obtained by L. Euler in 1735. As one can expect from a field which is almost 300 years old, it is absolutely impossible to even give a slight hint of the whole, rich theory. A nice, modern, and reasonably complete treatment can be found in several books, including [**3, 4**].

One can guess that some kind of compatibility condition has to be enjoyed by the graph's nodes in order that the problem has a solution.

DEFINITION 2.8. *Let me introduce some classes of oriented graphs.*

(1) *An oriented subgraph* $\tilde{\mathsf{G}} = (\tilde{\mathsf{V}}, \tilde{\mathsf{E}}, \tilde{\phi})$ *of* G *is a graph such that* $\tilde{\mathsf{G}} \subset \mathsf{G}$, $\tilde{\mathsf{E}} \subset \mathsf{E}$, *and* $\phi_{|\mathsf{E}} = \tilde{\phi}$.
(2) *An* n-path *in* G *is an oriented subgraph* $\tilde{\mathsf{G}} = (\tilde{\mathsf{V}}, \tilde{\mathsf{E}}, \tilde{\phi})$ *with nodes* $\tilde{\mathsf{V}} = \{\mathsf{v}_{k_1}, \mathsf{v}_{k_2}, \ldots, \mathsf{v}_{k_{n+1}}\}$ *and* $\tilde{\mathsf{E}} = \{\mathsf{e}_{k_1}, \ldots, \mathsf{e}_{k_n}\}$, *such that* $\phi(\mathsf{e}_{k_j}) = (\mathsf{v}_{k_j}, \mathsf{v}_{k_{j+1}})$, $j = 1 \ldots, n$. *If additionally* $\mathsf{v}_{k_1} = \mathsf{v}_{k_n}$, *then* $\tilde{\mathsf{G}}$ *is called* n-cycle.
(3) G *is* orientedly simple *for any two nodes* v, w *there is at most one* $\mathsf{e} \in \mathsf{E}$ *such that* $\phi(\mathsf{e}) = (\mathsf{v}, \mathsf{w})$.
(4) G *is an* outbound tree *(resp., an* inbound tree*)* with root v *if for each node* w *there is a unique path that connects* v *to* w *(resp.,* w *to* v*). Any node with outdegree 0 (resp., indegree 0) is called* leaf *of the outbound (resp., inbound) tree.* G *is called an* oriented tree *if it is an inbound or an outbound tree.*
(5) *G is an* outbound star *(resp., an* inbound star*)* with center v *if* v *is the initial (resp., terminal) endpoint of each edge. G is called an* oriented star *if it is an inbound or an outbound star.*
(6) G *is* orientedly bipartite *if* V *is disjoint union of two subsets* $\mathsf{V}_0, \mathsf{V}_1$ *such that each node in* V_0 *(resp. in* V_1*) is initial (resp. terminal) endpoint of all the links incident to it.*
(7) G *is* orientedly Eulerian *if there exists a cycle* $\tilde{\mathsf{G}} = (\tilde{\mathsf{V}}, \tilde{\mathsf{E}}, \tilde{\phi})$ *in* G, *called* Eulerian tour, *such that* $\tilde{\mathsf{E}} = \mathsf{E}$.

By the way, with the above terminology the solution to Problem 2.7 can be stated as follows.

THEOREM 2.9 (Euler 1735, Hierholzer 1873). *A connected graph* G *is orientedly Eulerian if and only if each node has equal indegree and outdegree.*

In particular, the answer to Problem 2.7 is negative.

All information about the topology of a graph can be essentially encoded in a matrix.

Definition 2.10. *The* incidence matrix *of the graph* G *is defined by* $\mathcal{I} := \mathcal{I}^+ - \mathcal{I}^-$, *where* $\mathcal{I}^+ := (\iota^+_{ij})$ *and* $\mathcal{I}^- := (\iota^-_{ij})$ *are given by*

$$\iota^+_{ij} := \begin{cases} 1, & \textit{if } \mathsf{e}_j(0) = \mathsf{v}_i, \\ 0, & \textit{otherwise}, \end{cases} \qquad \textit{and} \qquad \iota^-_{ij} := \begin{cases} 1, & \textit{if } \mathsf{e}_j(1) = \mathsf{v}_i, \\ 0, & \textit{otherwise.} \end{cases}$$

In other words, $\iota_{ij} = 1$, $\iota^+_{ij} = 1$, *or* $\iota^-_{ij} = 1$, *respectively, if* v_i *is endpoint, initial endpoint, or terminal endpoint of* e_j, *respectively.*

Observe that $|\Gamma(\mathsf{v}_i)|$ is the number of non-zero entries in the i^{th} column of $\mathcal{I}$. Moreover, $\sum_{i\in\mathbb{N}} \iota_{ij} = 0$ for all $j \in \mathbb{N}$, whereas $\sum_{j\in\mathbb{N}} \iota_{ij} = 0$ for all $i \in \mathbb{N}$ if and only if G is orientedly Eulerian.

CHAPTER 3

Basics on strongly continuous semigroups of operators

Main aim of this survey is to discuss a possible approach to the study of partial differential equations on networks. To this aim, I will follow an abstract approach based on the theory of strongly continuous semigroup of operators. A good introduction to this theory can be found in [**5**].

Definition 3.1. *A* strongly continuous semigroup *(in the following: C_0-*semigroup*) is a family $(T(t))_{t\geq 0}$ of bounded linear operators on a Banach space X such that*

$$T(t)T(s) = T(t+s), \qquad t,s \geq 0, \qquad \text{and} \qquad T(0) = I,$$

and moreover

$$\lim_{t\to 0^+} T(t)x = x \qquad \text{for all } x \in X.$$

If $(T(t))_{t\geq 0}$ extends to a family of bounded linear operators $(T(t))_{t\in\mathbb{R}}$ that satisfies the semigroup law for all $t,s \in \mathbb{R}$, then it is called a strongly continuous group *(or C_0-*group*).*

It is possible to associate each strongly continuous semigroup with a closed operator on X in the following way.

Definition 3.2. *A* generator *of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X is an operator A with domain*

$$D(A) = \left\{ x \in X : \lim_{t\to 0^+} \frac{T(t)x - x}{t} \ \text{exists} \right\}$$

such that

$$Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t}.$$

Generators of strongly continuous semigroups are always closed and densely defined.

Definition 3.3. *Let A be a closed operator on a Banach space X, $\lambda \in \mathbb{C}$. If $\lambda - A$ is an invertible operator, then one says that λ is in the* resolvent set *$\rho(A)$ of A. The inverse of $\lambda - A$ is denoted by $R(\lambda, A)$, the* resolvent operator *of A at λ. The* spectrum *$\sigma(A)$ of A is $\mathbb{C} \setminus \rho(A)$.*

By the closed graph theorem, resolvent of closed operators are always bounded.

Probably the main reason for studying operator semigroups is their connection to evolution equations and Cauchy problems in ∞-dimensional spaces.

Proposition 3.4. *Let A be a densely defined, closed operator on a Banach space X. The following assertions are equivalent.*

(a) A is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$.

(b) the abstract Cauchy problem

$$\text{(ACP)} \qquad \left\{ \begin{array}{rcll} \dot{u}(t) &=& Au(t), & t \geq 0, \\ u(0) &=& u_0, & \end{array} \right.$$

is well-posed*, i.e., it has a unique solution* $u \in C^1(\mathbb{R}_+, X) \cap C(\mathbb{R}_+, D(A))$ *that continuously depends on the initial value* $u_0 \in D(A)$*. Such a solution is given by* $u(t) := T(t)u_0$, $t \geq 0$.

Since $\|u(t)\|$ usually represents some physically relevant value (e.g., the system's total heat), then one can expect that in many cases it is non-increasing in time, i.e., that $\|T(t)\|_{\mathcal{L}(X)} \leq 1$ for all $t \geq 0$. In this case, $(T(t))_{t\geq 0}$ is called a *contraction* semigroups. It is possible to characterise generators of contraction semigroups by the celebrated theorems of Hille–Yosida and Lumer–Phillips.

Theorem 3.5 (Hille–Yosida 1948). *Let* A *be a closed, densely defined operator on a Banach space* X*. Then the following assertions are equivalent.*

(a) A *generates a strongly continuous contraction semigroup* $(T(t))_{t\geq 0}$.
(b) $\lambda \in \rho(A)$ *and* $\|\lambda R(\lambda, A)\|_{\mathcal{L}(X)} \leq 1$ *for all* $\lambda > 0$.

Theorem 3.6 (Lumer–Phillips 1961). *Let* A *be a densely defined operator on a Hilbert space* H*. Then the following assertions are equivalent.*

(a) the closure $\overline{A}$ *of* A *generates a strongly continuous contraction semigroup* $(T(t))_{t\geq 0}$.
(b) $\mathrm{Re}(Ax|x)_H \leq 0$ *for all* $x \in D(A)$ *and the range* $\mathrm{Range}(\lambda - \overline{A})$ *is dense in* H *for all all* $\lambda > 0$.

Proposition 3.7. *Let* A *be a densely defined operator on a Hilbert space* H*. If* $\mathrm{Re}(Ax|x)_H \leq 0$ *for all* $x \in D(A)$ *and* $\mathrm{Re}(A'x|x)_H \leq 0$ *for all* $x \in D(A')$*, then* $\overline{A}$ *generates a contraction semigroup.*

If in particular $(T(t))_{t\in\mathbb{R}}$ is a C_0-group such that $T(t)$ is a unitary operator for all $t \in \mathbb{R}$, then $(T(t))_{t\in\mathbb{R}}$ is called a *unitary* group. The following result is crucial in quantum mechanics.

Theorem 3.8 (Stone 1936). *Let* A *be a densely defined operator on a Hilbert space. Then the following assertions are equivalent.*

(a) iA *is the generator of a unitary group* $(T(t))_{t\in\mathbb{R}}$.
(b) A *is self-adjoint.*

Strongly continuous semigroups are tightly related to resolvent of their generators by means of the Laplace Transform and of the backward Euler scheme.

Proposition 3.9. *Let* $(T(t))_{t\geq 0}$ *be a strongly continuous semigroup on a Banach space* X *and* A *be its generator. Then the following assertions hold.*

(1) *For* λ *large enough one has* $\lambda \in \rho(A)$ *and*
$$R(\lambda, A)x = \lim_{t\to\infty} \int_0^t e^{-\lambda s} T(s)x\,ds, \qquad x \in X$$
(2) *On the other hand, if* $(T(t))_{t\geq 0}$ *is a contraction semigroup, then*
$$T(t)x = \lim_{n\to\infty} R\left(1, \frac{t}{n}A\right)^n x, \qquad x \in X.$$

As a consequence of Proposition 3.9, the following holds, see [**6**, Prop. 2.1]. Observe that while characterizing well-posedness of a Cauchy problem by the theorem of Hille–Yosida is often a realistic task, it is usually hopeless to look for an explicit solution to the problem. Thus, it is most useful to characterise qualitative properties of solutions. These can often be formulated as invariance of certain suitable subsets of the state space.

Proposition 3.10. *Let* $(T(t))_{t\geq 0}$ *be a contraction semigroup on a Banach space* X *with generator* A*. Let* C *be a closed convex set of* X*. Then the following assertions are equivalent.*

(a) $T(t)C \subset C$ *for all* $t > 0$.
(b) $\lambda R(\lambda, A)C \subset C$ *for all* $\lambda > 0$.

COROLLARY 3.11. *Let $(T(t))_{t\geq 0}$ be a semigroup on a Banach space X with generator A. Let Y be a closed subspace of X. Then the following assertions are equivalent.*

(a) $T(t)Y \subset Y$ *for all* $t > 0$.
(b) $R(\lambda, A)Y \subset Y$ *for some* $\lambda > 0$.

PROOF. By Proposition 3.10 and a simple rescaling argument one sees that Y is invariant under $(T(t))_{t\geq 0}$ if and only if $R(\lambda, A)Y \subset Y$ for all $\lambda > 0$. Assume now Y to be invariant under $R(\lambda_0, A)$ for some $\lambda_0 > 0$ and let $\lambda > 0$. Developing $R(\lambda, A)$ as a power series centered at λ_0 (possibly using the path connectedness of the resolvent set of A) one obtains that $R(\lambda, A)$ leaves Y invariant too, and the claim follows. $\square$

CHAPTER 4

Flows on networks

In this section I am going to discuss some elementary properties of a simple transport process on a network. This could be considered as a toy traffic model: much more elaborate problems can be investigated, see e.g. [**7**]. The approach in this section is based on [**8, 9, 10**].

In contrast to graph theoretical questions like Problem 2.7, where spatial issues of the model may (and ought to!) be neglected, for the scope of this section it is necessary to assign a length to all links of the graph. Thus, I consider a network represented by a countable geometric graph G. The structure of the network is given by the incidence matrix considered in Section 2. In this section, a transport process

$$\text{(Tr)} \qquad \frac{\partial u_j}{\partial t}(t,x) = c_j(x)\frac{\partial u_j}{\partial x}(t,x) - p_j(x)u_j(t,x), \qquad t \geq 0,\ x \in (0,1),\ j \in \mathbb{N},$$

taking place on each link e_j of the network will be discussed, where the velocities c_j are functions whose regularity will be specified below.

Boundary conditions still have to be precised: a Kirchhoff-type rule

$$\text{(Kr)} \qquad \iota^+_{ij} u_j(t,\mathsf{v}_i) = \omega_{ij} \sum_{k\in\mathbb{N}} \iota^-_{ik} u_k(t,\mathsf{v}_i), \qquad t \geq 0,\ i \in \mathbb{N}$$

will be imposed throughout. Here $\Omega = (\omega_{ij})$ is a suitable row-stochastic matrix whose entries ω_{ij} vanish whenever $\iota^-_{ij} = 0$. In other words, a ratio $1/\omega_{ij}$ of the total mass flowing into v_i is oriented into e_j: this condition prescribes that at each node v_i only a fraction $1/\omega_{ij}$ of the incoming mass is flowing into the outgoing link e_j. In particular, due to stochasticity of Ω one is imposing conservation of mass in each node v_i – i.e., a classical Kirchhoff law

$$\sum_{j\in\mathbb{N}} \iota^+_{ij} u_j(t,\mathsf{v}_i) = \sum_{k\in\mathbb{N}} \iota^-_{ik} u_k(t,\mathsf{v}_i), \qquad t \geq 0,\ i \in \mathbb{N},$$

too. In order to deal with general coefficients, consider the weighted state space $X := L^1((0,1;\frac{dx}{c_j});\ell^1)$, i.e.,

$$\|f\|_X := \sum_{j\in\mathbb{N}} \int_0^1 \frac{|f_j(x)|}{c_j(x)} dx.$$

This defines a norm that is equivalent to the canonical one of $L^1(0,1;\ell^1)$ under the general assumption that $c \in L^\infty(0,1;\ell^1)$, with $c_j(x) \geq \epsilon > 0$ for a.e. $x \in (0,1)$ and all j. This will be imposed throughout this section.

Remark 4.1. Different node conditions may also be imposed, modelling different kinds of phenomena. E.g., time-dependent conditions have been considered in [**11**].

All results in this section still hold if G is a finite graph, up to replacing ℓ^1 by $\mathbb{C}^m$ if the graph only has m links.

Proposition 4.2. *Consider the operator defined by*

$$Au := \operatorname{diag}\left(c_j \frac{du_j}{dx} - p_j u_j\right)_{j=1,\dots,m}$$

with domain

$$D(A) := \{u \in W^{1,1}(0,1;\ell^1) : \exists d \in \ell^1 : u(1) = (\Omega^\top \cdot \mathcal{I}^+)d\}.$$

Then the initial value problem associated with the system (Tr) − (Kr) *is equivalent to the abstract Cauchy problem* (ACP) *considered in Proposition 3.4.*

Thus, the above network transport problem can be studied by means of semigroup theory. The following result, which can be proved directly, is [**9**, Prop. 1.2.1 and Cor. 3.2.5].

Proposition 4.3. *Let $c_j \equiv 1$ and $p_j \equiv 0$ for all $j \in \mathbb{N}$. Then A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X given by*

$$T(t)f(s) := (\Omega^\top \cdot \mathcal{I}^+)^k f(t+s-k) \qquad \text{if } t+s \in [k,k+1),\ k \in \mathbb{N},$$

for all $f \in X$.

Remark 4.4. Although transport equations usually boast a hyperbolic character, the system (Tr) − (Kr) is *not* backward well-posed. This is intuitively due to the fact that (Kr) only prescribes the behaviour of the flow while *leaving* a node, but not while *entering* it. If the flow would be reversed, the system would then lack proper node conditions.

What happens if not all links have unitary length? If the link e_j has length $\ell_j > 0$, then for all $j \in \mathbb{N}$ one can simply replace by the new unknown

$$v_j(x) := u_j\left(\frac{x}{\ell_j}\right), \qquad x \in (0,\ell_j),$$

the old one $u(x)$, $x \in (0,1)$. Observe that $\frac{dv_j}{dx} = \ell_j^{-1}\frac{dv_j}{dx}$. Then, it is possible to consider on the same state space X a similar abstract Cauchy problem, where A is this time, more generally, the operator matrix given by $A = \operatorname{diag}(\ell_j^{-1}\frac{d}{dx} - p_j)$ and introduced in Proposition 4.2.

The proof of the following result is based on that presented in [**9**, page 44].

Proposition 4.5. *Let $0 \leq p \in L^\infty(0,1;\ell^\infty)$. Then the operator A generates a contraction C_0-semigroup $(T(t))_{t\geq 0}$ on X.*

Proof. It is easy to see that A is closed and densely defined. Let now $f \in D(A)$ and $g \in X$ with $R(\lambda,A)g = f$, for any λ in the right open half-plain. Then

$$\lambda|f_j(x)| - c_j(x)\frac{d}{dx}|f_j(x)| + p_j(x)|f_j(x)| = \operatorname{sign} f_j(x)\cdot g_j(x), \qquad x \in [0,1],\ j \in \mathbb{N}.$$

Dividing by c_j, integrating over $(0,1)$, and summing over j one obtains

$$\begin{aligned}
\lambda\|f\|_X &= \sum_{j\in\mathbb{N}}\int_0^1 \frac{d}{dx}|f_j(x)|dx - \frac{p_j(x)}{c_j(x)}|f_j(x)|dx + \sum_{j\in\mathbb{N}}\int_0^1 \frac{\operatorname{sign} f_j(x)}{c_j(x)} g_j(x)dx \\
&\leq \sum_{j\in\mathbb{N}}(|f_j(1)| - |f_j(0)|) + \sum_{j\in\mathbb{N}}\int_0^1 \frac{\operatorname{sign} f_j(x)}{c_j(x)} g_j(x)dx,
\end{aligned}$$

since the functions p_j are positive. Taking into account the node conditions satisfied by f it is possible to further estimate

$$\begin{aligned}\lambda\|f\|_X &\leq \sum_{j\in\mathbb{N}}(|(\Omega^\top\mathcal{I}^+)f_j(0)|-|f_j(0)|)+\sum_{j\in\mathbb{N}}\int_0^1\frac{\mathrm{sign} f_j(x)}{c_j(x)}g_j(x)dx\\ &\leq \sum_{j\in\mathbb{N}}(\|\Omega^\top\mathcal{I}^+\|-1)|f_j(0)|+\sum_{j\in\mathbb{N}}\int_0^1\frac{|g_j(x)|}{c_j(x)}dx\\ &= (\|\Omega^\top\mathcal{I}^+\|-1)\|f_j(0)\|_{\ell^1}+\|g\|_X=\|g\|_X.\end{aligned}$$

This is justified: due to the stochasticity of Ω, the matrix $\Omega^\top\mathcal{I}^+$ has norm 1, as can be checked directly. By the theorem of Hille–Yosida, this concludes the proof. □

The case of general (i.e., not necessarily row-stochastic) Ω, corresponding to absorption and/or generation phenomena in the nodes, can be treated in a similar way using the idea presented in [**9**, Rem. at page 45].

For the semigroup whose generation has been proved in Proposition 4.5 there is no explicit formula, but the resolvent operator of A has been computed explicitly in [**8**, Prop. 3.3] in the case of a finite graph. The following notation will be used whenever the graph G has m links and n nodes: for $\mathrm{Re}\lambda>0$, $\epsilon_\lambda(s)$ denotes the $m\times m$ matrix

$$\epsilon_\lambda(s):=\mathrm{diag}\left(e^{\frac{\lambda(s-1)}{c_j}}\right),\qquad s\in[0,1],$$

and $D_\lambda:\mathbb{C}^n\to D(A)$ is the operator defined by

$$D_\lambda d(s):=\epsilon_\lambda(s)\cdot(\Omega^-)^\top,\qquad d\in\mathbb{C}^n,\ s\in(0,1).$$

Finally, $C:=\mathrm{diag}(c_j)$, while $M:D(A)\to\mathbb{C}^n$ is the operator defined by

$$Mf:=\mathcal{I}^+f(0),\qquad f\in D(A).$$

Proposition 4.6. *Let G be finite. If the coefficients c_j are constant, then for all $f\in X$*

$$R(\lambda,A)f(s)=(I+D_\lambda(1-MD_\lambda)^{-1}M)\int_s^1\epsilon_\lambda(s-\tau+1)C^{-1}f(\tau)d\tau,\quad s\in[0,1]. \tag{4.1}$$

A more involved formula has been obtained in [**10**, Lemma 3.4] in the case of variable coefficients.

By Propositions 3.10–4.6 one sees that the semigroup is positive.

Remark 4.7. Observe that if all coefficients $c_j\equiv1$, then $\epsilon_\lambda(s-\tau+1)C^{-1}f(\tau)$ is a diagonal matrix for all $s\in[0,1]$ and all $\tau\in(s,1)$, hence the integral term in (4.1) leaves any subspace Y of X invariant. Let e.g. K be an orthogonal projection of $\mathbb{C}^m$ and consider the subspace

$$Y:=\{f\in L^1(0,1;\ell^1):f(x)\in\mathrm{Range}\,K\text{ for a.e. }x\in(0,1)\}.$$

Then, by Corollary 3.11 one sees that Y is invariant under the semigroup $(T(t))_{t\geq0}$ introduced in Proposition 4.3 if and only if

$$D_\lambda(1-MD_\lambda)^{-1}MY\subset Y$$

for some real λ. By definition of M,D_λ such a condition only depends on the topology of the graph. Invariance of this kind of subspaces under the action of diffusion semigroups will be thoroughly investigated in Chapter 10, but it seems that no investigation has been carried on in the context of flows on networks.

CHAPTER 5

Strings with acoustic node conditions

As a second application, let me give another example of hyperbolic system on a network. More precisely, I discuss well-posedness of a wave equation on a finite network of strings. The underlying graphs has m links and n nodes, on which conditions of *acoustic type* are imposed.

The system takes the form

$$\text{(NABC)} \qquad \begin{cases} \ddot{u}_j(t,x) &=& u_j''(t,x) - u_j(t,x), & t \in \mathbb{R},\ x \in (0,1),\ j = 1,\ldots,m, \\ u_j(t,\mathsf{v}_i) &=& u_\ell(t,\mathsf{v}_i) =: d_i^u(t), & t \in \mathbb{R},\ j,\ell \in \Gamma(\mathsf{v}_i),\ i = 1,\ldots,n, \\ \dot{\phi}_i(t) &=& \sum_{j=1}^m \iota_{ij} u_j'(t,\mathsf{v}_i), & t \in \mathbb{R},\ i = 1,\ldots,n, \\ \ddot{\phi}_i(t) &=& -\gamma_i\dot{\phi}_i(t) - \kappa_i\phi_i(t) - \rho\dot{d}_i^u(i), & t \in \mathbb{R},\ i = 1,\ldots,n, \end{cases}$$

Boundary conditions of acoustic type are a special form of dynamic boundary conditions. They have been introduced by Krasil'nikov and, independently, by Morse. In fact, in [**12, 13**] and following papers Krasil'nikov and Beale–Rosencrans have proposed a mathematical approach and started the development of a spectral theory for a wave equation with such boundary conditions on domains of $\mathbb{R}^n$. Network of strings with dynamic boundary conditions (of different type) have also been considered in [**14**] and [**15**, § 2.7].

In order to show well-posedness of the initial value problem associated with (NABC), one can apply the theorem of Lumer–Phillips. To this aim, I will borrow the ideas presented in [**13, 16**] for the case of a wave equation on a smooth domain of $\mathbb{R}^n$. It is assumed throughout this section that the coefficients $\gamma_i, \kappa_i, \sigma_i, \rho$ are positive constants. Consider the Hilbert product space $H := H^1(0,1;\mathbb{C}^m) \times L^2(0,1;\mathbb{C}^m) \times \mathbb{C}^n \times \mathbb{C}^n$ endowed with the weighted inner product

$$\begin{aligned} \big((f,g,\phi,\psi)|(r,s,\zeta,\xi)\big)_H \;:=\; & \rho\sum_{j=1}^m \int_0^1 \Big(f_j'(x)\overline{r_j'}(x) + f_j(x)\overline{r_j}(x)\Big)\,dx \\ & + \sum_{j=1}^m \int_0^1 \frac{\rho}{c_j} g_j(x)\overline{s_j}(x)dx + \sum_{i=1}^n \kappa_i\phi_i\overline{\zeta_i} + \sum_{i=1}^n \psi_i\overline{\xi_i}, \end{aligned}$$

which is equivalent to the canonical one. This can be interpreted as an energy norm given the sum of potential and kinetic energies represented by the integral terms, plus boundary terms associated with the damping effect in the nodes.

