Second order abstract initial-boundary value problems

Dissertation
der Fakultät für Mathematik und Physik
der Eberhard-Karls-Universität Tübingen
zur Erlangung des Grades eines
Doktors der Naturwissenschaften

Vorgelegt von
Delio Mugnolo
aus Bari

2004

Tag der mündlichen Qualifikation	**24. Juni 2004**
Dekan:	**Prof. Dr. Herbert Müther**
1. Berichterstatter	**Prof. Dr. Rainer Nagel**
2. Berichterstatter	**Prof. Dr. Frank Loose**

Introduction

Partial differential equations on bounded domains of $\mathbb{R}^n$ have traditionally been equipped with homogeneous boundary conditions (usually Dirichlet, Neumann, or Robin). However, other kinds of boundary conditions can also be considered, and for a number of concrete application it seems that dynamic (i.e., time-dependent) boundary conditions are the right ones.

Motivated by physical problems, numerous partial differential equations with dynamic boundary conditions have been studied in the last decades: H. Amann and J.L. Lions, among others, have investigated elliptic equations (see, e.g., [Li61, Chapt. VI.6], [Hi89], [Gu94], [AF97], and references therein); J. Escher has investigated parabolic problems (see [Es93] and references therein); and J.T. Beale and V.N. Krasil'nikov, among others, have investigated second order hyperbolic equations *with dynamical boundary conditions* (see [Be76], [Kr61], [Be00], and references therein).

In recent years, a systematic study of problems of this kind has been performed mainly by A. Favini, J.A. Goldstein, G.R. Goldstein, and S. Romanelli, who in a series of papers (see [FGGR02], [FGG+03], and references therein) have convincingly shown that dynamic boundary conditions are the natural L^p-counterpart to the well-known (generalized) Wentzell boundary conditions. On the other side, K.-J. Engel has introduced a powerful abstract technique to handle this kind of problems, reducing them in some sense to usual, perturbed evolution equations with homogeneous, time-independent boundary conditions (see [En99], [CENN03], and [KMN03]). Both schools reduce the problem to an abstract Cauchy problem associated to an operator matrix on a suitable product space.

We remark that more recently an abstract approach that in some sense unifies dynamic and static boundary value problems has been developed by G. Nickel, cf. [Ni04].

In the first chapter we introduce an abstract setting to consider what we call an *abstract initial boundary value problem*, i.e., a system of the form

$$\text{(AIBVP)} \qquad \begin{cases} \dot{u}(t) &= Au(t), & t \geq 0, \\ \dot{x}(t) &= Bu(t) + \tilde{B}x(t), & t \geq 0, \\ x(t) &= Lu(t), & t \geq 0, \\ u(0) &= f \in X, \\ x(0) &= g \in \partial X. \end{cases}$$

Here the first equation takes place on a Banach *state space* X (in concrete applications, this is often a space of functions on a domain $\Omega \subset \mathbb{R}^n$ with smooth, nonempty boundary $\partial\Omega$). The third equation represents a coupling relation between the variable in X and the variable in a Banach *boundary space* ∂X (in concrete applications, this is often a space of functions on $\partial\Omega$). Finally, the second equation represents an evolution equation on the boundary with a feedback term given by the operator B.

Following [KMN03, § 2], we first define reasonable notions of solution to, and well-posedness of (AIBVP). Then, we show the equivalence between its well-posedness and the well-posedness of the abstract Cauchy problem

$$(0.1) \qquad \begin{cases} \begin{pmatrix} \dot u \\ \dot x \end{pmatrix}(t) &= \begin{pmatrix} A & 0 \\ B & \tilde B \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix}(t), \quad t \geq 0, \\ \\ \begin{pmatrix} u \\ x \end{pmatrix}(0) &= \begin{pmatrix} f \\ g \end{pmatrix} \end{cases}$$

on the product space $X \times \partial X$. This formally justifes the semigroup techniques used, e.g., in [FGGR02], [AMPR03], and [CENN03]. It is crucial that the operator matrix that appears in (0.1) has a suitable, non-diagonal domain, as discussed in detail in Chapter 2. We refer to [Ni04] for a systematic treatment of these issues.

Then, it is natural to extend such results to second order problems like

$$\begin{cases} \ddot u(t) &= Au(t), & & t \in \mathbb{R}, \\ \ddot x(t) &= Bu(t) + \tilde B x(t), & & t \in \mathbb{R}, \\ u(0) &= f \in X, & \dot u(0) = g \in X, & \\ x(0) &= h \in \partial X, & \dot x(0) = j \in \partial X. & \end{cases}$$

However, we still need to impose a coupling relation between the variables $u(\cdot)$ and $x(\cdot)$. In fact, a second order abstract problem can be equipped with several kinds of dynamic boundary conditions, and they differ essentially in the coupling relation: motivated by applications we consider three kinds of them. We show that the well-posedness of such problems is related to the theory of cosine operator functions.

In the second chapter we consider a certain class of operator matrices $\mathcal{A}$ that arise naturally while transforming (AIBVP) into an abstract Cauchy problem. The peculiarity of such operator matrices is that their domain is not a diagonal subset of the product Banach space $X \times \partial X$ (say, $D(A) \times D(\tilde B)$); instead, following K.-J. Engel (see [En97], [En99], and [KMN03b]) we introduce the notion of *operator matrix with coupled domain*. We recall some known properties of such operator matrices and prove several new results: in particular, in Section 2.2 we are able to characterize boundedness of the semigroup generated by $\mathcal{A}$ and resolvent compactness of $\mathcal{A}$, to obtain a regularity result, and moreover to generalize some generation results obtained in [CENN03] and [KMN03]. The results obtained here are systematically exploited in the following chapters.

In the third chapter we consider a second order problem where the coupling relation is given by

$$\dot x(\cdot) = Lu(\cdot).$$

This is physically motivated by so-called *wave equations with acoustic boundary conditions*, first investigated in [MI68] and [BR74] (for bounded domains of $\mathbb{R}^3$), and more recently in [GGG03]. The traditional approach has been recently extended by C. Gal to bounded domains of $\mathbb{R}^n$. Gal's results concern well-posedness and compactness issues, and have been obtained simultaneously to, but independently of ours; they will appear in [Ga04]. The core of this chapter is [Mu04].

It is possible to say that, roughly speaking, wave equations with acoustic boundary conditions have been traditionally interpreted as wave equations equipped with (first order) dynamic Neumann-like boundary conditions, cf. Section 3.1. Instead, we argue that acoustic boundary conditions should be looked at as dynamic (first order) *Robin*-like boundary conditions. To our opinion, this accounts for several properties of such systems, including well-posedness and resolvent compactness of the associated operator matrix.

In the fourth chapter we investigate second order problems equipped with abstract second order dynamic boundary conditions, given by

$$x(\cdot) = Lu(\cdot) \tag{0.2}$$

or else

$$x(\cdot) = Lu(\cdot) \qquad \text{and} \qquad \dot{x}(\cdot) = L\dot{u}(\cdot). \tag{0.3}$$

As shown in [Mu04b], on which this chapter is essentially based, dynamic boundary conditions complemented with (0.2) or (0.3) represent quite different concrete problems, modelling, for example, in concrete applications second order Neumann (or Robin) and Dirichlet dynamic boundary conditons, respectively. We show that an abstract approach to these boundary conditions is necessarily different. In fact, we can show that the phase space associated to such problems depend on the assumed coupling relation. More precisely, if (0.2) holds, then the first coordinate-space of the phase space associated to the problem is a diagonal subspace of $X \times \partial X$, while if (0.3) holds, then the first coordinate-space of the phase space is shown to be a certain subpaces of $X \times \partial X$ that contains a coupling relation in its definition. This kind of non-diagonal spaces has been considered, e.g., in [En03] to discuss heat equations with dynamic boundary conditions on spaces of continuous functions.

In the fifth chapter we generalize the problem to *complete* second order problems, i.e., sytems where the first equation is

$$\ddot{u}(t) = Au(t) + C\dot{u}(t), \qquad t \in \mathbb{R}.$$

Also in this case we need to distinguish between cases that represent abstract versions of dynamic Dirichlet and Neumann boundary conditions. We also consider the case of *overdamped* complete problems, i.e., where C is more unbounded than A. Similar abstract problems have been investigated, by different means, in [XL04b]; concrete problems fitting into this framework have been considered, e.g., in [CENP04], equipped with both first and second order dynamic boundary conditions.

In Appendix A we recall some well-known facts about C_0-semigroups, including perturbation and almost periodicity results.

Appendix B contains basic results in the theory of cosine operator functions; most of them are well-known. Moreover, the boundedness of the H^∞-calculus associated to the invertible generator of a bounded cosine operator function on a UMD-space is established. We also briefly describe the well-posedness of some classes of complete second-order problems. Using a new Desch–Schappacher-type perturbation result, we can also obtain the well-posedness of a certain class of overdamped abstract wave equations, complementing known results stated in [EN00, § VI.3].

In Appendix C we collect some basic facts and relations about Dirichlet operators, i.e. solution operators of abstract (eigenvalue) Dirichlet problems of the form

$$\begin{cases} Au &= \lambda u, \\ Lu &= x. \end{cases}$$

Such operators, already investigated in [Gr87] and [GK91], play a key role in our approach.

Acknowledgement. It is my great pleasure to express my warmest thank to several people for their support.

First of all, my supervisor Rainer Nagel has invited me to Tübingen four years ago, patiently introduced me to semigroup theory, tried to moderate my impulsive mathematical character, co-written two papers, and taught me that a mathematician should always struggle to improve his results.

I am indebted to Giséle Ruiz Goldstein and Jerry Goldstein, as well as to Zoltán Sebestyén and Andras Bátkai, for their kind invitation to visit the University of Memphis and the Eötvös Loránd University of Budapest, respectively, and for their kind hospitality there.

The topic of [Mu04] was suggested to me by Jerry Goldstein in Memphis; during that visit he has also helped me to write [Mu04c], whose results are not included in this thesis, and reviewed a still unpublished note which later became Chapter 5 of this thesis.

I owe everything I know about wave equations and operator matrices to Jerry Goldstein and Klaus-Jochen Engel, who have also supported me and provided me with countless suggestions and remarks.

Silvia Romanelli has always warmly discussed with me whenever I have visited the University of Bari, where I have completed my undergraduate studies.

Writing my first papers with Marjeta Kramar has been a great fun and an important human contact, beside a nice mathematical experience.

I also thank all the members of the Arbeitsgemeischaft Funktionalanalysis of the University of Tübingen, and in particular Ulf Schlotterbeck, for making Mathematical Institute's sixth floor a magical place to do research at.

Last but not least, Maria Winkler has supported me with patience, understanding, and love. I can never possibly be grateful enough to her.

Contents

Chapter 1

Well-posedness of abstract initial-boundary value problems

1.1 Abstract initial-boundary value problems

The standard setting throughout this chapter is the following.

- X is a Banach space.
- ∂X is a Banach space.
- $A : D(A) \subset X \to X$ is a linear operator.
- $L : D(L) \subset X \to \partial X$ is a linear operator such that $D(A) \subset D(L)$.
- $B : D(B) \subset X \to \partial X$ is a linear operator such that $D(A) \subset D(B)$.
- $\tilde{B} : D(\tilde{B}) \subset \partial X \to \partial X$ is a linear operator.

We denote by

$$\mathcal{X} := X \times \partial X$$

the product space of X and ∂X, and by π_1 and π_2 the projections from $\mathcal{X}$ onto X and ∂X, respectively.

For these operators we consider what we call an *abstract initial-boundary value problem on the state space X and the boundary space ∂X*:

$$\text{(AIBVP}_{f,g}) \qquad \begin{cases} \dot{u}(t) &= Au(t), & t \geq 0, \\ \dot{x}(t) &= Bu(t) + \tilde{B}x(t), & t \geq 0, \\ x(t) &= Lu(t), & t \geq 0, \\ u(0) &= f \in X, \\ x(0) &= g \in \partial X. \end{cases}$$

If it is clear from the context which initial data f, g we are considering, we will simply write (AIBVP) instead of (AIBVP$_{f,g}$).

In order to tackle (AIBVP) by means of C_0-semigroups we consider the operator matrix on $\mathcal{X}$ given by

$$(1.1)\qquad \mathcal{A} := \begin{pmatrix} A & 0 \\ B & \tilde{B} \end{pmatrix}, \qquad D(\mathcal{A}) := \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in D(A) \times D(\tilde{B}) \,:\, Lu = x \right\}.$$

We emphasize that in general $\mathcal{A}$ does *not* have diagonal domain, i.e., $D(\mathcal{A})$ is not a diagonal subset of $\mathcal{X}$.

Our aim is to show that (AIBVP$_{f,g}$) is equivalent to the abstract Cauchy problem

$$(\mathcal{ACP}_{\mathfrak{f}})\qquad \begin{cases} \dot{\mathfrak{u}}(t) &= \mathcal{A}\,\mathfrak{u}(t), \quad t \geq 0, \\ \mathfrak{u}(0) &= \mathfrak{f} \in \mathcal{X}, \end{cases}$$

on the product space $\mathcal{X}$, where we have set

$$(1.2)\qquad \mathfrak{u}(t) := \begin{pmatrix} u(t) \\ Lu(t) \end{pmatrix}, \quad t \geq 0, \qquad \text{and} \quad \mathfrak{f} := \begin{pmatrix} f \\ g \end{pmatrix}.$$

Motivated by applications, we do *not* assume the operator A introduced above to be closed. Instead, we assume the operator

$$(1.3)\qquad \begin{pmatrix} A \\ L \end{pmatrix} : D(A) \ni u \mapsto \begin{pmatrix} Au \\ Lu \end{pmatrix} \in \mathcal{X}$$

to be closed. Under this assumption, we obtain a Banach space by endowing $D(A)$ with the graph norm of $\binom{A}{L}$, i.e.,

$$\|u\|_{\binom{A}{L}} := \|u\|_X + \|Au\|_X + \|Lu\|_{\partial X}.$$

We denote this Banach space by $[D(A)]_L$.

In many applications the restriction A_0 defined by

$$(1.4)\qquad A_0 u := Au \qquad \text{for all } u \in D(A_0) := D(A) \cap \ker(L)$$

plays an important role. If if is closed, then $D(A_0)$ becomes a Banach space $[D(A_0)]$ when equipped with the graph norm

$$\|u\|_{A_0} := \|u\|_X + \|A_0 u\|_X.$$

Remark 1.1.1. If the operators $\binom{A}{L}, \tilde{B}, \mathcal{A}$ are closed, then it follows by definition that also A_0 is closed and further $[D(A_0)] \hookrightarrow [D(A)]_L$ and $[D(\mathcal{A})] \hookrightarrow \Big([D(A)]_L \times [D(\tilde{B})]\Big)$.

We begin by showing some relations between operator theoretical properties of the operators $A, B, L, \tilde{B}, \mathcal{A}$ defined above.

Lemma 1.1.2. *The following assertions hold.*

(1) *Assume* $\binom{A}{L}$ *and* $\tilde{B}$ *to be closed, and* B *to be bounded from* $[D(A)]_L$ *to* ∂X*. Then* $\mathcal{A}$ *is closed.*

(2) *Assume* $\mathcal{A}$ *and* $\binom{A}{L}$ *to be closed and* $\tilde{B}$ *to be bounded. Then* B *is bounded from* $[D(A)]_L$ *to* ∂X*.*

(3) *Assume* $\mathcal{A}$ *to be closed,* $\tilde{B}$ *to be bounded, and* B *to be bounded from* $[D(A)]_L$ *to* ∂X*. Then* $\binom{A}{L}$ *is closed.*

(4) *Assume* $\mathcal{A}$ *and* $\binom{A}{L}$ *to be closed and* B *to be bounded from* $[D(A)]_L$ *to* ∂X*. If* A_0 *has nonempty resolvent set, then* $\tilde{B}$ *is closed.*

Proof. (1) Let

$$\begin{pmatrix} u_n \\ Lu_n \end{pmatrix}_{n\in\mathbb{N}} \subset D(\mathcal{A}), \quad \lim_{n\to\infty} \begin{pmatrix} u_n \\ Lu_n \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix},$$

$$\text{and} \quad \lim_{n\to\infty} \mathcal{A} \begin{pmatrix} u_n \\ Lu_n \end{pmatrix} = \lim_{n\to\infty} \begin{pmatrix} Au_n \\ Bu_n + \tilde{B}Lu_n \end{pmatrix} = \begin{pmatrix} w \\ z \end{pmatrix}$$

for some $u, w \in X$ and $v, z \in \partial X$. Since $(u_n)_{n\in\mathbb{N}} \subset D(A)$,

$$\lim_{n\to\infty} \begin{pmatrix} A \\ L \end{pmatrix} u_n = \lim_{n\to\infty} \begin{pmatrix} Au_n \\ v_n \end{pmatrix} = \begin{pmatrix} w \\ v \end{pmatrix},$$

i.e., the sequence (u_n) converges in the Banach space $[D(A)]_L$. Thus, the boundedness of B from $[D(A)]_L$ to ∂X implies that $\lim_{n\to\infty} Bu_n = Bu$ and consequently $\lim_{n\to\infty} \tilde{B}Lu_n = z - Bu$. Moreover, $(Lu_n)_{n\in\mathbb{N}} \subset D(\tilde{B})$ and since $\tilde{B}$ is closed, it follows that $v \in D(\tilde{B})$ and $\tilde{B}v = z - Bu$. Moreover, the closedness of $\binom{A}{L}$ now yields that $u \in D(A)$ and $Lu = v$, thus showing that $\binom{u}{v} \in D(\mathcal{A})$, and furthermore $Au = w$. Hence $\mathcal{A}\binom{u}{v} = \binom{w}{z}$. Thus, $\mathcal{A}$ is closed.

(2) Let

$$(u_n)_{n\in\mathbb{N}} \subset D(A), \quad [D(A)]_L - \lim_{n\to\infty} u_n = u, \quad \text{and} \quad \lim_{n\to\infty} Bu_n = z$$

for some $u \in D(A)$, $z \in \partial X$, where "$[D(A)]_L - \lim$" stands for the limit with respect to the norm of $[D(A)]_L$. It follows that $u \in D(A)$ and also $\lim_{n\to\infty} Au_n = Au$ and $\lim_{n\to\infty} Lu_n = Lu$. Consequently $\lim_{n\to\infty} \tilde{B}Lu_n = \tilde{B}Lu$. Thus,

$$\begin{pmatrix} u_n \\ Lu_n \end{pmatrix}_{n\in\mathbb{N}} \subset D(\mathcal{A}), \quad \lim_{n\to\infty} \begin{pmatrix} u_n \\ Lu_n \end{pmatrix} = \begin{pmatrix} u \\ Lu \end{pmatrix},$$

$$\text{and} \quad \lim_{n\to\infty} \mathcal{A} \begin{pmatrix} u_n \\ Lu_n \end{pmatrix} = \lim_{n\to\infty} \begin{pmatrix} Au_n \\ Bu_n + \tilde{B}Lu_n \end{pmatrix} = \begin{pmatrix} Au \\ z + \tilde{B}Lu \end{pmatrix}.$$

Due to the closedness of $\mathcal{A}$, we obtain in particular that $Bu = z$, thus showing the closedness of B as an operator from $[D(A)]_L$ to ∂X. The claim now follows by the closed graph theorem.

(3) Let

$$(u_n)_{n\in\mathbb{N}} \subset D(A), \quad \lim_{n\to\infty} u_n = u, \quad \text{and} \quad \lim_{n\to\infty} \begin{pmatrix} A \\ L \end{pmatrix} u_n = \lim_{n\to\infty} \begin{pmatrix} Au_n \\ Lu_n \end{pmatrix} = \begin{pmatrix} w \\ v \end{pmatrix}$$

for some $u, w \in X$, $v \in \partial X$. This means that $(u_n)_{n\in\mathbb{N}}$ converges with respect to the norm of $[D(A)]_L$, hence $\lim_{n\to\infty} Bu_n = Bu$. Moreover $\lim_{n\to\infty} \tilde{B}Lu_n = \tilde{B}v$, and it follows that

$$\begin{pmatrix} u_n \\ Lu_n \end{pmatrix}_{n\in\mathbb{N}} \subset D(\mathcal{A}), \quad \lim_{n\to\infty} \begin{pmatrix} u_n \\ Lu_n \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix},$$

$$\text{and} \quad \lim_{n\to\infty} \mathcal{A} \begin{pmatrix} u_n \\ Lu_n \end{pmatrix} = \lim_{n\to\infty} \begin{pmatrix} Au_n \\ Bu_n + \tilde{B}Lu_n \end{pmatrix} = \begin{pmatrix} w \\ Bu + \tilde{B}v \end{pmatrix}.$$

Finally, due to the closedness of $\mathcal{A}$,

$$\begin{pmatrix} u \\ v \end{pmatrix} \in D(\mathcal{A}) \quad \text{and} \quad \mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} Au \\ Bu + \tilde{B}v \end{pmatrix} = \begin{pmatrix} w \\ Bu + \tilde{B}v \end{pmatrix},$$

i.e., $u \in D(A)$, $Lu = v$, and $Au = w$.

(4) Let

$$(x_n)_{n\in\mathbb{N}} \subset D(\tilde{B}), \quad \lim_{n\to\infty} x_n = x, \quad \text{and} \quad \lim_{n\to\infty} \tilde{B}x_n = y$$

for some $x, y \in \partial X$. Take $\lambda \in \rho(A_0)$ and observe that by assumption Lemma C.1 applies and yields the existence of the Dirichlet operator $D_\lambda^{A,L}$ associated to the pair (A, L). Thus, for all $n \in \mathbb{N}$ there exists $u_n := D_\lambda^{A,L} x_n \in D(A)$ such that by definition $Au_n = \lambda u_n$ and $Lu_n = x_n$. Moreover, by Lemma C.4 $D_\lambda^{A,L}$ is bounded from ∂X to $[D(A)]_L$, and it follows that $\lim_{n\to\infty} Bu_n = BD_\lambda^{A,L} x$.

Summing up, we can consider

$$\begin{pmatrix} u_n \\ x_n \end{pmatrix}_{n\in\mathbb{N}} \subset D(\mathcal{A}) \quad \text{such that} \quad \lim_{n\to\infty} \begin{pmatrix} u_n \\ x_n \end{pmatrix} = \begin{pmatrix} D_\lambda^{A,L} x \\ x \end{pmatrix}$$

$$\text{and} \quad \lim_{n\to\infty} \mathcal{A} \begin{pmatrix} u_n \\ x_n \end{pmatrix} = \lim_{n\to\infty} \begin{pmatrix} Au_n \\ Bu_n + \tilde{B}x_n \end{pmatrix} = \begin{pmatrix} AD_\lambda^{A,L} x \\ BD_\lambda^{A,L} x + y \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} D_\lambda^{A,L} x \\ x \end{pmatrix} \in D(\mathcal{A}) \quad \text{and} \quad \mathcal{A} \begin{pmatrix} D_\lambda^{A,L} x \\ x \end{pmatrix} = \begin{pmatrix} AD_\lambda^{A,L} \\ BD_\lambda^{A,L} x + \tilde{B}x \end{pmatrix},$$

and we conclude that $x \in D(\tilde{B})$ and further $\tilde{B}x = y$. □

Lemma 1.1.3. *The following assertions hold.*

(1) Assume A_0 and $\tilde{B}$ to be densely defined in X and ∂X, respectively. If L is surjective from $D(A)$ to $D(\tilde{B})$, then $\mathcal{A}$ is densely defined.

(2) If $\mathcal{A}$ is densely defined, then A and $\tilde{B}$ are densely defined in X and ∂X, respectively.

Proof. (1) Let $x \in X$, $y \in \partial X$, $\epsilon > 0$. Take $z \in D(\tilde{B})$ such that $\|y - z\| < \epsilon$. The surjectivity of L ensures that there exists $u \in D(A)$ such that $Lu = z$. Take $\tilde{u}, \tilde{x} \in \ker(L) \cap D(A)$ such that $\|u - \tilde{u}\|_X < \epsilon$ and $\|x - \tilde{x}\|_X < \epsilon$. Let $w := \tilde{x} + u - \tilde{u} \in D(A)$. Then

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} w \\ z \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} x - \tilde{x} \\ 0 \end{pmatrix} \right\| + \left\| \begin{pmatrix} u - \tilde{u} \\ 0 \end{pmatrix} \right\| + \left\| \begin{pmatrix} 0 \\ y - z \end{pmatrix} \right\| < 3\epsilon.$$

Since $L(w) = L(u) = z$, we obtain $\binom{w}{z} \in D(\mathcal{A})$.

(2) The claim follows immediately by definition. □

1.2 Equivalence between $(\mathcal{ACP})$ and (AIBVP)

It is known that the well-posedness of abstract Cauchy problems is related to the theory of C_0-semigroups, as recalled in Appendix A. Thus, it is reasonable to impose, throughout this section, a set of minimal assumptions on the operators $A, B, L, \tilde{B}$ ensuring that the operator matrix $\mathcal{A}$ be closed and densely defined. The following are motivated by Lemma 1.1.2.