In order to deal with the involved structure of the considered problem, and in particular with the mixed node conditions, I am going to introduce two auxiliary incidence-type matrices. As usual, the graph is described by the ingoing and outgoing incidence matrices $\mathcal{I}^+ = (\iota_{ij}^+)$ and $\mathcal{I}^- = (\iota_{ij}^-)$. Define on H an operator A by

$$\begin{aligned} A(f,g,\phi,\psi) \;:=\; & \big(g_1,\ldots,g_m, f_1'' - f_1,\ldots,f_m'' - f_m, \\ & \psi_1,\ldots,\psi_n, -\gamma_1\psi_1 - \kappa_1\phi_1 - \rho d_1^g,\ldots,-\gamma_n\psi_n - \kappa_n\phi_n - \rho d_n^g\big) \end{aligned}$$

with domain

$$D(A) := \left\{ (f,g,\phi,\psi) \in H^2(0,1;\mathbb{C}^m) \times H^1(0,1;\mathbb{C}^m) \times \mathbb{C}^n \times \mathbb{C}^n : \begin{array}{l} \exists d^f,\ d^g \in \mathbb{C}^n \text{ s.t.} \\ (\mathcal{I}^+)^\top d^f = f(0),\ (\mathcal{I}^-)^\top d^f = f(1), \\ (\mathcal{I}^+)^\top d^g = g(0),\ (\mathcal{I}^-)^\top d^g = g(1), \\ \mathcal{I}^+ f'(0) - \mathcal{I}^- f'(1) = \psi, \end{array} \right\}.$$

With an appropriate choice of parameters, conservation of energy can now be showed.

PROPOSITION 5.1. *If $\gamma_i = 0$, $i = 1, \dots, n_0$, then A generates a unitary group on H.*

PROOF. The operator A is densely defined. In fact, set

$$X_1 := \left\{ (f,g,\phi) \in H^2(0,1;\mathbb{C}^m) \times H^1(0,1;\mathbb{C}^m) \times \mathbb{C}^{n_0} : \begin{array}{l} \exists d^f,\ d^g \in \mathbb{C}^n \text{ s.t.} \\ (\mathcal{I}^+)^\top d^f = f(0),\ (\mathcal{I}^-)^\top d^f = f(1), \\ (\mathcal{I}^+)^\top d^g = g(0),\ (\mathcal{I}^-)^\top d^g = g(1), \end{array} \right\}$$

and

$$X_2 := \left\{ (f,g,\phi) \in H^1(0,1;\mathbb{C}^m) \times L^2(0,1;\mathbb{C}^m) \times \mathbb{C}^{n_0} : \begin{array}{l} \exists d^f \in \mathbb{C}^n \text{ s.t.} \\ (\mathcal{I}^+)^\top d^f = f(0),\ (\mathcal{I}^-)^\top d^f = f(1) \end{array} \right\}.$$

Define the operator

$$Kf := \mathcal{I}^+ f'(0) - \mathcal{I}^- f'(1),$$

i.e.,

$$Kf \equiv \left(\sum_{j=1}^m \iota_{1j} f_j'(\mathsf{v}_1), \dots, \sum_{j=1}^m \iota_{nj} f_j'(\mathsf{v}_n \right).$$

Then, K is surjective and $\operatorname{Ker}(K)$ is the space of H^2-functions over the graph that satisfy pure Kirchhoff conditions in the n nodes. Since such a space is dense in $L^2(0,1;\mathbb{C}^m)$, the claim follows by [**17**, Lemma B.1].

Due to the 1-dimensional structure of the problem, it is also easy to see that the operator is closed. The range condition in the theorem of Lumer–Phillips can be checked applying the representation theorem of Riesz and mimicking the proof of [**16**, Thm. 2.1].

Further, integrating by parts and taking into account the definition of incidence matrix, one sees that for $\mathfrak{f} = (f,g,\phi,\psi)^\top \in D(A)$ there holds

$$\begin{aligned} \left(A\mathfrak{f} \mid \mathfrak{f}\right)_H &= \sum_{j=1}^m \int_0^1 \rho \left(g_j'(x)\overline{f_j'(x)} + g_j(x)\overline{f_j(x)} \right) dx + \sum_{j=1}^m \int_0^1 \rho \left(f_j''(x) - f_j(x) \right) \overline{g_j(x)} dx \\ &\quad + \sum_{i=1}^n \kappa_i \psi_i \overline{\phi_i} - \sum_{i=1}^n \kappa_i \phi_i \overline{\psi_i} - \sum_{i=1}^n \rho d_i^g \overline{\psi_i} \\ &= \sum_{j=1}^m \int_0^1 \rho \left(g_j'(x)\overline{f_j'(x)} + g_j(x)\overline{f_j(x)} \right) dx + \sum_{i=1}^n \sum_{j=1}^m \rho \iota_{ij} f_j'(\mathsf{v}_i) \overline{d_i^g} \\ &\quad - \sum_{j=1}^m \int_0^1 \rho \left(f_j'(x)\overline{g_j'(x)} + f_j(x)\overline{g_j(x)} \right) dx \\ &\quad + \sum_{i=1}^n \kappa_i \psi_i \overline{\phi_i} - \sum_{i=1}^n \kappa_i \phi_i \overline{\psi_i} - \sum_{i=1}^n \sum_{j=1}^m \rho \iota_{ij} d_i^g \overline{f_j'(\mathsf{v}_i)}. \end{aligned}$$

Thus, $\operatorname{Re}\left(A\mathfrak{f} \mid \mathfrak{f}\right)_H = 0$, the theorem of Stone applies and A generates a unitary group. □

Reasoning as in the proof of Theorem 5.1 and applying the theorem of Lumer–Phillips, the following can be proved directly.

COROLLARY 5.2. *If* $\gamma_i \geq 0$*,* $i = 1, \ldots, n_0$*, then* A *generates a strongly continuous, contractive semigroup on* H*.*

It is easy to see that A has compact resolvent (basically due to the compactness of the embedding $X_1 \hookrightarrow X_2$ in the proof of Proposition 5.1). More refined spectral results can be obtained mimicking the methods developed in [**16**].

In this and the previous chapters I have considered two examples of evolutionary physical system displaying a non-parabolic behaviour. Yet more problem of this kind could be considered: let me mention the investigations on Dirac operators on graphs performed in [**18**] and references therein.

CHAPTER 6

Basics on sesquilinear forms and analytic semigroups

First consider a σ-finite measure space (X,μ) and the Hilbert space $H := L^2(X)$, endowed with inner product $(\cdot \mid \cdot)_H$.

Let V be another Hilbert space such that V is densely and continuously embedded in H. A sesquilinear (not necessarily symmetric) mapping $a : V \times V \to \mathbb{C}$ will be considered in the following.

Definition 6.1. *A sesquilinear mapping $a : V \times V \to \mathbb{C}$, also called a* sesquilinear form *is said to be H-*elliptic, continuous, accretive*, and* symmetric *if there exist $\alpha > 0$, $\omega \in \mathbb{R}$, and $M \geq 0$ such that for all $f, g \in V$ it enjoys the properties*

- $\mathrm{Re} a(f,f) \geq \alpha \|f\|_V^2 - \omega \|f\|_H^2$,
- $|a(f,g)| \leq M \|f\|_V \|g\|_V$,
- $\mathrm{Re} a(f,f) \geq 0$,
- $a(f,g) = \overline{a(g,f)}$,

respectively. The operator A defined by

$$\begin{aligned} D(A) &:= \left\{ f \in V : \exists h \in H \ s.t.\ a(f,g) = (h \mid g)_H \ \forall g \in V \right\}, \\ Af &:= -h. \end{aligned}$$

is said to be associated with a.

Due to the density of V in H, one sees that the operator associated with a is uniquely determined. Observe that A is self-adjoint if and only if a is symmetric.

The assumptions under which most of the results below are formulated are by no means sharp. The reader is referred to [**6, 19, 20**] for further details, although the first significant advances in the theory of sesquilinear forms go back to Kato and Lions.

In the following, Σ_θ will denote a sector of angle $\theta > 0$, i.e., $\Sigma_\theta := \{z \in \mathbb{C} : |\mathrm{arange} z| < \theta\}$.

Definition 6.2. *A strongly continuous semigroup is said to be* analytic *if it admits a holomorphic extension $(T(z))_{z \in \Sigma_\theta}$ such that $T(z) \in \mathcal{L}(H)$ for all $z \in \Sigma_\theta$. It is said to be* bounded analytic *if it is analytic and moreover for all $\omega \in (0,\theta)$ there exists $M_\omega > 0$ such that $\|T(z)\|_{\mathcal{L}(H)} \leq M_\omega$ for all $z \in \Sigma_\omega$.*

The property of analyticity of a semigroup is an important one. In fact, generators A of C_0-semigroups that are analytic enjoy several distinctive features, and in particular their spectral theory is richer. E.g., their spectrum not only lies in a closed half-plain, but in fact even a sector thereof. Moreover, their spectral bound $s(A) := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$ is strictly negative if and only if the semigroup is uniformly exponentially stable, in the following sense.

Remarks 6.3. (1) Let f be an eigenfunction of the operator A associated with a form a. If $\|f\|_H = 1$ and λ denotes the associated eigenvalue, then $a(f,f) = -\lambda$. If A is self-adjoint, it is possible to interpret A as the observable of a physical system and $a(f,f)$ as the system's energy: more precisely, if H is an L^2-space on a finite measure space, then the set of all quantized energy levels of the system (i.e., the (positive) spectrum of the system – or equivalently the spectrum of $-A$), is contained in the numerical range

$$W(a) := \{a(f,f) \in \mathbb{C} : f \in V \text{ and } \|f\|_H = 1\}.$$

(2) Also in the context of a common heat equation (say, with Dirichlet boundary conditions) $a(u,u) = \|\nabla u\|_2^2$ represents the energy of the system. This can be generalised to abstract parabolic problems. Therefore, approaches to evolution equations based on the tool of sesquilinear forms are often referred to as *energy methods*.

DEFINITION 6.4. *A strongly continuous semigroup is said to be* uniformly exponentially stable *if* $\|T(t)\|_{\mathcal{L}(H)} \le Me^{-\epsilon t}$ *for some* $M, \epsilon > 0$ *and all* $t > 0$.

Beside asymptotics, a highly desirable property of solutions to physical problems is that of *smoothing of initial data*. Also in this context, analyticity is a crucial property: if $(T(t))_{t\ge 0}$ is an analytic semigroup, then $T(t)u_0 \in D(A^k)$ for all $k \in \mathbb{N}$, $u_0 \in H$, and $t > 0$.

The following assertion is a direct consequence of [**6**, Prop. 1.51 and Thm. 1.52].

PROPOSITION 6.5. *Let* $a : V \times V \to \mathbb{C}$ *be a continuous,* H*-elliptic sesquilinear form on* H*. Then the associated operator* A *generates an analytic contraction semigroup* $(T(t))_{t\ge 0}$ *on* H*. Such a semigroup is contractive if and only if* a *is accretive. Finally, this semigroup is compact if and only if* V *is compactly embedded in* H*, and self-adjoint if and only if* a *is symmetric.*

REMARK 6.6. Observe that if $a - \epsilon$ is accretive for $\epsilon > 0$, then $(T(t))_{t\ge 0}$ is uniformly exponentially stable: more precisely, it satisfies $\|T(t)\|_{\mathcal{L}(X)} \le e^{-\epsilon t}$ for all $t \ge 0$.

EXAMPLE 6.7. Consider an open bounded domain $\Omega \subset \mathbb{R}^n$ with C^1-boundary $\partial\Omega$. In order to discuss the heat equation with Dirichlet boundary conditions the operator $A := \Delta$, $D(A) := H^2(\Omega) \cap H^1_0(\Omega)$ on the complex space $H = L^2(\Omega)$ is introduced. Let $V = H^1_0(\Omega)$. Then, by the Gauss-Green formulae one has for all $u \in D(A)$ and $v \in V$

$$(Au \mid v)_H = \int_\Omega \Delta u \overline{v} dx = -\int_\Omega \nabla u \overline{\nabla v} dx,$$

due to the Dirichlet boundary conditions satisfied by v. This suggests to define a sesquilinear form $a : V \times V \to \mathbb{C}$ by

$$a(u,v) := \int_\Omega \nabla u \overline{\nabla v} dx.$$

Poincaré's inequality ensures that $(u \mid v) := \int_\Omega \nabla u \overline{\nabla u} dx$ defines an equivalent inner product on $H^1_0(\Omega)$, which $a - 1$ is accretive with respect to. The form a is also continuous, as a consequence of the Cauchy–Schwarz inequality applied to the inner product of V. It follows that the operator associated with a is the generator of an analytic, contractive, uniformly exponentially stable semigroup.

Thus, it is crucial to show that A is actually the operator associated with a in the sense of Definition 6.1. To this aim, take $u \in V$ such that there exists $w \in H$ with $a(u,v) = \int_\Omega \nabla u \overline{\nabla v} dx = \int_\Omega w \overline{v} dx$ for all $v \in V$. Then, by definition of weak derivative $\nabla u \in H^1$ and $w = \Delta u \in L^2(\Omega)$. By standard boundary regularity results, one has in fact $u \in H^2(\Omega)$. □

Once a generation result has been proved for an operator, it is often interesting to generalise it to its lower-order and/or boundary perturbations. To this aim, the following perturbation lemma often proves useful. It is the form equivalent of a well-known perturbation result for operators due to Desch–Schappacher. In the following, let me denote by H_α any interpolation space between V and H, i.e., any linear subspace such that $V \hookrightarrow H_\alpha \hookrightarrow H$ and such that additionally the interpolation inequality

$$\|f\|_{H_\alpha} \le M_\alpha \|f\|_V^\alpha \|f\|_H^{1-\alpha}, \qquad f \in V, \tag{6.1}$$

is satisfied. The following is [**21**, Lemma 2.1]. It is the form analogon of known perturbation results due to Desch–Schappacher and Greiner–Kuhn.

LEMMA 6.8. *Let $a : V \times V \to \mathbb{C}$ be an H-elliptic form. Let $\alpha \in (0,1)$ such that $b : V \times H_\alpha \to \mathbb{C}$ and $c : H_\alpha \times V \to \mathbb{C}$ be continuous sesquilinear mappings. Then $a+b+c : V \times V \to \mathbb{C}$ is H-elliptic.*

REMARK 6.9. Consider now time-dependent perturbations of a form, i.e., families $(b_t)_{t\in[0,T]}$ and $(c_t)_{t\in[0,T]}$. If $b_\cdot \in L^\infty(0,T;(V\times H_\alpha)')$ and $c_\cdot \in L^\infty(0,T;(H_\alpha \times V)')$, then there exists $M>0$ such that

$$|b_t(f,g)| \le M\|f\|_V\|g\|_{H_\alpha} \quad \text{and} \quad |c_t(g,f)| \le M\|g\|_V\|f\|_{H_\alpha}, \qquad f\in V,\ g\in H_\alpha,$$

holds uniformly in $t \ge 0$. It follows that the family of densely defined forms $(a+b_t+c_t)_{t\ge 0}$ is equi-H-elliptic and equicontinuous, too. Then, it is possible to apply the theory presented in [**22**, § XVII]. If A is the operator associated with a and $B(t)$ is the operator associated with $b_t + c_t$, $t \ge 0$, then it can be shown that the non-autonomous abstract Cauchy problem

$$\text{(nACP)} \qquad \begin{cases} \dot u(t) &= Au(t) + B(t)u(t), \qquad t\ge 0,\\ u(0) &= u_0, \end{cases}$$

is well-posed in a suitable sense.

The following is a direct consequence of a beautiful result due to Crouzeix, cf. [**23**, p. 204].

THEOREM 6.10 (Crouzeix 2003). *Let $a : V\times V \to \mathbb{C}$ be an H-elliptic, continuous sesquilinear form. Assume that there exists $M>0$ such that $|\mathrm{Im} a(f,f)| \le M\|f\|_V\|f\|_H$ for all $f\in V$. Then for all $B \in \mathcal{L}(H)$ the second order abstract Cauchy problem*

$$\begin{cases} \ddot u(t) &= Au(t) + B\dot u(t), \qquad t\in\mathbb{R},\\ u(0) &= u_0,\\ \dot u(0) &= u_1, \end{cases}$$

is well-posed*, i.e., it has a unique solution $u \in C^2(\mathbb{R},H)\cap C^1(\mathbb{R},V)\cap C(\mathbb{R}_+,D(A))$ that continuously depends on the initial values $u_0\in D(A)$, $u_1 \in V$.*

EXAMPLE 6.11. Let $V = H^1(0,1)$, $H = L^2(0,1)$, and $a : V\times V \to \mathbb{C}$ be an H-elliptic and continuous form. Since $C[0,1]$ satisfies (6.1) with $\alpha = \frac{1}{2}$ (see [**24**, Cor. 4.11]), it is an interpolation space between V and H. One concludes that $a+b+c : V\times V\to\mathbb{C}$ is H-elliptic and continuous whenever $b : H^1(0,1)\times C^[0,1] \to \mathbb{C}$ and $c : C[0,1]\times C[0,1]\to\mathbb{C}$ are continuous sesquilinear mapping. In particular, this allows for any perturbative form b,c representing a first order bounded drift and boundary terms that involves point evaluations, respectively. If for example

$$\begin{aligned} a(f,g) &:= \int_0^1 f'(x)\overline{g'(x)}dx,\\ b(f,g) &:= \int_0^1 \Big(\beta(x)f'(x)+\gamma(x)f(x)\Big)\overline{g(x)}dx, \quad \text{and}\\ c(f,g) &:= m_{11}f(0)\overline{g(0)} + m_{12}f(2)\overline{g(1)} + m_{21}f(0)\overline{g(1)} + m_{22}f(2)\overline{g(2)}, \end{aligned}$$

where $\beta\in L^2(0,1)$, $\gamma\in L^1(0,1)$, and m_{ij} are arbitrary complex numbers, then the operator associated with $a+b$ is the second derivative with nonlocal boundary conditions of Robin type. Since $a+b+c$ is densely defined, continuous and by Lemma 6.8 also H-elliptic, it follows that the initial-boundary value problem

$$\begin{cases} \dot u(t,x) &= u''(t,x)+\beta(x)u'(t,x)+\gamma(x)u(t,x), \qquad t\ge 0,\ x\in(0,1),\\ u'(t,0) &= m_{11}u(t,0)+m_{12}u(t,1), \qquad t\ge 0,\\ -u'(t,1) &= m_{21}u(t,0)+m_{22}u(t,1), \qquad t\ge 0,\\ u(0,x) &= u_0(x), \qquad x\in(0,1), \end{cases}$$

is well-posed. In fact, it is governed by an analytic semigroup.
Finally, observe that if $\beta \equiv 0$, then $\mathrm{Im} a(f,f) = m_{12} f(2)\overline{g(1)} + m_{21} f(1)\overline{g(2)}$. Accordingly,

$$|\mathrm{Im} a(f,f)| \le \max\{|m_{12}|, |m_{21}|\}\|f\|_\infty^2 \le \max\{|m_{12}|, |m_{21}|\}\|f\|_V \|f\|_H,$$

again because $C[0,1]$ is an interpolation space of order $\frac{1}{2}$ between $H^1(0,1)$ and $L^2(0,1)$. By Theorem 6.10 also the second order initial-boundary value problem

$$\begin{cases} \ddot{u}(t,x) &=& u''(t,x) + \gamma(x) u(t,x), & t \ge 0,\ x \in (0,1),\\ -u'(t,0) &=& m_{11} u(t,0) + m_{12} u(t,1), & t \ge 0,\\ u'(t,1) &=& m_{21} u(t,0) + m_{22} u(t,1), & t \ge 0,\\ u(0,x) &=& u_0(x), & x \in (0,1),\\ \dot{u}(0,x) &=& v_0(x), & x \in (0,1), \end{cases}$$

is well-posed. ☐

Under the assumptions of Proposition 6.10, it follows by a result due to McIntosh that V is a complex interpolation space between $D(A)$ and H, cf. [**19**, § 5.6.6]. By the theory developed in [**25**, Chapt. 7] the following result about nonlinear well-posedness holds at once.

Corollary 6.12. *Let $a : V \times V \to \mathbb{C}$ be a continuous, H-elliptic sesquilinear form with associated operator A. Assume that there exists $M > 0$ such that $|\mathrm{Im} a(f,f)| \le M\|f\|_V\|f\|_H$ for all $f \in V$. Let $F : [0,T] \times V \to H$ be a continuous nonlinear operator that is locally Lipschitz in the second variable. Then the nonlinear abstract Cauchy problem*

$$\begin{cases} \dot{u}(t) &=& Au(t) + F(t,u(t)), & t \ge 0,\\ u(0) &=& u_0, \end{cases}$$

is globally well-posed, i.e., there exists a function $u \in C([0,\infty); H)$ that satisfies

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(s,u(s))ds, \qquad t \ge 0,$$

where $(T(t))_{t\ge 0}$ denotes as usual the semigroup generated by A.

Using form methods allows to deduce simple, almost purely algebraical criteria in order to characterise crucial properties of semigroups, and thus of solution to Cauchy problems. The following is an easy but important consequence of Proposition 3.10.

Theorem 6.13 (Ouhabaz 1992). *Let $a : V \times V \to \mathbb{C}$ be an H-elliptic, accretive, continuous sesquilinear form. Let C be a closed convex subset of H and Π denote the orthogonal projection of H onto C. Then the following assertions are equivalent.*

(a) $(T(t))_{t\ge 0}$ leaves C invariant, i.e., $T(t)C \subset C$ for all $t > 0$.
(b) $\Pi V \subset V$ and $\mathrm{Re} a(\Pi u, u - \Pi u) \ge 0$ for all $u \in V$.

Theorem 6.13 allows to characterise several relevant invariant subsets of H in a very effective way. The following results come from [**6**, § 2], possibly after applying a simple rescaling argument in order to drop the accretivity assumption.

Proposition 6.14. *Let $a : V \times V \to \mathbb{C}$ be an H-elliptic, continuous sesquilinear form. Denote by $(T(t))_{t\ge 0}$ the semigroup on H generated by the associated operator A. Then the following assertions hold.*

(i) $(T(t))_{t\ge 0}$ is real *(i.e., $T(t)f$ is real-valued for all $t \ge 0$ and all real valued f) if and only if for all $f \in V$ one has $\mathrm{Re} f \in V$ and $a(\mathrm{Re} f, \mathrm{Im} f) \in \mathbb{R}$.*
(ii) $(T(t))_{t\ge 0}$ is positive *(i.e., $T(t)f$ is positive for all $t \ge 0$ and all positive valued f) if and only if it is real and additionally for all $f \in V$ one has $\mathrm{Re} f^+ \in V$ and $a(\mathrm{Re} f^+, \mathrm{Re} f^-) \le 0$.*

(iii) *Let $b : V \times V \to \mathbb{C}$ be another H-elliptic, continuous sesquilinear form, and denote by $(S(t))_{t\geq 0}$ the associated semigroup. Assume both a and b to be accretive, and $(T(t))_{t\geq 0}$ to be positive. Then $(T(t))_{t\geq 0}$* dominates *$(S(t))_{t\geq 0}$* in the sense of positive semigroups *(i.e., $|S(t)f| \leq T(t)|f|$ for all $t \geq 0$ and $f \in H$) if and only if for all $f, g \in V$ such that $f\overline{g} \geq 0$ one has $\mathrm{Re} b(f,g) \leq a(|f|,|g|)$.*

(iv) *$(T(t))_{t\geq 0}$ is L^∞-*quasicontractive *(i.e., there exists $\epsilon \in \mathbb{R}$ such that $T(t)f \in L^\infty(X,\mu)$ and $\|T(t)f\|_\infty \leq e^{\epsilon t}$, $t \geq 0$, for all $f \in L^2(X,\mu) \cap L^\infty(X,\mu)$ with $\|f\|_\infty \leq 1$) if and only if for all $f \in V$ one has $(1 \wedge |f|)\mathrm{sign} f \in V$ and*

$$\mathrm{Re} a((1 \wedge |f|)\mathrm{sign} f, (|f|-1)^+ \mathrm{sign} f) + \epsilon \int_{\{|f|\geq 1\}} (|f(x)|-1) d\mu \geq 0.$$

In (iv) above, $L^\infty(X,\mu)$ denotes the space of all μ-essentially bounded functions from X to $\mathbb{C}$, which is embedded in H since by assumption the X has finite measure. Moreover, the mapping $\mathrm{sign} f : X \to \mathbb{C}$ is defined by $\mathrm{sign} f(x) := \frac{f(x)}{|f(x)|}$, for $f \in H$ and $x \in X$. An L^∞-quasicontractive semigroup with constant $\epsilon = 0$ is called contractive.

Remark 6.15. *If Theorem 6.14.(iv) applies, then a direct application of the Riesz-Thorin interpolation theorem yields that $(T(t))_{t\geq 0}$ defines a contraction semigroup on $L^p(X,\mu)$, for all $p \in [2,\infty]$, which is denoted by $(T_p(t))_{t\geq 0}$ or, if no confusion is possible, again by $(T(t))_{t\geq 0}$. The same is true for the adjoint semigroup $(T_p(t)^*)_{t\geq 0}$ on $L^p(X,\mu)$, for all $p \in [1,2]$. All these semigroup are strongly continuous, except for that on $L^\infty(X,\mu)$. All these semigroup are analytic, with the possible exception of those on $L^\infty(X,\mu)$ and $L^1(X,\mu)$. They are consistent, in the sense that*

$$T_p(t)f = T_q(t)f, \qquad t \geq 0,$$

for all $f \in L^p(X,\mu) \cap L^q(X,\mu)$ and all $p, q \in [2,\infty]$.

To conclude this section, the case of analytic semigroups that enhance integrability (and not only regularity) properties is considered.

Proposition 6.16. *Let $a : V \times V \to \mathbb{C}$ be an H-elliptic, continuous sesquilinear form. Denote by $(T(t))_{t\geq 0}$ the semigroup on H generated by the associated operator A. If the inequality*

$$\|f\|_2^{2+\frac{4}{n}} \leq M \|f\|_V^2 \|f\|_1^{\frac{4}{n}} \tag{6.2}$$

holds for some constant M and all $f \in V$, or if for $n \geq 2$ the embedding

$$V \hookrightarrow L^{\frac{2n}{n-2}}(X) \tag{6.3}$$

holds, then the semigroup $(T(t))_{t\geq 0}$ on H associated with a is ultracontractive of dimension *n, i.e., it satisfies the estimate*

$$\|T(t)f\|_\infty \leq c t^{-\frac{n}{4}} \|f\|_2, \quad t \in (0,1],, \tag{6.4}$$

for some constant $c > 0$ and all $f \in L^2$.