Assumptions 1.2.1.

1. *A_0 is densely defined.*
2. $\begin{pmatrix} A \\ L \end{pmatrix}$ *is closed.*
3. *L is surjective.*
4. *B is bounded from $[D(A)]_L$ to $[D(\tilde{B})]$.*
5. *$\tilde{B}$ is closed and densely defined.*

Under these assumptions we now make precise what we understand by a solution to an abstract initial-boundary value problem.

Definition 1.2.2. *A* classical solution (in $(X, \partial X)$) to (AIBVP$_{f,g}$) *is a function* $u(\cdot)$ *such that*

- $u(\cdot) \in C^1(\mathbb{R}_+, X)$,
- $u(t) \in D(A)$ *for all* $t \geq 0$,
- $Lu(\cdot) \in C^1(\mathbb{R}_+, \partial X)$,
- $Lu(t) \in D(\tilde{B})$ *for all* $t \geq 0$, *and*
- $u(\cdot)$ *satisfies* (AIBVP$_{f,g}$).

Moreover, (AIBVP) *is called* well-posed on $(X, \partial X)$ *if*

- (AIBVP$_{f,g}$) *admits a unique classical solution* $u = u(\cdot, f, g)$ *for all initial data* $f \in D(A)$, $g \in D(\tilde{B})$ *satisfying the compatibility condition* $Lf = g$, *and*

- *for all sequences of initial data* $(f_n, g_n)_{n\in\mathbb{N}} \subset D(A) \times D(\tilde{B})$ *tending to* 0 *and satisfying the compatibility condition* $Lf_n = g_n$, *one has* $\lim_{n\to\infty} u(t, f_n, g_n) = 0$ *and* $\lim_{n\to\infty} Lu(t, f_n, g_n) = 0$ *uniformly for* t *in compact intervals.*

Our aim is to show that the well-posedness of the abstract initial-boundary value problem (AIBVP) is equivalent to the well-posedness of the abstract Cauchy problem ($\mathcal{ACP}$), with $\mathcal{A}$ defined as in (1.1). To this purpose we relate the solutions of the two problems. We recall that π_1, π_2 denote the first and second projection of the product space $\mathcal{X} = X \times \partial X$, respectively.

Lemma 1.2.3. *The following assertions hold.*

(1) If $u(\cdot)$ *is a classical solution to* $(\text{AIBVP}_{f,g})$, *then*

$$\mathfrak{u}(\cdot) := \begin{pmatrix} u(\cdot) \\ Lu(\cdot) \end{pmatrix}$$

is a classical solution to $(\mathcal{ACP}_{\binom{f}{g}})$.

(2) Conversely, if $\mathfrak{u}(\cdot)$ *is a classical solution to* $(\mathcal{ACP}_{\mathfrak{f}})$, *then*

$$u(\cdot) := \pi_1 \mathfrak{u}(\cdot)$$

is a classical solution to $(\text{AIBVP}_{\pi_1\mathfrak{f}, \pi_2\mathfrak{f}})$.

Proof. (1) Let $u(\cdot)$ be a classical solution to $(\text{AIBVP}_{f,g})$. It follows that $u \in C^1(\mathbb{R}_+, X)$ and $Lu \in C^1(\mathbb{R}_+, \partial X)$, and therefore $\mathfrak{u} \in C^1(\mathbb{R}_+, \mathcal{X})$. Moreover, $u(t) \in D(A)$ and $Lu(t) \in D(\tilde{B})$ for all $t \geq 0$, thus $\mathfrak{u}(t) \in D(\mathcal{A})$ for all $t \geq 0$. Finally, one can see that $(\mathcal{ACP}_{\binom{f}{g}})$ is fulfilled.

(2) Assume now $\mathfrak{u}(\cdot)$ to be a classical solution to $(\mathcal{ACP}_{\mathfrak{f}})$. Then $\mathfrak{u}(t) \in D(\mathcal{A})$ for $t \geq 0$ and hence $u(t) := \pi_1\mathfrak{u}(t) \in D(A)$ and $\pi_2\mathfrak{u}(t) = L\pi_1\mathfrak{u}(t) = Lu(t) \in D(\tilde{B})$ for all $t \geq 0$. It also follows from $\mathfrak{u} \in C^1(\mathbb{R}_+, \mathcal{X})$ that $u \in C^1(\mathbb{R}_+, X)$ and $Lu \in C^1(\mathbb{R}_+, \partial X)$. One can see that $(\text{AIBVP}_{\pi_1\mathfrak{f}, \pi_2\mathfrak{f}})$ is fulfilled, and the claim follows. □

Finally, we can show that the well-posedness of ($\mathcal{ACP}$), i.e., the generator property of $\mathcal{A}$ is equivalent to the well-posedness of the corresponding abstract initial-boundary value problem (AIBVP). This motivates the strategy of our later investigation.

Theorem 1.2.4. *The operator matrix* $\mathcal{A}$ *generates a* C_0*-semigroup* $(e^{t\mathcal{A}})_{t\geq 0}$ *on the product space* $\mathcal{X} = X \times \partial X$ *if and only if* (AIBVP) *is well-posed on* $(X, \partial X)$.

In this case, the unique classical solution to $(\text{AIBVP}_{f,g})$ *is given by*

$$u = u(t, f, g) := \pi_1 e^{t\mathcal{A}} \begin{pmatrix} f \\ g \end{pmatrix}, \qquad t \geq 0,$$

for all initial data $f \in D(A)$ *and* $g \in D(\tilde{B})$ *such that* $Lf = g$.

Proof. Let $\mathcal{A}$ generate a C_0-semigroup on $\mathcal{X}$. Then $\mathcal{A}$ is closed, thus by Lemma A.3 the associated abstract Cauchy problem ($\mathcal{ACP}$) is well-posed. By Lemma 1.2.3, $u = u(t,(f,g)) = \pi_1 \mathfrak{u}(t, \binom{f}{g}) = \pi_1 e^{t\mathcal{A}}\binom{f}{g}$, $t \geq 0$, yields a classical solution to ($\mathrm{AIBVP}_{f,g}$) for all $(f,g) \in D(A) \times D(\tilde{B})$ such that $Lf = g$, i.e., for all $\binom{f}{g} \in D(\mathcal{A})$. This classical solution is unique, again by Lemma 1.2.3.

Let now $t_0 > 0$ and $(f_n, g_n)_{n\in\mathbb{N}}$ be a sequence of initial data satisfying $Lf_n = g_n$ and tending to 0. Note that $\mathfrak{f}_n := \binom{f_n}{g_n} \in D(\mathcal{A})$ and $\mathfrak{u}(t,\mathfrak{f}_n) \in D(\mathcal{A})$, for all $t \geq 0$ and $n \in \mathbb{N}$. Hence we have $\lim_{n\to\infty} \mathfrak{u}(t,\mathfrak{f}_n) = 0$ uniformly for $t \in [0,t_0]$ if and only if

$$\lim_{n\to\infty} u(t,f_n,g_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} Lu(t,f_n,g_n) = 0,$$

both uniformly for $t \in [0,t_0]$. Since ($\mathcal{ACP}$) is well-posed, the assertion follows.

Assume now (AIBVP) to be well-posed. Since under the standing Assumptions 1.2.1 $\mathcal{A}$ is closed and densely defined, it suffices by Lemma A.3 to show that the associated ($\mathcal{ACP}$) admits a unique classical solution for all $\mathfrak{f} \in D(\mathcal{A})$, continuously depending on the initial data. Let $\mathfrak{f} \in D(\mathcal{A})$. Well-posedness of (AIBVP) yields, by Lemma 1.2.3, existence and uniqueness of a classical solution to ($\mathcal{ACP}_{\mathfrak{f}}$).

To show continuous dependence on initial data, let $t_0 > 0$ and $(\mathfrak{f}_n)_{n\in\mathbb{N}} \subset D(\mathcal{A})$ be a sequence of initial data tending to 0. Then $(\pi_1\mathfrak{f}_n, \pi_2\mathfrak{f}_n)_{n\in\mathbb{N}}$ is a sequence of initial data for (AIBVP) tending to 0 and such that $L\pi_1\mathfrak{f}_n = \pi_2\mathfrak{f}_n$. Then there holds $\lim_{n\to\infty} u(t,\pi_1\mathfrak{f}_n,\pi_2\mathfrak{f}_n) = 0$ and $\lim_{n\to\infty} Lu(t,\pi_1\mathfrak{f}_n,\pi_2\mathfrak{f}_n) = 0$ (both uniformly for $t \in [0,t_0]$). Also,

$$\mathfrak{u} = \mathfrak{u}(t,\mathfrak{f}_n) = \begin{pmatrix} u(t,\pi_1\mathfrak{f}_n,\pi_2\mathfrak{f}_n) \\ Lu(t,\pi_1\mathfrak{f}_n,\pi_2\mathfrak{f}_n) \end{pmatrix}, \qquad t \geq 0,$$

is the (unique) classical solution to ($\mathcal{ACP}_{\mathfrak{f}_n}$) for each $n \in \mathbb{N}$, and we finally obtain $\lim_{n\to\infty} \mathfrak{u}(t,\mathfrak{f}_n) = 0$ uniformly for $t \in [0,t_0]$. □

The following regularity result for the solution to ($\mathrm{AIBVP}_{f,g}$) holds, by Corollary 2.3.4 below.

Proposition 1.2.5. *The following assertions hold.*

(1) Let B map $D(A^{k+1})$ into $D(\tilde{B}^k)$, $k \in \mathbb{N}$. Assume $\mathcal{A}$ to generate a C_0-semigroup on $\mathcal{X}$. If for some $n = 1,2,\dots$ the initial data f belongs to

$$\bigcap_{h=0}^{n-1} \left\{ u \in D(A^n) : LA^h u = BA^h u = 0 \right\}$$

and moreover $g = 0$, then the unique classical solution $u = u(t)$ to ($\mathrm{AIBVP}_{f,g}$) belongs to $D(A^n)$ for all $t \geq 0$ and $n = 1,2,\dots$.

(2) Let $\mathcal{A}$ generate an analytic semigroup on $\mathcal{X}$. Then the unique classical solution $u = u(t)$ to ($\mathrm{AIBVP}_{f,g}$) belongs to

$$D^\infty(A) := \bigcap_{n\in\mathbb{N}} D(A^n)$$

for all $t > 0$ and all $f \in X$, $g \in \partial X$.

As in the case of abstract Cauchy problems, we can relax the notion of classical solution and introduce the following.

Definition 1.2.6. *A* mild solution (in $(X, \partial X)$) to $(\mathrm{AIBVP}_{f,g})$ *is a function* $u(\cdot)$ *such that*

- $u(\cdot) \in C(\mathbb{R}_+, X)$,
- $\int_0^t u(s)ds \in D(A)$ *for all* $t \geq 0$,
- $L\int_0^t u(s)ds \in D(\tilde{B})$ *for all* $t \geq 0$, *and*
- *there exists a function* $x(\cdot) \in C(\mathbb{R}_+, \partial X)$ *such that the integrated identities*

$$(1.5) \qquad \begin{cases} u(t) &= f + A\int_0^t u(s)ds, & t \geq 0, \\ x(t) &= g + B\int_0^t u(s)ds + \tilde{B}L\int_0^t u(s)ds, & t \geq 0, \\ \int_0^t x(s)ds &= L\int_0^t u(s)ds, & t \geq 0, \end{cases}$$

are satisfied.

Lemma 1.2.7. *The following assertions hold.*

(1) Let $u(\cdot)$ *be a mild solution to* $(\mathrm{AIBVP}_{f,g})$*, and let* x *be the function introduced in Definition 1.2.6. Then* $\mathfrak{u}(\cdot) := \binom{u(\cdot)}{x(\cdot)}$ *is a mild solution to* $(\mathcal{ACP}_{\binom{f}{g}})$.

(2) Conversely, let $\mathfrak{u}(\cdot)$ *be a mild solution to* $(\mathcal{ACP}_{\mathfrak{f}})$*. Then* $u(\cdot) := \pi_1\mathfrak{u}(\cdot)$ *is a mild solution to* $(\mathrm{AIBVP}_{\pi_1\mathfrak{f},\pi_2\mathfrak{f}})$.

Proof. (1) Let $u(\cdot)$ be a mild solution to $(\mathrm{AIBVP}_{f,g})$. Since $u(\cdot) \in C(\mathbb{R}_+, X)$, one has $x(\cdot) \in C(\mathbb{R}_+, \partial X)$, and therefore $\mathfrak{u} \in C(\mathbb{R}_+, \mathcal{X})$. Moreover, $u(\cdot)$ and $x(\cdot)$ satisfy the system (1.5), hence in particular $L\int_0^t u(s)ds = \int_0^t x(s)ds$ and therefore $\int_0^t \mathfrak{u}(s)ds \in D(\mathcal{A})$ for all $t \geq 0$. Finally, the equalities in (1.5) show that the integrated identity

$$(1.6) \qquad \mathfrak{u}(t) = \begin{pmatrix} f \\ g \end{pmatrix} + \mathcal{A}\int_0^t \mathfrak{u}(s)\,ds, \qquad t \geq 0,$$

is satisfied.

(2) Assume now $\mathfrak{u}(\cdot)$ to be a mild solution to $(\mathcal{ACP}_{\mathfrak{f}})$. Since $\mathfrak{u}(\cdot) \in C(\mathbb{R}_+, \mathcal{X})$ and $\int_0^t \mathfrak{u}(s)ds \in D(\mathcal{A})$, there holds $u(\cdot) := \pi_1\mathfrak{u}(\cdot) \in C(\mathbb{R}_+, X)$ and $\int_0^t u(s)ds \in D(A)$. Set now $x(\cdot) := \pi_2\mathfrak{u}(\cdot)$. Again because $\mathfrak{u}(\cdot) \in C(\mathbb{R}_+, \mathcal{X})$ and $\int_0^t \mathfrak{u}(s)ds \in D(\mathcal{A})$, there holds $x(\cdot) \in C(\mathbb{R}_+, \partial X)$ and $\int_0^t x(s)ds \in D(\tilde{B})$ with $L\int_0^t u(s)ds = \int_0^t x(s)\,ds$. Finally, $\mathfrak{u}(\cdot)$ satisfies (1.6). Considering its components yields the first two identities in (1.5). □

1.3 Equivalence between $(\mathcal{ACP})$ and (aAIBVP^2)

For $X, \partial X$ as in Section 1.1, instead of Assumptions 1.2.1 we impose the following, where Y is a further Banach space such that $Y \hookrightarrow X$.

Assumptions 1.3.1.

1. *$D(A) \subset Y$.*
2. *$A_0 := A|_{\ker(R)}$ is densely defined.*
3. *$\begin{pmatrix} A \\ R \end{pmatrix}$ is closed as an operator from Y to $X \times \partial X$.*
4. *$R : D(A) \to \partial X$ is surjective.*
5. *B_1 is bounded from $[D(A)]_R^Y$ to ∂X.*
6. *B_2 is bounded from Y to ∂X.*
7. *B_3, B_4 are bounded on ∂X.*

In the Assumption 1.3.1.5 we have denoted by $[D(A)]_R^Y$ the Banach space obtained by endowing $D(A)$ with the graph norm of $\binom{A}{R}$. Moreover, it will be convenient to define a new operator L by

$$L := R + B_2, \qquad L : D(A) \to \partial X.$$

We discuss in this section well-posedness issues of abstract second order initial-boundary value problems of the form

$$(\text{aAIBVP}^2_{f,g,h,j}) \quad \begin{cases} \ddot{u}(t) &= Au(t), & t \in \mathbb{R}, \\ \ddot{x}(t) &= B_1 u(t) + B_2 \dot{u}(t) + B_3 x(t) + B_4 \dot{x}(t), & t \in \mathbb{R}, \\ \dot{x}(t) &= Lu(t), & t \in \mathbb{R}, \\ u(0) &= f \in X, \qquad \dot{u}(0) = g \in X, \\ x(0) &= h \in \partial X, \qquad \dot{x}(0) = j \in \partial X, \end{cases}$$

on X and ∂X

Definition 1.3.2. *A* classical solution (in $(Y, X, \partial X)$) to (aAIBVP^2) *is a function $u(\cdot)$ such that*

- *$u(\cdot) \in C^2(\mathbb{R}, X) \cap C^1(\mathbb{R}, Y)$,*
- *$u(t) \in D(A)$ for all $t \in \mathbb{R}$,*
- *$Lu(\cdot) \in C^1(\mathbb{R}, \partial X)$, and*
- *$u(\cdot)$ satisfies* (aAIBVP^2).

Moreover, (aAIBVP^2) *is called* well-posed on $(Y, X, \partial X)$ *if*

- $(\text{aAIBVP}^2_{f,g,h,j})$ *admits a unique classical solution $u = u(\cdot, f, g, h, j)$ for all initial data $f \in D(A)$, $g \in Y$, $h, j \in \partial X$ satisfying the compatibility condition $Lf = j$, and*

- *for all sequences of initial data* $(f_n, g_n, h_n, j_n)_{n\in\mathbb{N}} \subset D(A)\times Y\times\partial X\times\partial X$ *tending to* 0 *and satisfying the compatibility condition* $Lf_n = j_n$*, one has* $\lim_{n\to\infty} u(t, f_n, g_n, h_n, j_n) = 0$ *and* $\lim_{n\to\infty} Lu(t, f_n, g_n, h_n, j_n) = 0$ *uniformly for* t *in compact intervals.*

In order to tackle (aAIBVP^2) by means of the results of the previous section, we consider the operator matrix $\mathbf{A}$ on the product Banach space

$$\mathbf{X} := Y \times X \times \partial X$$

defined by

$$\mathbf{A} := \begin{pmatrix} 0 & I_Y & 0 \\ A & 0 & 0 \\ L & 0 & 0 \end{pmatrix}, \qquad D(\mathbf{A}) := D(A)\times Y \times \partial X.$$

Further, $\mathbf{L}$ and $\mathbf{B}$ are the operators

$$\mathbf{L} := \begin{pmatrix} R & 0 & 0 \end{pmatrix}, \qquad D(\mathbf{L}) := D(\mathbf{A}),$$

and

$$\mathbf{B} := \begin{pmatrix} B_1 + B_4 B_2 & 0 & B_3 \end{pmatrix}, \qquad D(\mathbf{B}) := D(\mathbf{A}),$$

respectively, both from $\mathbf{X}$ to $\partial\mathbf{X} := \partial X$. Moreover, $\tilde{\mathbf{B}}$ is the operator

$$\tilde{\mathbf{B}} := B_4, \qquad D(\tilde{\mathbf{B}}) := D(B_4),$$

on $\partial\mathbf{X}$.

Lemma 1.3.3. *The operator matrices* $\mathbf{A}$ *on* $\mathbf{X}$*, and* $\binom{\mathbf{A}}{\mathbf{L}}$ *from* $\mathbf{X}$ *to* $\mathbf{X}\times\partial\mathbf{X}$*, are both closed and their graph norms are equivalent.*

Proof. We show that the operator matrix $\mathbf{A}$ is closed. Let

$$\begin{pmatrix} u_n \\ v_n \\ x_n \end{pmatrix}_{n\in\mathbb{N}} \subset D(\mathbf{A}), \quad \lim_{n\to\infty} \begin{pmatrix} u_n \\ v_n \\ x_n \end{pmatrix} = \begin{pmatrix} u \\ v \\ x \end{pmatrix} \quad \text{in } \mathbf{X},$$

$$\begin{aligned} \text{and} \quad \lim_{n\to\infty} \mathbf{A} \begin{pmatrix} u_n \\ v_n \\ x_n \end{pmatrix} &= \lim_{n\to\infty} \begin{pmatrix} 0 & I_Y & 0 \\ A & 0 & 0 \\ L & 0 & 0 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \\ x_n \end{pmatrix} \\ &= \lim_{n\to\infty} \begin{pmatrix} v_n \\ Au_n \\ Lu_n \end{pmatrix} = \begin{pmatrix} v \\ w \\ z \end{pmatrix} \quad \text{in } \mathbf{X}. \end{aligned}$$

Since $\lim_{n\to\infty} u_n = u$ holds with respect to the norm of Y, it follows from the Assumption 1.3.1.6 that $\lim_{n\to\infty} B_2 u_n = Bu$ and accordingly $\lim_{n\to\infty} Ru_n = z - B_2 u$. Moreover, by Assumption 1.3.1.3 we obtain $u \in D(A)$ and $Au = w$. This completes the proof of the closedness of $\mathbf{A}$. The closedness of $\binom{\mathbf{A}}{\mathbf{L}}$ and the equivalence of the graph norms can be proven likewise. ☐

Lemma 1.3.4. *The operators* $\mathbf{A}, \mathbf{B}, \mathbf{L}, \tilde{\mathbf{B}}$ *satisfy the Assumptions 1.2.1.*

Proof. First observe that, by Lemma 1.3.3 and the closed graph theorem, $[D(\mathbf{A})]_{\mathbf{L}} \hookrightarrow [D(A)]_R^Y \times Y \times \partial X$. Hence the boundedness of $\mathbf{B}$ from $[D(\mathbf{A})]_{\mathbf{L}}$ to $\partial\mathbf{X}$ is a consequence of Assumptions 1.3.1.5–7. The remaining Assumptions 1.2.1 are clearly satisfied by $\mathbf{A}, \mathbf{L}, \tilde{\mathbf{B}}$ under the Assumptions 1.3.1. $\square$

By virtue of the above lemma, we can apply the results of Section 1.2, and in particular obtain that

$$(\mathbf{AIBVP_{f,g}}) \qquad \begin{cases} \dot{\mathbf{u}}(t) &= \mathbf{A}\mathbf{u}(t), & t \in \mathbb{R}, \\ \dot{\mathbf{x}}(t) &= \mathbf{B}\mathbf{u}(t) + \tilde{\mathbf{B}}\mathbf{x}(t), & t \in \mathbb{R}, \\ \mathbf{x}(t) &= \mathbf{L}\mathbf{u}(t), & t \in \mathbb{R}, \\ \mathbf{u}(0) &= \mathbf{f} \in \mathbf{X}, \\ \mathbf{x}(0) &= \mathbf{g} \in \partial\mathbf{X}, \end{cases}$$

on the Banach spaces $\mathbf{X}$ and $\partial\mathbf{X}$ is well posed (in the sense of Theorem 1.2.4) if and only if the operator matrix

$$(1.7) \qquad \mathcal{A} := \begin{pmatrix} \mathbf{A} & 0 \\ \mathbf{B} & \tilde{\mathbf{B}} \end{pmatrix}, \qquad D(\mathcal{A}) := \left\{ \begin{pmatrix} \mathbf{u} \\ \mathbf{x} \end{pmatrix} \in D(\mathbf{A}) \times D(\tilde{\mathbf{B}}) \, : \, \mathbf{L}\mathbf{u} = \mathbf{x} \right\},$$

generates a C_0-group on $\mathbf{X} \times \partial\mathbf{X}$. (Here we have set

$$(1.8) \quad \mathbf{u}(t) := \begin{pmatrix} u(t) \\ v(t) \\ x(t) \end{pmatrix}, \qquad t \in \mathbb{R}, \qquad \text{and} \qquad \mathbf{f} := \begin{pmatrix} f \\ g \\ h \end{pmatrix}, \quad \mathbf{g} := j - B_2 f.)$$

Thus, our goal becomes to prove the equivalence between (aAIBVP^2) and (**AIBVP**). In the following we use the terminology of Definitions 1.2.2 and 1.3.2.

Lemma 1.3.5. *The following assertions hold.*

(1) Let $u(\cdot)$ be a classical solution in $(Y, X, \partial X)$ to (aAIBVP$^2_{f,g,h,j}$). *Then*

$$\mathbf{u}(\cdot) := \begin{pmatrix} u(\cdot) \\ \dot{u}(\cdot) \\ h + \int_0^{\cdot} Lu(s)ds \end{pmatrix}$$

is a classical solution in $(\mathbf{X}, \partial\mathbf{X})$ to $(\mathbf{AIBVP_{f,g}})$, with $\mathbf{f}, \mathbf{g}$ defined as in (1.8).

(2) Conversely, let $\mathbf{u}(\cdot)$ be a classical solution in $(\mathbf{X}, \partial\mathbf{X})$ to $(\mathbf{AIBVP_{f,g}})$. Then

$$u(\cdot) := \pi_1 \mathbf{u}(\cdot)$$

is a classical solution in $(Y, X, \partial X)$ to (aAIBVP$^2_{f,g,h,j}$), *with f, g, h, j defined as in* (1.8).

Proof. To begin with, observe that for all $u(\cdot) \in C^1(\mathbb{R}, Y)$ there holds $B_2 u(\cdot) \in C^1(\mathbb{R}, \partial X)$ and

$$(1.9) \qquad \begin{aligned} B_2 \tfrac{du}{dt}(\cdot) &= B_2 \left(Y - \lim_{h \to 0} \tfrac{u(\cdot + h) - u(\cdot)}{h} \right) \\ &= \partial X - \lim_{h \to 0} B_2 \left(\tfrac{u(\cdot + h) - u(t)}{h} \right) = \tfrac{d(B_2 u)}{dt}(\cdot), \end{aligned}$$

by Assumption 1.3.1.6, where "$Y - \lim$" stands for the limit with respect to the norm of Y. Note that this argument does not hold for L.

(1) Let $u(\cdot)$ be classical solution to (aAIBVP$^2_{\mathrm{f,g,h,j}}$). Observe that $x(\cdot) := \big(h + \int_0^\cdot Lu(s)ds\big) \in C^1(\mathbb{R}, \partial X)$, hence $\mathbb{u}(\cdot) \in C^1(\mathbb{R}, \mathbf{X})$. Moreover, $\mathbf{Ru}(\cdot) \in C^1(\mathbb{R}, \partial\mathbf{X})$, since $\mathbf{Ru}(\cdot) = Lu(\cdot) - B_2 u(\cdot)$. Further, $\mathbf{u}(t) \in D(\mathbf{A})$ for all $t \in \mathbb{R}$ because $u(t) \in D(A)$ for all $t \in \mathbb{R}$, and one can check directly that $\mathbf{u}(\cdot)$ satisfies $(\mathbf{AIBVP_{f,g}})$.

(2) Let

$$\mathbf{u}(\cdot) = \begin{pmatrix} u(\cdot) \\ v(\cdot) \\ x(\cdot) \end{pmatrix} \in C^1(\mathbb{R}, \mathbf{X})$$

be a classical solution to $(\mathbf{AIBVP_{f,g}})$. Hence $\mathbf{Lu}(\cdot) \in C^1(\mathbb{R}, \partial\mathbf{X})$, i.e. $y(\cdot) := Ru(\cdot) = Lu(\cdot) - B_2 u(\cdot) \in C^1(\mathbb{R}, \partial X)$. Thus, there holds

$$(1.10) \qquad \begin{cases} \dot u(t) &= v(t), & t \in \mathbb{R}, \\ \dot v(t) &= Au(t), & t \in \mathbb{R}, \\ \dot x(t) &= Lu(t), & t \in \mathbb{R}, \\ \dot y(t) &= (B_1 + B_4 B_2)u(t) + B_3 x(t) + B_4 y(t), & t \in \mathbb{R}, \\ y(t) &= (L - B_2)u(t), & t \in \mathbb{R}, \\ u(0) &= f, \qquad v(0) = g, & \\ x(0) &= h, \qquad y(0) = j - B_2 f, & \end{cases}$$

with $v(t) \in Y$, $t \in \mathbb{R}$, or, equivalently,

$$(1.11) \qquad \begin{cases} \ddot u(t) &= Au(t), & t \in \mathbb{R}, \\ \dot x(t) &= Lu(t), & t \in \mathbb{R}, \\ \ddot x(t) &= B_1 u(t) + B_2 \dot u(t) + B_3 x(t) + B_4 \dot x(t), & t \in \mathbb{R}, \\ u(0) &= f, \qquad \dot u(0) = g, & \\ x(0) &= h, \qquad \dot x(0) = j. & \end{cases}$$

Here we have used the fact that $\dot u(t) \in Y$, $t \in \mathbb{R}$, and the last step is justified by (1.9). Moreover, $u(\cdot) \in C^2(\mathbb{R}, X) \cap C^1(\mathbb{R}, Y)$ and $Lu(\cdot) \in C^1(\mathbb{R}, \partial X)$. This shows that $u(\cdot)$ is a classical solution to (aAIBVP$^2_{f,g,h,j}$) on $(Y, X, \partial X)$. □

Summing up, we obtain the following.

Theorem 1.3.6. *The operator matrix $\mathcal{A}$ in* (1.7) *generates a C_0-group $(e^{t\mathcal{A}})_{t\in\mathbb{R}}$ on the product space $\mathcal{X} = \mathbf{X} \times \partial\mathbf{X}$ if and only if* (aAIBVP^2) *is well-posed on* $(Y, X, \partial X)$.

In this case, the unique classical solution to (aAIBVP$^2_{f,g,h,j}$) *is given by*

$$u = u(t, f, g, h, j) := \pi_1 e^{t\mathcal{A}} \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix}, \qquad t \in \mathbb{R},$$

for all initial data $f \in D(A)$, $g \in Y$, and $h, j \in \partial X$ such that $Rf = j$. Here $\mathbf{f}, \mathbf{g}$ *are defined as in* (1.8).

Proof. Taking into account Lemma 1.3.5, one can check directly that (aAIBVP^2) is well-posed on $(Y, X, \partial X)$ if and only if $(\mathbf{AIBVP})$ is well-posed on $(\mathbf{X}, \partial\mathbf{X})$. Now the claims follows directly by Theorem 1.2.4. □

The following holds by Proposition 1.2.5 and Lemma 3.3.1 below.

Corollary 1.3.7. *Assume the operator matrix $\mathcal{A}$ in (1.7) to generate a C_0-group on $\mathcal{X}$. Let the initial data f, g be in*

$$\bigcap_{k=0}^{\infty} \left\{ u \in D^\infty(A) \, : \, LA^k u = B_2 A^k u = 0 \right\}. \tag{1.12}$$

If further $j = 0$, then the classical solution $u = u(t)$ to $(\text{aAIBVP}^2_{f,g,h,j})$ is in $D^\infty(A)$ for all $t \in \mathbb{R}$ and $h \in \partial X$.

1.4 Equivalence between $(\mathcal{ACP}^2)$ and (dAIBVP^2)

Throughout this section we consider two Banach spaces $Y, \partial Y$ such that $D(A) \subset Y \hookrightarrow X$ and $D(\tilde{B}) \subset \partial Y \hookrightarrow \partial X$. We impose the Assumptions 1.2.1 and discuss well-posedness issues for a second order abstract initial-boundary value problem of the form

$$(\text{dAIBPV}^2_{f,g,h,j}) \qquad \begin{cases} \ddot{u}(t) &= Au(t), & & t \in \mathbb{R}, \\ \ddot{x}(t) &= Bu(t) + \tilde{B}x(t), & & t \in \mathbb{R}, \\ x(t) &= Lu(t), & & t \in \mathbb{R}, \\ u(0) &= f \in X, & \dot{u}(0) = g \in X, & \\ x(0) &= h \in \partial X, & \dot{x}(0) = j \in \partial X, & \end{cases}$$

on X and ∂X.

Definition 1.4.1. *A* classical solution to (dAIBVP^2) in $(Y, X, \partial Y, \partial X)$ *is a function $u(\cdot)$ such that*

- *$u(\cdot) \in C^2(\mathbb{R}, X) \cap C^1(\mathbb{R}, Y)$,*
- *$u(t) \in D(A)$ for all $t \in \mathbb{R}$,*
- *$Lu(\cdot) \in C^2(\mathbb{R}, \partial X) \cap C^1(\mathbb{R}, \partial Y)$,*
- *$Lu(t) \in D(\tilde{B})$ for all $t \in \mathbb{R}$, and*
- *$u(\cdot)$ satisfies (dAIBVP^2).*

Moreover, (dAIBVP^2) is called well-posed on $(Y, X, \partial Y, \partial X)$ *if*

- *$(\text{dAIBVP}^2_{f,g,h,j})$ admits a unique classical solution $u = u(\cdot, f, g, h, j)$ for all initial data $f \in D(A)$, $g \in Y$, $h \in D(\tilde{B})$, and $j \in \partial Y$ satisfying the compatibility condition $Lf = h$, and*
- *for all sequences of initial data $(f_n, g_n, h_n, j_n)_{n \in \mathbb{N}} \subset D(A) \times Y \times D(\tilde{B}) \times \partial Y$ tending to 0 and satisfying the compatibility condition $Lf_n = h_n$, one has $\lim_{n \to \infty} u(t, f_n, g_n, h_n, j_n) = 0$ and $\lim_{n \to \infty} Lu(t, f_n, g_n, h_n, j_n) = 0$ uniformly for t in compact intervals.*

Define the operator matrix $\mathcal{A}$ as in (1.1) and consider the second order abstract Cauchy problem

$$(\mathcal{ACP}^2_{\mathfrak{f},\mathfrak{g}}) \qquad \begin{cases} \ddot{\mathfrak{u}}(t) &= \mathcal{A}\,\mathfrak{u}(t), \quad t \in \mathbb{R}, \\ \mathfrak{u}(0) &= \mathfrak{f} \in \mathcal{X}, \quad \dot{\mathfrak{u}}(0) = \mathfrak{g} \in \mathcal{X}, \end{cases}$$

on the product space $\mathcal{X} = X \times \partial X$. Like in (1.2) here we have set

$$\mathfrak{u}(t) := \begin{pmatrix} u(t) \\ Lu(t) \end{pmatrix}, \quad t \in \mathbb{R}, \qquad \text{and} \quad \mathfrak{f} := \begin{pmatrix} f \\ h \end{pmatrix}, \; \mathfrak{g} := \begin{pmatrix} g \\ j \end{pmatrix}.$$

Similarly to what we have done in Section 1.2, in the remainder of this section we show that the well-posedness of the abstract initial-boundary value problem (dAIBVP^2) is equivalent to the well-posedness of the abstract Cauchy problem ($\mathcal{ACP}^2$).

The following can be proven exactly like Lemma 1.2.3.

Lemma 1.4.2. *The following assertions hold.*

(1) If $u(\cdot)$ is a classical solution to (dAIBVP$^2_{f,g,h,j}$)*, then*

$$\mathfrak{u}(\cdot) := \begin{pmatrix} u(\cdot) \\ Lu(\cdot) \end{pmatrix}$$

is a classical solution to $(\mathcal{ACP}^2_{\binom{f}{h},\binom{g}{j}})$.

(2) Conversely, if $\mathfrak{u}(\cdot)$ is a classical solution to $(\mathcal{ACP}^2_{\mathfrak{f},\mathfrak{g}})$*, then*

$$u(\cdot) := \pi_1 \mathfrak{u}(\cdot)$$

is a classical solution to (dAIBVP$^2_{\pi_1\mathfrak{f},\pi_1\mathfrak{g},\pi_2\mathfrak{f},\pi_2\mathfrak{g}}$).

We can finally relate the property of cosine operator function generator of $\mathcal{A}$ and the well-posedness of (dAIBVP^2).

Theorem 1.4.3. *The operator matrix $\mathcal{A}$ generates a cosine operator function $(C(t,\mathcal{A}))_{t\in\mathbb{R}}$ with associated phase space $\mathcal{Y}\times\mathcal{X} := (Y \times \partial Y) \times (X \times \partial X)$ if and only if* (dAIBVP^2) *is well-posed on $(Y, X, \partial Y, \partial X)$.*

In this case, the unique classical solution to (dAIBVP$^2_{f,g,h,j}$) *is given by*

$$u = u(t,f,g,h,j) := \pi_1 C(t,\mathcal{A}) \begin{pmatrix} f \\ h \end{pmatrix} + \pi_1 S(t,\mathcal{A}) \begin{pmatrix} g \\ j \end{pmatrix}, \qquad t \in \mathbb{R},$$

for all initial data $f \in D(A)$, $g \in Y$, $h \in D(\tilde{B})$, and $j \in \partial Y$ such that $Lf = h$.

Proof. Let $\mathcal{A}$ generate a cosine operator function with associated phase space $\mathcal{Y}\times\mathcal{X}$. Then $\mathcal{A}$ is closed, thus by Proposition B.11 the associated second order abstract Cauchy problem ($\mathcal{ACP}^2$) is well-posed. By Lemma 1.4.2,

$$\begin{aligned} u = u(t,(f,g,h,j)) &= \pi_1 \mathfrak{u}(t, \tbinom{f}{h}, \tbinom{g}{j}) \\ &= \pi_1 C(t,\mathcal{A})\tbinom{f}{h} + \pi_1 S(t,\mathcal{A})\tbinom{g}{j}, \qquad t \in \mathbb{R}, \end{aligned}$$

yields a classical solution to (dAIBVP$^2_{f,g,h,j}$) for all $f \in D(A)$, $g \in Y$, $h \in D(\tilde{B})$, and $j \in \partial Y$ such that $Lf = h$, i.e., for all $\binom{f}{h} \in D(\mathcal{A})$ and $\binom{g}{j} \in \mathcal{Y}$. This classical solution is unique, again by Lemma 1.4.2.

Let now $t_0 > 0$ and $(f_n, g_n, h_n, j_n)_{n\in\mathbb{N}}$ be a sequence of initial data satisfying $Lf_n = g_n$ and tending to 0. Note that $\mathfrak{f}_n := \binom{f_n}{h_n} \in D(\mathcal{A})$, $\mathfrak{g}_n := \binom{g_n}{j_n} \in \mathcal{Y}$, and $\mathfrak{u}(t, \mathfrak{f}_n, \mathfrak{g}_n) \in D(\mathcal{A})$, for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Hence we have $\lim_{n\to\infty} \mathfrak{u}(t, \mathfrak{f}_n, \mathfrak{g}_n) = 0$ uniformly for $t \in [0, t_0]$ if and only if

$$\lim_{n\to\infty} u(t, f_n, g_n, h_n, j_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} Lu(t, f_n, g_n, h_n, j_n) = 0,$$

both uniformly for $t \in [0, t_0]$. Since ($\mathcal{ACP}^2$) is well-posed, the assertion follows.

Assume now (dAIBVP^2) to be well-posed. Since under the standing Assumptions 1.2.1 $\mathcal{A}$ is closed and densely defined, it suffices by Proposition B.11 to show that the associated second order abstract Cauchy problem ($\mathcal{ACP}^2$) admits a unique classical solution for all $\mathfrak{f} \in D(\mathcal{A})$ and $\mathfrak{g} \in \mathcal{Y}$, continuously depending on the initial data. Let $\mathfrak{f} \in D(\mathcal{A})$, $\mathfrak{g} \in \mathcal{Y}$. Well-posedness of (dAIBVP^2) yields, by Lemma 1.4.2, existence and uniqueness of a classical solution to ($\mathcal{ACP}^2_{\mathfrak{f},\mathfrak{g}}$).

To show continuous dependence on initial data, let $t_0 > 0$ and $(\mathfrak{f}_n, \mathfrak{g}_n)_{n\in\mathbb{N}} \subset D(\mathcal{A}) \times \mathcal{Y}$ be a sequence of initial data for ($\mathcal{ACP}^2$) that tends to 0. Then $(\pi_1\mathfrak{f}_n, \pi_1\mathfrak{g}_n, \pi_2\mathfrak{f}_n, \pi_2\mathfrak{g}_n)_{n\in\mathbb{N}}$ is a sequence of initial data for (dAIBVP^2) that tends to 0 and such that $L\pi_1\mathfrak{f}_n = \pi_2\mathfrak{f}_n$, $n \in \mathbb{N}$. Then there holds

$$\lim_{n\to\infty} u(t, \pi_1\mathfrak{f}_n, \pi_1\mathfrak{g}_n, \pi_2\mathfrak{f}_n, \pi_2\mathfrak{g}_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} Lu(t, \pi_1\mathfrak{f}_n, \pi_1\mathfrak{g}_n, \pi_2\mathfrak{f}_n, \pi_2\mathfrak{g}_n) = 0$$

(both uniformly for $t \in [0, t_0]$). Also,

$$\mathfrak{u} = \mathfrak{u}(t, \pi_1\mathfrak{f}_n, \pi_1\mathfrak{g}_n) = \begin{pmatrix} u(t, \pi_1\mathfrak{f}_n, \pi_1\mathfrak{g}_n, \pi_2\mathfrak{f}_n, \pi_2\mathfrak{g}_n) \\ Lu(t, \pi_1\mathfrak{f}_n, \pi_1\mathfrak{g}_n, \pi_2\mathfrak{f}_n, \pi_2\mathfrak{g}_n) \end{pmatrix}, \qquad t \in \mathbb{R},$$

is the (unique) classical solution to ($\mathcal{ACP}^2_{\mathfrak{f}_n,\mathfrak{g}_n}$) for each $n \in \mathbb{N}$, and we finally obtain $\lim_{n\to\infty} \mathfrak{u}(t, \mathfrak{f}_n, \mathfrak{g}_n) = 0$ uniformly for $t \in [0, t_0]$. ☐

The following regularity result for the solution to (dAIBVP$^2_{f,g,h,j}$) holds, by Proposition 4.4.1 below.

Proposition 1.4.4. *Let B map $D(A^{k+1})$ into $D(\tilde{B}^k)$, $k \in \mathbb{N}$. Assume $\mathcal{A}$ to generate a cosine operator function on $\mathcal{X}$. If the initial data f, g belong to*

$$\bigcap_{h=0}^{\infty} \left\{ u \in D^\infty(A) : LA^h u = BA^h u = 0 \right\}$$

and moreover $h = j = 0$, then the unique classical solution $u = u(t)$ to (dAIBVP$^2_{f,g,h,j}$) *belongs to $D^\infty(A)$, for all $t \in \mathbb{R}$.*

As in the first order case, we can relax the notion of classical solution and introduce the following.

Definition 1.4.5. *A* mild solution (in $(Y, X, \partial Y, \partial X)$) to (dAIBVP$^2_{f,g,h,j}$) *is a function $u(\cdot)$ such that*

- $u(\cdot) \in C^1(\mathbb{R}, X) \cap C(\mathbb{R}, Y)$,
- $\int_0^t \int_0^s u(r)drds = \int_0^t (t-s)u(s)ds \in D(A)$ *for all* $t \in \mathbb{R}$,
- $L \int_0^t \int_0^s u(r)dr = L \int_0^t (t-s)u(s)ds \in D(\tilde{B})$ *for all* $t \in \mathbb{R}$, *and*
- *there exists a function* $x(\cdot) \in C^1(\mathbb{R}, \partial X) \cap C(\mathbb{R}, \partial Y)$ *such that the integrated identities*

$$\begin{cases} u(t) &= f + tg + A \int_0^t (t-s)u(s)ds, & t \in \mathbb{R}, \\ x(t) &= h + tj + B \int_0^t (t-s)u(s)ds \\ & \qquad + \tilde{B} \int_0^t (t-s)x(s)ds, & t \in \mathbb{R}, \\ \int_0^t (t-s)x(s)ds &= L \int_0^t (t-s)u(s)ds, & t \in \mathbb{R}, \end{cases}$$

are satisfied.

The following can be proven like Lemma 1.2.7.

Lemma 1.4.6. *The following assertions hold.*

(1) Let $u(\cdot)$ *be a mild solution to* $(\text{dAIBVP}^2_{f,g,h,j})$*, and let* x *be the function introduced in Definition 1.4.5. Then* $\mathfrak{u}(\cdot) := \binom{u(\cdot)}{x(\cdot)}$ *is a mild solution to* $(\mathcal{ACP}^2_{\binom{f}{h},\binom{g}{j}})$.

(2) Conversely, let $\mathfrak{u}(\cdot)$ *be a mild solution to* $(\mathcal{ACP}^2_{\mathfrak{f},\mathfrak{g}})$*. Then* $u(\cdot) := \pi_1 \mathfrak{u}(\cdot)$ *is a mild solution to* $(\text{dAIBVP}^2_{\pi_1\mathfrak{f},\pi_1\mathfrak{g},\pi_2\mathfrak{f},\pi_2\mathfrak{g}})$.

1.5 Equivalence between $(\mathcal{ACP}^2)$ and (bAIBVP^2)

Throughout this section we consider two Banach spaces $Y, \partial Y$ such that $D(A) \subset Y \hookrightarrow X$ and $D(\tilde{B}) \subset \partial Y \hookrightarrow \partial X$. We then complement the Assumptions 1.2.1 by imposing that L is well-defined on all of Y. We discuss well-posedness issues for a second order abstract initial-boundary value problem of the form

$$(\text{bAIBPV}^2_{f,g,h,j}) \qquad \begin{cases} \ddot{u}(t) &= Au(t), & & t \in \mathbb{R}, \\ \ddot{x}(t) &= Bu(t) + \tilde{B}x(t), & & t \in \mathbb{R}, \\ x(t) &= Lu(t), & \dot{x}(t) = L\dot{u}(t), & t \in \mathbb{R}, \\ u(0) &= f \in X, & \dot{u}(0) = g \in X, & \\ x(0) &= h \in \partial X, & \dot{x}(0) = j \in \partial X, & \end{cases}$$

on X and ∂X. Most definitions and assertions are only slighlty different from those in Section 1.4, hence we do not give any detail.

Definition 1.5.1. *A* classical solution *to* (bAIBVP^2) *in* $(Y, X, \partial Y, \partial X)$ *is a function* $u(\cdot)$ *such that*

- $u(\cdot) \in C^2(\mathbb{R}, X) \cap C^1(\mathbb{R}, Y)$,
- $u(t) \in D(A)$ *for all* $t \in \mathbb{R}$,

- $Lu(\cdot) \in C^2(\mathbb{R}, \partial X) \cap C^1(\mathbb{R}, \partial Y)$,
- $Lu(t) \in D(\tilde{B})$ *for all* $t \in \mathbb{R}$, *and*
- $u(\cdot)$ *satisfies* (bAIBVP^2).

Moreover, (bAIBVP^2) *is called* well-posed on $(Y, X, \partial Y, \partial X)$ *if*

- (bAIBVP$^2_{f,g,h,j}$) *admits a unique classical solution* $u = u(\cdot, f, g, h, j)$ *for all initial data* $f \in D(A)$, $g \in Y$, $h \in D(\tilde{B})$, *and* $j \in \partial Y$ *satisfying the compatibility conditions* $Lf = h$ *and* $Lg = j$, *and*
- *for all sequences of initial data* $(f_n, g_n, h_n, j_n)_{n \in \mathbb{N}} \subset D(A) \times Y \times D(\tilde{B}) \times \partial Y$ *tending to* 0 *and satisfying the compatibility conditions* $Lf_n = h_n$ *and* $Lg_n = j_n$, *one has* $\lim_{n\to\infty} u(t, f_n, g_n, h_n, j_n) = 0$ *and moreover* $\lim_{n\to\infty} Lu(t, f_n, g_n, h_n, j_n) = 0$ *uniformly for* t *in compact intervals.*

Lemma 1.5.2. *The following assertions hold.*

(1) If $u(\cdot)$ *is a classical solution to* (bAIBVP$^2_{f,g,h,j}$), *then*

$$\mathfrak{u}(\cdot) := \begin{pmatrix} u(\cdot) \\ Lu(\cdot) \end{pmatrix}$$

is a classical solution to $(\mathcal{ACP}^2_{\binom{f}{h},\binom{g}{j}})$.

(2) Conversely, if $\mathfrak{u}(\cdot)$ *is a classical solution to* $(\mathcal{ACP}^2_{\mathfrak{f},\mathfrak{g}})$, *then*

$$u(\cdot) := \pi_1 \mathfrak{u}(\cdot)$$

is a classical solution to (bAIBVP$^2_{\pi_1\mathfrak{f},\pi_1\mathfrak{g},\pi_2\mathfrak{f},\pi_2\mathfrak{g}}$).

Theorem 1.5.3. *The operator matrix* $\mathcal{A}$ *generates a cosine operator function* $(C(t, \mathcal{A}))_{t \in \mathbb{R}}$ *with associated phase space*

$$\mathcal{V} \times \mathcal{X} := \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in Y \times \partial Y : Lu = x \right\} \times (X \times \partial X)$$

if and only if (bAIBVP^2) *is well-posed on* $(Y, X, \partial Y, \partial X)$.

In this case, the unique classical solution to (bAIBVP$^2_{f,g,h,j}$) *is given by*

$$u = u(t, f, g, h, j) := \pi_1 C(t, \mathcal{A}) \begin{pmatrix} f \\ h \end{pmatrix} + \pi_1 S(t, \mathcal{A}) \begin{pmatrix} g \\ j \end{pmatrix}, \qquad t \in \mathbb{R},$$

for all initial data $f \in D(A)$, $g \in Y$, $h \in D(\tilde{B})$, *and* $j \in \partial Y$ *such that* $Lf = h$ *and* $Lg = j$.

Definition 1.5.4. *A* mild solution (in $(Y, X, \partial Y, \partial X)$) to (dAIBVP$^2_{f,g,h,j}$) *is a function* $u(\cdot)$ *such that*

- $u(\cdot) \in C^1(\mathbb{R}, X) \cap C(\mathbb{R}, Y)$,

- $Lu(\cdot) \in C^1(\mathbb{R}, \partial X) \cap C(\mathbb{R}, \partial Y)$,
- $\int_0^t \int_0^s u(r)drds = \int_0^t (t-s)u(s)ds \in D(A)$ *for all* $t \in \mathbb{R}$,
- $L \int_0^t \int_0^s u(r)dr = L \int_0^t (t-s)u(s)ds \in D(\tilde{B})$ *for all* $t \in \mathbb{R}$, *and*
- *the integrated identities*

$$\begin{cases} u(t) &= f + tg + A \int_0^t (t-s)u(s)ds, & t \in \mathbb{R}, \\ Lu(t) &= h + tj + B \int_0^t (t-s)u(s)ds + \tilde{B} \int_0^t (t-s)Lu(s)ds, & t \in \mathbb{R}, \end{cases}$$

  are satisfied.