Remark 6.17. Let $H = L^2(\Omega)$, where Ω is a bounded open domain of $\mathbb{R}^n$, and let the operator A associated with a satisfy $D(A^k) \hookrightarrow C(\overline{\Omega})$ for some $k \in \mathbb{N}$. Under the assumptions of Proposition 6.16 and because of the smoothing effect due to analyticity, the semigroup $(T(t))_{t\geq 0}$ maps $L^2(\Omega)$ into $C(\overline{\Omega})$ and satisfies the estimate in (6.4). By duality, the adjoint semigroup maps the space $\mathcal{M}(\Omega)$ of Radon measures on $\overline{\Omega}$ into $L^2(\Omega)$ and satisfies

$$\|T(t)\nu\|_2 \leq c t^{-\frac{n}{4}} \|\nu\|_{\mathcal{M}} \quad \text{for all } t \in (0,1] \text{ and all } \mu \in \mathcal{M}(\Omega),$$

If in particular the semigroup is self-adjoint (or, more generally, if the adjoint semigroup is ultracontractive with same dimension), then $L^2(\Omega)$ acts as a pivot space and one concludes that $(T(t))_{t\geq 0}$ maps $\mathcal{M}(\Omega)$ in $C(\overline{\Omega})$ with

$$\|T(t)\nu\|_\infty \leq ct^{-\frac{n}{2}}\|\nu\|_{\mathcal{M}} \quad \text{for all } t\in(0,1] \text{ and all } \mu\in\mathcal{M}(\Omega). \tag{6.5}$$

To conclude, recall that the adjoint of a strongly-continuous semigroup needs not be strongly-continuous.

Example 6.18. Consider the setting introduced in Example 6.7. Several qualitative properties of the semigroup generated by A, the Laplacian with Dirichlet boundary conditions, can be proved by means of Proposition 6.14. Let $u\in V$. Then, also $\mathrm{Re}u$ is weakly differentiable as well as $\mathrm{Re}u^+$ and in fact $\nabla\mathrm{Re}u^+ = \nabla\mathrm{Re}u\cdot\mathbf{1}_{\{u\geq 0\}}$. Likewise, $(1\wedge|f|)\mathrm{sign}f$ is weakly differentiable with $\nabla((1\wedge|f|)\mathrm{sign}f) = \nabla u\cdot\mathbf{1}_{\{|u|\leq 1\}}$. E.g.,

$$a(\mathrm{Re}u^+,\mathrm{Re}u^-) = \int_\Omega |\nabla\mathrm{Re}u|^2\mathbf{1}_{\{u\geq 0\}}\mathbf{1}_{\{u\leq 0\}}dx = 0,$$

and similarly $\mathrm{Re}a((1\wedge|f|)\mathrm{sign}f,(|f|-1)^+\mathrm{sign}f)=0$. By Proposition 6.14 the semigroup generated by A is real, positive, and L^∞-contractive. Furthermore, by interpolation and then by duality the semigroup extends to a family of strongly continuous semigroups $(T_p(t))_{t\geq 0}$ on all spaces $L^p(\Omega)$, $1<p<\infty$. It follows from the Nash inequality or the Sobolev embedding of $H^1(\Omega)$ that the semigroup is also ultracontractive of dimension n. (In fact, applying more refined results one could show that such semigroups are also strongly continuous and analytic on all L^p spaces – including $L^1(\Omega)$: see [**6**].) Finally, the estimate (6.5) holds by Remark 6.17. □

Remark 6.19. Let an operator A be associated with a continuous H-elliptic form a with domain V on $H=L^2(X)$. If X has finite measure and $V\hookrightarrow L^\infty(X)$, then both $(e^{ta})_{t\geq 0}$ and its adjoint $(e^{ta^*})_{t\geq 0}$ map H into $L^\infty(X)$, and by duality we see that $(e^{ta})_{t\geq 0}$ maps $L^1(X)$ into $L^\infty(X)$. By the theorem of Dunford–Pettis (cf. [**19**, § 7.3.1]) the operator e^{ta} is given by a suitable integral kernel $K_t\in L^\infty(X\times X)$, i.e.,

$$e^{ta}f(x) = \int_X K_t(x,y)f(y)dy, \qquad t\geq 0,\ x\in X.$$

Proving suitable estimates on such *heat kernels* usually yields interesting properties of the semigroup $(e^{ta})_{t\geq 0}$, cf. [**6**] for interesting results in this direction.

CHAPTER 7

Parabolic equations on networks

As a relevant application of the above introduced form techniques, and central topic in this survey, I wish to introduce a class of differential equations on a network whose underlying graph $\mathsf{G} = (\mathsf{E}, \mathsf{V})$ is countable. The mathematical analysis of elliptic operators acting on spaces of functions on networks was started by Lumer in [**26, 27**]. It has been subsequently continued by many authors, both in mathematics (in the context of *network diffusion problems*, see e.g. [**28, 29, 30**]) and in physics (leading to the theory of *quantum graphs* presented in Chapter 13, see e.g. [**31, 32, 33**]). In this section, a network diffusion equation equipped with a class of node conditions of fairly general type is introduced. Several properties of such a system are going to be discussed. Although several authors have discussed diffusion equations on networks by means of methods based on sesquilinear forms, the approach presented here is directly based on [**34, 35**].

Let $c \in C^1([0,1], \ell^\infty)$ and $p \in L^\infty(0,1;\ell^\infty)$, and consider the system of countably many diffusion problems

$$\text{(Di)} \qquad \frac{\partial u_j}{\partial t}(t,x) = \frac{\partial}{\partial x}\left(c_j \frac{\partial u_j}{\partial x}\right)(t,x) - p_j(x) u_j(t,x), \qquad t > 0,\ x \in (0,1),\ j \in \mathbb{N}.$$

It is assumed throughout this section that the general diffusion coefficients c_j satisfy

$$(7.1) \qquad 0 < \gamma := \inf_{j \in \mathbb{N}} \min_{x \in [0,1]} |c_j(x)|, \qquad \Gamma := \sum_{j \in \mathbb{N}} \|c_j\|_\infty^2 < \infty, \qquad \text{and} \qquad P := \sum_{j \in \mathbb{N}} \|p_j\|_\infty^2 < \infty.$$

Observe that in (Di) no kind of coupling between diffusion problems is assumed. In fact, one wants to impose some kind of boundary interaction. To this aim, the continuity condition

$$\text{(Cc)} \qquad u_j(t, \mathsf{v}_i) = u_\ell(t, \mathsf{v}_i) =: d_i^u(t), \qquad j, \ell \in \Gamma(\mathsf{v}_i),\ i \in \mathbb{N},\ t > 0,$$

are imposed in the nodes. In the motivating examples of a system of equation describing temperature or voltage diffusion, this means that endpoints of incident links are equally hot or equally loaded. Heat or electric current might be flowing through the nodes, though: this phenomenon is assumed to satisfy the transmission conditions

$$\text{(Tc)} \qquad \sum_{j \in \mathbb{N}} \iota_{ij} c_j(\mathsf{v}_i) u_j'(t, \mathsf{v}_i) + \sum_{k \in \mathbb{N}} m_{ik} d_k^u(t) = 0, \qquad i \in \mathbb{N},\ t > 0,$$

where $M = (m_{ik})$ is some infinite matrix whose entries describe absorption or generation phenomena and satisfy $M := \sum_{h,k \in \mathbb{N}} |m_{hk}|^2 < \infty$, i.e., M is a bounded operator on ℓ^2.

These conditions are in general *non-local* unless M is diagonal. If in particular $M = 0$, then the transmission conditions reduce to assuming that in each node the total incoming flux equals the total outgoing flux: if u represents voltage, and thus u' is an electric current, this is nothing but the celebrated Kirchhoff's first law, formulated in 1845 in the context of electric circuits. In analogy with the boundary conditions of *third type*, also known as *Robin* boundary conditions, considered in the theory of partial differential equations, I will sometimes refer to the conditions defined by a diagonal but non-zero matrix M as *Kirchhoff-Robin node conditions*.

REMARKS 7.1. (1) The operators considered above play a central rôle in approximation theory, too. Of course, networks (in the sense of 1-dimensional structures) do not exist in the physical world: they are only the idealization of higher-dimensional structures, if one size is largely predominant. While it is comparatively easy to study a 1-dimensional operator's behaviour, there is in general no reason why it should give precise information about the real, higher-dimensional structure that ideally shrinks to the considered network.

Approximation of domains in higher dimensional euclidean spaces whose limiting cases are 1-dimensional networks have been thoroughly treated, cf. [**36, 37, 38**] and references therein. It is interesting that in both 2- and 3-dimensional approximation schemes the limiting system varies in dependence on the ratio between the area/volume of the link neighborhoods (strips/cylinders) and the area/volume of the node neighborhoods, and that in fact in both cases a critical threshold arises as this ratio tends to 1. Convergence of resolvents has been discussed in [**39, 40**].

A vast interest about such topics has been arising in the last decade: quite surprisingly, this is more true in the community of mathematical physics rather than in that of numerical analysis.

(2) More general system of diffusion processes can be considered, describing nonlocal interactions that take place not only in the nodes, but also in the links. More precisely, (Di) can be replaced by

$$\frac{\partial u_j}{\partial t}(t,x) = \sum_{i\in\mathbb{N}} \frac{\partial}{\partial x}\left(c_{ji}\frac{\partial u_i}{\partial x}\right)(t,x) - \sum_{i\in\mathbb{N}} p_{ji}u_i, \qquad t>0,\ x\in(0,1),\ j\in\mathbb{N}. \tag{7.2}$$

This may be interpreted as a form of external control, perhaps with the aim of stabilizing the system. It has to be complemented by (Tc) and further suitable node conditions, which appear less intuitive than usual Kirchhoff-type ones. Most results presented in this and the following sections can be extended to such a general setting, under the basic assumption that the functions c_{ij} are sufficiently regular ($C^1[0,1]$ will do) and satisfy the ellipticity condition

$$\operatorname{Re}\sum_{i,j\in\mathbb{N}} c_{ij}(x)\xi_j\overline{\xi_i} \geq \gamma|\xi|^2_{\ell^2}$$

for some $\gamma > 0$ and all $x \in [0,1]$. Remarkably, (7.2) lacks most of the regularity properties that are typical of diffusion problems, in spite of its parabolic nature (shown e.g. in [**41**]): e.g., in the nontrivial case of $c_{ij} \not\equiv 0$ for $i \neq j$ the solution is not of class C^∞, no parabolic maximum principle holds, the system is not dissipative with respect to the L^1 and L^∞-norms. A thorough investigation of this kind of problems has been carried on in [**42**].

In order to prove well-posedness of the motivating problem, observe first that $L^2(0,m)$ is isomorphic to $L^2(0,1;\mathbb{C}^m)$ for all $m\in\mathbb{N}$, and similarly $L^2(0,\infty)$ is isomorphic to $L^2(0,1;\ell^2)$. More precisely, for given functions $f_j:[0,1]\to\mathbb{C}$, $j=1,\dots,n$, define a mapping $Uf:[0,m]\to\mathbb{C}$ by

$$Uf(x) := \tilde{f}(x) := f_j(x-j+1) \qquad \text{if } x\in(j-1,j),\ j=1,\dots,m.$$

With this notation, the following holds.

LEMMA 7.2. *For all $p\in[1,\infty]$ the mapping U is one-to-one from $L^p(0,m)$ is isomorphic to $L^p(0,1;\mathbb{C}^m)$ for all $m\in\mathbb{N}$, and similarly $L^p(0,\infty)$ is isomorphic to $L^p(0,1;\ell^2)$. In fact, it is an isometry if $L^p(0,1;\mathbb{C}^m)$ is endowed with the canonical l^p-norm, i.e.,*

$$\|f\|_p := \Big(\sum_{j=1}^m \|f_j\|^p_{L^p(0,1)}\Big)^{\frac{1}{p}} \qquad \textit{(resp., } \|f\|_p := \Big(\sum_{j\in\mathbb{N}} \|f_j\|^p_{L^p(0,1)}\Big)^{\frac{1}{p}}\textit{)}, \qquad 1\leq p<\infty,$$

or

$$\|f\|_\infty := \max_{1\leq j\leq m}\|f_j\|_{L^\infty(0,1)} \qquad \textit{(resp., } \|f\|_\infty := \max_{j\in\mathbb{N}}\|f_j\|_\infty\textit{)}.$$

In particular, one can consider the Hilbert spaces $L^2(0,1;\mathbb{C}^m)$ or $L^2(0,1;\ell^2)$ of vector-valued functions functions as usual scalar-valued L^2-spaces and apply the theory presented in Chapter 6.

Upon introducing the matrices $\Omega^+ = (\omega^+_{ij})$ and $\Omega^- = (\omega^-_{ij})$ defined by

$$\omega^+_{ij} := c_j(\mathsf{v}_i)\iota^+_{ij} \qquad \text{and} \qquad \omega^-_{ij} = c_j(\mathsf{v}_i)\iota^-_{ij},$$

condition (Tc) can be rewritten in a more compact form as $\Omega^+ u'(0) - \Omega^- u'(1) + M d^u = 0$. Similarly, condition (Cc) is equivalent to the existence of $d^f \in \ell^2$ such that

$$(\mathcal{I}^+)^\top d^f = f(0) \qquad \text{and} \qquad (\mathcal{I}^-)^\top d^f = f(1).$$

The coordinates of such a vector d^f are by definition the nodal values of f.

The initial value problem associated with the system arising from (Di)–(Cc)–(Tc) can thus be recasted as in Proposition 3.4 in the abstract form of a Cauchy problem (ACP) over the Hilbert space $H := L^2(0,1;\ell^2)$, where

$$A := \operatorname{diag}\left(\frac{d}{dx} c_j \frac{d}{dx}\right) - p_j u_j$$

with domain

$$D(A) := \left\{ f \in H^2(0,1;\ell^2) : \begin{array}{c} \exists d^f \in \ell^2 \text{ s.t.} \\ (\mathcal{I}^+)^\top d^f = f(0),\ (\mathcal{I}^-)^\top d^f = f(1), \\ \text{and } \Omega^+ f'(0) - \Omega^- f'(1) + M d^f = 0 \end{array} \right\}.$$

It can be proved that the matrices $\mathcal{I}^+, \mathcal{I}^-$ are bounded if and only if the network is uniformly locally finite, i.e., $\max_{v \in \mathsf{V}} |\Gamma(\mathsf{v})| < \infty$. In order to avoid technicalities, we impose this assumption throughout.

Remark 7.3. The above formulation of the node conditions goes back to [**29**]. In fact, taking into account concrete applications it is natural to impose continuity and Kirchhoff-type laws: they have been considered already in [**26**]. However, there are different ways of formulating them in a compact form: let me mention the alternative descriptions given in [**33**] and [**43**]. While less intuitive, these both approaches allow direct characterizations of several analytic properties in terms of simple algebraical conditions on suitable matrices.

In order to develop a form approach to this network problem, define the space V of H^1-functions that are continuous over the network, i.e.,

$$V := \left\{ f \in H^1(0,1;\ell^2) : \exists d^f \in \ell^2 \text{ s.t. } (\mathcal{I}^+)^\top d^f = f(0),\ (\mathcal{I}^-)^\top d^f = f(1) \right\}. \tag{7.3}$$

Since $H^1_0(0,1;\ell^2) \subset V$, V is densely embedded in H – the embedding is also compact if and only if the network is in fact finite, i.e., if ℓ^2 is replaced by $\mathbb{C}^m$. Let $u \in D(A)$ and $v \in V$ and observe that

$$\begin{aligned} (Au \mid v)_H &= \sum_{j\in\mathbb{N}} \int_0^1 \Big((c_j u_j')'(x) - p_j(x) u_j(x)\Big) \overline{v_j(x)} dx \\ &= \sum_{j\in\mathbb{N}} \Big[c_j u_j' \overline{v_j}\Big]_0^1 - \sum_{j\in\mathbb{N}} \int_0^1 \Big(c_j(x) u_j'(x) \overline{v_j'(x)} - p_j(x) u_j(x) \overline{v_j(x)}\Big) dx. \end{aligned}$$

Using the incidence matrix $\mathcal{I}$ one can write

$$\sum_{j\in\mathbb{N}} \Big[c_j u_j' \overline{v_j}\Big]_0^1 = \sum_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} c_j(\mathsf{v}_i)(\iota^+_{ij} - \iota^-_{ij}) u_j'(\mathsf{v}_i) \overline{d^v_i} = \sum_{i\in\mathbb{N}} \overline{d^v_i} \sum_{j\in\mathbb{N}} \omega_{ij} f_j'(\mathsf{v}_i).$$

Taking into account the transmission condition (Tc) satisfied by u one thus obtains

$$\sum_{j\in\mathbb{N}} \left[c_j u_j' \overline{v_j}\right]_0^1 = -\sum_{i,h\in\mathbb{N}} m_{ih} d_h^u \overline{d_i^v},$$

and accordingly

$$(Au \mid v)_H = -\sum_{j\in\mathbb{N}} \int_0^1 \left(c_j(x)u_j'(x)\overline{v_j'(x)} + p_j(x)u_j(x)\overline{v_j(x)}\right) dx - \sum_{i,h\in\mathbb{N}} m_{ih} d_h^u \overline{d_i^v}.$$

This leads to the introduction of a sesquilinear form $a : V \times V \to \mathbb{C}$ defined by

$$a(u,v) := \sum_{j\in\mathbb{N}} \int_0^1 \left(c_j(x)u_j'(x)\overline{v_j'(x)} + p_j(x)u_j(x)\overline{v_j(x)}\right) dx + \sum_{i,h\in\mathbb{N}} m_{ih} d_h^u \overline{d_i^v}. \tag{7.4}$$

LEMMA 7.4. *The operator associated with the form a is A.*

PROOF. Take $u \in V$ such that there exists $w \in H$ satisfying

$$a(u,v) = (w \mid v)_H = \sum_{j\in\mathbb{N}} \int_0^1 w_j(x)\overline{v_j(x)}dx \qquad \text{for all } v \in V. \tag{7.5}$$

Hence, (7.5) is satisfied in particular for all functions v of the form

$$v = \begin{pmatrix} 0 \\ \vdots \\ v_j \\ 0 \\ \vdots \end{pmatrix} \leftarrow j^{\text{th}} \text{ row}, \quad v_j \in H_0^1(0,1),$$

for any $j \in \mathbb{N}$. From this follows that for all $v_j \in H_0^1(0,1)$

$$\int_0^1 \left(c_j(x)u_j'(x)\overline{v_j'(x)} + p_j(x)u_j(x)\overline{v_j(x)}\right) dx = \int_0^1 w_j(x)\overline{v_j(x)}dx, \qquad j \in \mathbb{N}. \tag{7.6}$$

By definition of weak derivative, this means that $c_j \cdot u_j' \in H^1(0,1)$ with weak derivative $w_j - p_j u_j$, for all $j \in \mathbb{N}$. As moreover (7.6) holds in particular for all functions $v \in V$ such that $v_j' \equiv 1$ for any $j \in \mathbb{N}$ on arbitrarily large closed subsets of $(0,1)$, one sees that $c \cdot u' \in H^1(0,1;\ell^2)$. Since by (7.1) the functions that appear in the vector c are uniformly bounded from below, i.e., $0 < \gamma \le c_j \in C^1[0,1]$ for all $j \in \mathbb{N}$, there follows that $u' \in H^1(0,1;\ell^2)$ for all $j \in \mathbb{N}$. Accordingly, $u \in H^2(0,1;\ell^2)$. Moreover, taking into account (7.5) one sees that

$$\sum_{i\in\mathbb{N}} \overline{d_i^v} \sum_{j\in\mathbb{N}} \omega_{ij} u_j'(\mathsf{v}_i) = -\sum_{i,h\in\mathbb{N}} m_{ih} d_h^u \overline{d_i^v}.$$

Since $v \in V$ is arbitrary, this means that

$$\sum_{j\in\mathbb{N}} \omega_{ij} u_j'(\mathsf{v}_i) = -\sum_{h\in\mathbb{N}} m_{ih} d_h^u \qquad \text{for all } i \in \mathbb{N}.$$

Thus, u satisfies (Tc) and therefore $u \in D(A)$. Furthermore,

$$-\sum_{j\in\mathbb{N}} \int_0^1 \left(c_j(x)u_j'(x)\overline{v_j'(x)} - p_j(x)u_j(x)\overline{v_j(x)}\right) dx = \sum_{j\in\mathbb{N}} \int_0^1 w_j(x)\overline{v_j(x)}dx$$

holds for all $v \in V$ and this implies that $Au = -w$. ☐

Well-posedness of the original network problem can be proved by showing that the associated abstract Cauchy problem (ACP) is governed by a C_0-semigroup.

THEOREM 7.5. *The operator A generates a strongly continuous, bounded analytic semigroup $(T(t))_{t\geq 0}$ on H, which is given by an integral kernel. This semigroup is compact if and only if the graph G is finite. It is self-adjoint if and only if M is self-adjoint. Moreover, the following holds.*

If moreover the matrix M is positive semidefinite and $p_j(x) \geq 0$ for all j and a.e. $x \in (0,1)$, then the semigroup is contractive. If additionally M is positive definite or $p_j(x) \geq p_0 > 0$ for all j and a.e. $x \in (0,1)$, then the semigroup is also uniformly exponentially stable.

PROOF. Since V is densely embedded in H (and compactly embedded if and only if G is finite), by Proposition 6.5 it is enough to show that a is continuous and H-elliptic. In fact, let $u, v \in V$. Then by the Cauchy–Schwarz inequality one has

$$|a(u,v)| \leq \max\{\Gamma, P\}\|u\|_V\|v\|_V + M|d^u|_{\ell^2}|d^v|_{\ell^2}.$$

In order to check H-ellipticity of a, consider the special case of $m_{hk} = 0$ for all h, k and p_j for all j. Observe that for $\alpha = \omega = \gamma$ (for γ defined in (7.1)) the ellipticity inequality is trivially satisfied. It follows from Lemma 6.8 that the form is H-elliptic in general case, too. As observed in Remark 6.19, the semigroup is given by a heat kernel as a consequence of the theorem of Dunford–Pettis.

Finally, $(T(t))_{t\geq 0}$ is contractive if and only if a is accretive, i.e., if and only if

$$\mathrm{Re}a(f,f) = \mathrm{Re}\left(\sum_{j\in\mathbb{N}}\int_0^1 \left(c_j(x)|f_j'(x)|^2 + p_j(x)|f_j(x)|^2\right)dx + \sum_{i,h\in\mathbb{N}} m_{ih} d_h^f \overline{d_i^f}\right) \geq 0$$

for all $f \in V$. Of course, this holds whenever M is positive semidefinite and $p_j(x) \geq 0$ for all j and a.e. $x \in (0,1)$. Exponential stability of $(T(t))_{t\geq 0}$ can be established in a similar way, taking into account Remark 6.6. □

REMARKS 7.6. (1) It should be mentioned that in the case of a finite graph, and if the coefficients satisfy $c_j = p_j \equiv 1$ for all j as well as $M = 0$, an (involved) integral formula for the semigroup generated by A has been deduced in [**30**]. This formula has been extended to infinite graphs in [**44**]. For general coefficients, no formula is available; thus, it is crucial to investigate qualitative properties of the network diffusion equation by abstract methods. This is the aim of next section.

(2) Deciding whether a matrix M is positive (semi)definite is in general no easy task. However, it is possible to obtain a reasonably simple sufficient condition by means of Gershgorin's theorem, provided that the network G is finite. More precisely, if G has m links and

$$\mathrm{Re}m_{ii} > \sum_{h\neq i}\frac{|m_{ih} + \overline{m_{hi}}|}{2}, \qquad i = 1, \ldots, m, \tag{7.7}$$

then it follows from Gershgorin's circle theorem that all eigenvalues of $\frac{M+M^*}{2}$ are contained in the open right half plain of $\mathbb{C}$. Since positive (semi)definiteness of M is equivalent to the positive (semi)definiteness of $\frac{M+M^*}{2}$, it follows that condition (7.7) is sufficient for $(T(t))_{t\geq 0}$ to be uniformly exponentially stable. Observe that in the infinite case this is not true anymore, since Gershgorin's theorem only helps to locate the point spectrum of a matrix – which in the ∞-dimensional case does not necessarily agree with its spectrum.

(3) More qualitative properties can be deduced from Theorem 6.13. E.g., let G an inbound star with 2 links and consider the closed convex set

$$C := \{u \in L^2(0,1;\mathbb{C}^2) : |u_1(x)| \leq u_2(x) \text{ for a.e. } x \in (0,1)\}.$$

Then it has been shown in [**6**, Prop. 2.20] that the orthogonal projection Π onto C is given by

$$Pu = \frac{1}{2}\left(\left(|u_1| + \min\{|u_1|, \mathrm{Re}u_2\}\right)^+ \mathrm{sign}u_1, \left(\max\{|u_1|, \mathrm{Re}u_2\} + \mathrm{Re}u_2\right)^+\right).$$

It is easy to see that Theorem 6.13 applies, hence a linkwise domination result can be deduced: if u is the solution of the diffusion problem (Di)–(Cc)–(Tc) and the initial data u_0 verifies $|u_{01}(x)| \leq u_{02}(x)$ for a.e. $x \in (0,1)$, then the solution u satisfies the inequality $|u_1(t,x)| \leq u_2(t,x)$ for a.e. $x \in (0,1)$ and all $t \geq 0$.