Lemma 1.5.5. *The following assertions hold.*

(1) Let $u(\cdot)$ *be a mild solution to* $(\mathrm{bAIBVP}^2_{f,g,h,j})$. *Then* $\mathfrak{u}(\cdot) := \binom{u(\cdot)}{Lu(\cdot)}$ *is a mild solution to* $(\mathcal{ACP}^2_{\binom{f}{h},\binom{g}{j}})$.

(2) Conversely, let $\mathfrak{u}(\cdot)$ *be a mild solution to* $(\mathcal{ACP}^2_{\mathfrak{f},\mathfrak{g}})$. *Then* $u(\cdot) := \pi_1\mathfrak{u}(\cdot)$ *is a mild solution to* $(\mathrm{bAIBVP}^2_{\pi_1\mathfrak{f},\pi_1\mathfrak{g},\pi_2\mathfrak{f},\pi_2\mathfrak{g}})$.

Chapter 2

Operator matrices with coupled domain

Throughout this chapter we stick to the notation introduced in Section 1.1. In particular, we consider Banach spaces $X, \partial X$ and linear operators $A : D(A) \subset X \to X$, $B : D(B) \subset X \to \partial X$, $L : D(L) \subset X \to \partial X$, and $\tilde{B} : D(\tilde{B}) \subset \partial X \to \partial X$ such that $D(A) \subset D(L)$ and $D(A) \subset D(B)$. We consider the operator matrix

$$\mathcal{A} := \begin{pmatrix} A & 0 \\ B & \tilde{B} \end{pmatrix}, \qquad D(\mathcal{A}) := \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in D(A) \times D(\tilde{B}) \, : \, Lu = x \right\},$$

on the product space $\mathcal{X} := X \times \partial X$. We emphasize that the domain of $\mathcal{A}$ is *not* a product set, but rather a *strict* subset of the product set $D(A) \times D(\tilde{B})$, due to the relation $Lu = x$ inserted in the definition of $D(\mathcal{A})$. Thus, we call $\mathcal{A}$ an *operator matrix with coupled domain.*

2.1 Decoupling an operator matrix

The following assumptions are motivated by Lemma 1.1.2, and will be imposed throughout this chapter.

Assumptions 2.1.1.

1. *A_0 has nonempty resolvent set.*
2. *$\begin{pmatrix} A \\ L \end{pmatrix}$ is closed (as an operator from X to $X \times \partial X$).*
3. *L is surjective from $D(A)$ to ∂X.*
4. *$\tilde{B}$ is closed.*

By Lemma C.1 and Lemma C.4, the above assumptions ensure the existence of the Dirichlet operator $D_\lambda^{A,L}$, as a bounded operator from ∂X to any Banach space Z satisfying $D^\infty(A) \subset Z \hookrightarrow X$, for all $\lambda \in \rho(A_0)$.

In the Appendix C we have considered the Dirichlet operators only for their analytic property of yielding solutions to an abstract Dirichlet problem.

However, it is worth to remark the algebraic counterpart of the same property, viz., that for $\lambda \in \rho(A_0)$ a Dirichlet operator $D_\lambda^{A,L}$ is, by definition, a right inverse of L – or rather the inverse of the restriction $L_{|\ker(\lambda-A)}$. This allows us to perform some very useful matrix analysis of the operator matrix $\mathcal{A}$ with coupled domain defined in (1.1). This idea has been thoroughly developed in [En99] (see also [KMN03b]) where the following Lemma 2.1.2 has been proven. We mention its proof to illustrate our matrix methods.

In the following, the family of operators defined by

$$\tilde{B}_\lambda := \tilde{B} + BD_\lambda^{A,L}, \qquad D(\tilde{B}_\lambda) := D(\tilde{B}) \quad \text{for all } \lambda \in \rho(A_0), \tag{2.1}$$

will play an important role. Such operators are well-defined since the Dirichlet operators map ∂X into $\ker(\lambda - A) \subset D(A) \subset D(B)$.

Lemma 2.1.2. *Let $\lambda \in \rho(A_0)$. Then the factorization*

$$\begin{aligned} \lambda - \mathcal{A} &= \mathcal{L}_\lambda \mathcal{A}_\lambda \mathcal{M}_\lambda \\ &:= \begin{pmatrix} I_X & 0 \\ -BR(\lambda, A_0) & I_{\partial X} \end{pmatrix} \begin{pmatrix} \lambda - A_0 & 0 \\ 0 & \lambda - \tilde{B}_\lambda \end{pmatrix} \begin{pmatrix} I_X & -D_\lambda^{A,L} \\ 0 & I_{\partial X} \end{pmatrix} \end{aligned} \tag{2.2}$$

holds, and for all $\mu \in \mathbb{C}$ we have

$$\begin{aligned} \mu - \mathcal{A} &= \mathcal{L}_\lambda \begin{pmatrix} \mu - A_0 & 0 \\ 0 & \mu - \tilde{B}_\lambda \end{pmatrix} \mathcal{M}_\lambda \\ &\quad + (\mu - \lambda) \begin{pmatrix} 0 & D_\lambda^{A,L} \\ BR(\lambda, A_0) & -BR(\lambda, A_0) D_\lambda^{A,L} \end{pmatrix}. \end{aligned} \tag{2.3}$$

Proof. Let $\lambda \in \rho(A_0)$ and take $\mathfrak{u} := \binom{u}{v} \in \mathcal{X}$. Observe first that $\mathfrak{u}$ is in the domain of the operator matrix $\mathcal{L}_\lambda \mathcal{A}_\lambda \mathcal{M}_\lambda$ if and only if $u - D_\lambda^{A,L} v \in D(A_0)$ and $v \in D(\tilde{B}_\lambda)$, that is, if and only if $u \in D(A)$, $L\big(u - D_\lambda^{A,L} v\big) = Lu - v = 0$, and $v \in D(\tilde{B})$. This shows that the domains of the operators in (2.2) agree. Moreover, we obtain

$$\begin{aligned} &\begin{pmatrix} I_X & 0 \\ -BR(\lambda, A_0) & I_{\partial X} \end{pmatrix} \begin{pmatrix} \lambda - A_0 & 0 \\ 0 & \lambda - \tilde{B}_\lambda \end{pmatrix} \begin{pmatrix} I_X & -D_\lambda^{A,L} \\ 0 & I_{\partial X} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ &\qquad = \begin{pmatrix} I_X & 0 \\ -BR(\lambda, A_0) & I_{\partial X} \end{pmatrix} \begin{pmatrix} \lambda - A_0 & 0 \\ 0 & \lambda - \tilde{B}_\lambda \end{pmatrix} \begin{pmatrix} u - D_\lambda^{A,L} v \\ v \end{pmatrix} \\ &\qquad = \begin{pmatrix} I_X & 0 \\ -BR(\lambda, A_0) & I_{\partial X} \end{pmatrix} \begin{pmatrix} (\lambda - A_0)(u - D_\lambda^{A,L} v) \\ \lambda v - \tilde{B}_\lambda v \end{pmatrix} \\ &\qquad = \begin{pmatrix} (\lambda - A)(u - D_\lambda^{A,L} v) \\ -Bu + BD_\lambda^{A,L} v + \lambda v - \tilde{B}_\lambda v \end{pmatrix} \\ &\qquad = \begin{pmatrix} \lambda - A & 0 \\ -B & \lambda - \tilde{B} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \end{aligned}$$

where we have used (2.1) and the fact that $D_\lambda^{A,L}$ maps ∂X into $\ker(\lambda - A)$ by definition.

To show (2.3), take $\mu \in \mathbb{C}$ and observe that

$$\begin{aligned}\mu - \mathcal{A} &= (\mu-\lambda)I_{\mathcal{X}} + \mathcal{L}_\lambda \mathcal{A}_\lambda \mathcal{M}_\lambda \\ &= (\mu-\lambda)I_{\mathcal{X}} + \mathcal{L}_\lambda \left(\begin{pmatrix} \mu - A_0 & 0 \\ 0 & \mu - \tilde{B}_\lambda \end{pmatrix} - (\mu-\lambda) I_{\mathcal{X}} \right) \mathcal{M}_\lambda \\ &= \mathcal{L}_\lambda \begin{pmatrix} \mu - A_0 & 0 \\ 0 & \mu - \tilde{B}_\lambda \end{pmatrix} \mathcal{M}_\lambda + (\mu-\lambda)(I_{\mathcal{X}} - \mathcal{L}_\lambda \mathcal{M}_\lambda).\end{aligned}$$

One can check that

$$\mathcal{L}_\lambda \mathcal{M}_\lambda = \begin{pmatrix} I_X & -D_\lambda^{A,L} \\ -BR(\lambda, A_0) & I_{\partial X} + BR(\lambda, A_0) D_\lambda^{A,L} \end{pmatrix}, \tag{2.4}$$

and the claim follows. □

Proposition 2.1.3. *Let $\lambda \in \rho(A_0)$. Then $\mathcal{A} - \lambda$ is similar to the operator matrix*

$$\tilde{\mathcal{A}}_\lambda := \begin{pmatrix} A_0 - D_\lambda^{A,L} B - \lambda & -D_\lambda^{A,L}(\tilde{B}_\lambda - \lambda) \\ B & \tilde{B}_\lambda - \lambda \end{pmatrix}, \tag{2.5}$$

with diagonal domain

$$D(\tilde{\mathcal{A}}_\lambda) := D(A_0) \times D(\tilde{B}).$$

The similarity transformation is performed by means of the operator $\mathcal{M}_\lambda$ defined in (2.2)*, which is an isomorphism on $\mathcal{X}$.*

Proof. Take $\lambda \in \rho(A_0)$ and consider the factorisation in (2.2). By Lemma C.4 the Dirichlet operator $D_\lambda^{A,L}$ is bounded from ∂X to X, thus the operator matrix $\mathcal{M}_\lambda$ is bounded. Moreover, $\mathcal{M}_\lambda$ is invertible with bounded inverse

$$\mathcal{M}_\lambda^{-1} = \begin{pmatrix} I_X & D_\lambda^{A,L} \\ 0 & I_{\partial X} \end{pmatrix} = \begin{pmatrix} I_X & -D_\lambda^{A,-L} \\ 0 & I_{\partial X} \end{pmatrix},$$

thus $\lambda - \mathcal{A}$ is similar to $\mathcal{M}_\lambda \mathcal{L}_\lambda \mathcal{A}_\lambda \mathcal{M}_\lambda \mathcal{M}_\lambda^{-1} = \mathcal{M}_\lambda \mathcal{L}_\lambda \mathcal{A}_\lambda$. A direct matrix computation finally shows that $\mathcal{M}_\lambda \mathcal{L}_\lambda \mathcal{A}_\lambda = -\tilde{\mathcal{A}}_\lambda$. □

2.2 Generator and spectral properties

Most operator theoretical properties are invariant under similarity transformation, and the operator matrix introduced in (2.5) is much easier to handle. This is due to the fact that it has diagonal domain (see, e.g., [Na89] and [Na96]).

Proposition 2.2.1. *Let A_0, $\tilde{B}$, and $\mathcal{A}$ generate C_0-semigroups. If $B = 0$, then*

$$e^{t\mathcal{A}} = \begin{pmatrix} e^{tA_0} & (\lambda - A_0) \int_0^t e^{(t-s)A_0} D_\lambda^{A,L} e^{s\tilde{B}} ds \\ 0 & e^{t\tilde{B}} \end{pmatrix}, \qquad t \geq 0, \tag{2.6}$$

for all $\lambda \in \rho(A_0)$ (up to considering the extension of the upper-right entry from ∂X to X).

Observe that by (C.2) the upper-right entries of (2.6) actually agree for all $\lambda \in \rho(A_0)$.

Proof. Take $\lambda \in \rho(A_0)$. Then by Proposition 2.1.3 there holds

$$\mathcal{A} - \lambda = \mathcal{M}_\lambda^{-1} \tilde{\mathcal{A}}_\lambda \mathcal{M}_\lambda = \mathcal{M}_\lambda^{-1} \begin{pmatrix} A_0 - \lambda & -D_\lambda^{A,L}(\tilde{B} - \lambda) \\ 0 & \tilde{B} - \lambda \end{pmatrix} \mathcal{M}_\lambda.$$

We can compute the semigroup generated by $\tilde{\mathcal{A}}_\lambda$ by applying [Na89, Prop. 3.1] and obtain

$$\begin{aligned} e^{t\tilde{\mathcal{A}}_\lambda} &= \begin{pmatrix} e^{t(A_0-\lambda)} & -\int_0^t e^{(t-s)(A_0-\lambda)} D_\lambda^{A,L}(\tilde{B} - \lambda) e^{s(\tilde{B}-\lambda)} ds \\ 0 & e^{t(\tilde{B}-\lambda)} \end{pmatrix} \\ &= e^{-\lambda t} \begin{pmatrix} e^{tA_0} & -\int_0^t e^{(t-s)A_0} D_\lambda^{A,L}(\tilde{B} - \lambda) e^{s\tilde{B}} ds \\ 0 & e^{t(\tilde{B}-\lambda)}, \end{pmatrix} \end{aligned} \tag{2.7}$$

.

up to considering the extension of the upper-right entry from ∂X to X. By Lemma A.4

$$e^{-\lambda t} e^{t\mathcal{A}} = \mathcal{M}_\lambda^{-1} e^{t\tilde{\mathcal{A}}_\lambda} \mathcal{M}_\lambda,$$

and integrating by parts we obtain (2.6). □

The following seems to be new.

Corollary 2.2.2. *Let the assumptions of Proposition 2.2.1 hold. Then the following assertions hold.*

(1) If $(e^{t\mathcal{A}})_{t\geq 0}$ is bounded, then so are $(e^{tA_0})_{t\geq 0}$ and $(e^{t\tilde{B}})_{t\geq 0}$.

(2) Let $(e^{tA_0})_{t\geq 0}$ and $(e^{t\tilde{B}})_{t\geq 0}$ be bounded, with $\tilde{B} \in \mathcal{L}(\partial X)$. If either of these semigroups is uniformly exponentially stable, then $(e^{t\mathcal{A}})_{t\geq 0}$ is bounded.

(3) Let $\tilde{B} \in \mathcal{L}(\partial X)$. If $(e^{tA_0})_{t\geq 0}$ and $(e^{t\tilde{B}})_{t\geq 0}$ are both uniformly exponentially stable, then also $(e^{t\mathcal{A}})_{t\geq 0}$ is uniformly exponentially stable.

Proof. (1) follows directly from (2.6). To check (2) and (3), take $\lambda \in \rho(A_0)$ and observe that the upper-right entry of (2.7) can be seen as the convolution

$$\left(e^{\cdot A_0} D_\lambda^{A,L}\right) * \left((\tilde{B} - \lambda) e^{\cdot \tilde{B}}\right).$$

By the Datko–Pazy theorem, a C_0-semigroup is uniformly exponentially stable if and only if it is of class $L^1(\mathbb{R}_+, \mathcal{L}(X))$. Now the claims follows by well-known results on convolution of operator-valued functions on $\mathbb{R}_+$, cf. [ABHN01, Prop. 1.3.5.(c)–(d)]. □

The following is a direct consequence of [EN00, Prop. II.4.25].

Corollary 2.2.3. *Assume the operator matrix $\mathcal{A}$ to have nonempty resolvent set. Then $\mathcal{A}$ has compact resolvent if and only if the embeddings $[D(A_0)] \hookrightarrow X$ and $[D(\tilde{B})] \hookrightarrow \partial X$ are both compact.*

In particular, if $\tilde{B}$ is bounded on ∂X, then a necessary condition for $\mathcal{A}$ to have compact resolvent is that ∂X be finite dimensional.

The following is a consequence of Lemma A.8.(1) and Corollaries 2.2.2 and 2.2.3. Observe that [CENP04, Thm. 2.7] is a special case of it.

Corollary 2.2.4. *Let A_0, $\tilde{B}$, and $\mathcal{A}$ generate C_0-semigroups, with $B = 0$ and $\tilde{B} \in \mathcal{L}(\partial X)$. Assume $(e^{tA_0})_{t\geq 0}$ and $(e^{t\tilde{B}})_{t\geq 0}$ to be bounded, and either of them to be uniformly exponentially stable. If the embedding $[D(A_0)] \hookrightarrow X$ is compact, then $(e^{t\mathcal{A}})_{t\geq 0}$ is asymptotically almost periodic.*

The spectrum and the point spectrum (denoted by σ and $P\sigma$, respectively) of the operator matrix $\mathcal{A}$ on the product space $\mathcal{X} = X \times \partial X$ can be (partially) characterized by means of the operator pencils $(\tilde{B}_\lambda)_{\lambda\in\rho(A_0)}$ on ∂X.

The following is an immediate consequence of the theory developed in [En99], and earlier in [Na96].

Lemma 2.2.5. *Let B be bounded from $[D(A_0)]$ to ∂X. For $\lambda \in \rho(A_0)$ the equivalences*

$$\lambda \in \sigma(\mathcal{A}) \iff \lambda \in \sigma(\tilde{B}_\lambda) \qquad \text{and} \qquad \lambda \in P\sigma(\mathcal{A}) \iff \lambda \in P\sigma(\tilde{B}_\lambda) \tag{2.8}$$

hold. If moreover the set $\Gamma := \left\{\lambda \in \mathbb{C} : \lambda \in \rho(A_0) \cap \rho(\tilde{B}_\lambda)\right\} \subset \rho(\mathcal{A})$ is nonempty, then for $\lambda \in \Gamma$ the resolvent operator of $\mathcal{A}$ is given by

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} R(\lambda, A_0) + D_\lambda^{A,L} R(\lambda, \tilde{B}_\lambda) B R(\lambda, A_0) & D_\lambda^{A,L} R(\lambda, \tilde{B}_\lambda) \\ R(\lambda, \tilde{B}_\lambda) B R(\lambda, A_0) & R(\lambda, \tilde{B}_\lambda) \end{pmatrix}. \tag{2.9}$$

Proof. Let $\lambda \in \rho(A_0)$. Then the factorization (2.2) holds. Observe that the operators $\mathcal{L}_\lambda$, $\mathcal{M}_\lambda$ are isomorphism on $\mathcal{X}$, hence $\lambda - \mathcal{A}$ is invertible if and only if the diagonal matrix $\mathcal{A}_\lambda$ is invertible. We conclude that $\lambda \in \sigma(\mathcal{A})$ if and only if $\lambda \in \sigma(\tilde{B}_\lambda)$, and likewise that $\lambda \in P\sigma(\mathcal{A})$ if and only if $\lambda \in P\sigma(\tilde{B}_\lambda)$.

Finally, taking again into account (2.2) we obtain that for $\lambda \in \Gamma$ there holds $R(\lambda, \mathcal{A}) = \mathcal{M}_\lambda^{-1} \mathcal{A}_\lambda^{-1} \mathcal{L}_\lambda^{-1}$. A direct computation now yields (2.9). □

Lemma 2.2.6. *Let $B = 0$. If $\lambda \notin P\sigma(\tilde{B})$, then the equivalence*

$$\lambda \in P\sigma(\mathcal{A}) \iff \lambda \in P\sigma(A_0).$$

holds.

Proof. Take $\mathfrak{u} = \binom{u}{Lu} \in D(\mathcal{A})$ and let

$$(\mathcal{A} - \lambda)\mathfrak{u} = \begin{pmatrix} Au - \lambda u \\ (\tilde{B} - \lambda) Lu \end{pmatrix} = 0.$$

We obtain that $Lu = 0$, hence $(A_0 - \lambda)u = 0$ and the claim follows. □

The equivalences in (2.8) hold in fact not only for the spectrum and point spectrum, but also for the essential spectrum. However, under additional assumptions we can obtain a more precise characterisation.

Proposition 2.2.7. *The following assertions hold.*

(1) Let ∂X be finite dimensional. If B is bounded from $[D(A_0)]$ to ∂X, then the essential spectrum of $\mathcal{A}$ is given by

$$\sigma_{\text{ess}}(\mathcal{A}) = \sigma_{\text{ess}}(A_0),$$

and for the Fredholm index we have

$$\operatorname{ind}(\mathcal{A} - \mu) = \operatorname{ind}(A_0 - \mu) \qquad \text{for all } \mu \notin \sigma_{\text{ess}}(A_0).$$

(2) Let Y be a Banach space such that $D(A) \subset Y \hookrightarrow X$. Assume the embeddings $[D(A_0)] \hookrightarrow [D(A)]_L \hookrightarrow Y$ to be both compact. If B is bounded from Y to ∂X, then

$$\sigma_{\text{ess}}(\mathcal{A}) = \sigma_{\text{ess}}(\tilde{B})$$

and for the Fredholm index we have

$$\operatorname{ind}(\mathcal{A} - \mu) = \operatorname{ind}(\tilde{B} - \mu) \qquad \text{for all } \mu \notin \sigma_{\text{ess}}(\tilde{B}).$$

Proof. To begin with, recall that the essential spectrum does neither change under compact additive perturbations, nor under similarity transformations. Moreover, observe that by Remark C.5.(a) under the assumptions in both (1) and (2) $D_\lambda^{A,L}$ and $BR(\lambda, A_0)$ are compact operators (from ∂X to X and from X to ∂X, respectively) for all $\lambda \in \rho(A_0)$.

(1) Fix $\lambda \in \rho(A_0)$, take into account (2.3), and observe that by (2.4) $I_{\mathcal{X}} - \mathcal{L}_\lambda \mathcal{M}_\lambda$ is a compact operator on $\mathcal{X}$. Moreover, $\mathcal{L}_\lambda$ and $\mathcal{M}_\lambda$ are isomorphisms on $\mathcal{X}$. Thus, to decide whether a given $\mu \in \mathbb{C}$ is in the essential spectrum of $\mathcal{A}$ it suffices to check whether 0 is in the essential spectrum of the operator matrix

$$\begin{pmatrix} \mu - A_0 & 0 \\ 0 & \mu - \tilde{B}_\lambda \end{pmatrix} = \begin{pmatrix} \mu - A_0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mu - \tilde{B}_\lambda \end{pmatrix}. \tag{2.10}$$

The second addend is a bounded operator with finite dimensional range, hence it does not affect the essential spectrum of the operator matrix on the left-hand side, and the claim follows.

(2) Take $\lambda \in \rho(A_0)$ and reason as in the proof of (1) to obtain that $\mu \in \sigma_{\text{ess}}(\mathcal{A})$ if and only if 0 lies in the essential spectrum of the operator matrix in (2.10). Observe that A_0 has empty essential spectrum (because $[D(A_0)]$ is compactly embedded in X) and that $\sigma_{\text{ess}}(\tilde{B}_\lambda) = \sigma_{\text{ess}}(\tilde{B})$ (because BD_λ is a compact operator on ∂X) to conclude. □

Theorem 2.2.8.(1) and (2) (with a different proof) are [KMN03, Prop. 4.3] and [CENN03, Cor. 2.8], respectively.

Theorem 2.2.8. *The following assertions hold.*

(1) Let $B \in \mathcal{L}([D(A)]_L, \partial X)$ and $\tilde{B} \in \mathcal{L}(\partial X)$. Then the operator matrix $\mathcal{A}$ generates a C_0-semigroup on $\mathcal{X}$ if and only if the operator $A_0 - D_\lambda^{A,L} B$ generates a C_0-semigroup on X for some $\lambda \in \rho(A_0)$ if and only if $A_0 - D_\lambda^{A,L} B$ generates a C_0-semigroup on X for all $\lambda \in \rho(A_0)$.

(2) *Let $B \in \mathcal{L}(X, \partial X)$. Then $\mathcal{A}$ generates an analytic semigroup on $\mathcal{X}$ if and only if A_0 and $\tilde{B}$ generate analytic semigroups on X and ∂X, respectively.*

(3) *Let A_0 and $\tilde{B}$ generate anayltic semigroups on X and ∂X, respectively. If for some $0 < \alpha < 1$ there holds $[D(A)]_L \hookrightarrow [D(A_0), X]_\alpha$ and further $B \in \mathcal{L}([D(A)]_L, \partial X) \cap \mathcal{L}\left([D(A_0)], [D(\tilde{B}), \partial X]_\alpha\right)$, then $\mathcal{A}$ generates an analytic semigroup on $\mathcal{X}$.*

Proof. Take $\lambda \in \rho(A_0)$. It has been proven in Proposition 2.1.3 that $\mathcal{A} - \lambda$ is similar to the operator matrix $\tilde{\mathcal{A}}_\lambda$ defined in (2.5). Thus, $\mathcal{A}$ is a generator if and only if $\tilde{\mathcal{A}}_\lambda$ is a generator.

(1) We decompose

$$\tilde{\mathcal{A}}_\lambda = \begin{pmatrix} A_0 - D_\lambda^{A,L} B & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} + \begin{pmatrix} -\lambda & D_\lambda^{A,L}(\lambda - \tilde{B}_\lambda) \\ 0 & \tilde{B}_\lambda - \lambda \end{pmatrix}$$

with diagonal domain $D(\tilde{\mathcal{A}}_\lambda) = D(A_0) \times \partial X$.

Since $B \in \mathcal{L}([D(A)]_L, \partial X)$, by Remark 1.1.1 also $B \in \mathcal{L}([D(A_0)], \partial X)$, hence the second operator on the right-hand side is bounded on the product Banach space $[D(\tilde{\mathcal{A}}_\lambda)] = [D(A_0)] \times \partial X$. Moreover, the third operator on the right-hand side is bounded on $\mathcal{X}$ as a direct consequence of Lemma C.4. Taking into account Lemma A.6.(1)-(2) the claim follows.

(2) We decompose

$$\tilde{\mathcal{A}}_\lambda = \begin{pmatrix} A_0 & -D_\lambda^{A,L} \tilde{B} \\ 0 & \tilde{B} \end{pmatrix} + \begin{pmatrix} -D_\lambda^{A,L} B - \lambda & D_\lambda^{A,L}(\lambda - B D_\lambda^{A,L}) \\ B & B D_\lambda^{A,L} - \lambda \end{pmatrix}$$

with diagonal domain $D(\tilde{\mathcal{A}}_\lambda) = D(A_0) \times D(\tilde{B})$.

Since $B \in \mathcal{L}(X, \partial X)$, by Lemma C.4 the second operator on the right hand side is bounded on $\mathcal{X}$. Hence, by Lemma A.6.(1) $\tilde{\mathcal{A}}_\lambda$ generates an analytic semigroup on $\mathcal{X}$ if and only if

$$\begin{pmatrix} A_0 & -D_\lambda^{A,L} \tilde{B} \\ 0 & \tilde{B} \end{pmatrix} \qquad \text{with domain } D(A_0) \times D(\tilde{B})$$

generates an analytic semigroup on $\mathcal{X}$. Since $D_\lambda^{A,L} \tilde{B} \in \mathcal{L}([D(\tilde{B})], X)$, and the claim follows by [Na89, Cor. 3.3].

(3) We decompose

$$\tilde{\mathcal{A}}_\lambda = \begin{pmatrix} A_0 & 0 \\ 0 & \tilde{B} \end{pmatrix} + \begin{pmatrix} -D_\lambda^{A,L} B - \lambda & -D_\lambda^{A,L} \tilde{B} \\ B & 0 \end{pmatrix} + \begin{pmatrix} 0 & D_\lambda^{A,L}(\lambda - B D_\lambda^{A,L}) \\ 0 & B D_\lambda^{A,L} - \lambda \end{pmatrix}$$

with diagonal domain $D(\tilde{\mathcal{A}}_\lambda) = D(A_0) \times D(\tilde{B})$.

The first addend on the right-hand side generates an analytic semigroup on $\mathcal{X}$ and the corresponding interpolation space is $[D(\tilde{\mathcal{A}}_\lambda), \mathcal{X}]_\alpha = [D(A_0), X]_\alpha \times [D(\tilde{B}), \partial X]_\alpha$.

Thus, by assumption the second addend on the right-hand side is bounded from $[D(\tilde{\mathcal{A}}_\lambda)]$ to $[D(\tilde{\mathcal{A}}_\lambda), \mathcal{X}]_\alpha$, while the third one is bounded on $\mathcal{X}$. Hence, by Lemma A.6.(3) $\tilde{\mathcal{A}}_\lambda$ generates an analytic semigroup on $\mathcal{X}$. $\square$

Remarks 2.2.9. (a) A very special case is that of inhomogeneous, time-independent boundary conditions. That is, we consider an abstract problem of the form

$$\left\{\begin{array}{rcll} \dot v(t) &=& Av(t), & t\geq 0,\\ Lv(t) &=& g, & t\geq 0,\\ v(0) &=& f\in X. \end{array}\right. \tag{2.11}$$

for $g \in \partial X$. Then we can differentiate the second equation with respect to t, rewrite such a problem as $(\mathrm{AIBVP}_{f,g})$ with $B = \tilde B = 0$, and finally apply Theorem 1.2.4 and Theorem 2.2.8.(1) to obtain that (2.11) admits a unique mild solution for all $f \in X$, $g \in \partial X$ if and only if A_0 generates a C_0-semigroup on X. By Proposition 2.2.1 such a solution $v = v(t)$ is given by

$$v(t) = e^{tA_0}f + (\lambda - A_0)\int_0^t e^{(t-s)A_0} D_\lambda^{A,L} g ds, \qquad t\geq 0.$$

for any $\lambda \in \rho(A_0)$, and it is classical if $f \in D(A)$ and if moreover the compatibility condition $Lf = g$ is satisfied.

(b) It is worth to remark that if $B \in \mathcal{L}(X, \partial X)$ and $\tilde B \in \mathcal{L}(\partial X)$, then by Lemma A.6.(1) the estimate

$$\|u(t) - v(t)\|_X \leq tM, \qquad 0\leq t\leq 1,$$

holds for the mild solution u to $(\mathrm{AIBVP}_{f,g})$, where $M > 0$ is a suitable constant, and v is the mild solution to (2.11) obtained in (a).

We can now revisit a problem considered in [CENN03, § 3] and slightly improve the result obtained therein.

Example 2.2.10. Let Ω be a bounded open domain of $\mathbb{R}^n$ with boundary $\partial\Omega$ smooth enough. Set

$$X := L^2(\Omega) \qquad \text{and} \qquad \partial X := L^2(\partial\Omega).$$

Define

$$A := \Delta, \qquad D(A) := \left\{u \in H^{\frac32}(\Omega) : \Delta u \in L^2(\Omega)\right\},$$

$$L := \frac{\partial}{\partial\nu}, \qquad D(L) := D(A),$$

$$\tilde B := \Delta, \qquad D(\tilde B) := H^2(\partial\Omega),$$

i.e., $\tilde B$ is the Laplace–Beltrami operator on $\partial\Omega$.

Then, it has been shown in [CENN03, § 3] that A and L satisfy the Assumptions 2.1.1, and one sees that the restriction A_0 of A to $\ker(L)$ is the Neumann Laplacian. Since A_0 and $\tilde B$ generate analytic semigroups, Theorem 2.2.8.(3) applies and the operator matrix $\mathcal{A}$ defined in (4.2) generates an analytic semigroup on $L^2(\Omega)\times L^2(\partial\Omega)$ for any operator B that is bounded from $H^{\frac32}(\Omega)$ to $L^2(\partial\Omega)$ as well as from $H^2(\Omega)$ to $H^{\frac12}(\partial\Omega)$ (in [CENN03] only the case $B \in \mathcal{L}(L^2(\Omega), L^2(\partial\Omega))$ has been considered).

Remark 2.2.11. Lemma 2.2.8.(1) shows that the well-posedness of the abstract Cauchy problem ($\mathcal{ACP}$) on the product space $\mathcal{X}$ is in some sense equivalent to the well-posedness of the (perturbed) abstract Cauchy problem on X associated to

$$\dot{u}(t) = A_0 u(t) - D_\lambda^{A,L} B u(t), \qquad t \geq 0,$$

for some/all $\lambda \in \rho(A_0)$. Observe that if $B \in \mathcal{L}(X, \partial X)$, then $A_0 - D_\lambda^{A,L} B$ is a generator if and only if A_0 is (since in this case $D_\lambda^{A,L} B \in \mathcal{L}(X)$ for all $\lambda \in \rho(A_0)$).

2.3 Powers of an operator matrix with coupled domain

The following can be checked by a direct computation.

Lemma 2.3.1. *If the operator $B : D(A) \to \partial X$ maps $D(A^2)$ into $D(\tilde{B})$, then the square of the operator matrix $\mathcal{A}$ is given by*

$$\mathcal{A}^2 = \begin{pmatrix} A^2 & 0 \\ BA + \tilde{B}B & \tilde{B}^2 \end{pmatrix}$$

with domain

$$D(\mathcal{A}^2) = \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in D(A^2) \times D(\tilde{B}^2) : Lu = x,\ LAu = Bu + \tilde{B}Lu \right\}.$$

Remark 2.3.2. It is remarkable that $\mathcal{A}^2$ can be seen as an operator matrix with coupled domain

$$D(\mathcal{A}^2) = \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in D(A_w^2) \times D(\tilde{B}^2) : Lu = x \right\},$$

where A_w^2 is given by

$$A_w^2 u := A^2 u \qquad \text{for all} \quad u \in D(A_w^2) := \left\{ u \in D(A^2) : LAu = Bu + \tilde{B}Lu \right\}.$$

This can be looked at as an abstract formulation of a generalized Wentzell boundary condition on the operator A in the sense of [FGGR00], cf. also [En04b].

More generally, the following can be proven by induction on n.

Lemma 2.3.3. *Let the operator $B : D(A) \to \partial X$ map $D(A^{k+1})$ into $D(\tilde{B}^k)$, $k \in \mathbb{N}$. Define the family $(B_h)_{h \in \mathbb{N}}$ of operators from X to ∂X by*

$$B_h := \begin{cases} 0 & \text{for } h = 0, \\ \sum_{k=0}^{h-1} \tilde{B}^k B A^{h-k-1} & \text{for } h = 1, 2, \ldots. \end{cases}$$

Then the n-th power of $\mathcal{A}$ is given by

$$\mathcal{A}^n = \begin{pmatrix} A^n & 0 \\ B_n & \tilde{B}^n \end{pmatrix}$$

with domain

$$D(\mathcal{A}^n) = \bigcap_{h=0}^{n-1} \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in D(A^n) \times D(\tilde{B}^n) : LA^h u = B_h u + \tilde{B}^h x \right\}.$$

We can now deduce a regularity result for the solutions of the Cauchy problem governed by the semigroup generated by $\mathcal{A}$. We first need to define the class

$$\mathcal{D}_0^n := \bigcap_{h=0}^{n-1} \left\{ u \in D(A^n) : LA^h u = BA^h u = 0 \right\}, \qquad n = 1, 2, \dots,$$

which in concrete applications contains the class of test functions on a bounded domain, cf. e.g. Example 3.3.2 below.

Corollary 2.3.4. *Let the operator* $B : D(A) \to \partial X$ *map* $D(A^{k+1})$ *into* $D(\tilde{B}^k)$, $k \in \mathbb{N}$. *If* $\mathcal{A}$ *generates a* C_0*-semigroup, then* $e^{t\mathcal{A}}$ *maps* $\mathcal{D}_0^n \times \{0\}$ *into* $D(A^n) \times D(\tilde{B}^n)$ *for all* $t \geq 0$ *and* $n = 1, 2, \dots$. *If further this* C_0*-semigroup is analytic, then in fact* $e^{t\mathcal{A}}$ *maps* $\mathcal{X}$ *into* $D^\infty(A) \times D(\tilde{B}^\infty)$ *for all* $t > 0$.

Proof. It is evident that $\mathcal{D}_0^n \times \{0\} \subset D(\mathcal{A}^n) \subset D(A^n) \times D(\tilde{B}^n)$ for all $n \in \mathbb{N}$. Then, we just need to apply Lemma A.5. □

Chapter 3

Second order abstract problems with acoustic boundary conditions

Certain investigations have led theoretical physicists, cf. [MI68], to wave equations equipped with *acoustic* (or *absorbing*) boundary conditions, which can be written in the form

$$\text{(ABC)} \qquad \begin{cases} \ddot{\phi}(t,x) &= c^2\Delta\phi(t,x), & t\in\mathbb{R},\ x\in\Omega,\\ m(z)\ddot{\delta}(t,z) &= -d(z)\dot{\delta}(t,z)-k(z)\delta(t,z) & \\ & \qquad -\rho(z)\dot{\phi}(t,z), & t\in\mathbb{R},\ z\in\partial\Omega,\\ \dot{\delta}(t,z) &= \frac{\partial\phi}{\partial\nu}(t,z), & t\in\mathbb{R},\ z\in\partial\Omega. \end{cases}$$

Here ϕ is the velocity potential of a fluid filling an open domain $\Omega\subset\mathbb{R}^n$, either bounded or exterior; δ is the normal displacement of the (sufficiently smooth) boundary $\partial\Omega$ of Ω; m, d, and k are the mass per unit area, the resistivity, and the spring constant of the boundary, respectively; finally, ρ and c are the unperturbed density of, and the constant speed of sound in the medium, respectively. It is reasonable to assume all these physical quantities to be modelled by essentially bounded functions with ρ, m real valued, $\rho\geq 0$, $\inf_{z\in\partial\Omega} m(z)>0$.

Quoting Beale and Rosencrans in [BR74] (who denote by G our domain Ω), we point out that 'the physical model giving rise to these conditions is that of a gas undergoing small irrotational perturbations from rest in a domain G with smooth compact boundary', assuming that 'each point of the surface ∂G acts like a spring in response to the excess pressure in the gas, and that there is no transverse tension between neighboring points of ∂G, i.e., the “springs” are independent of each other'.

3.1 The direct approach: Beale's results

The mathematical formulation of, and the first well-posedness results for the initial value problem associated to (ABC) have been presented in the 1970s, in a series of papers mainly by Beale ([BR74], [Be76], and [Be77]). Beale was in fact already using techniques based on operator matrices and C_0-semigroups – this is one of the earliest historical appearences of such a method we are aware of. Due to their historical interest, and as a warm-up for our own investigation, we explain in this section Beale's ideas.

Under somehow stricter assumptions on the parameters m, d, ρ, k (in particular, ρ is now a constant), Beale considered the weighted product space

$$\mathcal{Y} := \left(H^1(\Omega;\rho)/ <\mathbb{1}> \right) \times L^2\left(\Omega; \frac{\rho}{c^2}\right) \times L^2(\partial\Omega; k) \times L^2(\partial\Omega; m)$$

and the operator matrix

$$\mathcal{G} := \begin{pmatrix} 0 & I_{H^1(\Omega)/<\mathbb{1}>} & 0 & 0 \\ \Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & -\frac{\rho}{m}\cdot_{|\partial\Omega} & -\frac{k}{m} & -\frac{d}{m} \end{pmatrix}$$

with maximal domain

$$D(\mathcal{G}) := \left\{ \begin{pmatrix} u \\ v \\ x \\ y \end{pmatrix} \in \mathcal{Y} : \Delta u \in L^2(\Omega; \frac{\rho}{c^2}), v \in \left(H^1(\Omega;\rho)/ <\mathbb{1}>\right), \frac{\partial u}{\partial \nu} = y \right\}.$$

Taking the quotient of $H^1(\Omega;\rho)$ (i.e., considering the functions of class $H^1(\Omega;\rho)$ modulo constants) in the first factor of $\mathcal{Y}$ is needed by Beale in order to endow $H^1(\Omega;\rho)$ with the equivalent Dirichlet norm

$$|||u||| := \int_\Omega \rho(x) \|\nabla u(x)\|^2 dx.$$

Because of boundary regularity, one can check that the domain of $\mathcal{G}$ agrees with

$$\left\{ \begin{pmatrix} u \\ v \\ x \\ y \end{pmatrix} \in \left(H^{\frac{3}{2}}(\Omega;\rho)/ <\mathbb{1}>\right) \times \left(H^1(\Omega;\rho)/ <\mathbb{1}> \cap L^2(\Omega; \tfrac{\rho}{c^2})\right) \right.$$
$$\left. \times L^2(\partial\Omega; k) \times L^2(\partial\Omega; m) : \Delta u \in L^2(\Omega; \tfrac{\rho}{c^2}) \text{ and } \tfrac{\partial u}{\partial \nu} = y \right\}.$$

Since (ABC) is formally equivalent to

$$\dot{\mathfrak{u}}(t) = \mathcal{G}\mathfrak{u}(t), \qquad t \in \mathbb{R},$$

the issue becomes to investigate the generator property of $\mathcal{G}$.

The following is [Be76, Thm. 2.1] (see also [FG00], where a more general nonlinear problem is considered).

Proposition 3.1.1. *Assume m, k, d to be continuous positive (m strictly positive) functions, ρ a positive constant. Then the operator matrix $\mathcal{G}$ generates a C_0-group on $\mathcal{Y}$. Moreover, the semigroup $(e^{t\mathcal{G}})_{t\geq 0}$ is contractive. If $d \equiv 0$, then $(e^{t\mathcal{G}})_{t\in\mathbb{R}}$ is unitary.*

Proof. The crucial point is that, by the Gauss–Green formulae, for $\mathfrak{u} \in D(\mathcal{G})$ there holds

$$\mathrm{Re}\,\langle \mathcal{G}\mathfrak{u}, \mathfrak{u}\rangle = -\int_{\partial\Omega} d(z)|y(z)|^2 d\sigma(z),$$

where y is the fourth component of $\mathfrak{u}$. If $d \equiv 0$, then Stone's theorem applies and $\mathcal{G}$ generates a unitary group. The general case can be considered as a bounded perturbation of the unitary case. Finally, due to the positivity of d, $\mathcal{G}$ is always dissipative, and the Lumer–Phillips theorem applies as well. □

Beale explicitly considered only the particular cases of a bounded (or exterior) domain Ω of $\mathbb{R}^3$, but in fact Proposition 3.1.1 holds whenever Ω is a bounded (or exterior) domain of $\mathbb{R}^n$ with boundary smooth enough that the formulae of Gauss–Green hold. The reason why Beale was restricting to the 3-dimensional case is perhaps that there explicit computations can be performed in order to show that $\mathcal{G}$ does not have compact resolvent and to describe the essential spectrum of $\mathcal{G}$. However, Beale's techniques are very technical and can be hardly applied to problems on domains of $\mathbb{R}^n$, $n \neq 3$. For an extension of Beale's results to arbitrary bounded domains of $\mathbb{R}^n$ see [Ga04].

We can fit Beale's setting into the abstract framework introduced in Chapter 2: in fact, we can see

$$\mathcal{G} = \left(\begin{array}{ccc|c} 0 & I_{H^1(\Omega)/<\mathbb{1}>} & 0 & 0 \\ \Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ \hline 0 & -\frac{\rho}{m}\cdot_{|\partial\Omega} & -\frac{k}{m} & -\frac{d}{m} \end{array}\right)$$

as an operator matrix with coupled domain

$$D(\mathcal{G}) = \left\{ \begin{pmatrix} \begin{pmatrix} u \\ v \\ x \end{pmatrix} \\ y \end{pmatrix} \in \Big(H^{\frac{3}{2}}(\Omega;\rho)/<\mathbb{1}> \times \big(H^1(\Omega;\rho)/<\mathbb{1}> \cap L^2(\Omega;\tfrac{\rho}{c^2})\big) \right.$$
$$\left. \times L^2(\Omega;k) : \Delta u \in L^2(\Omega;\tfrac{\rho}{c^2})\Big) \times L^2(\partial\Omega) : \begin{pmatrix} \frac{\partial}{\partial\nu} & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ x \end{pmatrix} = y \right\}.$$

Without going into details, one can check that the Assumptions 2.1.1 are satisfied, and in particular the upper-left block entry of $\mathcal{G}$, restricted to the null space of the operator $\begin{pmatrix} \frac{\partial}{\partial\nu} & 0 & 0 \end{pmatrix}$, becomes

$$\begin{pmatrix} 0 & I_{H^1(\Omega)/<\mathbb{1}>} & 0 \\ \Delta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with domain

$$\big\{u \in H^2(\Omega) \cap H^1(\Omega;\rho)/<\mathbb{1}>:\ \Delta u \in L^2(\Omega;\tfrac{\rho}{c^2}), \tfrac{\partial u}{\partial\nu} = 0\big\}$$
$$\times \big(H^1(\Omega;\rho)/<\mathbb{1}> \cap L^2(\Omega;\tfrac{\rho}{c^2})\big) \times L^2(\partial\Omega).$$

Hence, by Corollary 2.2.3 we are able to prove the following, which was formulated as a conjecture in the first draft of [GGG03, § 5]. J.A. Goldstein has informed us that his and G.R. Goldstein's student C. Gal has obtained, by different methods, the same result, which will appear in [Ga04].

Proposition 3.1.2. *The operator matrix $\mathcal{G}$ has compact resolvent if and only if Ω is a bounded interval of $\mathbb{R}$. In this case, $(e^{t\mathcal{G}})_{t\geq 0}$ is asymptotically almost periodic and, if $d \equiv 0$, then $(e^{t\mathcal{G}})_{t\in\mathbb{R}}$ is almost periodic.*

Proof. Recall that for a domain $\Omega \subset \mathbb{R}^n$ the embeddings $H^2(\Omega) \hookrightarrow H^1(\Omega) \hookrightarrow L^2(\Omega)$ are compact if and only if Ω is bounded. Moreover, the multiplication operator

$$L^2(\partial\Omega) \ni f \longmapsto \frac{df}{m} \in L^2(\partial\Omega)$$

is bounded under our assumption on d, m. Then Corollary 2.2.3, Lemma B.22, and Lemma B.25 yield the claim. □

Applying Propositions 2.2.5 and 2.2.7, one also obtains information about the spectrum and essential spectrum of $\mathcal{G}$ for Ω in arbitrary dimension. However, if we apply Theorem 2.2.8.(1) in order to re-prove the generator property by the methods we have introduced in Chapter 2, we conclude that $\mathcal{G}$ generates a C_0-group if

$$\begin{pmatrix} 0 & I_{H^1(\Omega)/<\mathbb{1}>} + D_{\lambda^2}^{\Delta,\frac{\partial}{\partial\nu}}\left(\frac{\rho}{m}\cdot_{|\partial\Omega}\right) \\ \Delta & 0 \end{pmatrix}$$

with domain

$$\left\{u \in H^2(\Omega) \cap H^1(\Omega;\rho)/<\mathbb{1}>: \Delta u \in L^2(\Omega; \frac{\rho}{c^2}), \frac{\partial u}{\partial \nu} = 0\right\} \times H^1(\Omega;\rho)/<\mathbb{1}>$$

generates a C_0-semigroup on $H^1(\Omega;\rho)/<\mathbb{1}> \times L^2(\Omega; \frac{\rho}{c^2})$ for some $\lambda > 0$. Seemingly, no known perturbation result for C_0-semigroups can be applied, so we are not able to show the well-posedness of our motivating problem in this way.

Motivated by this failure, our purpose becomes to develop a more abstract approach to tackle this and more general problems as an application of the theory introduced in the previous chapter. This will reduce the need for formal computations and allow more general cases.

3.2 General setting and well-posedness

We impose the following throughout the rest of this chapter.

Assumptions 3.2.1.

1. *X, Y, and ∂X are Banach spaces with $Y \hookrightarrow X$.*
2. *$A : D(A) \to X$ is linear with $D(A) \subset Y$.*

3. *$R : D(A) \to \partial X$ is linear and surjective.*

4. *B_1, B_2 are linear and bounded from Y to ∂X.*

5. *B_3, B_4 are linear and bounded on ∂X.*

6. *$\begin{pmatrix} A \\ R \end{pmatrix} : D(A) \subset Y \to X \times \partial X$ is closed.*

7. *$A_0 := A|_{\ker(R)}$ generates a cosine operator function with associated phase space $Y \times X$.*

Moreover, it will be convenient to define a new operator L by

$$L := R + B_2, \qquad L : D(A) \to \partial X.$$

We will see that in some applications the operator L is in some sense "more natural" than R. E.g., when we discuss the motivating equation (ABC), the operator B_2 will be the trace operator and L the normal derivative, while R is a linear combination of the two. This shows that the operator $A_0 = A_{|\ker(R)}$ can be considered as an abstract version of a operator equipped with Robin boundary conditions.

Remarks 3.2.2. (a) Observe in particular that since $B_2 \in \mathcal{L}(Y, \partial X)$ the Assumption 3.2.1.6 is satisfied if (and only if) also the operator

$$\begin{pmatrix} A \\ L \end{pmatrix} = \begin{pmatrix} A \\ R \end{pmatrix} + \begin{pmatrix} 0 \\ B_2 \end{pmatrix} : D(A) \subset Y \to X \times \partial X$$

is closed.

(b) Reasoning as in Lemma C.1 and Lemma C.4 one can see that under the Assumptions 3.2.1 the Dirichlet operators associated to (A, L) exist as bounded operators from ∂X to Z for every Banach space Z satisfying $D^\infty(A) \subset Z \hookrightarrow Y$.

Of concern in this chapter are abstract second order initial-boundary value problems equipped with *abstract acoustic boundary conditions* of the form

$$(\text{aAIBVP}^2_{f,g,h,j}) \quad \begin{cases} \ddot{u}(t) &= Au(t), & t \in \mathbb{R}, \\ \ddot{x}(t) &= B_1 u(t) + B_2 \dot{u}(t) + B_3 x(t) + B_4 \dot{x}(t), & t \in \mathbb{R}, \\ \dot{x}(t) &= Lu(t), & t \in \mathbb{R}, \\ u(0) &= f \in X, \qquad \dot{u}(0) = g \in X, & \\ x(0) &= h \in \partial X, \qquad \dot{x}(0) = j \in \partial X, & \end{cases}$$

on X and ∂X, where the operators A, B_1, B_2, B_3, B_4, R satisfy the Assumptions 3.2.1.