(4) It follows from the smoothing property of analytic semigroups that the solution u of the initial value problem associated with (Di)–(Cc)–(Tc) automatically satisfies additional compatibility conditions in the nodes. More precisely, since $T(t)u_0 \in \bigcup_{k\in\mathbb{N}} D(A_k)$ for all $t > 0$, the derivatives of u of even and odd order satisfy a continuity and Kirchhoff-type condition, respectively. If e.g. $c(x) = p(x) \equiv 1$ for a.e. $x \in (0,1)$, then for all $N \in \mathbb{N}$ a direct computation shows that for all $N \in \mathbb{N}$

$$\begin{aligned} u_i^{(2N)}(t, \mathsf{v}_\ell) &= u_j^{(2N)}(t, \mathsf{v}_\ell) =: d_\ell^{u^{(2N)}}(t), \qquad t > 0,\ i,j \in \Gamma(\mathsf{v}_\ell),\ \ell = 1, \dots, n, \\ \sum_{h=1}^n m_{ih} d_h^{u^{(2N)}}(t) &= \sum_{j=1}^m \sum_{j=1}^n \omega_{ij} u_j^{(2N+1)}(t, \mathsf{v}_i), \qquad t \geq 0,\ i = 1, \dots, n, \dots . \end{aligned}$$

(5) It should be observed that while $D(A)$ changes upon reorienting the graph underlying the network, this is not the case for the form domain V. This ensures that all the properties of the system that depend on the energy methods presented in Chapter 6 do not depend on the orientation.

Since A generates an analytic semigroup, it is already known that its spectrum $\sigma(A)$ lies in a sector, and that $\sigma(A) \subset \mathbb{R}$ if M is self-adjoint.

Proposition 7.7. *(1) If G is finite, then $\sigma(A)$ consists of eigenvalues only.*

(2) If $M\mathbf{1} = 0$ and $p_j \equiv 0$ for all j, then 0 is an eigenvalue of A.

(3) If M is positive semidefinite, $p_j(x) \geq 0$ for all j and a.e. $x \in (0,1)$, and 0 is an eigenvalue of A, then $\sum_{i,k\in\mathbb{N}} m_{ik} = 0$.

Proof. (1) If G is finite, then V is compactly embedded in H, and therefore A has compact resolvent. Thus, its point spectrum agrees with $\sigma(A)$.

(2) Observe that $\mathbf{1} \in D(A)$ if and only if $\mathbf{1}$ satisfies the generalised Kirchhoff-type rule (Tc′), i.e., if and only if

$$\sum_{k\in\mathbb{N}} m_{ik} d_k^{\mathbf{1}} = \sum_{k\in\mathbb{N}} m_{ik} = 0, \qquad i \in \mathbb{N}.$$

In this case, it is clear that $A\mathbf{1} = 0$.

(3) Assume now M to be accretive and let $u \in D(A)$ such that $Au = 0$. Then one has

$$0 = (-Au \mid u)_H = a(u,u) = \sum_{j\in\mathbb{N}} \int_0^1 \left(c_j(x)|u_j'(x)|^2 + p_j(x)|u_j(x)|^2\right) dx + (Md^u \mid d^u)_{\ell^2}.$$

If M is accretive, it follows that u is linkwise constant. By the continuity condition (Cc), d_i^u is a constant independent of $i \in \mathbb{N}$ and denoted by δ^u. It follows that

$$0 = (Md^u \mid d^u)_{\ell^2} = \sum_{i,k\in\mathbb{N}} m_{ik} d_k^u \overline{d_i^u} = \delta_u^2 \sum_{i,k\in\mathbb{N}} m_{ik}, \qquad i \in \mathbb{N}.$$

This concludes the proof. ☐

Remark 7.8. If G is finite and M is self-adjoint, then the spectrum of A is real and discrete and, as observed in Remark 6.3, the spectrum of the physical system is related to the numerical range of the associated form a. Studying the distribution of energy levels of a quantum system in a comparatively easy but non-trivial setting has been the main motivation that has led to the introduction of *quantum graphs* in the context of the physical theory of quantum chaos, see [**45, 46, 32**].

In fact, a much more elaborate spectral theory on networks has been performed by many authors in the case of constant coefficients: see e.g. [**29, 30, 34**] for spectral analysis of diffusion operators on finite graphs. The case of the Laplacian on infinite graphs has been considered e.g. in [**44, 47, 48**]. Let me only mention that the spectrum of the Laplacian can be directly related to the spectrum of so-called *admittance* matrix of the graph and, more generally, the spectral properties of elliptic operators can be shown to depend on the topology of the underlying graph. Also observe that, as in the case of domains of $\mathbb{R}^n$ with non-smooth boundary, Kac's famous question "Can one hear the shape of a drum?", addressed in [**49**], (that is, *mutatis mutandis*, "Can one hear the shape of a guitar?") has been answered in the negative in the case of differential operators on networks: in fact, several authors have proposed pairs of non-congruent graphs $\mathsf{G}_1, \mathsf{G}_2$ that are isospectral, i.e., such that the Laplacian on G_1 has the spectrum of the Laplacian on G_2, cf. [**50, 51**]. A beautiful general method for constructing isospectral graphs, relating spectral issues and (geometric) symmetries of a network, has been presented in [**52**].

Observe that there is a direct relation between spectral and asymptotic properties of a diffusion equation, since compact semigroups are known to converge toward a projection at a speed which is given by the second largest nonzero eigenvalue of the generator. In this way it is possible to obtain sharp estimates for the speed of convergence toward equilibrium of diffusion processes on networks, only depending on graph theoretical results, see e.g. [**34**, § 5]. For example, if $M = 0$, then among all networks with n nodes the fastest convergence is attained when the underlying graph is complete, and the slowest when it is a cycle. Further estimates can be obtained in terms of certain parameters that are relevant in graph theory, like the diameter and the edge connectivity parameter of a graph.

Since the coefficient matrix M is a bounded operator on ℓ^2, it generates a group $(e^{tM})_{t\in\mathbb{R}}$. Aim of this section is to show that several features of $(T(t))_{t\geq 0}$ can be characterised by those of $(e^{-tM})_{t\geq 0}$, and hence of M. Material in this section is mostly taken from [**35**]. A characterization of those boundary conditions inducing a positive semigroup has also been obtained in [**48**], in larger generality and in terms of more abstract algebraic notions.

THEOREM 7.9. *Let the functions p_j be real valued. Then the semigroup $(T(t))_{t\geq 0}$ is real if and only if the matrix M has real entries. It is positive if and only if the matrix M has real entries that are negative off-diagonal.*

PROOF. As shown in the proof of Theorem 7.5 a is continuous and H-elliptic. Thus, by Proposition 6.14 $(T(t))_{t\geq 0}$ is real and positive if and only if

- $f \in V \Rightarrow \mathrm{Re} f \in V$ and $a(\mathrm{Re} f, \mathrm{Im} f) \in \mathbb{R}$, and
- $f \in V \Rightarrow (\mathrm{Re} f)^+ \in V,\ a(\mathrm{Re} f, \mathrm{Im} f) \in \mathbb{R},\ a((\mathrm{Re} f)^+, (\mathrm{Re} f)^-) \leq 0,$

If $\phi \in H^1(0,1)$, then it is clear that Re $\phi \in H^1(0,1)$ as well as $(\mathrm{Re}\ \phi)^+ \in H^1(0,1)$. Furthermore, $(\mathrm{Re}\ \phi)' = \mathrm{Re}(\phi')$ and $((\mathrm{Re}\ \phi)^+)' = \mathrm{Re}(\phi')\mathbf{1}_{\{\phi\geq 0\}}$.

By definition, the subspace V contains exactly those functions on the network that are continuous in the nodes. Then for every $f \in V$ one has $\mathrm{Re}(f_j) = (\mathrm{Re} f)_j$, $j \in \mathbb{N}$. It follows from the above arguments that $\mathrm{Re} f \in H^1(0,1;\ell^2)$, and one can see that the continuity of the values attained by f in the nodes is preserved after taking the real part $\mathrm{Re} f$. All in all, $\mathrm{Re} f \in V$. Recall that by assumption the coefficients c_j, p_j are real-valued, positive functions. Since $d^{\mathrm{Re} f}, d^{\mathrm{Im} f} \in \ell^2$, it follows that $a(\mathrm{Re} f, \mathrm{Im} f) \in \mathbb{R}$ if and only if

$$\sum_{i,h\in\mathbb{N}} m_{ih} d_h^{\mathrm{Re} f} d_i^{\mathrm{Im} f} \in \mathbb{R}.$$

This holds for all $f \in V$ if and only if $m_{ih} \in \mathbb{R}$.

Moreover, if $f \in V$, then $((\mathrm{Re} f)^+)_j = (\mathrm{Re}(f_j))^+$, $j \in \mathbb{N}$, and one sees as above that $(\mathrm{Re} f)^+ \in V$. In particular, for all $i \in \mathbb{N}$ there holds

$$d_i^{(\mathrm{Re} f)^+} = \begin{cases} 0 & \text{if } \mathrm{Re} d_i^f \leq 0, \\ \mathrm{Re} d_i^f & \text{if } \mathrm{Re} d_i^f \geq 0, \end{cases} \quad \text{and} \quad d_i^{(\mathrm{Re} f)^-} = \begin{cases} -\mathrm{Re} d_i^f & \text{if } \mathrm{Re} d_i^f \leq 0, \\ 0 & \text{if } \mathrm{Re} d_i^f \geq 0. \end{cases}$$

Accordingly,

$$a((\mathrm{Re} f)^+, (\mathrm{Re} f)^-) \quad = \quad \sum_{i,h \in \mathbb{N}} m_{ih} d_h^{(\mathrm{Re} f)^+} d_i^{(\mathrm{Re} f)^-}.$$

Since the numbers $d_i^{(\mathrm{Re} f)^+}, d_i^{(\mathrm{Re} f)^-}$ are arbitrary, $i \in \mathbb{N}$, one has that $a((\mathrm{Re} f)^+, (\mathrm{Re} f)^-) \leq 0$ for all $f \in V$ if and only if $m_{ih} xy \leq 0$ for all $x, y \geq 0$ and all $i \neq h \in \mathbb{N}$, and the claim follows. □

A result similar to the following, yielding a *sufficient* condition in order that a network diffusion equation (with more genera node conditions than mine) admits L^∞-contractivity has been obtained in [**53**].

Theorem 7.10. *Let $p_j(x) \geq 0$ for all j and a.e. $x \in (0,1)$, M be positive semidefinite, and $M\mathbf{1} \in \ell^2$. If*

$$\mathrm{Re} m_{ii} \geq \sum_{h \neq i} |m_{ih}|, \qquad i \in \mathbb{N}, \tag{7.8}$$

then $(T(t))_{t \geq 0}$ is L^∞-contractive. The converse is true if additionally $p_j \equiv 0$ for all j.

If furthermore also the transpose $M^\top$ satisfies $M^\top \mathbf{1} \in \ell^2$ and (7.8)*, then $(T(t))_{t \geq 0}$ is contractive on all spaces L^p, $1 \leq p \leq \infty$.*

Proof. If M is positive semidefinite, then a is accretive and by Proposition 6.14 the semigroup $(T(t))_{t \geq 0}$ is L^∞-contractive if and only if

$$f \in V \Rightarrow (1 \wedge |f|)\mathrm{sign} f \in V \text{ and } \mathrm{Re} a((1 \wedge |f|)\mathrm{sign} f, (|f| - 1)^+ \mathrm{sign} f) \geq 0.$$

Take $\phi \in V$. Then the functions defined by

$$((1 \wedge |\phi|)\mathrm{sign}\phi)(x) = \begin{cases} \phi(x) & \text{if } |\phi(x)| \leq 1, \\ \frac{\phi(x)}{|\phi(x)|} & \text{if } |\phi(x)| \geq 1, \end{cases}$$

as well as

$$((|\phi| - 1)^+ \mathrm{sign}\phi)(x) = \begin{cases} 0 & \text{if } |\phi(x)| \leq 1, \\ \phi(x) - \frac{\phi(x)}{|\phi(x)|} & \text{if } |\phi(x)| \geq 1 \end{cases}$$

are in $H^1(0,1)$, with $((1 \wedge |\phi|)\mathrm{sign}\phi)' = \phi' \mathbf{1}_{\{|\phi| \leq 1\}}$ and $((|\phi| - 1)^+ \mathrm{sign}\phi)' = \phi' \mathbf{1}_{\{|\phi| \geq 1\}}$.

Accordingly, if $f \in V$, then $(1 \wedge |f|)\mathrm{sign} f \in H^1(0,1;\ell^2)$. As in the proof of Theorem 7.9 the continuity of f in the nodes imposes the same property to the function $(1 \wedge |f|)\mathrm{sign} f$. In fact

$$d_i^{(1 \wedge |f|)\mathrm{sign} f} = (1 \wedge |d_i^f|)\mathrm{sign} d_i^f = \begin{cases} d_i^f & \text{if } |d_i^f| \leq 1, \\ \mathrm{sign} d_i^f & \text{if } |d_i^f| > 1, \end{cases}$$

as well as

$$d_i^{(|f| - 1)^+ \mathrm{sign} f} = (|d_i^f| - 1)^+ \mathrm{sign} d_i^f = \begin{cases} 0 & \text{if } |d_i^f| \leq 1, \\ d_i^f - \mathrm{sign} d_i^f & \text{if } |d_i^f| > 1, \end{cases}$$

for all $i \in \mathbb{N}$. Now a direct computation yields

$$\begin{aligned} a((1\wedge |f|)\mathrm{sign} f, (|f|-1)^+\mathrm{sign} f) &= \sum_{j\in\mathbb{N}} \int_{\{f_j(x)\geq 1\}} p_j(x)(|f_j(x)|-1)dx \\ &\quad + \sum_{i,h\in\mathbb{N}} m_{ih}(1\wedge |d^f_h|)\mathrm{sign} d^f_h \overline{(|d^f_i|-1)^+\mathrm{sign} d^f_i} \\ &= -m((1\wedge |d^f|)\mathrm{sign} d^f, (|d^f|-1)^+\mathrm{sign} d^f), \end{aligned}$$

where m denotes the sesquilinear, accretive form on ℓ^2 associated with the matrix M. Since the nodal values of f are arbitrary, again by Proposition 6.14 one sees that the property of L^∞-contractivity for $(T(t))_{t\geq 0}$ is equivalent to that of ℓ^∞-contractivity for the semigroup $(e^{-tM})_{t\geq 0}$. Now the claim follows by Lemma 7.11 below, since each scalar matrix semigroup admits a modulus. In fact, it is known that the matrix $M^\sharp = (m^\sharp_{ih})$ that generates the modulus semigroup of $(e^{-tM})_{t\geq 0}$ is given by

$$m^\sharp_{ih} := \begin{cases} \mathrm{Re} m_{ii} & \text{if } h = i, \\ |m_{ih}| & \text{if } i \neq h. \end{cases}$$

Let now also $M^\top$ and hence M^* satisfy (7.8). Then $(e^{tM})_{t\geq 0}$ is both ℓ^∞- and ℓ^1-contractive, and reasoning as above so is $(T(t))_{t\geq 0}$. The claim follows by interpolation. □

The following result has been needed in the proof of Theorem 7.10. Its elegant proof is essentially due to W. Arendt, cf. [**35**, Lemma 6.1].

LEMMA 7.11. *Let B be an operator on $L^2(X,\mu)$ such that $\mathbf{1} \in D(B)$. Assume B to generate a semigroup $(S(t))_{t\geq 0}$ which also admits a modulus, i.e., a unique positive semigroup $(S^\sharp(t))_{t\geq 0}$ that is dominated by any other semigroup that also dominates $(S(t))_{t\geq 0}$. Then $(S(t))_{t\geq 0}$ is L^∞-contractive if and only if $B^\sharp\mathbf{1} \leq 0$, where $B^\sharp$ denotes the generator of $(S^\sharp(t))_{t\geq 0}$.*

PROOF. If the semigroup $(S(t))_{t\geq 0}$ is positive, then it is known that $(S(t))_{t\geq 0}$ is L^∞-contractive if and only if $B\mathbf{1} \leq 0$.

Consider now the case of a general B. Let us first assume that $B^\sharp\mathbf{1} \leq 0$ holds. Since $(S^\sharp(t))_{t\geq 0}$ is positive, it is also L^∞-contractive. Now, since $(S^\sharp(t))_{t\geq 0}$ dominates $(S(t))_{t\geq 0}$, it follows that $(S(t))_{t\geq 0}$ is L^∞-contractive as well.

Conversely, let $(S(t))_{t\geq 0}$ be L^∞-contractive. In order to show that $B^\sharp\mathbf{1} \leq 0$ holds, consider the modulus semigroup $(S^\sharp(t)))_{t\geq 0}$, which is positive and, by [**6**, Prop. 2.26], also L^∞-contractive. One can check directly that also the adjoint $(S^\sharp(t)^*)_{t\geq 0}$ of the modulus semigroup dominates the adjoint $(S(t)^*)_{t\geq 0}$, which by duality is L^1-contractive. Now one can check that also the semigroup $(S^\sharp(t)^*)_{t\geq 0}$ is L^1-contractive, and by duality the positive semigroup $(S^\sharp(t))_{t\geq 0}$ is L^∞-contractive. Thus, as in the positive case, one finally obtains $B\mathbf{1} \leq 0$. This concludes the proof. □

In fact, even more can be shown. The technical proof, based on a Nash-type inequality for network functions and on Remark 6.17, is omitted.

THEOREM 7.12. *If both M and its transpose $M^\top$ satisfy (7.8), then $(T(t))_{t\geq 0}$ is ultracontractive of dimension 1. In fact, the semigroup extends to the space $\mathcal{M}$ of Radon measures over the network and maps such a space into*

$$C(G) := \left\{ f \in \big(C[0,1]\big)^m : \begin{array}{l} \exists d^f \in \mathbb{C}^m \text{ s.t.} \\ (\mathcal{I}^+)^\top d^f = f(0) \text{ and } (\mathcal{I}^-)^\top d^f = f(1) \end{array} \right\} \subset D(A)$$

of continuous functions over the network, and it satisfies

$$\|T(t)\nu\|_\infty \leq ct^{-\frac{n}{2}}\|\nu\|_{\mathcal{M}} \quad \text{for all } t \in (0,1] \text{ and all } \mu \in \mathcal{M}. \tag{7.9}$$

Example 7.13. The above discussed energy methods can be also successfully applied to the case on non-parabolic problems. Transmission of electricity along a wire is, to a good approximation, described by the *telegraph equation*

$$\frac{\partial^2 v}{\partial t^2}v(t,x)+\alpha\frac{\partial}{\partial t}v(t,x)=\frac{\partial^2 v}{\partial x^2}(t,x). \tag{7.10}$$

Here, $v(t,x)$ represents electric charge at time t and at point x of the wire. More realistic models should include nonlinear terms, cf. [**14**].

If instead of a single wire one considers an electric distribution network, suitable node conditions are also needed. Using the notations of the previous sections, I will denote by $\mathsf{e}_1,\dots,\mathsf{e}_m$ the individual power lines, and by $\mathsf{v}_1,\dots,\mathsf{v}_n$ the node points. These will represent network branching points, generators (e.g, power plants), and final ends (e.g., consumers). More or less realistically, they are assumed to inject into the system, or to absorb from it, respectively, a quantity of electric charge which is related to the available current in accordance to a given allowed proportion. I wish to let the generation/absorption features depend on the load $d_1^v,\dots,d_n^v$ at each end $\mathsf{v}_1,\dots,\mathsf{v}_n$ of the whole network. In order to simplify the model generators are assumed to be able estimate the network's load – and to react to it – with a very short delay. I thus neglect such delay effect and in each generator node v_i consider the general Kirchhoff-type conditions

$$\sum_{j=1}^m \iota_{ij}v_j'(t,\mathsf{v}_i)=\sum_{h=1}^n m_{ih}d_h^v(t).$$

Observe that although the standard telegraph equation has finite speed of propagation, reasoning as in [**54**] one deduces that the considered network wave equation has in fact infinite speed of propagation. In order to discuss the solvability of the system, observe that for arbitrary coefficients m_{ih} one has $|\mathrm{Im}a(f,f)|\le\|M\|\|f\|^2_{C(G)}\le\|M\|\|f\|_V\|f\|_H$ and apply Proposition 6.10. Thus, the system is (backward as well as forward) well-posed in the sense of Proposition 6.10.

Observe that if $m_h=0$ for all h, and hence if the node conditions are of purely Kirchhoff type, an explicit formula for the solution to the (undamped) wave equation, i.e., for (7.10) with $\alpha=0$, has been obtained in [**55**]. A formula for the general case of $m_h\neq 0$ could probably be obtained by combining the ideas of [**55**] and of [**56**]. Then, in the general case of $\alpha\neq 0$ the solution to the problem can be (at least theoretically) obtained by the Dyson–Phillips series, cf. [**5**, Cor. III.3.2]. □

CHAPTER 8

An alternative setting

What happens if node conditions that differ from those imposed so far are considered? In this section I consider a set of node conditions that is in some sense dual to that considered so far in this section. The material presented in this section has been inspired by several discussions with Rainer Nagel and Stefano Cardanobile.

Throughout Chapter 7, continuity of the unknown function u in the nodes along with a Kirchhoff-type rule for the associated flux have been assumed. Let me now discuss the case of a continuity condition imposed on the flux in the nodes, along with a Kirchhoff-type rule for the unknown itself – these have sometimes been called *Anti-Kirchhoff node conditions* in the literature. Observe that the latter conditions bear certain similarities with the node condition imposed in the case of transport equations.

REMARK 8.1. Such node conditions have abstractly arisen in a classification program pursued in [**33**], and have been discussed in detail in [**57**] in the case of an outbound star. An interesting relation between the trace of the semigroups generated by the Laplacian with Kirchhoff and anti-Kirchhoff node conditions and the Euler characteristic of the underlying graph has been proved in [**43**] .

More precisely, let us complement the system of diffusion equations (Di) by a continuity assumption

$$u'_j(t,\mathsf{v}_i) = u'_\ell(t,\mathsf{v}_i) =: d_i^{u'}(t), \qquad j,\ell \in \Gamma(\mathsf{v}_i),\ i\in\mathbb{N},\ t>0, \tag{Cc'}$$

on the derivative in the nodes. In order to determine a suitable second set of boundary condition, let me proceed heuristically. Consider a network G consisting of the union of two links $\mathsf{e}_1, \mathsf{e}_2$ with one joint node v_2, and a diffusion process taking place on the graph described by a differential operator $A(f_1,f_2) := (f_1'', f_2'')$. For the sake of simplicity, let me impose Dirichlet boundary conditions on the boundary nodes $\mathsf{v}_1, \mathsf{v}_3$. Consider $H := L^2(0,1;\mathbb{C}^2)$ as state space. If functions in the domain of A satisfy (Cc′), then taking $f \in D(A)$, $g \in H$, and integrating by parts one obtains

$$\begin{aligned}(Af|g)_H &= \int_0^1 \left(f_1''(x)\overline{g_1(x)} + f_2''(x)\overline{g_2(x)}\right)dx \\ &= \left[f_1'\overline{g_1}\right]_0^1 + \left[f_2'\overline{g_2}\right]_0^1 - \int_0^1 \left(f_1'(x)\overline{g_1'(x)} + f_2'(x)\overline{g_2'(x)}\right)dx.\end{aligned}$$

Now, two cases are possible.

(1) If G is orientedly Eulerian, say $\mathsf{e}_1(1) = \mathsf{v}_2 = \mathsf{e}_2(0)$, then $f_1'(1) = f_2'(0) =: d_2^{f'}$. Accordingly, and due to Dirichlet boundary conditions, one has

$$\begin{aligned}(Af|g)_H &= f_1'(1)\overline{g_1(1)} - f_2'(0)\overline{g_2(0)} - \int_0^1 \left(f_1'(x)\overline{g_1'(x)} + f_2'(x)\overline{g_2'(x)}\right)dx \\ &= d_2^{f'}\left(\overline{g_1(1)} - \overline{g_2(0)}\right) - \int_0^1 \left(f_1'(x)\overline{g_1'(x)} + f_2'(x)\overline{g_2'(x)}\right)dx.\end{aligned}$$

If $f = g$, the latter term represents the energy of the system and it is highly desirable to let it agree with $(Af|f)_H$. In fact, there holds $(Af|g)_H = -\int_0^1 \left(f_1'(x)\overline{g_1'(x)} + f_2'(x)\overline{g_2'(x)}\right) dx$ if and only if $g_1(1) = g_2(0)$.

(2) Let now G be orientedly bipartite, so that for example $\mathsf{e}_1(1) = \mathsf{v}_2 = \mathsf{e}_2(1)$. Then $f_1'(1) = f_2'(1) =: d_2^{f'}$. Again due to Dirichlet boundary conditions one has

$$\begin{aligned}(Af|g)_H &= f_1'(1)\overline{g_1(1)} + f_2'(1)\overline{g_2(1)} - \int_0^1 \left(f_1'(x)\overline{g_1'(x)} + f_2'(x)\overline{g_2'(x)}\right) dx \\ &= d_2^{f'}\left(\overline{g_1(1)} + \overline{g_2(1)}\right) - \int_0^1 \left(f_1'(x)\overline{g_1'(x)} + f_2'(x)\overline{g_2'(x)}\right) dx.\end{aligned}$$

Thus, there holds $(Af|g)_H = -\int_0^1 \left(f_1'(x)\overline{g_1'(x)} + f_2'(x)\overline{g_2'(x)}\right) dx$ if and only if $g_1(1) = -g_2(1)$.

This motivates to imposing a set of Kirchhoff-like transmission conditions given by

(Tc') $$\sum_{j\in\mathbb{N}} \iota_{ij} c_j(\mathsf{v}_i) u_j(t, \mathsf{v}_i) = 0, \qquad i \in \mathbb{N},\ t > 0.$$

Like in Chapter 7, (Cc$'$) can be rewritten as

$$\exists d^{u'} \in \ell^2 \text{ s.t. } (\mathcal{I}^+)^\top d^{u'} = u'(t,0) \text{ and } (\mathcal{I}^-)^\top d^{u'} = u'(t,1) \text{ for all } t \geq 0,$$

while (Tc$'$) becomes

$$\Omega^+ u(t,0) = \Omega^- u(t,1) \text{ for all } t \geq 0.$$

It is possible to mimic the techniques applied in Chapter 7 in order to discuss well-posedness and further properties of such a system.