Following the approach of Section 1.3, we consider the operator matrix

$$(3.1) \qquad \mathbf{A} := \begin{pmatrix} 0 & I_Y & 0 \\ A & 0 & 0 \\ L & 0 & 0 \end{pmatrix}, \qquad D(\mathbf{A}) := D(A) \times Y \times \partial X,$$

on the product Banach space

$$\mathbf{X} := Y \times X \times \partial X,$$

the operators

$$\mathbf{L} := \begin{pmatrix} R & 0 & 0 \end{pmatrix}, \qquad D(\mathbf{L}) := D(\mathbf{A}),$$

and

$$\mathbf{B} := \begin{pmatrix} B_1 + B_4 B_2 & 0 & B_3 \end{pmatrix}, \qquad D(\mathbf{B}) := D(\mathbf{A}),$$

both from $\mathbf{X}$ to $\partial\mathbf{X} := \partial X$, and finally the operator

$$\tilde{\mathbf{B}} := B_4, \qquad D(\tilde{\mathbf{B}}) := D(B_4) = \partial X,$$

on $\partial\mathbf{X}$. Observe that we have identified

$$\mathbf{u}(t) := \begin{pmatrix} u(t) \\ v(t) \\ x(t) \end{pmatrix}, \quad t \in \mathbb{R}, \qquad \text{and} \qquad \mathbf{f} := \begin{pmatrix} f \\ g \\ h \end{pmatrix}, \quad \mathbf{g} := j - B_2 f.$$

By Proposition 1.2.4, the well-posedness of (aAIBVP^2) is equivalent to the generator property of the operator matrix with coupled domain

$$\text{(3.2)} \qquad \mathcal{A} := \begin{pmatrix} \mathbf{A} & 0 \\ \mathbf{B} & \tilde{\mathbf{B}} \end{pmatrix}, \qquad D(\mathcal{A}) := \left\{ \begin{pmatrix} \mathbf{u} \\ \mathbf{x} \end{pmatrix} \in D(\mathbf{A}) \times \partial\mathbf{X} \,:\, \mathbf{L}\mathbf{u} = \mathbf{x} \right\},$$

on the product Banach space $\mathcal{X} := \mathbf{X} \times \partial\mathbf{X}$. In order to apply the results on operator matrices with coupled domain obtained in Chapter 2 we need the following.

Lemma 3.2.3. *The following assertions hold.*

(1) The restriction $\mathbf{A}_0$ of $\mathbf{A}$ to $\ker(\mathbf{L})$ generates a C_0-group on $\mathbf{X}$.

(2) The operator $\mathbf{L}$ is surjective from $D(\mathbf{A})$ to $\partial\mathbf{X}$.

(3) The operator $\mathbf{B}$ is bounded from $\mathbf{X}$ to $\partial\mathbf{X}$.

(4) The operator $\tilde{\mathbf{B}}$ is bounded on $\partial\mathbf{X}$.

(5) The operator $\begin{pmatrix} \mathbf{A} \\ \mathbf{L} \end{pmatrix} : D(\mathbf{A}) \subset \mathbf{X} \to \partial\mathbf{X}$ is closed.

Proof. Observe first that $\ker(\mathbf{L}) = \{u \in D(A) \,:\, Lu = B_2 u\} \times Y \times \partial X$, thus the operator $\mathbf{A}_0$ takes the form

$$\text{(3.3)} \qquad \mathbf{A}_0 = \left(\begin{array}{cc|c} 0 & I_Y & 0 \\ A_0 & 0 & 0 \\ \hline B_2 & 0 & 0 \end{array}\right).$$

Observe that the perturbation $\begin{pmatrix} B_2 & 0 \end{pmatrix}$ is bounded from $Y \times X$ to ∂X, and the only non-zero diagonal block of $\mathbf{A}_0$ generates by Lemma B.11 a C_0-group on $Y \times X$. Therefore, $\mathbf{A}_0$ generates a C_0-group on $\mathbf{X}$, and (1) is proven. The remaining claims follow directly by Assumptions 3.2.1. $\square$

Therefore, $\mathbf{A}, \mathbf{L}, \tilde{\mathbf{B}}$ satisfy the Assumptions 2.1.1, and by Theorem 2.2.8.(1) and Remark 2.2.11 the following result is immediate.

Theorem 3.2.4. *The operator matrix with coupled domain $\mathcal{A}$ defined in* (3.2) *generates a C_0-group on $\mathcal{X}$.*

Hence, it follows by Theorem 1.3.6 that (aAIBVP^2) is well-posed.

Example 3.2.5. The abstract initial–boundary value problem associated to the wave equation with acoustic boundary conditions (ABC) on an open bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$ is well-posed. In particular, for all initial data

$$\phi(0,\cdot) \in H^{\frac{3}{2}}(\Omega), \quad \dot{\phi}(0,\cdot) \in H^1(\Omega), \quad \delta(0,\cdot), \dot{\delta}(0,\cdot) \in L^2(\partial\Omega)$$
$$\text{such that} \quad \Delta\phi(0,\cdot) \in L^2(\Omega) \quad \text{and} \quad \frac{\partial \phi}{\partial \nu}(0,\cdot) = \dot{\delta}(0,\cdot)$$

there exists a classical solution to (ABC) on $(H^1(\Omega), L^2(\Omega), L^2(\partial\Omega))$ continuously depending on them.

Take first

$$X := L^2(\Omega), \qquad Y := H^1(\Omega), \qquad \partial X := L^2(\partial\Omega).$$

We set

$$A := c^2\Delta, \qquad D(A) := \left\{ u \in H^{\frac{3}{2}}(\Omega) \,:\, \Delta u \in L^2(\Omega) \right\},$$

$$(Rf)(z) = \frac{\partial f}{\partial \nu}(z) + \frac{\rho(z)}{m(z)} f(z), \qquad f \in D(R) = D(A),\ z \in \partial\Omega,$$

$$B_1 = 0, \qquad (B_2 f)(z) := -\frac{\rho(z)}{m(z)} f(z), \qquad f \in H^1(\Omega),\ z \in \partial\Omega,$$

$$(B_3 g)(z) := -\frac{k(z)}{m(z)} g(z), \qquad (B_4 g)(z) := -\frac{d(z)}{m(z)} g(z), \qquad g \in L^2(\partial\Omega),\ z \in \partial\Omega.$$

By Theorem 3.2.4 and Theorem 1.3.6, it suffices to check that the Assumptions 3.2.1 are satisfied in the above setting.

To check the Assumption 3.2.1.3, we apply [LM72, Vol. I, Thm. 2.7.4] and obtain that for all $g \in L^2(\partial\Omega)$ there exists $u \in H^{\frac{3}{2}}(\Omega)$ such that $\Delta u = 0$ and $\frac{\partial u}{\partial \nu} + \frac{\rho}{m} u_{|\partial\Omega} = g$. The Assumption 3.2.1.4 holds because the trace operator is bounded from $H^1(\Omega)$ to $L^2(\partial\Omega)$ and because $\frac{\rho}{m} \in L^\infty(\partial\Omega)$, while the Assumption 3.2.1.5 is satisfied because $\frac{d}{m}, \frac{k}{m} \in L^\infty(\partial\Omega)$.

The Assumption 3.2.1.6 is satisfied because the closedness of $\binom{A}{L}$ holds by interior estimates for elliptic operators, (a short proof of this can be found in [CENN03, § 3]), and $B_2 \in \mathcal{L}(Y, \partial X)$, cf. Remark 3.2.2.(a).

To check Assumption 3.2.1.7, observe that

$$A_0 u = c^2 \Delta u, \qquad D(A_0) = \left\{ u \in H^2(\Omega) \,:\, \frac{\partial u}{\partial \nu} + \frac{\rho}{m} u_{|\partial\Omega} = 0 \right\},$$

i.e., A_0 is (up to the constant c^2) the Laplacian with Robin boundary conditions. By [Fa85, Thm. IV.5.1], this operator generates a cosine operator function with associated phase space $H^1(\Omega) \times L^2(\Omega) = Y \times X$.

Our Assumptions 3.2.1 are satisfied by a variety of other operators and spaces. We discuss a biharmonic wave equation with acoustic-type boundary conditions.

Example 3.2.6. Let $p, q, r, s \in L^\infty(\partial\Omega)$, $s \leq 0$. Then the initial value problem associated to

$$\begin{cases} \ddot{\phi}(t,x) &= -\Delta^2\phi(t,x), & t \in \mathbb{R},\ x \in \Omega, \\ \ddot{\delta}(t,z) &= p(z)\delta(t,z) + q(z)\dot{\delta}(t,z) & \\ & \qquad +r(z)\frac{\partial\phi}{\partial\nu}(t,z) + s(z)\frac{\partial\dot{\phi}}{\partial\nu}(t,z), & t \in \mathbb{R},\ z \in \partial\Omega, \\ \dot{\delta}(t,z) &= \Delta\phi(t,z), & t \in \mathbb{R},\ z \in \partial\Omega, \\ \phi(t,z) &= 0, & t \in \mathbb{R},\ z \in \partial\Omega, \end{cases}$$

is well-posed. In particular, for all initial data

$$\phi(0,\cdot) \in H^4(\Omega) \cap H^1_0(\Omega), \qquad \dot{\phi}(0,\cdot) \in H^2(\Omega) \cap H^1_0(\Omega), \qquad \delta(0,\cdot) \in L^2(\partial\Omega),$$

$$\text{and} \qquad \dot{\delta}(0,\cdot) \in L^2(\partial\Omega) \qquad \text{such that} \qquad \Delta\phi(0,z) = \dot{\delta}(0,z),\ z \in \partial\Omega,$$

there exists a classical solution continuously depending on them.

Take

$$X := L^2(\Omega), \qquad Y := H^2(\Omega) \cap H^1_0(\Omega), \qquad \partial X := L^2(\partial\Omega),$$

and consider the operators

$$A := -\Delta^2, \qquad D(A) := \left\{ u \in H^{\frac{5}{2}}(\Omega) \cap H^1_0(\Omega) : \Delta^2 u \in L^2(\Omega) \right\},$$

$$Ru := (\Delta u)_{|\partial\Omega} - s\frac{\partial u}{\partial\nu}, \qquad \text{for all} \quad u \in D(R) := D(A),$$

$$B_1 := r\frac{\partial}{\partial\nu}, \qquad B_2 := s\frac{\partial}{\partial\nu}, \qquad D(B_1) := D(B_2) := Y,$$

$$B_3 x := px, \qquad B_4 := qx, \qquad \text{for all} \quad x \in \partial X.$$

We are only going to prove that A_0, i.e., the restriction of $-\Delta^2$ to

$$D(A_0) := \ker(R) = \left\{ u \in H^4(\Omega) \cap H^1_0(\Omega) \ :\ (\Delta u)_{|\partial\Omega} = s\frac{\partial u}{\partial\nu} \right\},$$

generates a cosine operator function with associated phase space $\big(H^2(\Omega) \cap H^1_0(\Omega)\big) \times L^2(\Omega) = Y \times X$, the other Assumptions 3.2.1 being satisfied trivially.

Take $u, v \in D(A_0)$ and observe that applying the Gauss–Green formulae twice yields

$$\langle A_0 u, v\rangle = -\int_\Omega \Delta^2 u \cdot \overline{v}\, dx = -\int_\Omega \Delta u \cdot \overline{\Delta v}\, dx + \int_{\partial\Omega} s\frac{\partial u}{\partial\nu} \cdot \overline{\frac{\partial v}{\partial\nu}}\, d\sigma. \tag{3.4}$$

Since $s \leq 0$, it is immediate that A_0 is dissipative and self-adjoint, hence by Remark B.4.(a) the generator of a cosine operator function on X. We claim that the associated Kisyński space (see Definition B.13) is actually isomorphic to $Y = H^2(\Omega) \cap H^1_0(\Omega)$.

We first show that A_0 is injective. To see this, let $u \in D(A_0)$ such that $A_0 u = 0$. By (3.4) one obtains

$$0 = -\langle A_0 u, u\rangle = \|\Delta u\|^2_{L^2(\Omega)} - \int_{\partial\Omega} s \left|\frac{\partial u}{\partial \nu}\right|^2 d\sigma \geq \|\Delta u\|^2_{L^2(\Omega)}.$$

Hence, u is a harmonic function of class $H^1_0(\Omega)$. The fact that the Dirichlet Laplacian is invertible implies that $u = 0$, thus that 0 is not an eigenvalue of A_0. Since A_0 has compact resolvent, we conclude that it is a self-adjoint, strictly negative definite operator. Therefore, by Remark B.18 we deduce that the Kisyński space is isomorphic to $[D(A_0), L^2(\Omega)]_{\frac{1}{2}}] = \big(H^2(\Omega) \cap H^1_0(\Omega)\big)$ as claimed.

Remarks 3.2.7. (a) Among further operators and spaces fitting into our abstract framework we list the following. In both cases, the operator B_2 is defined as in Example 3.2.5.

a) $X := L^2(\Omega),\ Y := H^1(\Omega),\ \partial X := L^2(\partial\Omega),$

$Au(x) := \nabla(a(x)\nabla u(x)),\ x \in \Omega$, with the function $a \geq 0$ sufficiently regular on $\overline{\Omega}$,

$Lu(z) = \langle a(z)\nabla u(z), \nu(z)\rangle,\ z \in \partial\Omega,$

for $u \in D(A) := \{H^{\frac{3}{2}}(\Omega) : Au \in L^2(\Omega)\}$.

b) $X := L^2(\Omega),\ Y = H^2(\Omega),\ \partial X := L^2(\partial\Omega),$

$Au := -\Delta^2 u,$

$Lu := -\frac{\partial \Delta u}{\partial \nu},$

for $u \in D(A) := \left\{H^{\frac{7}{2}}(\Omega) : \Delta^2 u \in L^2(\Omega),\ (\Delta u)_{|\partial\Omega} = 0\right\}$.

In both cases one can check that A_0 is self-adjoint and dissipative by the Gauss–Green formulae, and this ensures that A_0 generates a cosine operator function.

(b) Remarkably, it seems to be still unknown whether the (realization of the) second derivative with Robin boundary conditions generates a cosine operator function on $L^p(0,1)$, $p \neq 2$. If this would be the case, however, a wave equation with acoustic boundary condition in $L^p(0,1)$ could be easily fitted into the setting presented in this section.

Remark 3.2.8. The main drawback of our approach, in comparison with Beale's, is that the group that governs the motivating equation (ABC) on a bounded domain $\Omega \subset \mathbb{R}^n$ is not contractive, as it can be seen already in the case of $n = 1$ – in other words, we fail to produce an energy space.

However, our approach has other advantages: e.g., the entries $\mathbf{B}$ and $\tilde{\mathbf{B}}$ of $\mathcal{A}$ defined in (3.2) are bounded from $\mathbf{X}$ to $\partial\mathbf{X}$ and on $\partial\mathbf{X}$, respectively. Thus, by Remark 2.2.9.(b) the estimate

$$\|\phi(t,\cdot) - \psi(t,\cdot)\|_{L^2(\Omega)} \leq tM, \qquad 0 \leq t \leq 1,$$

holds for some constant M. Here ϕ is the solution to the initial value problem associated to (ABC) and ψ is the solution to the inital value problem associated to the wave equation with inhomogeneous (static) Robin boundary conditions

$$\begin{cases} \ddot{\psi}(t,x) = c^2 \Delta \psi(t,x), & t \in \mathbb{R},\ x \in \Omega, \\ \frac{\partial \psi}{\partial \nu}(t,z) + \frac{\rho(z)}{m(z)} \psi(t,z) = \frac{\partial \phi}{\partial \nu}(0,z) + \frac{\rho(z)}{m(z)} \phi(0,z), & t \in \mathbb{R},\ z \in \partial\Omega. \end{cases}$$

3.3 Regularity and spectral theory

The following, which is crucial to prove Proposition 1.3.7, is a direct consequence of Lemma B.20.

Lemma 3.3.1. *For the operator* $\mathbf{A}$ *defined in* (3.1) *we obtain*

$$D(\mathbf{A}^{2k-1}) = D(A^k) \times D((A^{k-1})_{|Y}) \times \partial X \qquad \textit{and}$$

$$D(\mathbf{A}^{2k}) = D((A^k)_{|Y}) \times D(A^k) \times \partial X \qquad \textit{for all } k \in \mathbb{N}.$$

In particular, $D(\mathbf{A}^\infty) = D^\infty(A) \times D^\infty(A) \times \partial X$*.*

Example 3.3.2. Proposition 1.3.7 yields a regularity result that is analogous to [Be76, Thm. 2.2]. In the framework introduced in Example 3.2.5 to treat (ABC) the set defined in (1.12) is

$$\bigcap_{h \in \mathbb{N}} \left\{ u \in C^\infty(\Omega) \cap H^{\frac{3}{2}}(\Omega) \,:\, \gamma_h u = 0 \right\}$$

where γ_h denotes the normal derivative of h-th order, cf. [LM72, Vol. 1, § 1.8.2]. Hence, if in particular the initial value $\phi(0,\cdot), \dot{\phi}(0,\cdot)$ are functions of class $C^\infty(\Omega) \cap H^{\frac{3}{2}}(\Omega)$ that vanish in a suitable neighborhood of $\partial\Omega$, thus belonging to the set defined above, and if $\dot{\delta}(0,\cdot) \equiv 0$, then the unique classical solution to the initial value problem associated to (ABC) is contained in $D^\infty(A) \subset C^\infty(\Omega)$, no matter how rough $\delta(0,\cdot)$ is.

Due to the important role played by the operator matrix $\mathcal{A}$ defined in (3.2), we are interested in obtaining some spectral results about it. To begin with, we generalize Proposition 3.1.2.

Proposition 3.3.3. *The operator matrix* $\mathcal{A}$ *has compact resolvent if and only if* ∂X *is finite dimensional and moreover the embeddings* $[D(A_0)] \hookrightarrow Y \hookrightarrow X$ *are both compact.*

Proof. Take into account Corollary 2.2.3 and obtain that $\mathcal{A}$ has compact resolvent if and only if the embedding $[D(\mathbf{A}_0)] \hookrightarrow \mathbf{X}$ is compact and $\partial\mathbf{X}$ is finite dimensional. This yields the claim, since $D(\mathbf{A}_0) = D(A_0) \times Y \times \partial X$ and $\mathbf{X} = Y \times X \times \partial X$. □

Taking into account Lemma B.22 and [Na89, Thm. 2.4], we obtain the following.

Lemma 3.3.4. *The resolvent set of the operator matrix* $\mathbf{A}_0$ *as in* (3.3) *is given by*

$$\rho(\mathbf{A}_0) = \left\{\lambda \in \mathbb{C} \, : \, \lambda \neq 0, \; \lambda^2 \in \rho(A_0)\right\}.$$

For $\lambda \in \rho(\mathbf{A}_0)$ *there holds*

$$R(\lambda, \mathbf{A}_0) = \begin{pmatrix} \lambda R(\lambda^2, A_0) & R(\lambda^2, A_0) & 0 \\ A_0 R(\lambda^2, A_0) & \lambda R(\lambda^2, A_0) & 0 \\ -B_2 R(\lambda^2, A_0) & -\frac{1}{\lambda} B_2 R(\lambda^2, A_0) & \frac{1}{\lambda} I_{\partial X} \end{pmatrix}.$$

By Lemma 3.2.3, Lemma C.1, and Lemma C.4 we obtain the existence of the Dirichlet operator associated to the pair $(\mathbf{A}, \mathbf{L})$. More precisely, the following representation holds.

Lemma 3.3.5. *Let* $\lambda \in \rho(\mathbf{A}_0)$. *Then the Dirichlet operator* $D_\lambda^{\mathbf{A},\mathbf{L}}$ *exists and is given by*

$$D_\lambda^{\mathbf{A},\mathbf{L}} = \begin{pmatrix} D_{\lambda^2}^{A,R} \\ \lambda D_{\lambda^2}^{A,R} \\ \frac{1}{\lambda} L D_{\lambda^2}^{A,R} \end{pmatrix}.$$

Proof. To obtain the claimed representation, take $\mathbf{x} := y \in \partial X = \partial \mathbf{X}$. By definition the Dirichlet operator $D_\lambda^{\mathbf{A},\mathbf{L}}$ maps $\mathbf{x}$ into the unique vector

$$\mathbf{u} := \begin{pmatrix} u \\ v \\ x \end{pmatrix} \in D(\mathbf{A}) \quad \text{such that} \quad \begin{cases} \mathbf{Au} &= \lambda \mathbf{u}, \\ \mathbf{Lu} &= \mathbf{x}, \end{cases} \quad \text{or rather} \quad \begin{cases} v &= \lambda u, \\ Au &= \lambda v, \\ Lu &= \lambda x, \\ Ru &= y. \end{cases}$$

Thus, we see that $u = D_{\lambda^2}^{A,R} y$, and the claim follows. □

Observe that by definition of L

$$L D_\lambda^{A,R} = I_{\partial X} + B_2 D_\lambda^{A,R}.$$

Thus, the following holds.

Lemma 3.3.6. *Let* $\lambda \in \rho(\mathbf{A}_0)$. *Then the operator*

$$\tilde{\mathbf{B}}_\lambda := \tilde{\mathbf{B}} + \mathbf{B} D_\lambda^{\mathbf{A},\mathbf{L}}$$

is given by

$$\tilde{\mathbf{B}}_\lambda = B_1 D_{\lambda^2}^{A,R} + \left(\frac{1}{\lambda} B_3 + B_4\right) L D_{\lambda^2}^{A,R}.$$

With the operators introduced in Lemma 3.3.4, 3.3.5, and 3.3.6 we can exploit the spectral results of Section 2.2, and in particular apply Lemma 2.2.5 to describe the spectrum and the resolvent operator of the operator matrix $\mathcal{A}$ defined in (3.2).

To conclude, we briefly consider the essential spectrum and obtain the following, which in some sense complements the results of [Be76, § 3]: e.g., it also applies if we consider the motivating equation (ABC) to take place on the unbounded domain $\Omega = \mathbb{R}_+$. The following is just a consequence of Proposition 2.2.7.(1).

Proposition 3.3.7. *If ∂X is finite dimensional, then the essential spectrum of $\mathcal{A}$ is given by*

$$\sigma_{\text{ess}}(\mathcal{A}) = \sigma_{\text{ess}}(\mathbf{A}_0),$$

and for the Fredholm index we have

$$\operatorname{ind}(\mathcal{A} - \mu) = \operatorname{ind}(A_0 - \mu) \qquad \text{for all } \mu \notin \sigma_{\text{ess}}(A_0).$$

3.4 Implicit acoustic boundary conditions

Among the so-called *boundary contact problems* discussed by B.P. Belinsky in [Be00, § 3], a version of the Timoschenko model

$$\text{(TM)} \quad \begin{cases} \ddot{\phi}(t,x) &=& c^2\Delta\phi(t,x), & t \in \mathbb{R},\ x \in \Omega, \\ \frac{\partial\phi}{\partial\nu}(t,z) &=& 0, & t \in \mathbb{R},\ z \in \Gamma_0, \\ m(1-\Delta)\ddot{\delta}(t,z) &=& -\rho\dot{\phi}(t,z) - k(z)\delta(t,z) & \\ && \quad -d(z)\dot{\delta}(t,z), & t \in \mathbb{R},\ z \in \Gamma_1, \\ \dot{\delta}(t,z) &=& \frac{\partial\phi}{\partial\nu}(t,z), & t \in \mathbb{R},\ z \in \Gamma_1, \\ \delta(t,y) &=& 0, & t \in \mathbb{R},\ y \in \partial\Gamma_1, \end{cases}$$

is particularly interesting, because it can be seen as a wave equation equipped with implicit acoustic-type boundary conditions. For the geophysical explanation of this model we refer to [Be00]. We only mention that the system (TM) models an ocean waveguide Ω covered (on the part Γ_1 of his surface $\partial\Omega$) by a thin pack ice layer with inertia of rotation. B.P. Belinsky investigates such a system for $\Omega \subset \mathbb{R}^2$ and obtains some spectral properties.

Here the boundary $\partial\Omega$ is the disjoint union of Γ_0, Γ_1. Observe that, due to technical reasons, we consider the case of a medium of *homogeneous* density ρ filling a domain whose boundary has *homogeneous* mass m. However, we still allow k and d to be essentially bounded functions, whereas B.P. Belinsky assumes them to be constant.

To begin with, we introduce an operator M that will appear in the new implicit acoustic-type boundary conditions.

Assumption 3.4.1. *We complement the Assumptions 3.2.1 by the following.*

$M : D(M) \subset \partial X \to \partial X$ is linear and satisfies $1 \in \rho(M)$.

We can now consider the abstract second order initial-boundary value problem obtained by replacing the second equation in (aAIBVP^2) by

$$\ddot{x}(t) - M\ddot{x}(t) = B_1 u(t) + B_2\dot{u}(t) + B_3 x(t) + B_4\dot{x}(t), \qquad t \in \mathbb{R}.$$

Thus, our aim is to show the well-posedness of the problem

$$(\text{iaAIBVP}^2) \quad \begin{cases} \ddot{u}(t) &=& Au(t), & t \in \mathbb{R}, \\ \ddot{x}(t) &=& B_1^\diamond u(t) + B_2^\diamond\dot{u}(t) + B_3^\diamond x(t) + B_4^\diamond\dot{x}(t), & t \in \mathbb{R}, \\ \dot{x}(t) &=& Lu(t), & t \in \mathbb{R}, \\ u(0) &=& f \in X, \qquad \dot{u}(0) = g \in X, & \\ x(0) &=& h \in \partial X, \qquad \dot{x}(0) = j \in \partial X, & \end{cases}$$

on X and ∂X, where

$$B_i^\diamond := R(1,M)B_i, \qquad i = 1,2, \tag{3.5}$$

are bounded operators from Y to ∂X, and

$$B_i^\diamond := R(1,M)B_i, \qquad i = 3,4, \tag{3.6}$$

are bounded operators on ∂X. Similarly, we consider the operator

$$R^\diamond := L - B_2^\diamond. \tag{3.7}$$

Observe now that, after replacing R by $R^\diamond$ and B_i by $B_i^\diamond$, $i = 1,2,3,4$, all the Assumptions 3.2.1 are satisfied, except for 3 and 7.

Assumptions 3.4.2. *We replace the corresponding Assumptions 3.2.1 by the following.*

> $3'$. $R^\diamond : D(A) \to \partial X$ *is surjective.*
>
> $7'$. $A_0^\diamond := A|_{\ker(R^\diamond)}$ *generates a cosine operator function with associated phase space* $Y \times X$.

Proposition 3.4.3. *Under the Assumptions 3.2.1, 3.4.1, and 3.4.2 the problem* (iaAIBVP^2) *with abstract implicit acoustic-type boundary conditions is well-posed.*

Proof. Consider the operator matrix $\mathbf{A}$ introduced in (3.1) and define the operators

$$\mathbf{L}^\diamond := \begin{pmatrix} R^\diamond & 0 & 0 \end{pmatrix}, \qquad D(\mathbf{L}^\diamond) := D(\mathbf{A}),$$
$$\mathbf{B}^\diamond := \begin{pmatrix} B_1^\diamond + B_4^\diamond B_2^\diamond & 0 & B_3^\diamond \end{pmatrix}, \qquad D(\mathbf{B}^\diamond) := \mathbf{X},$$
$$\tilde{\mathbf{B}}^\diamond := B_4^\diamond, \qquad D(\tilde{\mathbf{B}}^\diamond) := \partial\mathbf{X}.$$

We can now directly check that properties analogous to those in Lemma 3.2.3 are satisfied. Therefore, the well-posedness of

$$(i\mathbf{AIBVP}) \qquad \begin{cases} \dot{\mathbf{u}}(t) = \mathbf{A}\mathbf{u}(t), & t \in \mathbb{R}, \\ \dot{\mathbf{x}}(t) = \mathbf{B}^\diamond\mathbf{u}(t) + \tilde{\mathbf{B}}^\diamond\mathbf{x}(t), & t \in \mathbb{R}, \\ \mathbf{x}(t) = \mathbf{L}^\diamond\mathbf{u}(t), & t \in \mathbb{R}, \\ \mathbf{u}(0) = \mathbf{f} \in \mathbf{X}, \\ \mathbf{x}(0) = \mathbf{g} \in \partial\mathbf{X}, \end{cases}$$

follows like in Theorem 3.2.4. Finally, reasoning like in Lemma 1.3.5 one obtains the equivalence between (i**AIBVP**) and (iaAIBVP^2), and the claim follows. □

We revisit Example 3.2.5.

Example 3.4.4. The initial value problem associated to the wave equation with implicit acoustic-type boundary conditions (TM) on an open bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, is well-posed. In particular, for all initial data

$$\phi(0,\cdot) \in H^{\frac{3}{2}}(\Omega), \quad \dot{\phi}(0,\cdot) \in H^1(\Omega), \quad \delta(0,\cdot), \dot{\delta}(0,\cdot) \in L^2(\Gamma_1)$$
$$\text{such that} \quad \Delta\phi(0,\cdot) \in L^2(\Omega) \quad \text{and} \quad \frac{\partial \phi}{\partial \nu}(0,\cdot)_{|\Gamma_1} = \dot{\delta}(0,\cdot)$$

there exists a classical solution on $(H^1(\Omega), L^2(\Omega), L^2(\Gamma_1))$ continuously depending on them.

We adapt the setting introduced in Example 3.2.5 to the current problem. We let

$$X := L^2(\Omega), \qquad Y := H^1(\Omega), \qquad \partial X := L^2(\Gamma_1).$$

Moreover, we set

$$A := c^2\Delta, \qquad D(A) := \left\{ u \in H^{\frac{3}{2}}(\Omega) \, : \, \Delta u \in L^2(\Omega), \, \frac{\partial u}{\partial \nu}_{|\Gamma_0} = 0 \right\},$$

$$(Rf)(z) = \frac{\partial f}{\partial \nu}(z) + \frac{\rho(z)}{m(z)} f(z), \qquad f \in D(R) = D(A), \; z \in \Gamma_1,$$

$$B_1 = 0, \qquad (B_2 f)(z) := -\frac{\rho(z)}{m(z)} f(z), \qquad f \in H^1(\Omega), \; z \in \Gamma_1,$$

$$(B_3 g)(z) := -\frac{k(z)}{m(z)} g(z), \qquad (B_4 g)(z) := -\frac{d(z)}{m(z)} g(z), \qquad g \in L^2(\Gamma_1), \; z \in \Gamma_1.$$

Further, we introduce the operator

$$M := \Delta_{\Gamma_1}, \qquad D(M) := H^2(\Gamma_1) \cap H^1_0(\Gamma_1),$$

that is, the Laplace–Beltrami operator (on the precompact manifold Γ_1 with boundary $\partial\Gamma_1$) equipped with Dirichlet boundary conditions. We can now define the auxiliary operators $B_i^\diamond$, $i = 1, 2, 3, 4$, and $R^\diamond$ like in (3.5)–(3.7).

The operator M is self-adjoint and strictly negative definite, hence it satisfies the Assumption 3.4.1, hence only the Assumptions 3.4.2 still need to be checked.

To check the Assumption 3.4.2.3' we are going to apply Lemma C.7. First consider the normal derivative $L = R + B_2$ on Γ_1: L is surjective by [LM72, Vol. I, Thm. 2.7.4]. Moreover, observe that

$$\ker(L) = \left\{ u \in H^2(\Omega) \, : \, \frac{\partial u}{\partial \nu}_{|\partial\Omega} = 0 \right\},$$

that is, the restriction A_0 of A to $\ker(L)$ is the Neumann Laplacian Δ_N, which generates an analytic semigroup on X. Further, by [LM72, Vol. II, (4.14.32)], we obtain that

$$[D(\Delta_N), L^2(\Omega)]_\theta = H^{2(1-\theta)}(\Omega) \qquad \text{for all } \theta \in \left(\frac{1}{4}, 1 \right].$$

Therefore we obtain that

$$D(A) \subset H^{\frac{3}{2}}(\Omega) \subset [D(\Delta_N)], L^2(\Omega)]_{\frac{1}{2}-\epsilon} \subset [D(\Delta_N), L^2(\Omega)]_{\frac{1}{2}} = H^1(\Omega) = Y$$

for all $0 \leq \epsilon < \frac{1}{4}$. Since $L - R^\diamond = B_2^\diamond \in \mathcal{L}(H^1(\Omega), L^2(\Gamma_1))$, by Lemma C.7 the Assumption 3.4.2.3' holds.

Finally, to check the Assumption 3.4.2.7' we show that $A_0^\diamond$ is self-adjoint and strictly negative definite. Recall that we are assuming ρ and m to be real constants, and observe that

$$D(A_0^\diamond) = \left\{ f \in H^2(\Omega) \, : \, \frac{\partial u}{\partial \nu}_{|\Gamma_0} = 0, \; \frac{\partial u}{\partial \nu}_{|\Gamma_1} + \frac{\rho}{m} R(1,M)(u_{|\Gamma_1}) = 0 \right\}.$$

Take $u, v \in D(A_0^\diamond)$ and obtain that

$$\begin{aligned} \langle A_0^\diamond u, v \rangle &= \int_\Omega \Delta u \cdot \overline{v} dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot \overline{v} d\sigma - \int_\Omega \nabla u \cdot \overline{\nabla v} dx \\ &= -\frac{\rho}{m} \int_{\Gamma_1} R(1,M) u \cdot \overline{v} d\sigma - \int_\Omega \nabla u \cdot \overline{\nabla v} dx. \end{aligned}$$

Taking into account the positivity and the self-adjointness of the operator $R(1,M)$ (see [Gr99, § 2.4]), one obtains that $A_0^\diamond$ is self-adjoint and dissipative, hence it generates a cosine operator function. Letting $A_0^\diamond u = 0$ and multiplying by u one can likewise see that $A_0^\diamond$ is also invertible. Hence, apply Remark B.18 to obtain that the Kisyński space is isomorphic to $[D(A_0^\diamond), X]_{\frac{1}{2}}$. Since $[D(A_0^\diamond), X]_{\frac{1}{2}} = H^1(\Omega)$ by [LM72, Vol. II, (4.14.32)], the claim follows.

3.5 The special case of $B_3 = 0$: asymptotic behavior

After setting

$$y(t) \equiv \dot{x}(t), \qquad t \in \mathbb{R},$$

(aAIBVP^2) can equivalently be written as the second order problem with *integro-differential boundary conditions*

$$\left\{ \begin{array}{rcll} \ddot{u}(t) &=& Au(t), & t \in \mathbb{R}, \\ \dot{y}(t) &=& B_1 u(t) + B_2 \dot{u}(t) + B_3 \left(h + \int_0^t y(s) ds \right) + B_4 y(t), & t \in \mathbb{R}, \\ y(t) &=& Lu(t), & t \in \mathbb{R}, \\ u(0) &=& f \in X, \qquad \dot{u}(0) = g \in X, & \\ y(0) &=& j \in \partial X. & \end{array} \right.$$

In the special case of $B_3 = 0$, which we assume throughout this section, the initial value $x(0) = h$ is therefore superfluous, and we obtain an abstract second order problem with first order dynamic boundary conditions. Similar problems have been discussed, among others, in [CENP04], and in fact our Theorem 3.2.4 complements some well-posedness result obtained therein, cf. [CENP04, Thm. 2.2].

We can replace $\mathbf{X} = Y \times X \times \partial X$ by

$$\mathbf{X} = Y \times X,$$

and the operator matrix $\mathbf{A}$ as defined in (3.2) by

$$\mathbf{A} = \begin{pmatrix} 0 & I_Y \\ A & 0 \end{pmatrix}, \qquad D(\mathbf{A}) = D(A) \times Y.$$

Accordingly, the operators $\mathbf{L}$ and $\mathbf{B}$ become

$$\begin{aligned} \mathbf{L} &= \begin{pmatrix} R & 0 \end{pmatrix}, \qquad D(\mathbf{L}) = D(A) \times X, \\ \mathbf{B} &= \begin{pmatrix} B_1 + B_4 B_2 & 0 \end{pmatrix}, \qquad D(\mathbf{B}) = \mathbf{X}. \end{aligned}$$

Then the operator matrix $\mathcal{A}$ defined in (3.2) is replaced by

$$\mathcal{A} = \begin{pmatrix} 0 & I_Y & 0 \\ A & 0 & 0 \\ B_1 + B_4 B_2 & 0 & B_4 \end{pmatrix} \tag{3.8}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} u \\ v \\ y \end{pmatrix} \in D(A) \times Y \times \partial X \,:\, Ru = y \right\}. \tag{3.9}$$

The main difference with the general setting of Section 3.2 is that the resolvent of $\mathbf{A}_0$ as well as the Dirichlet operators associated to $(\mathbf{A}, \mathbf{L})$ can be compact also in the case of $\dim \partial X = \infty$.

Proposition 3.5.1. *Let $B_3 = 0$. Assume that there exists a Banach space Z such that $D(A) \subset Z \hookrightarrow Y$. If the embeddings $Z \hookrightarrow Y \hookrightarrow X$ are both compact, then the essential spectrum of $\mathcal{A}$ is given by*

$$\sigma_{\text{ess}}(\mathcal{A}) = \sigma_{\text{ess}}(B_4),$$

and for the Fredholm index we have

$$\operatorname{ind}(\mathcal{A} - \mu) = \operatorname{ind}(B_4 - \mu) \quad \textit{for } \lambda \notin \sigma_{\text{ess}}(B_4).$$

In particular, $\sigma_{\text{ess}}(\mathcal{A}) = \emptyset$ if and only if ∂X is finite dimensional.

Example 3.5.2. In the context of our motivating equation (ABC), the assumption $B_3 = 0$ means that $k \equiv 0$, hence the initial-boundary value problem becomes

$$\begin{cases} \ddot{\phi}(t,x) &= c^2 \Delta \phi(t,x), & t \in \mathbb{R},\ x \in \Omega, \\ \ddot{\delta}(t,z) &= -\frac{d(z)}{m(z)} \dot{\delta}(t,z) - \frac{\rho(z)}{m(z)} \dot{\phi}(t,z), & t \in \mathbb{R},\ z \in \partial\Omega, \\ \dot{\delta}(t,z) &= \frac{\partial \phi}{\partial \nu}(t,z), & t \in \mathbb{R},\ z \in \partial\Omega, \\ \phi(0,\cdot) &= f, \qquad \dot{\phi}(0,\cdot) = g, & \\ \dot{\delta}(0,\cdot) &= j, & \end{cases} \tag{3.10}$$

on a bounded open domain $\Omega \subset \mathbb{R}^n$. Observe that $D(A) \subset Z := H^{\frac{3}{2}}(\Omega)$, and for $Y = H^1(\Omega)$ the embeddings $H^{\frac{3}{2}}(\Omega) \hookrightarrow H^1(\Omega) \hookrightarrow L^2(\Omega)$ are compact by [LM72, Vol. I, Thm. 1.16.1]. Thus, by Proposition 3.5.1 the essential spectrum of the operator matrix $\mathcal{A}$ associated to (3.10) agrees with the essential spectrum of the (bounded) multiplication operator

$$(B_4 u)(z) = -\frac{d(z)}{m(z)} u(z), \qquad u \in L^2(\partial\Omega),\ z \in \partial\Omega.$$

The essential spectrum of B_4 cannot be empty unless ∂X is finite dimensional, thus the essential spectrum of $\mathcal{A}$ cannot be empty unless Ω is a (possibly unbounded) interval of $\mathbb{R}$.

In case $B_1 = -B_4 B_2$, the operator matrix $\mathcal{A}$ as in (3.8) has in fact only diagonal entries, i.e.

$$\mathcal{A} = \begin{pmatrix} \mathbf{A} & 0 \\ 0 & \tilde{\mathbf{B}} \end{pmatrix}.$$

Such an operator matrix is associated to the second order problem with first order dynamic boundary conditions

$$\begin{cases} \ddot{u}(t) &= Au(t), & t \in \mathbb{R}, \\ \dot{y}(t) &= B_4 y(t), & t \in \mathbb{R}, \\ y(t) &= Ru(t), & t \in \mathbb{R}, \\ u(0) &= f, \qquad \dot{u}(0) = g, & \\ y(0) &= j. & \end{cases}$$

Beyond well-posedness, we can now apply the results on boundedness and asymptotical almost periodicity obtained in Section 2.2.

Proposition 3.5.3. *Let $B_1 = -B_4 B_2$ and $B_3 = 0$. Assume $(C(t, A_0))_{t\in\mathbb{R}}$ to be bounded. If A_0 is invertible and $(e^{tB_4})_{t\geq 0}$ is uniformly exponentially stable, then the semigroup generated by $\mathcal{A}$ as in (3.8)–(3.9) is bounded.*

If moreover the embeddings $[D(A_0)] \hookrightarrow Y \hookrightarrow X$ are both compact and ∂X is finite dimensional, then $(e^{t\mathcal{A}})_{t\geq 0}$ is asymptotically almost periodic.

Proof. Under our assumptions, $(e^{t\mathbf{A}_0})_{t\geq 0}$ is a bounded C_0-semigroup on $\mathbf{X}$ and $(e^{t\tilde{\mathbf{B}}})_{t\geq 0}$ is a uniformly exponentially stable C_0-semigroup on $\partial\mathbf{X}$ with bounded generator. The claim follows by Corollaries 2.2.2 and 2.2.4. □

Example 3.5.4. We consider a version of the problem discussed in Example 3.5.2, which we modify by adding a new feedback term. Observe that by definition

$$(B_4 B_2 f)(z) = \frac{\rho(z) d(z)}{m^2(z)} f(z), \qquad f \in H^1(\Omega),\ z \in \partial\Omega.$$

Hence, the problem we want to consider is

$$
(3.11) \qquad \begin{cases}
\ddot{\phi}(t,x) &= c^2\Delta\phi(t,x), & t\in\mathbb{R},\ x\in\Omega,\\
\ddot{\delta}(t,z) &= -\frac{d(z)}{m(z)}\dot{\delta}(t,z) - \frac{\rho(z)d(z)}{m^2(z)}\phi(t,z) & \\
 & \qquad -\frac{\rho(z)}{m(z)}\dot{\phi}(t,z), & t\in\mathbb{R},\ z\in\partial\Omega,\\
\dot{\delta}(t,z) &= \frac{\partial\phi}{\partial\nu}(t,z), & t\in\mathbb{R},\ z\in\partial\Omega,\\
\phi(0,\cdot) &= f, \qquad \dot{\phi}(0,\cdot) = g, & \\
\dot{\delta}(0,\cdot) &= j, &
\end{cases}
$$

As seen before, A_0 is the Laplacian with Robin boundary conditions: if ρ is not identically 0, i.e., if the Robin boundary conditions do not reduce to Neumann, then A_0 is self-adjoint and strictly negative, and by Remark B.18 it generates a contractive cosine operator function.

Moreover, $(e^{tB_4})_{t\geq 0}$ is the multiplication semigroup given by

$$(e^{tB_4}g)(z) = e^{-t\frac{d(z)}{m(z)}}g(z), \qquad t\geq 0,\ g\in L^2(\partial\Omega),\ z\in\partial\Omega,$$

which is uniformly exponentially stable if (and only if) the closure of the essential range of $\frac{d}{m}$ lies in $\{z\in\mathbb{C} : \text{Rez} < 0\}$.

Summing up, if

$$\rho \not\equiv 0 \quad \text{and} \quad \overline{\left(\frac{d}{m}\right)_{\text{ess}}(\Omega)} \subset \{z\in\mathbb{C} : \text{Re(z)} < 0\},$$

then by Proposition 3.5.3 the solution $\phi = \phi(t)$ to (3.11) is bounded for $t\geq 0$. If moreover Ω is an interval of $\mathbb{R}$, then the compactness of the Sobolev embeddings yields asymptotical almost periodicity of the solution.

Chapter 4

Second order abstract problems with dynamic boundary conditions

Of concern in this chapter are second order abstract initial-boundary value problems with *dynamic boundary conditions* of the form

$$(\mathrm{dAIBPV}^2_{f,g,h,j}) \qquad \begin{cases} \ddot{u}(t) &= Au(t), & & t \in \mathbb{R}, \\ \ddot{x}(t) &= Bu(t) + \tilde{B}x(t), & & t \in \mathbb{R}, \\ x(t) &= Lu(t), & & t \in \mathbb{R}, \\ u(0) &= f \in X, & \dot{u}(0) = g \in X, & \\ x(0) &= h \in \partial X, & \dot{x}(0) = j \in \partial X, & \end{cases}$$

on X and ∂X.

As in the corresponding first order initial-boundary value problems (see [CENN03] and [En03] for the cases of $L \notin \mathcal{L}(X, \partial X)$ and $L \in \mathcal{L}(X, \partial X)$, respectively), we need to distinguish three different cases: for some given Banach space Y (somehow related to the Kisyński space associated to the problem

$$\begin{cases} \ddot{u}(t) &= Au(t), & t \in \mathbb{R}, \\ Lu(t) &= 0, & t \in \mathbb{R}, \end{cases}$$

in the sense of Definition B.13) such that $D(A) \subset Y \hookrightarrow X$ the operator L can be unbounded from Y to ∂X; unbounded from X to ∂X but bounded from Y to ∂X; or bounded from X to ∂X. In this chapter we only consider the first two cases in Sections 4.2 and 4.3, respectively. These cases occur, e.g., when we consider a wave equation on an L^p-space and L is the normal derivative or the trace operator, respectively.

As an example for the latter, we mention the following. P. Lancaster, A. Shkalikov, and Q. Ye [LSY93, § 5 and § 7]) and later C. Gal, G.R. Goldstein and J.A. Goldstein ([GGG03]), M. Kramar, R. Nagel, and the author ([KMN03b, Rem. 9.13]), and T.-J. Xiao and J. Liang ([XL04b, Ex. 6.1]) have already considered wave equations with second order dynamic boundary conditions in an

L^2-setting, using quite different methods. The following is a corollary of statements obtained in these papers. We mention it as a motivation for our investigations. All the proofs in the above mentioned papers deeply rely on the Hilbert space setting.

Example 4.0.1. The problem

$$\text{(4.1)} \qquad \begin{cases} \ddot{u}(t,x) &= u''(t,x), & t \in \mathbb{R},\ x \in (0,1), \\ \ddot{u}(t,j) &= (-1)^j u'(t,j) + \beta_j u(t,j), & t \in \mathbb{R},\ j = 0,1, \\ u(0,\cdot) &= f, \qquad \dot{u}(0,\cdot) = g, \end{cases}$$

admits a unique classical solution for all $f, g \in H^2(0,1)$ and $\beta_0, \beta_1 \in \mathbb{C}$, continuously depending on the initial data. If $(\beta_0, \beta_1) \in \mathbb{R}^2_- \setminus \{0,0\}$, then such a solution is uniformly bounded in time with respect to the L^2-norm.

We finally remark that the third case (i.e., $L \in \mathcal{L}(X, \partial X)$) is typical for wave equations with so-called Wentzell boundary conditions on spaces of continuous functions. Among those who have already treated such problems we mention A. Favini, G.R. Goldstein, J.A. Goldstein, and S. Romanelli ([FGGR01]), who considered plain Wentzell boundary conditions, and T.-J. Xiao and J. Liang ([XL04]), who treated *generalized* Wentzell boundary conditions. Later, A. Bátkai and K.-J. Engel ([BE04]) extended the above results to hyperbolic problems with arbitrary (possibly degenerate) second order differential operators and (possibly *non-local*) generalized Wentzell boundary conditions. Finally, K.-J. Engel ([En04b, § 5]) developed an abstract framework that includes all the above mentioned results as special cases. The results obtained in this chapter complement his investigation.

4.1 General setting

We impose the following throughout this chapter.

Assumptions 4.1.1.

1. *X and Y are Banach spaces such that $Y \hookrightarrow X$.*
2. *∂X and ∂Y are Banach spaces such that $\partial Y \hookrightarrow \partial X$.*
3. *$A : D(A) \to X$ is linear, with $D(A) \subset Y$.*
4. *$L : D(A) \to \partial X$ is linear and surjective.*
5. *$A_0 := A_{|\ker(L)}$ is densely defined and has nonempty resolvent set.*
6. *$\begin{pmatrix} A \\ L \end{pmatrix} : D(A) \subset X \to X \times \partial X$ is closed.*
7. *$B : [D(A)]_L \to \partial X$ is linear and bounded.*
8. *$\tilde{B} : D(\tilde{B}) \subset \partial X \to \partial X$ is linear and closed, with $D(\tilde{B}) \subset \partial Y$.*

Observe that the Assumptions 2.1.1 are satisfied whenever the Assumptions 4.1.1 are.

Remark 4.1.2. By Lemma C.1 and Lemma C.4, the Dirichlet operators associated to (A, L) exist and are bounded from ∂X to $[D(A)]_L$ as well as to Y. Moreover, the operator $BD_\lambda^{A,L}$ is bounded on ∂X for all $\lambda \in \rho(A_0)$.

To start our investigations on (dAIBVP^2), we re-write such problem as a more usual second order abstract Cauchy problem

$$(\mathcal{ACP}^2) \qquad \begin{cases} \ddot{\mathfrak{u}}(t) &= \mathcal{A}\mathfrak{u}(t), \qquad\qquad t \in \mathbb{R}, \\ \mathfrak{u}(0) &= \mathfrak{f}, \qquad \dot{\mathfrak{u}}(0) = \mathfrak{g}, \end{cases}$$

on the product Banach space

$$\mathcal{X} := X \times \partial X,$$

where

$$(4.2) \qquad \mathcal{A} := \begin{pmatrix} A & 0 \\ B & \tilde{B} \end{pmatrix}, \qquad D(\mathcal{A}) := \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in D(A) \times D(\tilde{B}) : Lu = x \right\},$$

is an operator matrix with coupled domain on $\mathcal{X}$.