Consider the space of H^1-functions that satisfy the transmission conditions (Tc$'$) over the network, i.e.,

(8.1) $$V := \left\{ f \in H^1(0,1;\ell^2) : \Omega^+ f(0) = \Omega^- f(1) \right\}.$$

It becomes a Hilbert space with inner product

$$(f \mid g)_V := \sum_{j\in\mathbb{N}} \int_0^1 f_j'(x)\overline{g_j'(x)} + f_j(x)\overline{g_j(x)}\, dx, \qquad f, g \in V.$$

Observe that $H^1_0(0,1;\ell^2) \subset V$, hence in particular V is densely embedded in H – the embedding is also compact if and only if the network is in fact finite, i.e., if ℓ^2 is replaced by $\mathbb{C}^m$. In sharp contrast to the space defined in (7.3), the space V in (8.1) apparently depends on the structure of the oriented graph and also on the nodal values of the diffusion coefficient c that appear in the matrices Ω^+, Ω^-. Taking into account Theorem 6.13, one can thus expect that also qualitative properties of the diffusion process will depend on the orientation of the graph. This seemingly makes the diffusion anisotropic.

Example 8.2. Assume for the sake of simplicity that c attains a unique (strictly positive) value c_0 in each node of G, i.e., $\Omega = c_0\mathcal{I}$.

(1) If G consists of a single interval, then $V = H^1_0(0,1)$. Thus, the setting presented here seems to be the correct extension of diffusion processes with Dirichlet boundary conditions to the network case.

(2) Let now G be a triangle, i.e., a complete graph with three nodes and no multiple link. Up to permutations, only two possible parametrizations are possible on G: either an incoming link and an outgoing one are incident to each node, or to exactly one, depending on whether G is

orientedly Eulerian or not. In the first case, each pair of functions defined on adjacent links attains a joint value in the common node, and an equivalent inner product on V is given by

$$\begin{aligned}((f \mid g))_V \quad &:= \quad \int_0^1 \left(f_1'(x)\overline{g_j'(x)} + f_2'(x)\overline{g_2'(x)} + f_3'(x)\overline{g_3'(x)}\right) dx \\ &\quad + f(\mathsf{v}_1)\overline{g(\mathsf{v}_1)} + f(\mathsf{v}_2)\overline{g(\mathsf{v}_2)} + f(\mathsf{v}_3)\overline{g(\mathsf{v}_3)}, \qquad f, g \in V.\end{aligned}$$

The function of constant value 1 belongs to V if and only if G is orientedly Eulerian.

(3) More generally, let the space V be constructed over an orientedly Eulerian graph G. Let f be a function on G which is of class H^1 over each link, and number its individual components following an Eulerian tour. If all pairs of successive components attain a joint value at the common endpoint, then $f \in V$. In particular, the function of constant value 1 belongs to V. Moreover, by Theorem 2.9 each node v_i has even degree $2N_i$. Denoting by $f(\mathsf{v}_i)_1, \ldots, f(\mathsf{v}_i)_{N_i}$ (with $N_i = \infty$, possibly) all the node values attained at v_i by pairs of successive components of f (each with multiplicity ≥ 2), an equivalent inner product on V is given by

$$((f \mid g))_V := \sum_{j \in \mathbb{N}} \int_0^1 f_j'(x)\overline{g_j'(x)}dx + \sum_{i \in \mathbb{N}} \sum_{k=1}^{N_i} f(\mathsf{v}_i)_k \, \overline{g(\mathsf{v}_i)_k}, \qquad f, g \in V.$$

(4) If the oriented graph G is a cycle, then each node v_i has degree 2, and V is simply the space of H^1-functions that are continuous over the graph, i.e., it agrees with the space introduced in (7.3). An equivalent inner product on V is given by

$$((f \mid g))_V := \sum_{j \in \mathbb{N}} \int_0^1 f_j'(x)\overline{g_j'(x)}dx + \sum_{i \in \mathbb{N}} f(\mathsf{v}_i)_k \, \overline{g(\mathsf{v}_i)_k}, \qquad f, g \in V.$$

(5) Let finally the space V be constructed over an orientedly bipartite graph G. Then no function whose node values are all equal belongs to V; in particular, $1 \notin V$.

□

Introduce a sesquilinear form $a : V \times V \to \mathbb{C}$ given by

$$a(f, g) := \sum_{j \in \mathbb{N}} \int_0^1 c_j(x) f_j'(x)\overline{g_j'(x)}dx. \tag{8.2}$$

In order to relate the above form with the motivating network diffusion problem (Cc′)–(Tc′), the following still has to be proved.

LEMMA 8.3. *The operator associated with the form a is given by*

$$Af := \left((c_1 f_1')', \ldots, (c_m(f_m'))'\right)^\top$$

with domain

$$D(A) := \left\{ f \in H^2(0,1;\ell^2) : \begin{array}{c} \exists d^f \in \ell^2 \text{ s.t.} \\ (\mathcal{I}^+)^\top d^{f'} = f'(0), \ (\mathcal{I}^-)^\top d^{f'} = f'(1), \\ \textit{and } \Omega^+ f(0) = \Omega^- f(1) \end{array} \right\}.$$

PROOF. Denote by B the operator associated with a, i.e.,

$$\begin{aligned} D(B) \quad &:= \quad \left\{ f \in V : \exists h \in H \text{ s.t. } a(f,g) = (h \mid g)_H \ \forall g \in V \right\}, \\ Bf \quad &:= \quad -h. \end{aligned}$$

In order to show that $A \subset B$, take $u \in D(A)$ and observe that for all $v \in V$

$$\begin{aligned} a(u,v) &= \sum_{j\in\mathbb{N}} \int_0^1 c_j(x) v_j'(x)\overline{v_j'(x)}dx \\ &= \sum_{j\in\mathbb{N}} \Big[c_j u_j' v_j\Big]_0^1 - \sum_{j\in\mathbb{N}} \int_0^1 (c_j u_j')'(x)\overline{v_j(x)}dx \\ &= \sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}} \iota_{ij} c_j(\mathsf{v}_i) u_j'(\mathsf{v}_i)\overline{v_j(\mathsf{v}_i)} - \sum_{j\in\mathbb{N}} \int_0^1 (c_j u_j')'(x)\overline{v_j(x)}dx \\ &= \sum_{i\in\mathbb{N}} d_i^{u'} \sum_{j\in\mathbb{N}} \iota_{ij} c_j(\mathsf{v}_i)\overline{v_j(\mathsf{v}_i)} - \sum_{j\in\mathbb{N}} \int_0^1 (c_j u_j')'(x)\overline{v_j(x)}dx \\ &= \sum_{i\in\mathbb{N}} d_i^{u'} \sum_{j\in\mathbb{N}} \overline{\omega_{ij} v_j(\mathsf{v}_i)} - \sum_{j\in\mathbb{N}} \int_0^1 (c_j u_j')'(x)\overline{v_j(x)}dx. \end{aligned}$$

Using the Kirchhoff-type law on h, which holds for all elements of V, one finally obtains

$$a(u,v) = -\sum_{j\in\mathbb{N}} \int_0^1 (c_j u_j')'(x)\overline{v_j(x)}dx = (w \mid v)_H, \tag{8.3}$$

with $w = -Au \in H$. The proof of the inclusion $A \subset B$ is completed.

To check the converse inclusion $B \subset A$, let $u \in V$ such that (8.3) holds for all $v \in V$ and some $w \in H$. As in the proof of Lemma 7.4 one can see that $c_j \cdot u_j' \in H^1(0,1)$ for all $j \in \mathbb{N}$, and in fact that $u \in H^2(0,1;\ell^2)$ and $(c_j u_j')' = w_j$. Moreover, since (8.3) holds for arbitrary $v \in V$, repeating the lenghty integration by parts performed in the proof of the converse inclusion one sees that

$$\sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}} \iota_{ij} c_j(\mathsf{v}_i) u_j'(\mathsf{v}_i)\overline{v_j(\mathsf{v}_i)} = 0$$

for all $v \in V$. Since $v \in V$, there also holds

$$\sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}} \iota_{ij} c_j(\mathsf{v}_i)\overline{v_j(\mathsf{v}_i)} = 0.$$

It remains to show that for all $i \in \mathbb{N}$ the boundary values $u_j(\mathsf{v}_i)$ are pairwise equal, $j \in \mathbb{N}$. Now, let $i \in \mathbb{N}$ and pick $v \in V$ in such a way that only $\omega_{i1}v_1(\mathsf{v}_i), \omega_{i2}v_2(\mathsf{v}_i)$ are non-zero. One concludes that the 2-vector

$$\begin{pmatrix}\omega_{i1}v_1(\mathsf{v}_i)\\ \omega_{i2}v_2(\mathsf{v}_i)\end{pmatrix} \text{ is orthogonal to } \begin{pmatrix}u_1'(\mathsf{v}_i)\\ u_2'(\mathsf{v}_i)\end{pmatrix} \text{ as well as to } \begin{pmatrix}1\\1\end{pmatrix}.$$

This promptly yields that $u_1'(\mathsf{v}_i) = u_2'(\mathsf{v}_i)$. One can similarly show that $u_2'(\mathsf{v}_i) = u_3'(\mathsf{v}_i)$. The proof is concluded by repeating the same argument countably many times. □

Computations similar to those performed in Theorem 7.5 promptly show that the following holds.

COROLLARY 8.4. *The operator A defined in Lemma 8.3 generates a strongly continuous, self-adjoint, bounded analytic semigroup $(T(t))_{t\geq 0}$ on H, which is given by an integral kernel. Such a semigroup is also compact if and only if G is finite.*

REMARK 8.5. Reasoning as in Remark 7.6.(3), one sees that the solution u of the initial value problem associated with (Di)–(Cc′)–(Tc′) automatically satisfies in the nodes the additional compatibility conditions

$$\begin{aligned} u_i^{(2N+1)}(t,\mathsf{v}_\ell) &= u_j^{(2N+1)}(t,\mathsf{v}_\ell) =: d_\ell^{u^{(2N+1)}}(t), \qquad t>0,\ i,j\in\Gamma(\mathsf{v}_\ell),\ \ell=1,\dots,n,\\ 0 &= \sum_{j=1}^m \sum_{j=1}^n \omega_{ij} u_j^{(2N+1)}(t,\mathsf{v}_i), \qquad t\ge 0,\ i=1,\dots,n,\dots. \end{aligned}$$

As already emphasized, a striking difference between the setting considered here and that presented in Chapter 7 is that most properties of the semigroup generated by A depend on the orientation of the graph. The following result is exemplary.

PROPOSITION 8.6. *Assume c to attain a unique value $c_0 > 0$ in each node of G. Then the semigroup $(T(t))_{t\ge 0}$ is real. The following assertions also hold.*

(1) Let G be a cycle. Then $(T(t))_{t\ge 0}$ is positive and L^∞-contractive.
(2) There exists an orientedly Eulerian graph G such that $(T(t))_{t\ge 0}$ is neither positive, nor L^∞-contractive.
(3) Assume G to be oriented in such a way that a node v_i with $\Gamma(\mathsf{v}_i)\ge 3$ and either vanishing indegree or vanishing outdegree exists. Then $(T(t))_{t\ge 0}$ is neither positive, nor L^∞-contractive.

PROOF. The proof is a direct application of Proposition 6.14. Due to the assumptions on c and because $a(u,v)=0$ whenever $u,v\in V$ have disjoint support, it is clear that the only conditions one has to check is whether for all $f\in V$ one has $\mathrm{Re}f$, $\mathrm{Re}f^+$, or $(1\wedge|f|)\mathrm{sign}f\in V$. More precisely, one has to check that the node conditions in V are satisfied, since reasoning as in the proof of Theorem 7.9 one sees that all these functions are of class H^1 over each link.

Let then $f\in V$ and $\iota_{ij}\ne 0$. Then $\sum_{j\in\mathbb{N}}\iota_{ij}c_j(\mathsf{v}_i)f_j(\mathsf{v}_i)=0$ for all $i\in\mathbb{N}$, and accordingly also $\sum_{j\in\mathbb{N}}\iota_{ij}c_j(\mathsf{v}_i)\mathrm{Re}f_j(\mathsf{v}_i)=\mathrm{Re}\sum_{j\in\mathbb{N}}\iota_{ij}c_j(\mathsf{v}_i)f_j(\mathsf{v}_i)=0$.

(1) By the considerations expressed in Example 8.2.(5), the claimed positivity and L^∞-contractivity result can be seen as special case of Proposition 7.9.

(2) Consider a butterfly-shaped graph G, i.e. a graph with 4 nodes with degree 2 and 1 central node v with degree 4 – say, with incoming and outgoing links $\mathsf{e}_1,\mathsf{e}_2$ and $\mathsf{e}_3,\mathsf{e}_4$, respectively. Let f be a linkwise smooth function that attains value 0 in the 4 lateral nodes and in the central one satisfies $f_1(\mathsf{v})=f_2(\mathsf{v})=1$, $f_3(\mathsf{v})=3$, $f_4(\mathsf{v})=-1$. Although f satisfies the Kirchhoff-type rule in v, it is clear that neither f^+ does, nor $(1\wedge|f|)\mathrm{sign}f$.

(3) Assume that all links incident in v are outgoing, i.e., $\iota_{ij}=1$ for all $j\in\Gamma(\mathsf{v}_i)=\{i_1,\dots,i_N\}$ (with $i_N=\infty$, possibly). Consider a linkwise smooth function that attains value 0 in all the remaining nodes, and with $f_{i_1}(\mathsf{v}_i)=f_{i_2}(\mathsf{v}_2)=1$, $f_{i_3}(\mathsf{v}_3)=-2$, $f_{i_4}(\mathsf{v}_4)=\ldots=f_{i_N}(\mathsf{v}_N)=0$. While clearly $f\in V$, one sees that neither f^+ does, nor $(1\wedge|f|)\mathrm{sign}f$. □

While a system of the above type is mathematically fine-tuned, it is still quite unclear what possible applications may exist for the above results. In other words:

PROBLEM 8.7. *Are there concrete systems (coming from physics, information theory, mechanics, or other) described by a system of equations that is solved by the semigroup $(T(t))_{t\ge 0}$ from Corollary 8.4?*

CHAPTER 9

Invariance properties

It is sometimes useful to discuss invariance of closed linear subspaces of the state space under the action of a semigroup. Moreover, invariance result for subspaces can be directly extended to a large class of nonlinear, strip-like subsets of H. In the following Π will denote an orthogonal projection on a Hilbert space H onto a closed subspace Y, and by $\mathcal{C}_{P,\alpha}$ the closed convex subset of H defined as the strip around Y of thickness 2α, i.e.,

$$\mathcal{C}_{P,\alpha} := \left\{ f \in H : \|f - \Pi f\| \leq \alpha \right\}.$$

The following result is a consequence of Theorem 6.13. It has been established in [**58**, § 5].

PROPOSITION 9.1. *Let $a : V \times V \to \mathbb{C}$ be an H-elliptic, continuous sesquilinear form, and denote by $(T(t))_{t\geq 0}$ the associated semigroup. Let Π be the orthogonal projection of H onto some closed subspace Y. Consider the following assertions.*

(a) $\mathcal{C}_{P,\alpha}$ is invariant under $(T(t))_{t\geq 0}$ for all $\alpha > 0$.
(b) $\mathcal{C}_{P,\beta}$ is invariant under $(T(t))_{t\geq 0}$ for some $M > 0$.
(c) Y is invariant under $(T(t))_{t\geq 0}$.
(d) There holds
(i) $\Pi V \subset V$ and
(ii) $a(g,h) = 0$ for all $g \in V \cap Y, h \in V \cap Y^\perp$.

Then (a)$\iff$(b)$\implies$(c)$\iff$(d). If $(T(t))_{t\geq 0}$ is contractive, then also (c) $\implies$ (b) holds.

Of course, condition (d.ii) above can be rephrased as $a(Pf, f) = a(Pf, Pf)$ for all $f \in V$.

REMARK 9.2. It is sometimes interesting to formulate invariance properties for rough initial data – say, for a Dirac delta or, more generally, for a measure.

Let Ω be an open bounded subset of $\mathbb{R}^n$. Consider a self-adjoint, ultracontractive semigroup $(T(t))_{t\geq 0}$ mapping $L^2(\Omega)$ into $C(\overline{\Omega})$. Let Y be a subspace of $L^2(\Omega)$ that is invariant under $(T(t))_{t\geq 0}$. As already remarked, the semigroup leaves $C(\overline{\Omega})$ invariant, hence also $Y \cap C(\overline{\Omega})$. A subspace of the space of Radon measure $\mathcal{M}(\Omega)$ can be naturally introduced as

$$Y_{\mathcal{M}} := \left\{ \mu \in \mathcal{M}(\Omega) : \int_\Omega f d\mu = 0 \text{ for all } f \in Y^\perp \cap C(\overline{\Omega}) \right\}.$$

Since $(T(t))_{t\geq 0}$ leaves Y invariant, one has

$$\int_\Omega f dT(t)\mu = \int_\Omega T(t) f d\mu = 0, \qquad t \geq 0,$$

for all $\mu \in Y_{\mathcal{M}}$ and all $f \in Y \cap C(\overline{\Omega})$, i.e., the extrapolated semigroup acting on $\mathcal{M}(\Omega)$ leaves $Y_{\mathcal{M}}$ invariant, too.

EXAMPLE 9.3. Consider the setting introduced in Example 6.7. As an example of application of Proposition 9.1, let me consider the subspace $Y \subset H$ of all functions with constant value over Ω. The projection of H onto Y is given by

$$(\Pi f)(x) \equiv \frac{1}{|\Omega|}\int_\Omega f d\lambda.$$

One can see that $\Pi V \not\subset V$, and therefore by Proposition 9.1 conclude that Y is not left invariant by the semigroup generated by the Laplacian with Dirichlet boundary conditions. (Of course, this can also be deduced by the fact that $T(t)$ maps Y into $D(A) \subset H^1_0(\Omega)$ for all $t > 0$, the only constant function contained in $H^1_0(\Omega)$ being 0.)

If furthermore Ω is a ball centered in the origin consider the subspace $Y \subset H$ of all radial functions, i.e., of all functions such that $u(x) = u(y)$ whenever $|x| = |y|$. One sees that the projection Π of H onto Y maps each function f in the function Πf whose value in x is the average value (in the sense of Hausdorff) of f over the sphere of radius $|x|$. Then one can deduce from Proposition 9.1 that Y is left invariant by the action of $T(t)$, $t \geq 0$. (Another equivalent, more usual way to prove this property is based on an application of the Laplacian's well-known invariance under rotations.) □

EXAMPLE 9.4. Let a be an H-elliptic continuous sesquilinear form with dense domain V and denote by $(T(t))_{t\geq 0}$ the associated semigroup and by A its generator. Let $0 \neq \phi \in H$ and consider the one-dimensional subspace Y spanned by ϕ: of course, the orthogonal projection onto Y is given by

$$Pf := (\phi|f)_H \phi, \qquad f \in H.$$

On one hand, a direct application of Proposition 9.1 shows that Y is invariant under the action of $(T(t))_{t\geq 0}$ if and only if $\phi \in V$ and

$$|(\phi|f)_H|^2 a(\phi,\phi) = a(Pf,Pf) = a(Pf,f) = (\phi|f)_H a(\phi,f), \qquad f \in V. \tag{9.1}$$

On the other hand, $T(t)Y \subset Y$ for all $t \geq 0$ if and only if there exists a function $\Lambda : \mathbb{R}_+ \to \mathbb{R}$ such that $T(t)\phi = \Lambda(t)\phi$, i.e., for all $t \geq 0$ ϕ is an eigenfunction of $T(t)$ with associated eigenvalue $\Lambda(t)$. One sees that Λ has to satisfy $\Lambda(t+s) = \Lambda(t)\Lambda(s)$ for all $t,s \geq 0$, and therefore Λ is necessarily an exponential function, say $\Lambda(t) = e^{\lambda t}$ for some $\lambda \in \mathbb{R}$. By [**5**, Cor. IV.3.8] there now holds

$$\mathrm{Ker}(\lambda - A) = \bigcap_{t\geq 0} \mathrm{Ker}(e^{\lambda t} - T(t)) \supset Y,$$

i.e., condition (9.1) is satisfied if and only if ϕ is an eigenfunction of A (with associated eigenvalue λ). Of course, this result is relevant in the case where the mathematical problem already has a variational formulation, and thus we know a and V but not necessarily A and/or its domain. □

Let me mention a tight relation between invariance of linear subspaces and the notion of "symmetry", which has been ubiquitous in the classical and quantum field theories (and more generally in theoretical physics) over the last century, beginning with the celebrated Noether's Theorem, cf. [**59**].

THEOREM 9.5 (Noether 1918). *Let G be a Lie group of dimension r acting on the phase space in terms of symmetries. Then there exist r independent conserved quantities.*

Although this result is directly applicable to classical systems, a tentative interpretation in terms of quantum mechanics and Schrödinger equations on networks has been proposed in [**42**, § 5.2] and [**60**].

In mathematical physics, and in particular in the Lagrangian formulation of classical field theory, one says that a given physical system exhibits a *symmetry* if some of its properties remain invariant under the action of a certain class of transformations. More precisely, one says that there exists a (global) symmetry of a given dynamical system if the Lagrange functional $\mathcal{L}(\phi)$ of the field ϕ is invariant under all (time- and space-independent) transformations O that belong to a (Lie) group $\mathcal{O}$, the so-called *gauge group* of the system, i.e., if $\mathcal{L}(\phi) = \mathcal{L}(O\phi)$. Such a Lagrange functional is usually obtained by applying a suitable multiplier to both sides of an evolution equations, and then integrating over space-time. Symmetry properties of systems of different kind can in fact be compared: in particular, I will discuss system of equations of diffusion, wave, and Schrödinger type on a network.

If for example A is a self-adjoint operator with associated sesquilinear form a, for parabolic problems associated with A the Lagrange functional is

$$\mathcal{L}(\phi) := \int_0^T \Big[(\phi(t)|\dot\phi(t))_H + a(\phi(t),\phi(t))\Big]dt, \tag{9.2}$$

whereas for a Schrödinger equation (possibly with electromagnetic potential, see Chapter 13) it is given by

$$\mathcal{L}(\phi) := \int_0^T \Big[i(\phi(t)|\dot\phi(t)) - a(\phi(t),\phi(t))\Big]dt, \tag{9.3}$$

and finally in the case of a wave equation with damping term α analogous to (7.10) it is defined by

$$\mathcal{L}(\phi) := \int_0^T \Big[\|\dot\phi(t)\|_H^2 - a(\phi(t),\phi(t)) - \alpha(\dot\phi(t)|\phi(t))_H)\Big]dt, \tag{9.4}$$

for $\phi \in C^1([0,T],H)\cap C([0,T],V)$.

Example 9.6. A prototypical example is the invariance under rotations of the Laplacian on $\mathbb{R}^n$, which has been shortly discussed in Example 9.3. This implies a symmetry for the heat, the Schrödinger, and the wave equations in $\mathbb{R}^n$, which is associated with the orthogonal group $\mathcal{O}=O_n$. In classical field theory, this is e.g. the reason for conservation of angular momentum. Another well-known symmetry is associated with the transformations given by a phase rotation of the value of a vector field. The associated *gauge group* is in this case the unitary group $\mathcal{O}=U(1)$, which in quantum mechanics yields conservation of electrical charge. □

Since each of the symmetries $O\in\mathcal{O}$ is time independent and thus commutes with the time derivative, in many relevant cases O defines a symmetry for the evolutionary problem if and only if it is a symmetry for the stationary one, i.e., if and only if the sesquilinear form satisfies $a(\phi,\phi)=a(O\phi,O\phi)$ for all states ϕ. In particular, consider an orthogonal projection Π on the Hilbert space H. Then, since K and Π are self-adjoint, taking into account Stone's theorem it is then natural to consider the unitary groups $(e^{isP_K})_{s\in R}$ and $(e^{isK})_{s\in R}$. The latter is in fact a (compact, simply connected) matrix Lie group of dimension 1 (see [**61**]) and $e^{isK}\mapsto e^{isP_K}$ is a unitary Lie group representation. By the general theory of Lie groups we know in particular that this representation is completely reducible, i.e., it is the direct sum or irreducible representations. In particular, the trajectories of the system are confined to a submanifold of the phase space H that is defined by the conserved quantities to take fixed values. This allows to reduce the complexity of the problem by discussing the behaviour on the system on (more) submanifold is of lower dimension.

Remark 9.7. Observe that due to the development of the group $(e^{z\Pi})_{z\in\mathbb{C}}$ as a power series, we have for all projections Π

$$e^{z\Pi} = \sum_{k\in\mathbb{N}} \frac{z^k}{k!}\Pi^k = K^0 + \sum_{k=1}^\infty \frac{z^k}{k!}\Pi^k = K^0 + \sum_{k=1}^\infty \frac{z^k}{k!}\Pi = K^0 + (e^z-1)\Pi = e^z\Pi + \Pi^\perp, \qquad z\in\mathbb{C},$$

where $\Pi^\perp$ denotes the orthogonal projection onto $\mathrm{Ker}(\Pi)$.

Thus, for all $s\in\mathbb{R}$ and all the Lagrangian functionals $\mathcal{L}$ considered above, one actually has that $a(\phi,\phi)=a(e^{is\Pi}\phi,e^{is\Pi}\phi)$ if and only if $\mathcal{L}(\phi)=\mathcal{L}(e^{is\Pi}\phi)$: this will be the (somewhat casual) definition of symmetry throughout the following. Thus, $(e^{is\Pi})_{s\in\mathbb{R}}$ can be considered as a a gauge group for the above systems. It has can be shown as in [**42**, § 5.2] that if Π is an orthogonal projection on H onto some closed subspace Y, then in fact $a(\phi,\phi)=a(e^{is\Pi}\phi,e^{is\Pi}\phi)$ if and only if Y is invariant under the solution families of any/all of the above evolution problems.

PROPOSITION 9.8. *Let* Π *be an orthogonal projection of the Hilbert space* H *onto a closed subspace* Y *and* a *be a self-adjoint accretive sesquilinear form with associated operator* A*. Then the following assertions are equivalent.*

(a) Π *define a symmetry of the network parabolic problem whose Lagrange functional is defined in* (9.2)*.*
(b) Π *define a symmetry of the quantum graph whose Lagrange functional is defined in* (9.2)*.*
(c) Π *define a symmetry of the network system of damped wave equations whose Lagrange functional is defined in* (9.2)*.*
(d) Y *is invariant under the* C_0*-semigroup generated by* A*.*
(e) Y *is invariant under the unitary group generated by* iA*.*
(f) $Y \times Y$ *is invariant under the* C_0*-group generated on* $V \times H$ *by*

$$\begin{pmatrix} 0 & I \\ A & -\alpha I \end{pmatrix} \qquad \text{with domain } D(A) \times V, \tag{9.5}$$

provided that Y *is a closed subspace of* V*, too.*

Observe that by Proposition 9.1 each of the above listed properties can be characterised in terms of properties of a and V.

PROOF. First of all, observe that the resolvent at point λ of the operator defined in (9.5) is given by

$$\begin{pmatrix} \lambda R(\lambda^2, A) & R(\lambda^2, A) \\ AR(\lambda^2, A) & \lambda R(\lambda^2, A) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & (I + \alpha\lambda R(\lambda^2, A))^{-1} \end{pmatrix}$$

for real λ large enough. One sees that $Y \times Y$ is invariant under this operator if and only if Y is invariant under $R(\lambda^2, Y)$ for real λ large enough. The equivalence of (a), (b), and (c) now follows from the possibility of developing both $R(\lambda^2, A)$ and $R(i\lambda, A)$ as a power series of $R(\lambda, A)$, for arbitrary real λ with $\mathrm{Re}\lambda > 0$ (and viceversa). The equivalence of (a), (b), and (c) with (d), (e), and (f) can now be showed mimicking the proof of [**42**, Prop. 5.3]. □

CHAPTER 10

Symmetry properties for network equations

Consider a graph G consisting of two links, both outgoing from a common node v, and consider the setting which has been introduced in Chapter 7. Denoting by $u_0 = (u_{01}, u_{02})$ the initial data of a diffusion problem on the network, does the solution u to the problem with initial data u_0 satisfy $\|u_1(t) - u_2(t)\|_H \leq \|u_{01} - u_{02}\|_H$ for all $t > 0$? In a certain sense, this property represents a form of *synchronization* (or at least of non-desynchronization), and can be rephrased as: Is the set $\left\{f \in H : \|f - \Pi f\| \leq \epsilon\right\}$ invariant under the semigroup that governs the problem for some $\epsilon > 0$? Here Π is the orthogonal projection on $H = L^2(0,1;\mathbb{C}^2)$ defined by

$$\Pi f(x) := \frac{1}{2}\begin{pmatrix} f_1(x) - f_2(x) \\ f_2(x) - f_1(x) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \qquad \text{for a.e. } x \in (0,1).$$

By Proposition 9.1 this is the case if and only if the subspace $Y := \operatorname{Range}\Pi$ is invariant under the semigroup that governs the diffusion problem. Because of the interplay between physical symmetry properties and invariances of linear subspaces discussed in Chapter 9, I will refer to this class of qualitative properties as "symmetries" throughout. Related notions of symmetries on quantum graphs have been discussed by several authors, cf. [**31**]–[**62**]–[**63**].

Let me reformulate the criteria in Proposition 9.1 in our special case. Rewrite the form a as

$$a(f,g) = (Cf' \mid g')_H + (Pf \mid f) + (Md^f \mid d^g)_{\mathbb{C}^n}, \qquad f, g \in V,$$

and observe that the condition (d.ii) of Proposition 9.1 holds if

$$(Cf' \mid g')_H = (Pf \mid g)_H = 0 \qquad \text{for all } f \in Y \cap V, g \in Y^\perp \cap V \tag{10.1}$$

and

$$(Md^f \mid d^g)_{\mathbb{C}^n} = 0 \qquad \text{for all } f \in Y \cap V, g \in Y^\perp \cap V. \tag{10.2}$$

In the following, condition (d.i) of Proposition 9.1 is referred to as to the *admissibility of the projection* Π, while (10.1)–(10.2) are the *orthogonality condition with respect to* Π of the coefficient matrices C, P, M.

For the sake of simplicity, throughout this section only the case of a finite, connected graph with m links and n nodes will be considered, although in fact most of the following results can be generalized to the case of an infinite, locally finite network. Moreover, let me assume that no node is isolated, i.e., $|\Gamma(\mathsf{v})| \neq 0$ for all $\mathsf{v} \in \mathsf{V}$.

As already sketched in Remark 4.7, a relevant class of subspaces of H can be more generally constructed as follows: Let K be an orthogonal projection on $\mathbb{C}^m$ and consider

$$Y := \left\{f \in H : f(x) \in \operatorname{Range} K \text{ for a. e. } x \in (0,1)\right\}. \tag{10.3}$$

Invariance of such subspaces will be the topic of this section. Most of the results presented are taken from [**42**, § 3–4].

Proposition 10.1. *The orthogonal projection Π_K of H onto the closed subspace Y defined in* (10.3) *is given by*

$$(\Pi_K f)(x) = K\big(f(x)\big) \qquad \textit{for a. e. } x \in (0,1). \tag{10.4}$$

Its kernel $\operatorname{Ker}\Pi_K$ *is isomorphic to* $L^2(0,1;\mathbb{C}^{\dim \operatorname{Ker} K})$*, whereas its range* $\operatorname{Range}\Pi_K$ *is isomorphic to* $L^2(0,1;\mathbb{C}^{\dim \operatorname{Range} K})$.

To begin with, a characterization of admissible orthogonal projections Π_K on H will be established. The following is a direct consequence of the continuity condition in the nodes satisfied by functions in V.

LEMMA 10.2. *If the projection* Π_K *is admissible, then* $\mathbf{1}$ *is an eigenvector of* K.

REMARK 10.3. Observe that $K\mathbf{1} \in \{0, \mathbf{1}\}$, since the only eigenvalues of an orthogonal projection are 0 and 1, and that $\mathbf{1} \in \ker(I-K)$ if $\mathbf{1} \in \operatorname{Range} K$. Moreover, K is admissible if and only if $I-K$ is admissible. Therefore, it can be assumed without loss of generality that $\mathbf{1} \in \operatorname{Range} K$.

Thus, one can already rule out a class of subspaces of H that correspond to subgraphs of the given graph.

EXAMPLE 10.4. No symmetry of the system takes the form of a restriction, i.e., there exists no proper subgraph G' of G such that the linear subspace $Y := \{f \in H : f_{|G'} = 0\}$ of the functions vanishing on G' is invariant under $(e^{tA})_{t\geq 0}$. Without loss of generality, the subgraph G' corresponds to the links $\mathsf{e}_{m'+1}, \ldots, \mathsf{e}_m$. The projection onto Y is given by Π_K as in (10.4), where K is the $m' \times (m-m')$ diagonal block matrix

$$K = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $\mathbf{1}$ is not an eigenvector of K, Y is not a symmetry of the system, i.e., it is not invariant under any network diffusion semigroup, independently of C and M. In other words: Assume the initial data u_0 to vanish on a link e_ℓ of G. Assume also that $u_{0j} \neq 0$ on some subset of e_j of positive measure. Then there exists $t_0 > 0$ such that the solution $u_\ell(t_0,\cdot) \neq 0$ on a subset of e_ℓ of positive measure. □

LEMMA 10.5. *Let the matrix* K *be an orthogonal projection of* $\mathbb{C}^d$ *and the let the set* Y *be a linear subspace of* $\mathbb{C}^d$*. Then the following assertions are equivalent.*

(a) $KY \subset Y$.
(b) $Y = \ker K \cap Y \oplus \operatorname{Range} K \cap Y$.

A characterization of admissibility of projections can then be obtained.

THEOREM 10.6. *The following assertions are equivalent.*

(a) The orthogonal projection Π_K *is admissible.*
(b) The range of $\tilde{\mathcal{I}}$ *is invariant under* $\tilde{K}$*, i.e.,* $\tilde{K}\operatorname{Range}\tilde{\mathcal{I}} \subset \operatorname{Range}\tilde{\mathcal{I}}$*, where the* $2m \times n$ *matrix* $\tilde{\mathcal{I}}$ *and the* $2m \times 2m$ *matrix* $\tilde{K}$ *are defined by*

$$\tilde{\mathcal{I}} := (\mathcal{I}^+, \mathcal{I}^-)^\top = \begin{pmatrix} (\mathcal{I}^+)^\top \\ (\mathcal{I}^-)^\top \end{pmatrix} \qquad \text{and} \qquad \tilde{K} := \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}. \tag{10.5}$$

(c) There exists a basis of $\operatorname{Range}\tilde{\mathcal{I}}$ *consisting of eigenvectors of* $\tilde{K}$.

PROOF. To begin with, let me prove that (a) and (b) are equivalent. By the continuity in the nodes of functions $f \in V$ there exists a vector $d^f \in \mathbb{C}^n$ such that

$$(\mathcal{I}^+)^\top d^f = f(0) \qquad \text{and} \qquad (\mathcal{I}^-)^\top d^f = f(1).$$

The admissibility of the projection is in turn equivalent to the fact that for every $f \in V$ there exists a vector $d^{\Pi f} \in \mathbb{C}^n$ such that

$$(\mathcal{I}^+)^\top d^{\Pi_K f} = \Pi_K f(0) = Kf(0) \qquad \text{and} \qquad (\mathcal{I}^-)^\top d^{\Pi_K f} = \Pi_K f(1) = Kf(1).$$

Combining these two couples of conditions and observing that for all $x \in \mathbb{C}^n$ there exists a function $f \in (H^1(0,1))^m$ that is continuous in the nodes and satisfies $d^f = x$, one obtains that (a) is equivalent to the fact that for all $x \in \mathbb{C}^n$ there exists $y \in \mathbb{C}^n$ such that

$$(\mathcal{I}^+)^\top y = K(\mathcal{I}^+)^\top x, \qquad (\mathcal{I}^-)^\top y = K(\mathcal{I}^-)^\top x,$$

i.e., such that $\tilde{K}\,\mathrm{Range}\,\tilde{\mathcal{I}} \subset \mathrm{Range}\,\tilde{\mathcal{I}}$.

To check the second equivalence, observe that the existence of the claimed basis is equivalent to the condition $\mathrm{Range}\,\tilde{\mathcal{I}} = (\ker \tilde{K} \cap \mathrm{Range}\,\tilde{\mathcal{I}}) \oplus (\mathrm{Range}\,\tilde{K} \cap \mathrm{Range}\,\tilde{\mathcal{I}})$. By Lemma 10.5, setting $Y := \mathrm{Range}\,\tilde{\mathcal{I}}$ and $K := \tilde{K}$ one obtains the claim. □

It is interesting to observe that some classes of oriented graphs can be characterised in terms of the admissibility of a special projection.

THEOREM 10.7. *Consider the orthogonal projection K defined by*

$$K := \frac{1}{m}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}. \tag{10.6}$$

Then Π_K is admissible if and only if G is orientedly bipartite or Eulerian.

PROOF. If $f \in V$, then $\Pi_K f$ lies in $(H^1(0,1))^m$ since every component is a linear combination of H^1 functions. Thus, V is invariant under Π_K if and only if $\Pi_k f$ is continuous in the nodes for all $f \in V$. Denote by V_0 (resp., by V_1) the subset of V consisting of all nodes with nonzero outdegree (resp., nonzero indegree).

Let first Π_K be admissible. If $\mathsf{V}_0 \cap \mathsf{V}_1 = \emptyset$, then G is by definition an orientedly bipartite graph. If conversely G is orientedly bipartite, then for an arbitrary $f \in V$ there holds

$$d_i^{\Pi_K f} = \sum_{i=1}^m \frac{f_i(0)}{m} \qquad \text{for all } \mathsf{v}_i \in \mathsf{V}_0,$$

and

$$d_i^{\Pi_K f} = \sum_{i=1}^m \frac{f_i(1)}{m} \qquad \text{for all } \mathsf{v}_i \in \mathsf{V}_1,$$

This shows continuity of $\Pi_K f$ in the nodes and hence admissibility of Π_K.

Let on the other hand Π_K be admissible with $\mathsf{V}_0 \cap \mathsf{V}_1 \neq \emptyset$. If $f \in V$, then a vector $d^{\Pi_K f} \in \mathbb{C}^n$ of joint node values exists if and only if

$$\sum_{j=1}^m \frac{f_j(0)}{m} = \sum_{j=1}^m \frac{f_j(1)}{m}. \tag{10.7}$$

For an arbitrary node $\mathsf{v}_i \in \mathsf{V}$ choose $f \in V$ in such a way that $d_i^f = 1$ and $d_k^f = 0$ for all $k \neq i$. Then $\sum_{j=1}^m f_j(0) = \sum_{j=1}^m \iota_{kj}^+ = |\Gamma^+(\mathsf{v}_k)|$ as well as $\sum_{j=1}^m f_j(1) = \sum_{j=1}^m \iota_{kj}^- = |\Gamma^-(\mathsf{v}_k)|$. Accordingly, it follows from (10.7) that $\left|\Gamma^-(\mathsf{v}_k)\right| = \left|\Gamma^+(\mathsf{v}_k)\right|$ for all $k = 1, \dots, n$, i.e., G is orientedly Eulerian. If conversely G is orientedly Eulerian, it can be proved likewise that (10.7) holds, and hence that Π_K is admissible. □

It is also possible to characterise stars within the class of the orientedly simple graphs in terms of symmetries. The lengthy proof of the following can be found in [**42**, § 3].

THEOREM 10.8. *The following assertions hold.*

(1) All nodes of G have degree 1 if and only if Π_K is admissible for all orthogonal projections K of $\mathbb{C}^m$.

(2) *Let* G *be an orientedly simple graph. Then* G *is a star if and only if* Π_K *is admissible for all orthogonal projections* K *with eigenvector* $\mathbf{1}$.

It is possible to characterise orthogonality of C, P with respect to Π_K by means of a linear algebraical condition.

THEOREM 10.9. *Let the sesquilinear form* a *on* H *be defined as in* (7.4)*, with* $M = 0$*. Then the following assertions are equivalent.*

(a) *The matrices* C, P *satisfy the orthogonality condition* (10.1) *with respect to* Π_K.

(b) *The range of* K *is invariant under* C *and* P*, i.e.,*

$$C(x)\,\text{Range}\,K \subset \text{Range}\,K \quad \textit{and} \quad P(x)\,\text{Range}\,K \subset \text{Range}\,K \qquad \textit{for a.e. } x \in (0,1). \tag{10.8}$$

PROOF. Since the space H can be decomposed into $H = \text{Range}\,\Pi_K \oplus \text{Range}(I - \Pi_K)$, the orthogonality condition (10.1) is equivalent to $a(\Pi_K u, (I - \Pi_K)v) = 0$ for all $u, v \in V$. Using the linearity of the derivative and the self-adjointness of the orthogonal projection K, one can compute

$$\begin{aligned} a(\Pi_K u, (I - \Pi_K)v) &= \big(C\Pi u' \mid (I-\Pi)v'\big)_H + \big(P\Pi u \mid (I-\Pi)v\big)_H \\ &= \int_0^1 \Big(\big((I-K)C(x)Ku'(x) \mid v'(x)\big)_{\ell^2} + \big((I-K)P(x)Ku(x) \mid v(x)\big)_{\ell^2} \Big) dx. \end{aligned}$$

One sees that this holds if and only if $(I-K)C(x)K = 0$ and $(I-K)P(x)K = 0$ for a.e. $x \in (0,1)$. Since K is a projection, this is equivalent to (10.8). □

Observe that condition (10.8) is satisfied if in particular both C, P are pointwise multiple of the identity matrix, i.e., if there exist functions $c \in C^1[0,1]$ and $p \in L^\infty(0,1)$ such that $C(x) \equiv c(x)\,\text{Id}$ and $P(x) \equiv p(x)\,\text{Id}$ for a.e. $x \in (0,1)$.

In order to characterise the orthogonality condition for M with respect to Π_K it is necessary to introduce the $2m \times 2m$ matrix

$$\mathcal{M} := \tilde{\mathcal{I}} D M D \tilde{\mathcal{I}}^\top,$$

where D denotes the diagonal matrix with entries $|\Gamma(\mathsf{v}_i)|^{-1}$, $i = 1, \dots, n$, the inverse of the node degrees. I refer to [**42**, § 3] for the quite technical proof of the following.

PROPOSITION 10.10. *Assume the orthogonal projection* Π_K *to be admissible. Then the matrix* M *satisfies the orthogonality condition with respect to* Π_K *if and only if*

$$\text{Range}\,\mathcal{M}\tilde{K}\tilde{\mathcal{I}} \subset \text{Range}\,\tilde{K}. \tag{10.9}$$

The lengthy proof of the following can be found in [**42**, § 3].

PROPOSITION 10.11. *Define* K *as in* (10.6) *and let the graph* G *be orientedly bipartite. Then* M *satisfies the orthogonality condition with respect to* Π_K *if and only if there exist numbers* $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}$ *such that*

$$\begin{aligned} &\alpha_{11}|\Gamma(\mathsf{v}_i)| = \textstyle\sum_{k=1}^{n_1} m_{ik}, \qquad \alpha_{12}|\Gamma(\mathsf{v}_i)| = \sum_{k=n_1+1}^{n} m_{ik} \quad \textit{for all } i = 1, \dots, n_1, \quad \textit{and} \\ &\alpha_{21}|\Gamma(\mathsf{v}_i)| = \textstyle\sum_{k=1}^{n_1} m_{ik}, \qquad \alpha_{22}|\Gamma(\mathsf{v}_i)| = \sum_{k=n_1+1}^{n} m_{ik} \quad \textit{for all } i = n_1+1, \dots, n. \end{aligned}$$

To conclude, let me state the existence of a large class of symmetries on oriented stars and trees. The involved proof can be found in [**42**, § 4].

In the following, the p^{th} *layer of an oriented tree* is the index set (denoted by Λ_p) of all links that come into the nodes at distance p from the root. An oriented tree is called *symmetric* if the number of links that go out of a node is constant for all the nodes at same distance p from the root.

PROPOSITION 10.12. *Assume that the coefficients* c_j *are constant functions. If* $P = 0$ *and* $M = 0$*, then the following assertions hold.*

(1) *Assume the coefficients* c_i *to be pairwise different. If* K *is a nontrivial orthogonal projection of* $\mathbb{C}^m$*, then*

$$Y_{i,j} := \{f \in H : f_i = f_j\} \tag{10.10}$$

is not invariant under $(e^{tA})_{t\geq 0}$ *for any pair* (i,j)*.*

(2) *Let* G *be an oriented star. If there exist* i_0, j_0 *such that* $c_{i_0} = c_{j_0}$*, then the subspace* Y_{i_0,j_0}*, defined as in* (10.10)*, is invariant under* $(e^{tA})_{t\geq 0}$*.*

(3) *Let* G *be a symmetric oriented tree. If* $c_1 = \ldots = c_m$*, then the subspace*

$$Y := \{f \in H : f_i = f_j \text{ for all } i,j \in \Lambda_p \text{ and all } p\} \tag{10.11}$$

is invariant under $(e^{tA})_{t\geq 0}$*.*

REMARKS 10.13. (1) In Proposition 10.12.(3), the orthogonal projection K associated with Y via (10.3) is the block-diagonal matrix whose p^{th} block is a $|\Lambda_p| \times |\Lambda_p|$ matrix all of whose $|\Lambda_p|^2$ entries equal $|\Lambda_p|^{-1}$.

(2) In Theorem 7.12 sufficient conditions have been given in order that the semigroup governing a network diffusion problem extends to the space of Radon measure. It is clear that symmetries on spaces of Radon measures, as introduced in Remark 9.2, can be studied in the context of diffusion on networks, too. For example, one can check that if G is a symmetric oriented tree with r layers, $c_1 = \ldots = c_m$, $P = 0$, $M = 0$, and the initial datum is a measure defined by

$$\mu(f) := \sum_{j\in\Lambda_r} f_j(1),$$

i.e., a Dirac delta in the leaves of the tree, then a natural symmetry is exhibited by solutions of the system. More precisely, the semigroup maps μ into the space Y defined in (10.11), since Y is invariant for the semigroup on H.

CHAPTER 11

A crash course in cortical modelling

Particularly relevant applications of the theory of evolution equations on networks arise in the context of theoretical neuroscience. In this section, I will introduce some elementary notions that are common in the modern neurobiology: my account is by no means complete or conclusive, since the development of neuroscience has been tumultuous over the last hundred years and promises to keep on for a long while. A beautiful and more detailed synopsis of the manifold of theories coexisting or competing in the mathematical and experimental neuroscience can be found in [**64**], see also [**65, 66, 67**] for complete surveys.

Neurons are the basic building blocks of the nervous system. They gather external impulses, and process them while transmitting to other neurons. A nervous system can be seen at at least two different levels: a microscopic and a mesoscopic one. At a mesoscopic level one usually neglects the spatial features of individual neurons and considers them as dimensionless points: this approach leads to *computational models* that are usual in the modern neuroinformatics, where networks of neurons are modelled as dynamical systems – i.e., as systems of coupled, possibly delayed ordinary differential equations.

I will rather concentrate on a description at the microscopic level of individual neurons and refer to the [**68**] for a brief survey of the interplay between microscopic and mesoscopic modelling.

A human brain contains approx. 10^{11} neurons, whose dimension can be considerably different. Each neuron is a cell. An individual neuron's basic structure can be described as the juxtaposition of three different entities: a *dendritical tree*[1], a *soma* (the cell's body), and an *axon* ending in an *axonal tree*. A dendrite is a linear fiber that can reach a length of 1 cm. On its surfaces several thousands (up to 100,000) of appendages (*spines*) can be usually found: they collect electric *synaptic* impulses from other neurons and transmit them toward the dendrite's ending and then, via branching points, to further dendrites.

Each collection of dendrites is a highly ramified structure. More precisely, it is tree-shaped – in a proper graph theoretical sense! – and therefore called *dendritical tree*. It propagates electric signal in a centripetal direction, i.e., toward the soma – the tree's top. The soma elaborates these inputs and transmits them to other neurons through the *axon*, a projection that eventually ramifies in an *axonal tree*. Each of its branching ends with an *axodendritic synapse* that links it to another neuron's dendritic tree, or more seldom with an *axoaxonal synapse* that links it to another neuron's axonal tree. Finally, *dendrodendritical synapses* also exist, which connect different dendrites: their existence has been universally accepted only in relatively recent years, and their functioning mechanism is not yet fully understood. A human brain contains approx. 10^{16} synapses, a part of the set of all branching points of dendritic and axonal trees. Synapses are usually divided in *electric* and *chemical* ones. The latter are much more common in higher invertebrates and mammalians: their mechanism is based on secretion of *neurotransmitters* by the pre-synaptic neuron, and their reception by the post-synaptic one; they act as gates, preventing synaptic signal from backpropagating in the pre-synaptic neuron. On the

[1]Although this name is now common, I find it quite nonsensical, as "dendrite" comes from "$\Delta\acute{\epsilon}\nu\delta\rho o\varsigma$", Greek for "plant" or "tree".

other hand, electric synapses are symmetric and can transmit signal in both directions. Both kinds of synapses feature some form of delayed reaction to impulses, due to the time required for release of the chemical transmitter substance or to other physical issue. I am not going to explain the origin and the possible advantages of these delays effects more precisely. Let me only mention that synaptic delay ranges between 0.1 and $0.3msec$ for chemical synapses, whereas electric synapses' delay is much smaller: around $0.05msec$ – in fact, delay of electric synapses is often neglected in neuronal models.

Dendrites are passive fibers: in most common models they passively transmit electric potential, without any form of self-excitation. According to *linear cable theory*, this process is modelled on each dendritic tract by the linear *cable equation*

$$\frac{\partial}{\partial t}v(t,x) = \frac{\partial^2}{\partial x^2}v(t,x) - v(t,x). \tag{11.1}$$

By means of chemical processes in the soma, the signal is elaborated and amplified until it is able to be transmitted to the axon. Such processes are, in the common Rall's lumped soma model, described by a dynamical boundary condition

$$\frac{\partial}{\partial t}v(t) = -\sum \iota_{ij} v_j'(t, \mathsf{v}_i) - \gamma v(t) \tag{11.2}$$

imposed on the node at the dendritic tree. Here γ is damping constant and ι_{ij} is the incidence matrix of the graph, thus one is summing over all the dendritical trees converging in the soma as well as over the outgoing axon. Observe that W. Rall formulated his theory under strong assumptions on the topology of the network. In particular, the dendritic architecture needs to define a symmetric oriented tree, in the sense explained before Proposition 10.12.

To close the model of dendritic trees, node conditions for the system of cable equations in the dendritic network are still needed. Synapses can be essentially divided in two large classes: *inhibitory* and *excitatory* ones, which transmit electric signals after having weakened and amplified it, respectively. This can be mathematically modeled by imposing in the branching point a Kirchhoff–Robin-type law of the form

$$\sum \iota_{ij} v_j'(t, \mathsf{v}_i) + \gamma_i v(t, \mathsf{v}_i) = 0, \tag{11.3}$$

where the parameter γ_i is negative or positive depending on whether the synapse v_i is of inhibitory or excitatory type (or $\gamma_i = 0$ if v_i is a branching point where axial currents are conserved). In (11.2) and (11.3) the sum term represents the total incoming and outgoing currents: in fact, due to the incidence matrix entries ι_{ij}, only the links e_j incident in the branching point v_i contribute. Observe that I have implicitly assumed the voltage to be homogeneous at branching points, i.e., that

$$v_j(t, \mathsf{v}_i) = v_\ell(t, \mathsf{v}_i) \tag{11.4}$$

for all links e_j incident in the branching point v_i. Of course, this does *not* imply that an analogous continuity condition is also satisfied by the incoming and outgoing currents $v_j'(t, \mathsf{v}_i)$ – in fact, this does not hold at all.

Transmission of signal in axons occurs by means of a short, intense wave of potential called *action potential* or *spike*. Spikes are initiated in the *axon hillock*, the junction between soma and axon, when the soma potential in an interval $[\xi_1, \xi_2]$ corresponding to the phenomenological thresholds of approx. $-50mV$ and $+40mV$. However, these thresholds may vary in time: right after a spike has been initiated they quickly increase (*refractory period*) and then slightly decrease (*enhancement period*). I will emphasize this variability by writing $\xi_1 = \xi_1(t)$ and $\xi_2 = \xi_2(t)$.

Although signals are essentially electric events, speed of signal propagation in vertebrates' axons is high, but fairly slower than that of light: usually approx. 30 m/s, and up to 100 m/s. Electric potential is transmitted along a neuron in a poor way, due to the bad conductivity properties of biological fibers.

Axon of vertebrated are usually covered with a continuous *myelin sheath* that dramatically speeds up the propagation of action potentials.

Transmission of potential in axons is usually mathematically described by a semilinear diffusion (or rather, *cable*) equation. However, axons spend energy in order to propagate potential. This would soon stop, had the neurons not developed a regenerative self-excitation mechanism based on activation of ionic current. A now common model has been formulated by FitzHugh and Nagumo. It describes the transmission of potential by the system of semilinear differential equations

$$\frac{\partial}{\partial t}v(t,x) = \frac{\partial^2}{\partial x^2}v(t,x) - v(t,x) - \frac{v(t,x)(v(t,x)-\xi_1(t))(v(t,x)-\xi_2(t))}{\xi_2(\xi_2(t)-\xi_1(t))} - R(t,x), \tag{11.5}$$

$$\frac{\partial}{\partial t}R(t,x) = v(t,x) - R(t,x) + \zeta(t), \tag{11.6}$$

where R is an ad-hoc variable whose rôle is to approximate and sum up the activity of ion pumps. In fact, axons often form bundles (commonly called *nerves*). There is increasing experimental evidence of (chemical and/or electric) *ephaptic* interaction among axons that form bundles. A possible model for this (not yet universally accepted) mode of interaction is given by the system

$$\frac{\partial}{\partial t}v_j(t,x) = \sum_{k=1}^{N} c_{jk}\frac{\partial^2}{\partial x^2}v_k(t,x) - v_j(t,x) - R_j(t,x) \tag{11.7}$$

$$-\frac{v_j(t,x)(v_j(t,x)-\xi_1)(v_j(t,x)-\xi_2)}{\xi_2(\xi_2-\xi_1)}, \tag{11.8}$$

$$\frac{\partial}{\partial t}R_j(t,x) = v_j(t,x) - R_j(t,x) + \zeta_j(t), \tag{11.9}$$

where (c_{jk}) is a positive definite matrix whose entries have biological significance. Here, v_j represents the membrane voltage along the j^{th} axon in a bundle of N.

It appears that ephaptic coupling, a comparatively usual mode of communications in the invertebrates' brain, appears more rarely in vertebrates: so far, evidence of human ephaptic connections has been produced only in the olfactory system and in certain syndromes (most notably, epilepsy and different forms of sclerosis). In fact, a possible explanation for ephaptic communication is based on electric excitability of neighboring neurons. According to experiments, it appears possible that ephaptic effects are due to pathological damages to myelin sheath, causing them to interact in a way (symmetry, lack of delay) that recalls the mechanism of electric synapses.

CHAPTER 12

Diffusion equations for neurobiological models

Our aim is to investigate the neuronal network model (11.1)–(11.2)–(11.3)–(11.4) described in Chapter 11. Material in this section is taken from [**69**], where the interplay with the neurobiological investigations carried on in [**70**] has also been discussed.

Consider a cortical model represented by a network that contains m among dendrites and axons and n branching nodes, n_0 of which are soma. Basing on Rall's model introduced in Chapter 11, the system reads

$$\text{(NDP)}\qquad \begin{cases} \dot{u}_j(t,x) = u_j''(t,x) - u_j(t,x), & t \geq 0,\ x \in (0,1),\ j = 1,\dots,m, \\ u_j(t,\mathsf{v}_i) = u_\ell(t,\mathsf{v}_i) =: d_i^u(t), & t \geq 0,\ j,\ell \in \Gamma(\mathsf{v}_i),\ i = 1,\dots,n, \\ \dot{d}_i^u(t) = -\sum_{j=1}^m \iota_{ij} u_j'(t,\mathsf{v}_i) & \\ \qquad\qquad -m_i d_i^u(t), & t \geq 0,\ i = 1,\dots,n_0, \\ m_i d_i^u(t) = -\sum_{j=1}^m \iota_{ij} u_j'(t,\mathsf{v}_i), & t \geq 0,\ i = n_0+1,\dots,n, \\ u_j(0,x) = u_{0j}(x), & x \in (0,1),\ j = 1,\dots,m, \\ u(0,\mathsf{v}_i) = \alpha_{0i}, & i = 1,\dots,n_0, \end{cases}$$

where $m_i \leq 0$, $i = 1,\dots,n$. Observe that a set of dynamic boundary conditions differentiates this system from (Di)–(Cc)–(Tc) introduced in Chapter 7.

As in Chapter 7.1, the conditions in the second and fourth equations can be reformulated as

$$(12.1)\qquad \exists d^u(t) \in \mathbb{C}^n \quad \text{s.t.} \quad \begin{array}{l} (\mathcal{I}^+)^\top d^u(t) = u(t,0),\ (\mathcal{I}^-)^\top d^u(t) = u(t,1), \quad \text{and} \\ \Omega^+ u'(t,0) - \Omega^- u'(t,1) = M d^u(t) \quad \text{for all } t \geq 0, \end{array}$$

where $M := \operatorname{diag}(0,\dots,0,m_{n_0+1},\dots,m_n)$.

It is possible to equivalently re-write the above problem as an abstract Cauchy problem (ACP) as in Proposition 3.4, for a suitable operator $\tilde{A}$. In this case, one has to consider the Hilbert space $\tilde{H} := L^2(0,1;\mathbb{C}^m) \times \mathbb{C}^{n_0}$ and the operator $\tilde{A}$ defined by

$$\tilde{A}u := \Big(u_1'' - u_1,\dots,u_m'' - u_m, -\sum_{j=1}^m \iota_{1j} f_j'(\mathsf{v}_1) - m_1 d_1^u,\dots,-\sum_{j=1}^m \iota_{n_0 j} f_j'(\mathsf{v}_{n_0}) - m_{n_0} d_{n_0}^u\Big)$$

with domain

$$D(\tilde{A}) := \left\{ (f,\alpha) \in H^2(0,1;\mathbb{C}^m) \times \mathbb{C}^{n_0} : \begin{array}{l} \exists d^f \in \mathbb{C}^n \text{ s.t.} \\ (\mathcal{I}^+)^\top d^f = f(0),\ (\mathcal{I}^-)^\top d^f = f(1), \\ (d_1^f,\dots,d_{n_0}^f)^\top = \alpha, \text{ and} \\ \Omega^+ f'(0) - \Omega^- f'(1) = M d^f \end{array} \right\}.$$

Taking into account [**17**, Lemma B.1], a direct computation shows that the following holds.

Lemma 12.1. *The linear subspace*

$$\tilde{V} := \left\{ (f,\alpha) \in H^1(0,1;\mathbb{C}^m) \times \mathbb{C}^{n_0} : \begin{array}{l} \exists d^f \in \mathbb{C}^n \textit{ s.t.} \\ (\mathcal{I}^+)^\top d^f = f(0),\ (\mathcal{I}^-)^\top d^f = f(1), \\ \textit{and } (d_1^f,\dots,d_{n_0}^f)^\top = \alpha \end{array} \right\}$$

is densely and compactly embedded in $\tilde{H}$.

A suitable form can now be introduced.

LEMMA 12.2. *Consider the densely defined sesquilinear form $\tilde{a} : \tilde{V} \times \tilde{V} \to \mathbb{C}$ given by*

$$\tilde{a}((f,\alpha),(g,\beta)) := \sum_{j=1}^{m} \int_0^1 \left(f_j'(x)\overline{g_j'(x)} + f_j(x)\overline{g_j(x)} \right) dx + \sum_{i=1}^{n} m_i d_i^f \overline{d_i^g}.$$

Then $\tilde{a}$ is $\tilde{H}$-elliptic and continuous. It is symmetric (resp., accretive) if and only if the coefficients m_i are real (resp., positive).

PROOF. In order to check $\tilde{H}$-ellipticity and continuity of $\tilde{a}$, first observe that $\tilde{V}$ becomes a Hilbert space when equipped with the inner product

$$((f,\alpha) \mid (g,\beta)) := \sum_{j=1}^{m} \int_0^1 \left(f_j'(x)\overline{g_j'(x)} + f_j(x)\overline{g_j(x)} \right) dx + \sum_{i=1}^{n_0} d_i^f \overline{d_i^g}.$$

Due to the continuous embedding of $H^1(0,1)$ into $C[0,1]$ there holds

$$|d^f|_{\mathbb{C}^{n_0}} \leq \max_{1\leq j\leq m} \max_{x\in[0,1]} |f_j(x)| =: \|f\|_{C(G)} \leq \|f\|_{H^1}, \tag{12.2}$$

where $C(G)$ is the space introduced in Theorem 7.12. Accordingly, the above inner product is in fact equivalent to

$$((f,\alpha) \mid (g,\beta))_{\tilde{V}} := \sum_{j=1}^{m} \int_0^1 \left(f_j'(x)\overline{g_j'(x)} + f_j(x)\overline{g_j(x)} \right) dx.$$

Ellipticity of $\tilde{a}$ follows from Lemma 6.8, since

$$\mathrm{Re}\tilde{a}((f,\alpha),(f,\alpha)) - \sum_{i=1}^{n} m_i |d_i^f|^2 = \|(f,\alpha)\|_{\tilde{V}}^2.$$

Finally, $\tilde{a}$ is continuous. Take $(f,\alpha),(g,\beta) \in \tilde{V}$, set $M := \max_{1\leq i\leq n}$, and observe that

$$\begin{aligned} |\tilde{a}((f,\alpha),(g,\beta))| &\leq \sum_{j=1}^{m} \int_0^1 \left| f_j'(x)\overline{g_j'(x)} \right| + \left| f_j(x)\overline{g_j(x)} \right| + M \sum_{i=1}^{n} |d_i^f||d_i^g| \\ &\leq 2\max\{1,M\} \|(f,\alpha)\|_{\tilde{V}} \|(g,\beta)\|_{\tilde{V}}. \end{aligned}$$

This concludes the proof. □

The following can be proved in a way similar to Lemma 7.4 and Theorem 7.9.

THEOREM 12.3. *The operator associated with the form $\tilde{a}$ is $\tilde{A}$ as defined above. It generates a strongly continuous, compact, bounded analytic semigroup, which is given by an integral kernel. Such a semigroup is self-adjoint and positive if and only if all m_i are real.*

The semigroup generated by $\tilde{A}$ is contractive with respect to the L^2 and the L^∞-norm if and only if all m_i are positive. In this case, the semigroup is uniformly exponentially stable. Moreover, it extrapolates to a consistent family of semigroups on $L^p(0,1;\ell^p) \times \mathbb{C}^{n_0}$. In fact, by Proposition 6.16 and reasoning as in Theorem 7.12 one obtains the following.

THEOREM 12.4. *If the coefficients m_i are positive numbers, then the semigroup generated by $\tilde{A}$ extends to $\mathcal{M} \times \mathbb{C}^{n_0}$, where $\mathcal{M}$ denotes the space of Radon measures over the network. It maps such a space into*

$$\tilde{C}(G) := \left\{ (f, \alpha) \in C([0,1]; \mathbb{C}^m) \times \mathbb{C}^{n_0} : \begin{array}{c} \exists d^f \in \mathbb{C}^n \ s.t. \\ (\mathcal{I}^+)^\top d^f = f(0),\ (\mathcal{I}^-)^\top d^f = f(1), \\ \text{and } (d_1^f, \dots, d_{n_0}^f)^\top = \alpha \end{array} \right\}$$

and satisfies an ultracontractivity estimate (with dimension 1) analogous to (7.9).

It is possible to extend the theory of symmetries to the setting in this section. As a motivating example, let me state the following, whose easy proof I omit. I denote by P the orthogonal projection on the subspace Y introduced in (10.11), cf. Remark 10.13.(1).

PROPOSITION 12.5. *Consider a diffusion process* (NDP) *on a symmetric oriented tree G, on whose root the sole dynamic condition of the system is imposed. Assume the coefficients m_i to be positive. Let $\tilde{Y} := \mathrm{Range}\tilde{P}$, where the projection $\tilde{P}$ on H is given by*

$$\tilde{P}(f, \alpha) := \big(Pf, \alpha\big)\,.$$

If the initial data is a measure belonging to $\tilde{Y}_{\mathcal{M}}$, where

$$\tilde{Y}_{\mathcal{M}} := \left\{ (\mu, \alpha) \in \mathcal{M}(\Omega) \times \mathbb{C} : \int_\Omega f d\mu + \alpha\phi = 0 \text{ for all } (f, \phi) \in \tilde{Y}^\perp \cap \tilde{C}(G) \right\},$$

then the solution to (NDP) *is a continuous function in $\tilde{Y}$ for all $t > 0$, i.e., it is layerwise equal.*

Observe that Rall's models actually assumes that dendritic trees are symmetric oriented trees.

REMARKS 12.6. (1) As a direct application of Corollary 6.12 one can show that the initial value problem associated with a semilinear version of (NDP) (consistent with the FitzHugh–Nagumo model (11.5)–(11.6) presented in Chapter 11) is well-posed. To this aim one has to introduce the sesquilinear form

$$\begin{aligned} a((f,R),(g,S)) \quad := \quad & \sum_{j\in\mathbb{N}} \int_0^1 \Big(f_j'(x)\overline{g_j'(x)} + f_j(x)\overline{g_j(x)} \\ & \qquad + R_j(x)\overline{g_j(x)} - f_j(x)S_j(x) + R_j(x)\overline{S_j(x)} \Big)\, dx, \end{aligned}$$

on $L^2(0,1;\ell^2) \times L^2(0,1;\ell^2)$, whose domain is a product space. Ellipticity and continuity of this sesquilinear form can be easily proved, so that the associated operator generates an analytic semigroup. In particular, Corollary 6.12 can be applied, since the nonlinear terms appearing in the FitzHugh–Nagumo models are Lipschitz continuous, cf. [**71**, § 6] and [**58**, § 4] for details. Since spikes have been observed to propagate at constant velocity along the axonal fibres, existence of travelling waves as solutions of the FitzHugh–Nagumo model is a particularly relevant issue. This has been shown to be possible under strong assumptions on the architecture of the neuronal network, cf. [**72**].

(2) Consider the model (11.7)–(11.8)–(11.9) based of neuronal ephaptic coupling briefly discussed in Chapter 11. Taking into account Remark 7.1.(1), it is clear that its linear part can be discussed by means of the methods presented above. More precisely, one considers as usual the weak formulation of the diffusion problem associated with a network of ephaptically coupled axons, which leads to introducing the form

$$\begin{aligned} a((f,R),(g,S)) \quad := \quad & \sum_{i,j\in\mathbb{N}} \int_0^1 \Big(c_{ij} f_j'(x)\overline{g_i'(x)} + f_j(x)\overline{g_j(x)} \\ & \qquad + R_j(x)\overline{g_j(x)} - f_j(x)S_j(x) + R_j(x)\overline{S_j(x)} \Big)\, dx, \end{aligned}$$

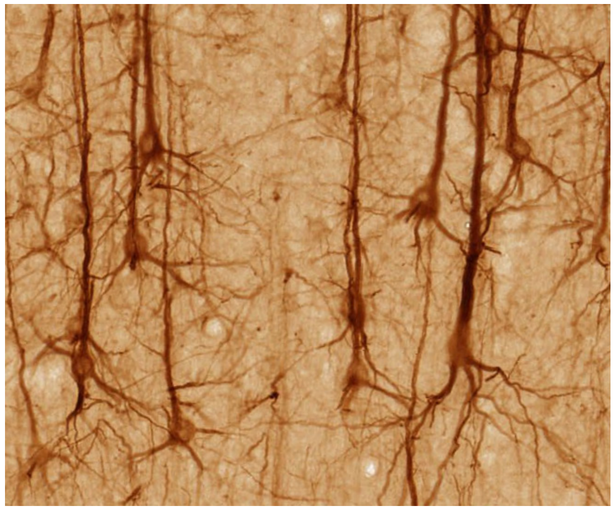

Figure 1. A pyramidal neuron in medial prefrontal cortex of macaque (Picture by brainmaps.org)

with same domain of the form considered in (1). The problem is well-posed if the coupling matrix (c_{ij}) is positive definite, and several qualitative and symmetry properties of this system have been studied in [**42**].

(3) Observe that some of the properties of the sesquilinear form a defined in (12.2) have also been obtained, by other methods and with different motivations, in [**73**], see also references therein. Moreover, the same form also appears in approximation theory, whenever one considers the limit of Laplacians on shrinking thin domains in a critical case where the volumes of 3-dimensional neighborhoods of the network nodes and links decay at the same rate. In this context, interesting spectral and convergence results and a priori estimates have been obtained in [**36**, § 3.3] and [**37**, § 7]. It is worth to observe that – according to the results of [**37**] – only in the case where the volume of the node neighborhoods is comparable to that of the link neighborhoods the limiting case gives rise to the form a in (12.2). In particular, Rall's *Ansatz* that a collapsed dendritical tree undergoes passive transmission in the branching nodes and a dynamic law in the soma should be carefully compared with what is currently known about geometry of neurobiological tissues, cf. Figure 1.

CHAPTER 13

Quantum graphs

Wave propagation in thin structures has been often discussed in the applied sciences, and in particular in physical chemistry and theoretical quantum mechanics. If in particular one is concerned with the issue of electromagnetic wave transport in circuits of quantum wires[1], then a fairly precise description of the system's behaviour is given by a system of general Schrödinger equations

$$\text{(Se)} \qquad i\frac{\partial u_j}{\partial t}(t,x) = -\left(i\frac{\partial}{\partial x}+a_j\right)^2 u_j(t,x)+v_j(x)u_j(t,x), \qquad t\in\mathbb{R},\ x\in(0,\ell_j),\ j\in\mathbb{N},$$

on each link of finite length ℓ_j in the network. This kind of Schrödinger equations on quantum graphs have been considered by many authors, see e.g. [**74, 36, 75, 76**] and references therein. This section is based on the treatment presented in [**60**].

As usual, I will rescale in order to replace ℓ_j by 1 for all j. To this aim, replace the spatial variable x by $\frac{x}{\ell_j}$ and obtain instead of (SE) the equivalent equation

$$\text{(SE')} \qquad i\frac{\partial u_j}{\partial t}(t,x) = -\left(i\frac{1}{\ell_j}\frac{\partial}{\partial x}+a_j\right)^2 u_j(t,x)+v_j(x)u_j(t,x), \qquad t\in\mathbb{R},\ x\in(0,1),\ j\in\mathbb{N}.$$

Assume for the sake of simplicity that $a\in L^2(0,1;\ell^2)$ and $v\in L^1(0,1;\ell^1)$. Then, in the Hilbert space $H=L^2(0,1;\ell^2)$ I consider the form

$$\text{(13.1)} \qquad a(f,g) := -\sum_{j\in\mathbb{N}}\int_0^1\left(\left(i\frac{1}{\ell_j}\frac{df_j}{dx}+a_jf_j\right)\overline{\left(i\frac{1}{\ell_j}\frac{dg_j}{dx}+a_jg_j\right)}+v_jf_j\overline{g_j}\right)dx+\sum_{i\in\mathbb{N}}m_id_i^f\overline{d_i^g}$$

defined on the domain V introduced in Chapter 7, i.e., on the space of H^1-functions that are continuous over the graph. This form corresponds to the operator A defined by

$$A := \operatorname{diag}\left(-\left(i\frac{1}{\ell_j}\frac{\partial}{\partial x}+a_j\right)^2+v_j\right)_{j\in\mathbb{N}}$$

with electric and magnetic potentials a and v, respectively, and Kirchhoff–Robin node conditions. By Lemma 6.8 one sees that the form a is elliptic. Thus, up to the bounded perturbation defined by the term v, A is self-adjoint and the generator of a unitary group, denoted by $(e^{itA})_{t\in\mathbb{R}}$. Accordingly, the following holds.

[1]In the literature, by *quantum wire* a 3-dimensional physical structure is usually meant, in which two dimensions are much smaller than the third one – possibly up to a few nanometers.

PROPOSITION 13.1. *If $a \in L^2(0,1;\ell^2)$, $v \in L^1(0,1;\ell^1)$, and $(m_i)_{i\in\mathbb{N}} \in \ell^2$, then the system*

$$\text{(QG)}\qquad \begin{cases} \dot u_j(t,x) &=& \left(-\left(i\frac{\partial}{\partial x}+a_j\right)^2+v_j\right)u_j, & t\in\mathbb{R},\ x\in(0,\ell_j),\ j\in\mathbb{N},\\ u_j(t,\mathsf{v}_i) &=& u_\ell(t,\mathsf{v}_i)=: d_i^u(t), & t\in\mathbb{R},\ j,\ell\in\Gamma(\mathsf{v}_i),\ i\in\mathbb{N},\\ m_i d_i^u(t) &=& \sum_{j=1}^m \iota_{ij}\left(i\frac{\partial}{\partial x}+a_j\right)u_j(t,\mathsf{v}_i), & t\in\mathbb{R},\ i\in\mathbb{N},\\ u_j(0,x) &=& u_{0j}(x), & x\in(0,\ell_j),\ j\in\mathbb{N}, \end{cases}$$

is governed by a C_0-group and hence it is well-posed. In fact, if $v_j(x) \in i\mathbb{R}$ for a.e. $x \in (0,1)$ and all j and if $m_i \in \mathbb{R}$ for all i, then the group is unitary.

As already mentioned in Remark 12.6.(3), in the context of approximation theory for thin quantum wires, dynamic node conditions naturally arise. The following can be proved in a way similar to Proposition 13.1, taking into account the results of Chapter 12 and applying direct perturbation arguments.

PROPOSITION 13.2. *If $a \in L^2(0,1;\ell^2)$, $v \in L^1(0,1;\ell^1)$, and $(m_i)_{i\in\mathbb{N}} \in \ell^2$, then the system*

$$\text{(QGD)}\qquad \begin{cases} \dot u_j(t,x) &=& \left(-\left(i\frac{\partial}{\partial x}+a_j\right)^2+v_j\right)u_j, & t\in\mathbb{R},\ x\in(0,\ell_j),\ j\in\mathbb{N},\\ u_j(t,\mathsf{v}_i) &=& u_\ell(t,\mathsf{v}_i)=: d_i^u(t), & t\in\mathbb{R},\ j,\ell\in\Gamma(\mathsf{v}_i),\ i\in\mathbb{N},\\ \dot d_i^u(t) &=& \sum_{j=1}^m \iota_{ij}\left(i\frac{\partial}{\partial x}+a_j\right)u_j(t,\mathsf{v}_i)-m_i d_i^u(t), & t\in\mathbb{R},\ i\in\mathbb{N},\\ u_j(0,x) &=& u_{0j}(x), & x\in(0,\ell_j),\ j\in\mathbb{N}, \end{cases}$$

is governed by a C_0-group and hence it is well-posed. In fact, if $v_j(x) \in i\mathbb{R}$ for a.e. $x \in (0,1)$ and all j and if $m_i \in \mathbb{R}$ for all i, then the group is unitary.

Consider for the remainder of this section the special case of a finite quantum graph, and hence replace the state space $L^2(0,1;\mathbb{C}^m)$ by $L^2(0,1;\mathbb{C}^m)$. Then, it is possible to discuss symmetry properties of quantum graphs by means of the techniques developed in Chapter 10.

PROPOSITION 13.3. *Let K be an orthogonal projection of $\mathbb{C}^m$ and define a linear subspace Y as in* (10.3)*. Assume the lengths ℓ_j, the incidence matrices $\mathcal{I}^+,\mathcal{I}^-$, the potential functions A,V, the nodal coefficients M, and the proportion matrix K to satisfy the conditions*

$$\begin{aligned} &A(x)\operatorname{Range}K \subset \operatorname{Range}K \quad \textit{and} \quad V(x)\operatorname{Range}K \subset \operatorname{Range}K \quad \textit{for all } x\in(0,1),\\ &C\operatorname{Range}\tilde K \subset \operatorname{Range}\tilde K,\quad \tilde K\operatorname{Range}\tilde{\mathcal{I}} \subset \operatorname{Range}\tilde{\mathcal{I}},\quad \textit{and}\quad \operatorname{Range}\mathcal{M}\tilde K\tilde{\mathcal{I}} \subset \operatorname{Range}\tilde K. \end{aligned}$$

Then the solution $u(t)$ of the time evolution of the quantum graph (QG) *belongs to Y for all $t \in \mathbb{R}$ provided that the initial data u_0 belongs to Y.*

In the statement of the proposition, $A(x), V(x), C$ denote the diagonal matrices with entries ℓ_j^{-1}, $a_j(x)$, $v_j(x)$, respectively, for $x \in (0,1)$.

PROOF. Consider the state space $H = L^2(0,1;\mathbb{C}^m)$ and the form a defined in (13.1). By the results of Chapter 10, one first has to check admissibility of the orthogonal projection Π_K associated with K via (10.4). Let $u \in V$. By the theorem of Pythagoras $|f(x)|^2 = |Kf(x)|^2 + |(I-K)f(x)|^2$ for all $x \in (0,1)$, and therefore

$$\sum_{j=1}^m \int_0^1 v_j(x)\left|\left(\Pi_K u\right)_j(x)\right|^2 dx \le \sum_{j=1}^m \int_0^1 v_j(x)\left|u_j(x)\right|^2 dx < \infty,$$

i.e., $\Pi u \in V$. Furthermore, due to the self-adjointness of Π_K and $V(x)$ there holds

$$\sum_{j=1}^m \int_0^1 \left(-\frac{i}{\ell_j}\left(\Pi_K \frac{du}{dx}\right)_j \overline{a_j\left((I-\Pi_K)u\right)_j} + a_j(\Pi_K u)_j \overline{\left(-\frac{i}{\ell_j}\left((I-\Pi_K)\frac{du}{dx}\right)_j\right)}\right) dx$$
$$= i\left(\frac{du}{dx} \mid \Pi_K CA(I-\Pi_K)u\right)_{L^2(0,1;\mathbb{C}^m)} + i\left((I-\Pi_K)CA\Pi_K u \mid \frac{du}{dx}\right)_{L^2(0,1;\mathbb{C}^m)} = 0,$$

because by assumptions $LA\mathrm{Range}\,\Pi_K \subset \mathrm{Ker}\,(I-\Pi_K)$ as well as $LA\mathrm{Range}\,(I-\Pi_K) \subset \mathrm{Ker}\,\Pi_K$. Thus, one sees like in Theorem 10.9 that $a(\Pi_K u, (I-\Pi_K)u) = 0$. Again by Lemma 6.13 one concludes that Y is left invariant by $(e^{itA})_{t\in\mathbb{R}}$. □

The above result can be applied in order to introduce and characterise a whole family of symmetries of quantum graphs. I do not go into details and refer to [**42**, § 4] for similar results.

Let me conclude this section by discussing a more general setting, related to the physical theory of gauge transformations.

In the prototypical case of the Schrödinger equation in $\mathbb{R}^n$, for example, the Lagrangian is given by

$$\mathcal{L}(\phi) = \int_0^T \int_{\mathbb{R}^n} i\dot\phi(t,x)\overline{\phi(t,x)} + |\nabla\phi(t,x)|^2 dx dt, \tag{13.2}$$

and introducing the Hilbert space $L^2(\mathbb{R}^n)$ and the sesquilinear form

$$a(f,g) := \int_{\mathbb{R}^n} \nabla f(x)\overline{\nabla g(x)} dx, \qquad f \in V := H^1(\mathbb{R}^n),$$

one has

$$\mathcal{L}(\phi) = \int_0^T i(\dot\phi(t)|\phi(t))_{L^2(\mathbb{R}^n} + a(\phi(t),\phi(t))dt.$$

Thus, if $\mathcal{O}$ is a (Lie) group of unitary operators, we conclude that the system admits a global symmetry if and only if

$$Of \in V \qquad \text{and} \qquad a(Of) = a(f) \qquad \text{for all } f \in V \text{ and } O \in \mathcal{O}. \tag{13.3}$$

In analogy to the class of global symmetries introduced in Section 10, consider those unitary groups whose generator is $i\Pi_K$, where Π_K is the orthogonal projection onto the closed subspace

$$Y := \left\{ f \in H : f(x) \in \mathrm{Range} K \text{ for a. e. } x \in \mathbb{R}^n \right\}. \tag{13.4}$$

of H. Of course, Π_K is given by

$$(\Pi_K f)(x) := K(f(x)) \qquad \text{for all } f \in H \text{ and a.e. } x \in \mathbb{R}^n,$$

where K denotes an orthogonal projection of $\mathbb{C}^m$. Then Π_K is self-adjoint, thus by Stone's theorem we can consider $(e^{is\Pi_K})_{s\in\mathbb{R}}$, the unitary group generated by $i\Pi_K$.

It follows from Remark 9.7 that

$$\begin{aligned}(e^{z\Pi_K} f)(x) &= e^z(\Pi_K f)(x) + (\Pi_K^\perp f)(x)\\ &= e^z K(f(x)) + K^\perp(f(x))\\ &= e^{zK}(f(x)) \qquad \text{for all } z\in\mathbb{C},\ f\in H \text{ and a.e. } x \in \mathbb{R}^n.\end{aligned}$$

Here $\Pi_K^\perp$ and $K^\perp$ denote the orthogonal projections onto $Y^\perp$ and $\mathrm{Ker}K^\perp$, respectively.

In fact, I wish to consider a more general setting. More precisely, consider an orthogonal-projection-valued mapping

$$x \mapsto K_x.$$

Let this mapping be of class $H^1(\mathbb{R}^n, M_n(\mathbb{C}))$. One can thus define in analogy to the constant case a subspace

$$Y := \left\{ f \in H : f(x) \in \operatorname{Range} K_x \text{ for a. e. } x \in \mathbb{R}^n \right\}. \tag{13.5}$$

of H. The orthogonal projection onto Y is given by

$$(\Pi_K f)(x) := K_x(f(x)) \qquad \text{for all } f \in H \text{ and a.e. } x \in \mathbb{R}^n. \tag{13.6}$$

REMARK 13.4. Observe that both K and $K^\perp$ are by assumption weakly differentiable and in fact

$$\nabla K^\perp = -\nabla K.$$

Let us also mention the expression

$$\nabla K_x = K_x^\perp (\nabla K_x) K_x + K_x (\nabla K_x) K_x^\perp \qquad \text{for a.e. } x \in \mathbb{R}^n. \tag{13.7}$$

It shows in particular that ∇K_x (which in general is still a hermitian matrix but in general *not* a projection) boasts an off-diagonal block structure that is complementary to that of K_x. This has been derived in [**77**, (1.15)] in a different context. It keeps its validity (with an analogous proof) in our framework though, and leads to

$$\nabla \Pi_K = \Pi_K^\perp (\nabla \Pi_K) \Pi_K + \Pi_K (\nabla \Pi_K) \Pi_K^\perp. \tag{13.8}$$

One can wonder whether the subspace Y defined in (13.4) can be left invariant under $(e^{tA})_{t\geq 0}$, provided that suitable conditions are verified by A, the (self-adjoint) operator associated with the symmetric sesquilinear form a. In order to answer this question, let us recall the following result yielding a characterization of invariant subspaces. It has been proved in [**42**, §5] in the case of an x-independent projection P_Y, but its proof carries over verbatim, since it only relies upon the fact that Π_K is an orthogonal projection.

PROPOSITION 13.5. *Let Π_K be the orthogonal projection onto Y defined in* (13.6). *The following assertions are equivalent.*

(a) The subspace Y is invariant under $(e^{tA})_{t\geq 0}$.
(b) If $f \in V$, then $\Pi_K f \in V$ and $a(\Pi_K f, f) = a(\Pi_K f, \Pi_K f)$.
(c) If $f \in V$, then $e^{is\Pi_K} f \in V$ and $a(f, f) = a(e^{is\Pi_K} f, e^{is\Pi_K} f)$.

PROOF. Due Proposition 9.1, (a) is equivalent to $\Pi_K V \subset V$ and $a(\Pi_K f, (I - \Pi_K) f) = 0$ for every $f \in V$. This is precisely (b).

By Remark 9.7, V is invariant under the action of Π_K if and only if it is invariant under the action of $(e^{is\Pi_K})_{s\in\mathbb{R}}$, and in fact

$$a\left(e^{is\Pi_K} f, e^{is\Pi_K} f\right) = |e^{is} - 1|^2 a(\Pi_K f, \Pi_K f) + 2 \operatorname{Re}(e^{is} - 1) a(\Pi_K f, f) + a(f, f).$$

On the one hand, the identity $|e^{is} - 1|^2 = 2 - 2 \operatorname{Re} e^{is}$ now shows that (b) implies (c). On the other hand, if (c) holds, then the above calculation implies

$$|e^{is} - 1|^2 a(\Pi_K f, \Pi_K f) = -2 \operatorname{Re}(e^{is} - 1) a(\Pi_K f, f)$$

for every $s \in \mathbb{R}$. This is (b) for $s = \pi$. □

Thus, the unitary group $(e^{is\Pi_K})_{s\in\mathbb{R}}$ defined in (13.6) is a global symmetry for a Schrödinger equation if and only if Y is invariant under the time evolution of the associated parabolic diffusion equation, and that this issue can be discussed applying energy methods typical of parabolic problems. In fact, using Proposition 13.5 one can see that this is not the case. The intuitive reason for this is that the symmetry defined by Y is not global, but only *local*: a so-called *gauge symmetry*.

Taking into account Remarks 9.7–13.4 one finds that

$$\begin{aligned}(\nabla e^{is\Pi_K} f)(x) &= \nabla(e^{is}\Pi_K f + \Pi_K^{\perp} f)(x)\\ &= e^{is}\nabla(K_x f(x)) + \nabla(K_x^{\perp} f(x))\\ &= e^{is}(\nabla K_x) f(x) + e^{is} K_x(\nabla f(x)) + (\nabla K_x^{\perp}) f(x) + K_x^{\perp}(\nabla f(x)).\\ &= (e^{is}-1)(\nabla K_x) f(x) + (e^{is}-1) K_x(\nabla f(x)) + \nabla f(x)\\ &= (\nabla f + (e^{is}-1)\nabla(K.f))(x)\\ &= (\nabla f + (e^{is}-1)\nabla(\Pi_K f))(x)\end{aligned}$$

for all $f \in H^1(\mathbb{R}^n)$ and $s \in \mathbb{R}$ and all $x \in \mathbb{R}^n$, and accordingly

$$a(e^{is\Pi_K} f, e^{is\Pi_K} f) = \|\nabla f + (e^{is}-1)\nabla(\Pi_K f)\|^2 =: a_s(f).$$

This shows that in the motivating example introduced in (13.2), the Lagrangian $\mathcal{L}(e^{is\mathcal{P}} f)$ stems from a suitable Schrödinger equation with magnetic potential that depend on s in a 2π-periodic fashion. In general, $a_s(f)$ does not agree with $a(f) = \|\nabla f\|^2$ for all $s \in \mathbb{R}$ and all $f \in H^1(\mathbb{R}^n)$ unless the projections K are x-independent. Thus, in the case of non-constant mapping $x \mapsto K_x$ one cannot hope for a global symmetry.

Let us now discuss the above observation in the context of a Schrödinger equation on a network. In fact, in the motivating example of a quantum graph the sesquilinear form associated with the system is given by

$$a(f,g) := \sum_{j\in E} \frac{1}{\ell_j^2} \int_0^1 f_j'(x)\overline{g_j'(x)}dx,$$

with domain

$$V := \{f \in H^1(0,1;\mathbb{C}^m) : \exists d^f \in \mathbb{C}^n : \Phi^{+\top} d^f = f(0) \text{ and } \Phi^{-\top} d^f = f(1)\},$$

in the simple case of vanishing potentials a, v. As above,

$$a(e^{is\Pi_K} f, e^{is\Pi_K} f) = \|f' + (e^{is}-1)(\Pi_K f)'\|^2,$$

so that in general a global symmetry result fails to hold.

CHAPTER 14

Mixed dynamics on networks

So far, I have only considered problems concerning networks whose ongoing dynamical processes are homogeneous: on each link the evolution equation is of transport, diffusion, wave, or Schrödinger type. However, many physical models consist of coexisting, interacting processes of different type. On different links a different kind of dynamics may take place; or else, one may introduce fictitious, auxiliary links in the model in order to describe certain phenomena in a more efficient way. In this section I will present a toy model of a synaptic connection with mixed parabolic and hyperbolic features. The content of this section is the product of several discussions with Stefano Cardanobile and Rainer Nagel.

Consider a simplified setting of a dendrodendritical chemical synapse: i.e., two dendrites $\mathsf{e}_1, \mathsf{e}_2$ are incident in the synapse v, which is terminal endpoint of e_1 and initial endpoint of e_2. The synaptic input coming from e_1 undergoes a delay of τ_{del} before reaching e_2 and cannot double back. For the sake of simplicity, I also impose *sealed end* conditions on the other endpoints of $\mathsf{e}_1, \mathsf{e}_2$. In other words, a network diffusion problem with boundary delay

$$\text{(BD)}\qquad \begin{cases} \dot{u}_1(t,x) &= u_1''(t,x), & t \geq 0,\ x \in (0,1),\\ \dot{u}_2(t,x) &= u_2''(t,x), & t \geq 0,\ x \in (0,1),\\ u_1(t,1) &= -2u_1'(t,1), & t \geq 0,\\ u_2(t,0) &= 2(u_2'(t,0) - 2u_1'(t-\tau_{\mathrm{del}},1)), & t \geq 0,\\ u_1'(t,0) &= 0, & t \geq 0,\\ u_2'(t,1) &= 0, & t \geq 0,\\ u_1(0,x) &= f_1(0,x), & x \in (0,1),\\ u_2(0,x) &= f_2(0,x), & x \in (0,1),\\ u_1(t,1) &= f_{del}(t), & t \in [-\tau_{\mathrm{del}}, 0], \end{cases}$$

is considered. For the sake of simplicity, I have neglected absorption phenomena at v, i.e., the third equation of (BD) is a classical Kirchhoff law. However, discussing general cable equations on more involved and extended ramied structures is only a lenghtier and more technical exercise, once the above problem is solved. This problem seems to be essentially different from the case of delay in a dynamic node condition. The latter has been often considered, cf. [**78**].

The first boundary condition can be interpreted by saying that the half of all ions reaching the presynaptic nerve terminal is reflected into the dendrite – the other half actually flows further, thus forcing the vescicles to release into the synaptic cleft the neurotransmitters they contain. Finally, it is easy to convince ourselves that the above problem is undetermined if the last initial condition on the delay term is not imposed.

In order to transform the above diffusion problem with node delay into a system of undelayed equations, let us introduce an auxiliary edge e_{del} where the signal coming out of e_1 is "stored" in before traversing v. This is mathematically accomplished by considering a transport equation

$$\text{(14.1)}\qquad \dot{u}_{del}(t,x) = -\tau_{del}^{-1} u_{del}'(t,x), \qquad t \geq 0,\ x \in (0,1),$$

on the edge e_{del}. Observe that the synaptic input needs a time τ_{del} to cross e_{del}. In order to implement the delay feature into the node conditions, I modify the usual continuity assumptions and impose that the presynaptic potential at time t satisfies the continuity condition

$$u_1(t,1) = u_{del}(t,0), \qquad t \geq 0. \tag{14.2}$$

Our aim is now to replace the fourth, delayed equation in (BD) by two node conditions in the endpoints of $\mathsf{e}_{\rm del}$. More precisely, I impose that

$$u_2'(t,0) = \frac{1}{2}u_2(t,0) - u_{\rm del}(t,1), \qquad t \geq 0. \tag{14.3}$$

This equation means that all neurotransmitters reaching the opposite side of the synaptic gap, as well as half the ions sitting in the postsynaptic nerve terminal, determine the flow of *postsynaptic potential.*

Observe that the above model is intrinsically non-symmetric: i.e., potential can only flow from dendrite e_1 to e_2, but not viceversa. This is a typical feature of *chemical synapses*, as opposed to electric synapses.

In other words, one is led to consider an (undelayed) initial boundary value problem

$$\text{(BD')} \qquad \begin{cases} \dot u_1(t,x) &=& u_1''(t,x), & t \geq 0,\ x \in (0,1), \\ \dot u_{\rm del}(t,x) &=& -\tau_{\rm del}^{-1} u_{\rm del}'(t,x), & t \geq 0,\ x \in (0,1), \\ \dot u_2(t,x) &=& u_2''(t,x), & t \geq 0,\ x \in (0,1), \\ u_1(t,1) &=& u_{del}(t,0), & t \geq 0, \\ u_1'(t,1) &=& -\frac{1}{2}u_1(t,1), & t \geq 0, \\ u_2'(t,0) &=& \frac{1}{2}u_2(t,0) - u_{\rm del}(t,1), & t \geq 0, \\ u_1'(t,0) &=& 0, & t \geq 0, \\ u_2'(t,1) &=& 0, & t \geq 0, \\ u_1(0,x) &=& f_1(0,x), & x \in (0,1), \\ u_2(0,x) &=& f_2(0,x), & x \in (0,1), \\ u_{\rm del}(0,x) &=& \tilde f_{\rm del}(0,x), & x \in (0,1), \end{cases}$$

i.e., I have got rid of the boundary delay by passing to the larger state space $H := L^2(0,1;\mathbb{C}^3)$. Here

$$\tilde f_{\rm del}(0,x) = f_{\rm del}(\tau_{\rm del}^{-1}(x-1)), \qquad x \in (0,1).$$

One can check that the problems (BD) and (BD′) are equivalent.

Consider the Hilbert space $H := L^2(0,1;\mathbb{C}^3)$. Then the initial boundary value problem (BD′) is equivalent to an abstract Cauchy problem (ACP), where the operator A is defined by

$$Au := (u_1'', -\tau_{del}^{-1}u_{del}', u_2'')$$

with domain

$$D(A) := \left\{ (u_1, u_{del}, u_2) \in H^2(0,1) \times H^1(0,1) \times H^2(0,1) : \quad \begin{array}{l} u_1'(0) = 0, \\ u_1(1) = u_{del}(0), \\ u'(1) = -\frac{1}{2}u(1), \\ u_2'(0) = \frac{1}{2}u_2(0) - u_{del}(1), \\ u_2'(1) = 0 \end{array} \right\}.$$

Unlike in usual, undelayed network diffusion problems, (ACP) cannot possibly be governed by an analytic semigroup because of the transport term in (BD′). Hence, there is no chance that A is associated with a form that is H-elliptic and continuous.

In order to show that A generates a C_0-semigroup, I am going to apply another strategy. For the sake of simplicity, let me assume in the following that $\tau_{\rm del} = 1$; the general case can be treated by considering a weighted state space.

Lemma 14.1. *The operator B defined by*

$$Bu := \begin{pmatrix} u_1'' & u_{del}' & u_2'' \end{pmatrix}^\top$$

with domain

$$D(B) := \left\{ (u_1, u_{del}, u_2) \in H^2(0,1) \times H^1(0,1) \times H^2(0,1) : \begin{array}{l} u_1'(0) = 0, \\ u_1'(1) = -\frac{1}{2}u_1(1) + u_{del}(0), \\ u_2'(0) = \frac{1}{2}u_2(0), \\ u_2(0) = u_{del}(1), \\ u_2'(1) = 0 \end{array} \right\},$$

is the adjoint A^* of the operator A.

Proof. The adjoint A^* of A is defined by

$$\begin{aligned} D(A^*) &:= \{u \in H : \exists v \in H \text{ such that } (Af \mid u) = (f \mid v) \text{ for all } f \in D(A)\}, \\ A^* u &:= v. \end{aligned}$$

In order to prove that $B \subset A^*$ let $u := (u_1, u_{del}, u_2) \in D(B)$ and $f := (f_1, f_{del}, f_2) \in D(A)$ and compute

$$\begin{aligned} (Af|u)_H &= \int_0^1 f_1''\overline{u_1}dx - \int_0^1 f_{del}'\overline{u_{del}}dx + \int_0^1 f_2''\overline{u_2}dx \\ &= -\int_0^1 f_1'\overline{u_1'}dx + \int_0^1 f_{del}\overline{u_{del}'}dx - \int_0^1 f_2'\overline{u_2'}dx + [f_1'\overline{u_1}]_0^1 - [f_{del}\overline{u_{del}}]_0^1 + [f_2'\overline{u_2}]_0^1 \\ &= \int_0^1 f_1\overline{u_1''}dx + \int_0^1 f_{del}\overline{u_{del}'}dx + \int_0^1 f_2\overline{u_2''}dx \\ &\quad +[f_1'\overline{u_1}]_0^1 - [f_1\overline{u_1'}]_0^1 - [f_{del}\overline{u_{del}}]_0^1 + [f_2'\overline{u_2}]_0^1 - [f_2\overline{u_2'}]_0^1 \\ &= (f|Bu)_H + f_1'(1)\overline{u_1(1)} - f_{del}(1)\overline{u_{del}(1)} + f_{del}(0)\overline{u_{del}(0)} \\ &\quad -f_2'(0)\overline{u_2(0)} - f_1(1)\overline{u_1'(1)} + f_2(0)\overline{u_2'(0)} \\ &= (f|Bu)_H \end{aligned}$$

by virtue of the node conditions satisfied by $f \in D(A)$ and $u \in D(B)$. This concludes the proof of the inclusion $B \subset A^*$. The converse inclusion can be proved likewise. □

Proposition 14.2. *The operator A generates a contractive C_0-semigroup $(T(t))_{t \geq 0}$ on H.*

Proof. One can easily see that the operator A is closed. Let $f = (f_1, f_{del}, f_2) \in D(A)$ and compute

$$\begin{aligned} \mathrm{Re}(Af \mid f)_H &= \mathrm{Re}\int_0^1 f''\overline{f}dx - \mathrm{Re}\int_0^1 g'\overline{g}dx + \mathrm{Re}\int_0^1 h''\overline{h}dx \\ &= -\int_0^1 |f'|^2dx - \int_0^1 |h'|^2dx + \mathrm{Re}\left[f'\overline{f}\right]_0^1 - \frac{1}{2}\left[|g|^2\right]_0^1 + \mathrm{Re}\left[h'\overline{h}\right]_0^1 \\ &\leq -\frac{1}{2}|g(0)|^2 - \frac{1}{2}|g(1)|^2 + \frac{1}{2}|g(0)|^2 + \mathrm{Re}\left(g(1) - \frac{1}{2}h(0)\right)\overline{h(0)} \\ &\leq -\left(\frac{1}{2}|g(1)|^2 - \mathrm{Re}g(1)\overline{h(0)} + \frac{1}{2}|h(0)|^2\right) \leq 0. \end{aligned}$$

This shows that the operator A is dissipative. In order to show that it generates a C_0-semigroup, by Proposition 3.7 it suffices to show that A^* is dissipative, too. Indeed, a direct computation yields

$$\begin{aligned}
\mathrm{Re}(A^*f \mid f)_H &= \mathrm{Re}\int_0^1 f''\overline{f}dx + \mathrm{Re}\int_0^1 g'\overline{g}dx + \mathrm{Re}\int_0^1 h''\overline{h}hdx \\
&= -\int_0^1 |f'|^2dx - \int_0^1 |h'|^2dx + \mathrm{Re}\left[f'\overline{f}\right]_0^1 + \frac{1}{2}\left[|g|^2\right]_0^1 + \mathrm{Re}\left[h'\overline{h}\right]_0^1 \\
&\leq -\frac{1}{2}|g(1)|^2 + \frac{1}{2}|g(1)|^2 - \frac{1}{2}|g(0)|^2 + \mathrm{Re}\left(g(0) - \frac{1}{2}f(1)\right)\overline{f(1)} \\
&\leq -\left(\frac{1}{2}|g(0)|^2 - \mathrm{Re}g(0)\overline{f(1)} + \frac{1}{2}|f(1)|^2\right) \leq 0
\end{aligned}$$

for all $\mathfrak{f} \in D(A^*)$. This concludes the proof. □

In fact, more can be said.

Proposition 14.3. *The semigroup $(T(t))_{t\geq 0}$ on the Hilbert lattice H is positive.*

Proof. By [**79**, § C-II.1] it suffices to show that A is dispersive, i.e., that for every $\mathfrak{f} \in D(A)$ there exists $0 \leq \phi_\mathfrak{f} \in H$ such that

- $\|\phi_\mathfrak{f}\|_H \leq 1$,
- $(\mathfrak{f} \mid \phi_\mathfrak{f})_H = \|\mathrm{Re}\mathfrak{f}^+\|$, and
- $\mathrm{Re}(A\mathfrak{f} \mid \phi_\mathfrak{f})_H \leq 0$.

Let in fact $\mathfrak{f} = \begin{pmatrix} f & g & h \end{pmatrix} \in D(A)$ and define $\phi_\mathfrak{f}$ as a vector in the positive cone of H by

$$\phi_\mathfrak{f} := \begin{pmatrix} \frac{\mathrm{Re}f^+}{\|\mathrm{Re}\mathfrak{f}^+\|} & \frac{\mathrm{Re}g^+}{\|\mathrm{Re}\mathfrak{f}^+\|} & \frac{\mathrm{Re}h^+}{\|\mathrm{Re}\mathfrak{f}^+\|} \end{pmatrix}^\top .$$

Then, one has

$$\|\phi_\mathfrak{f}\|_H^2 = \frac{1}{\|\mathrm{Re}\mathfrak{f}^+\|^2}\left(\int_0^1 |\mathrm{Re}f^+|^2dx + \int_0^1 |\mathrm{Re}g^+|^2dx + \int_0^1 |\mathrm{Re}h^+|^2dx\right) = 1.$$

Furthermore,

$$\begin{aligned}
(\mathfrak{f} \mid \phi_\mathfrak{f})_H &= \frac{1}{\|\mathrm{Re}\mathfrak{f}^+\|}\left(\int_0^1 f\ \mathrm{Re}f^+dx + \int_0^1 g\ \mathrm{Re}g^+dx + \int_0^1 h\ \mathrm{Re}h^+dx\right) \\
&= \frac{1}{\|\mathrm{Re}\mathfrak{f}^+\|}\left(\int_0^1 |\mathrm{Re}f^+|^2dx + \int_0^1 |\mathrm{Re}g^+|^2dx + \int_0^1 |\mathrm{Re}h^+|^2dx\right) \\
&= \|\mathrm{Re}\mathfrak{f}^+\|.
\end{aligned}$$

Finally,

$$\mathrm{Re}(A\mathfrak{f} \mid \phi_\mathfrak{f})_H = \frac{1}{\|\mathrm{Re}\mathfrak{f}^+\|}\left(\int_0^1 f''f\mathbf{1}_{\{\mathrm{Re}f\geq 0\}}dx - \int_0^1 g'g\mathbf{1}_{\{\mathrm{Re}g\geq 0\}}dx + \int_0^1 h''h\mathbf{1}_{\{\mathrm{Re}h\geq 0\}}dx\right) \leq 0.$$

This concludes the proof. □

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