Here the new variable $\mathfrak{u}(\cdot)$ and the inital data $\mathfrak{f}, \mathfrak{g}$ are defined by

$$\mathfrak{u}(t) := \begin{pmatrix} u(t) \\ Lu(t) \end{pmatrix} \qquad \text{for} \quad t \in \mathbb{R}, \qquad \mathfrak{f} := \begin{pmatrix} f \\ h \end{pmatrix}, \quad \mathfrak{g} := \begin{pmatrix} g \\ j \end{pmatrix}.$$

Taking the components of $(\mathcal{ACP}^2)$ in the factor spaces of $\mathcal{X}$ yields the first two equations in (AIBVP^2), while the coupling relation $Lu(t) = x(t)$, $t \in \mathbb{R}$, is incorporated in the domain of the operator matrix $\mathcal{A}$. By the results of Section 1.4 we can therefore equivalently investigate $(\mathcal{ACP}^2)$ instead of (AIBVP^2).

4.2 The case $L \notin \mathcal{L}(Y, \partial X)$

Having reformulated (dAIBVP^2) as $(\mathcal{ACP}^2)$, the issue becomes to decide whether $\mathcal{A}$ generates a cosine operator function on $\mathcal{X}$, and what is the associated Kisyński space.

Assumption 4.2.1. *We complement the Assumptions 4.1.1 by the following.*

B is bounded either from $[D(A_0)]$ to ∂Y, or from Y to ∂X.

It is intuitive to consider the product space

$$\mathcal{Y} := Y \times \partial Y$$

as a candidate Kisyński space for $(\mathcal{ACP}^2)$. This intuition is partly correct, as we show in this and the next section.

Theorem 4.2.2. *Under the Assumptions 4.1.1 and 4.2.1 the operator matrix $\mathcal{A}$ generates a cosine operator function with associated phase space $\mathcal{Y} \times \mathcal{X}$ if and only if A_0 and $\tilde{B}$ generate cosine operator functions with associated phase spaces $Y \times X$ and $\partial Y \times \partial X$, respectively.*

Proof. Take $\lambda \in \rho(A_0)$. It has been proven in Proposition 2.1.3 that $\mathcal{A}-\lambda$ is similar to the operator matrix $\tilde{\mathcal{A}}_\lambda$ defined in (2.5). The similarity transformation is performed by the matrix $\mathcal{M}_\lambda$ introduced in (2.2), which is an isomorphism not only on $\mathcal{X}$, but also, by Remark 4.1.2, on $\mathcal{Y}$. Thus, by Lemma B.14, $\mathcal{A}$ generates a cosine operator function with associated phase space $\mathcal{Y}\times\mathcal{X}$ if and only if the similar operator $\tilde{\mathcal{A}}_\lambda$ generates a cosine operator function with associated phase space $\mathcal{Y} \times \mathcal{X}$.

We decompose

$$\tilde{\mathcal{A}}_\lambda = \begin{pmatrix} A_0 & -D_\lambda^{A,L}\tilde{B} \\ 0 & \tilde{B} \end{pmatrix} + \begin{pmatrix} -D_\lambda^{A,L}B & 0 \\ B & 0 \end{pmatrix} + \begin{pmatrix} -\lambda & D_\lambda^{A,L}(\lambda - BD_\lambda^{A,L}) \\ 0 & BD_\lambda^{A,L} - \lambda \end{pmatrix}$$

with diagonal domain $D(\tilde{\mathcal{A}}_\lambda) = D(A_0) \times D(\tilde{B})$. Taking into account again Remark 4.1.2, one can check that the second operator on the right hand side is bounded either from $[D(\tilde{\mathcal{A}}_\lambda)]$ to $\mathcal{Y}$ or from $\mathcal{Y}$ to $\mathcal{X}$, while the third one is bounded on $\mathcal{X}$. Thus, by Lemma B.15 we conclude that $\mathcal{A}$ generates a cosine operator function with associated phase space $\mathcal{Y} \times \mathcal{X}$ if and only if

$$\begin{pmatrix} A_0 & -D_\lambda^{A,L}\tilde{B} \\ 0 & \tilde{B} \end{pmatrix} \qquad \text{with domain} \qquad D(A_0) \times D(\tilde{B})$$

generates a cosine operator function with phase space $\mathcal{Y} \times \mathcal{X}$. Since $D_\lambda^{A,L}\tilde{B} \in \mathcal{L}([D(\tilde{B})], Y)$, the claim follows by Corollary B.29. $\square$

Example 4.2.3. By virtue of Theorem 4.2.2 we can revisit the setting introduced in Example 2.2.10 and improve the result obtained therein.

It has been seen that A_0 is the Neumann Laplacian, which generates a cosine operator function with associated phase space $H^1(\Omega) \times L^2(\Omega)$, cf. [Fa85, Thm. IV.5.1]. Further, $\tilde{B}$ is the Laplace–Beltrami operator, which is self-adjoint and dissipative, hence by Remark B.4.(b) it generates a cosine operator function on $L^2(\partial\Omega)$. By [Ka95, Thm. VI.2.23], the associated Kisyński space agrees with the form domain of $\tilde{B}$, which is $H^1(\partial\Omega)$ by definition.

We conclude that $\mathcal{A}$ generates a cosine operator function (hence an analytic semigroup of angle $\frac{\pi}{2}$ as well) on $\mathcal{X}$ whenever B is a bounded operator either from $H^2(\Omega)$ to $H^1(\partial\Omega)$, or from $H^1(\Omega)$ to $L^2(\partial\Omega)$. For example, we can take

$$(4.3) \qquad (Bu)(z) := -u(z), \qquad u \in H^1(\Omega),\ z \in \partial\Omega,$$

which defines a bounded operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$. With this choice $\mathcal{A}$ becomes dissipative (this can be checked applying the Gauss–Green formulae), hence the analytic semigroup generated by $\mathcal{A}$ is contractive.

However, one can compute te adjoint of $\mathcal{A}$ and obtain that

$$\mathcal{A}^* = \begin{pmatrix} A & 0 \\ -B & \tilde{B} \end{pmatrix}, \qquad D(\mathcal{A}^*) = \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in D(A) \times D(B) : Lu = -x \right\}.$$

Thus, $\mathcal{A}$ is not self-adjoint, Remark B.4.(b) does not apply, and we cannot deduce the contractivity of $(C(t,\mathcal{A}))_{t\in\mathbb{R}}$.

Remarks 4.2.4. (a) By Lemma B.30 we can characterize the generator property of the reduction matrix associated to the second order complete problem

$$\begin{cases} \ddot{\mathfrak{u}}(t) &= \mathcal{A}\mathfrak{u}(t) + \mathcal{C}\dot{\mathfrak{u}}(t), \qquad t \in \mathbb{R}, \\ \mathfrak{u}(0) &= \mathfrak{f} \in \mathcal{X}, \qquad \dot{\mathfrak{u}}(0) = \mathfrak{g} \in \mathcal{X}, \end{cases}$$

for $\mathcal{C} \in \mathcal{L}(\mathcal{Y})$. The Kisyński space $\mathcal{Y} = Y \times \partial Y$ has the nice property that an operator matrix

$$\mathcal{C} := \begin{pmatrix} 0 & 0 \\ C & \tilde{C} \end{pmatrix}$$

is bounded on $\mathcal{Y}$ if (and only if) $C \in \mathcal{L}(Y, \partial Y)$ and $\tilde{C} \in \mathcal{L}(\partial Y)$. Thus, we can perturb our dynamic boundary conditions by a quite wide class of *unbounded* damping operators C and $\tilde{C}$.

(b) If $\tilde{B}$ is a bounded operator on ∂X, then the Assumption 4.2.1 is satisfied as soon as the Assumptions 4.1.1 are (with $\partial Y = \partial X$), and Theorem 4.2.2 applies as soon as A_0 generates a cosine operator function with associated phase space $Y \times X$.

Hence our operator matrix approach yields an abstract result that can be reformulated in the following intuitive way: For any $\tilde{B} \in \mathcal{L}(\partial X)$, $C \in \mathcal{L}(Y, \partial X)$, and $\tilde{C} \in \mathcal{L}(\partial X)$, the second order abstract equation

$$\ddot{u}(t) = Au(t), \qquad t \in \mathbb{R},$$

equipped with (damped) dynamic boundary conditions

$$(Lu)^{\cdot\cdot}(t) = Bu(t) + C\dot{u}(t) + \tilde{B}Lu(t) + \tilde{C}(Lu)^{\cdot}(t), \qquad t \in \mathbb{R},$$

has a unique mild solution for all initial data

$$u(0) \in Y, \qquad Lu(0) \in \partial Y, \qquad \dot{u}(0) \in X, \quad \text{and} \quad (Lu)^{\cdot}(0) \in \partial X,$$

if and only if the same equation equipped with homogeneous boundary conditions

$$Lu(t) = 0, \qquad t \in \mathbb{R},$$

has a unique mild solution for all initial data

$$u(0) \in Y \qquad \text{and} \qquad \dot{u}(0) \in X.$$

Example 4.2.5. Consider the second order concrete initial-boundary value problem

$$(4.4) \quad \begin{cases} \ddot{\phi}(t,x) &= c^2 \Delta\phi(t,x), & t \in \mathbb{R},\ x \in \Omega, \\ \ddot{\delta}(t,z) &= p(z)\dot{\phi}(t,z) + q(z)\delta(t,z) + r(z)\dot{\delta}(t,z), & t \in \mathbb{R},\ z \in \partial\Omega, \\ \delta(t,z) &= \frac{\partial\phi}{\partial\nu}(t,z), & t \in \mathbb{R},\ z \in \partial\Omega, \\ \phi(0,\cdot) &= f, \qquad \dot{\phi}(0,\cdot) = g, & \\ \delta(0,\cdot) &= h, \qquad \dot{\delta}(0,\cdot) = j, & \end{cases}$$

on a bounded open domain $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$. Observe that there is a damping in the boundary conditions, thus the problem (4.4) resembles the wave equation with acoustic boundary conditions discussed in Example 3.2.5.

We claim the following: If $c \in \mathbb{R}$ and $p, q, r \in L^\infty(\partial\Omega)$, then the problem (4.4) can be reduced to a first-order problem that is governed by a C_0-group on $\big(H^1(\Omega) \times L^2(\partial\Omega)\big) \times \big(L^2(\Omega) \times L^2(\partial\Omega)\big)$.

Set

$$X := L^2(\Omega), \qquad Y := H^1(\Omega), \qquad \partial X := L^2(\partial\Omega).$$

and define

$$A := c^2\Delta, \qquad D(A) := \left\{u \in H^{\frac{3}{2}}(\Omega) : \Delta u \in L^2(\Omega)\right\},$$

$$L := \frac{\partial}{\partial \nu}, \qquad D(L) := D(A),$$

$$(Bu)(z) := q(z)\frac{\partial u}{\partial \nu}(z), \qquad \text{for all } u \in D(B) := D(A),\ z \in \partial\Omega, \qquad \tilde{B} := 0,$$

$$(Cu)(z) := p(z)u(z), \qquad \text{for all } u \in H^1(\Omega),\ z \in \partial\Omega,$$

$$(\tilde{C}v)(z) := r(z)v(z), \qquad \text{for all } v \in L^2(\partial\Omega).$$

First consider the undamped case of $p = r \equiv 0$. We want to prove that (4.4) is well-posed in $(H^1(\Omega), L^2(\Omega), L^2(\partial\Omega), L^2(\partial\Omega))$, in the sense of Definition 1.4.1. By Theorem 1.4.3 it suffices to show that the operator matrix $\mathcal{A}$ as in (4.2) generates a cosine operator function with associated phase space $\big(H^1(\Omega) \times L^2(\partial\Omega)\big) \times \big(L^2(\Omega) \times L^2(\partial\Omega)\big)$.

To check the Assumptions 4.1.1 reason as in Example 3.2.5 and observe that $\|Bu\| \leq \|q\|_\infty \|Lu\|$. As seen in Example 4.2.3, the restriction A_0 of A to $\ker(L)$ is the generator of a cosine operator function with associated phase space $H^1(\Omega) \times L^2(\Omega)$. Further, by [LM72, Vol. 1, Thm. 2.7.4] we obtain $B \in \mathcal{L}([D(A)]_L, \partial X)$. Hence, by Theorem 4.3.4 the operator matrix with coupled domain associated to (4.4) generates a cosine operator function with phase space $\big(H^1(\Omega) \times L^2(\partial\Omega)\big) \times \big(L^2(\Omega) \times L^2(\partial\Omega)\big)$.

For arbitrary $p \in L^\infty(\partial\Omega)$ we can consider C as a multiplicative perturbation of the trace operator, which is bounded from $Y = H^1(\Omega)$ to $\partial X = L^2(\partial\Omega)$. Also the multiplication operator associated to r is bounded on $L^2(\partial\Omega)$, by assumption, and summing up we obtain that C and $\tilde{C}$ are bounded from Y to ∂X and on ∂X, respectively. By Remark 4.2.4.(b) we finally obtain that the problem (4.4) is governed by a C_0-group on $\big(H^1(\Omega) \times L^2(\partial\Omega)\big) \times \big(L^2(\Omega) \times L^2(\partial\Omega)\big)$.

Finally, observe that since the Neumann Laplacian generates a cosine operator function on $L^p(\Omega)$, $\Omega \subset \mathbb{R}^n$, only if $p = 2$ or $n = 1$ (cf. [KW03, Thm. 3.2]), it follows that (4.4) is well-posed in an L^p-setting if and only if $p = 2$ or $n = 1$.

4.3 The case $L \in \mathcal{L}(Y, \partial X)$

We now consider the case where L is bounded from Y to the boundary space ∂X.

Assumptions 4.3.1. *We complement the Assumptions 4.1.1 by the following.*

1. *V is a Banach space such that $V \hookrightarrow Y$.*
2. *L can be extended to an operator that is bounded from Y to ∂X, which we denote again by L, and such that* $\ker(L) = V$.
3. *$\tilde{B}$ is bounded on ∂X.*

Observe that it follows by Assumptions 4.1.1.8 and 4.3.1.3 that $\partial Y = \partial X$.

Remark 4.3.2. Observe that if $u \in C^1(\mathbb{R}, Y)$ is a solution to (dAIBVP^2), then necessarily

$$L\frac{du}{dt}(\cdot) = \frac{dLu}{dt}(\cdot) = \frac{dx}{dt}(\cdot),$$

where we have used the boundedness of L from Y to ∂X. Hence, under the Assumptions 4.1.1 and 4.3.1 a classical solution to (dAIBVP^2) automatically satisfies the additional compatibility condition $L\dot{u}(t) = \dot{x}(t)$, $t \in \mathbb{R}$, i.e., it solves in fact the problem (bAIBVP^2) considered in Section 1.5.

Lemma 4.3.3. *Consider the Banach space*

$$\mathcal{V} := \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in Y \times \partial X : Lu = x \right\}.$$

Then for all $\lambda \in \rho(A_0)$ the operator matrix $\mathcal{M}_\lambda$ defined in (2.2) *can be restricted to an operator matrix that is an isomorphism from $\mathcal{V}$ onto*

$$\mathcal{W} := V \times \partial X,$$

which we denote again by $\mathcal{M}_\lambda$. Its inverse is given by the operator matrix

$$\mathcal{M}_\lambda^{-1} = \begin{pmatrix} I_V & D_\lambda^{A,L} \\ 0 & I_{\partial X} \end{pmatrix}. \tag{4.5}$$

Proof. Take $\lambda \in \rho(A_0)$. The operator matrix $\mathcal{M}_\lambda$ is everywhere defined on $\mathcal{V}$, and for $\mathfrak{u} = \binom{u}{Lu} \in \mathcal{V}$ there holds

$$\mathcal{M}_\lambda \mathfrak{u} = \begin{pmatrix} I_Y & -D_\lambda^{A,L} \\ 0 & I_{\partial X} \end{pmatrix} \begin{pmatrix} u \\ Lu \end{pmatrix} = \begin{pmatrix} u - D_\lambda^{A,L} Lu \\ Lu \end{pmatrix}.$$

Now $u \in Y$ and also $D_\lambda^{A,L} Lu \in Y$, due to Remark 4.1.2. Thus, the vector $u - D_\lambda^{A,L} Lu \in V$, since also $L(u - D_\lambda^{A,L} Lu) = Lu - LD_\lambda^{A,L} Lu = Lu - Lu = 0$. This shows that $\mathcal{M}_\lambda \mathfrak{u} \in \mathcal{W}$.

Moreover, one sees that the operator matrix given in (4.5) is the inverse of $\mathcal{M}_\lambda$. To show that it maps $\mathcal{W}$ into $\mathcal{V}$, take $v \in V$, $x \in \partial X$. Then

$$\begin{pmatrix} I_V & D_\lambda^{A,L} \\ 0 & I_{\partial X} \end{pmatrix} \begin{pmatrix} v \\ x \end{pmatrix} = \begin{pmatrix} v + D_\lambda^{A,L} x \\ x \end{pmatrix}.$$

Now $v + D_\lambda^{A,L} x \in Y$ because $V \hookrightarrow Y$ and due to Remark 4.1.2. Moreover, $Lv = 0$ by definition of the space V, thus $L(v + D_\lambda^{A,L} x) = LD_\lambda^{A,L} x = x$, and this yields the claim. $\square$

Theorem 4.3.4. *Under the Assumptions 4.1.1 and 4.3.1 the operator matrix $\mathcal{A}$ generates a cosine operator function with associated phase space $\mathcal{V}\times\mathcal{X}$ if and only if $A_0 - D_\lambda^{A,L}B$ generates a cosine operator function with associated phase space $V \times X$, for some $\lambda \in \rho(A_0)$ if and only if $A_0 - D_\lambda^{A,L}B$ generates a cosine operator function with associated phase space $V \times X$, for all $\lambda \in \rho(A_0)$.*

Proof. Take $\lambda \in \rho(A_0)$. By Lemma B.14, Proposition 2.1.3 and Lemma 4.3.3, $\mathcal{A}$ generates a cosine operator function with associated phase space $\mathcal{V} \times \mathcal{X}$ if and only if the operator matrix $\tilde{\mathcal{A}}_\lambda$, defined in (2.5), generates a cosine operator function with associated phase space $\mathcal{W} \times \mathcal{X}$ for some $\lambda \in \rho(A_0)$ if and only if $\tilde{\mathcal{A}}_\lambda$ generates a cosine operator function with associated phase space $\mathcal{W}\times\mathcal{X}$ for all $\lambda \in \rho(A_0)$.

We decompose

$$\tilde{\mathcal{A}}_\lambda = \begin{pmatrix} A_0 - D_\lambda^{A,L}B & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} + \begin{pmatrix} -\lambda & D_\lambda^{A,L}(\lambda - \tilde{B}_\lambda) \\ 0 & \tilde{B}_\lambda - \lambda \end{pmatrix}$$

with diagonal domain $D(\tilde{\mathcal{A}}_\lambda) = D(A_0) \times \partial X$.

Now the proof goes exactly as in Theorem 2.2.8.(1), taking into account Lemma B.15. □

Remarks 4.3.5. (a) Since $\mathcal{V}$ is not a product space, it may be tricky to endow it with a "good" norm. More precisely, the canonical norms

$$\left\| \begin{pmatrix} u \\ Lu \end{pmatrix} \right\|_{\mathcal{V}} := \|u\|_Y + \|Lu\|_{\partial X}$$

or (in the Hilbert space case)

$$\left\| \begin{pmatrix} u \\ Lu \end{pmatrix} \right\|_{\mathcal{V}} := \left(\|u\|_Y^2 + \|Lu\|_{\partial X}^2\right)^{\frac{1}{2}}$$

may not be the most suitable – that is, they may not yield conservation of energy of the system (or equivalently, the reduction matrix associated to $\mathcal{A}$ may be non-dissipative with respect to these norms, cf. Theorem B.11). This will be explained in Example 4.5.5.

(b) Theorem 4.3.4 can be expressed in the following way: The second order abstract equation

$$\ddot{u}(t) = Au(t), \qquad t \in \mathbb{R},$$

equipped with dynamic boundary conditions

$$(Lu)^{\cdot\cdot}(t) = Bu(t) + \tilde{B}Lu(t), \qquad t \in \mathbb{R},$$

has a unique mild solution for all initial data

$$u(0) \in Y, \qquad Lu(0) \in \partial X, \qquad \dot{u}(0) \in X, \qquad \text{and} \quad (Lu)^{\cdot}(0) \in \partial X,$$

if and only if the perturbed second order equation

$$\ddot{u}(t) = Au(t) - D_\lambda^{A,L}Bu(t), \qquad t \in \mathbb{R},$$

equipped with homogeneous boundary conditions

$$Lu(t) = 0, \qquad t \in \mathbb{R},$$

has a unique mild solution for all initial data

$$u(0) \in V \qquad \text{and} \qquad \dot{u}(0) \in X.$$

(c) Let $\lambda \in \rho(A_0)$. It follows by Remark 4.1.2 that $D_\lambda^{A,L}B$ is bounded from $[D(A_0)]$ to Y (the Kisyński space in Section 4.2), while $D_\lambda^{A,L}B$ is *not* bounded from $[D(A_0)]$ to the current Kisyński space V, since $D(A)$ is in general not contained in V. This explains why the characterization obtained in Theorem 4.2.2 is less satisfactory than that obtained in Theorem 4.3.4.

Corollary 4.3.6. *Let A_0 generate a cosine operator function with associated phase space $V \times X$. Assume that*

$$\|D_\lambda^{A,L}\|_{\mathcal{L}(\partial X, X)} = O(|\lambda|^{-\epsilon}) \qquad \text{as} \quad |\lambda| \to \infty, \quad \mathrm{Re}\lambda > 0, \tag{4.6}$$

and moreover that

$$\int_0^1 \|BS(s, A_0)f\|_{\partial X}\, ds \;\le\; M\|f\|_X \tag{4.7}$$

holds for all $f \in D(A_0)$ and some $M > 0$. Then $\mathcal{A}$ generates a cosine operator function with associated phase space $\mathcal{V} \times \mathcal{X}$.

Proof. The basic tool for the proof is Lemma B.19, which yields that $A_0 - D_\lambda^{A,L}B$ generates a cosine operator function with associated phase space $V \times X$ whenever

$$\int_0^1 \|D_\lambda^{A,L}BS(s, A_0)f\|_X\, ds \;\le\; q\|f\|_X$$

holds for all $f \in D(A_0)$ and some $q < 1$. This condition is clearly satisfied under our assumptions for $\lambda \in \rho(A_0)$ sufficiently large. □

Remark 4.3.7. By Lemma C.6.(2) the condition (4.6) is in particular satisfied whenever $[D(A)]_L \hookrightarrow [D(A_0), X]_\alpha$ for some $0 < \alpha < 1$.

Example 4.3.8. Consider the second order problem with dynamic boundary conditions

$$\begin{cases} \ddot{u}(t,x) &= u''(t,x) + q(x)u'(t,x) + r(x)u(t,x), & t \in \mathbb{R},\ x \in (0,1), \\ \ddot{u}(t,j) &= \alpha_j u'(t,j) + \beta_j u(t,j), & t \in \mathbb{R},\ j = 0,1, \end{cases}$$

as a generalization of (4.1). Here q, r are functions on $(0,1)$ and $\alpha_0, \alpha_1, \beta_0, \beta_1$ are complex numbers.

We are interested in the well-posedness (in the sense of Definition 1.5.1) in $(W^{1,p}(0,1), L^p(0,1), \mathbb{C}^2, \mathbb{C}^2)$ of the abstract initial-boundary value problem associated to such a system. Then by Theorem 1.5.3 the initial value problem associated to the above system is equivalent to the second order abstract Cauchy problem ($\mathcal{ACP}^2$), where $\mathcal{A}$ is the operator matrix

$$\mathcal{A} := \begin{pmatrix} \frac{d^2}{dx^2} + q\frac{d}{dx} + rI & 0 \\ \begin{pmatrix} \alpha_0\delta_0' \\ \alpha_1\delta_1' \end{pmatrix} & \begin{pmatrix} \beta_0 & 0 \\ 0 & \beta_1 \end{pmatrix} \end{pmatrix} \tag{4.8}$$

with domain

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} u \\ \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \end{pmatrix} \in W^{2,p}(0,1) \times \mathbb{C}^2 : u(0) = x_0,\ u(1) = x_1 \right\}, \tag{4.9}$$

where $\delta_i' u := u'(i)$, $i = 0, 1$.

We claim the following: the operator matrix $\mathcal{A}$ generates a cosine operator function on $L^p(0,1) \times \mathbb{C}^2$, $1 \leq p < \infty$, for all $q, r \in L^\infty(0,1)$ and $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbb{C}$. The associated Kisyński space is

$$\left\{ \begin{pmatrix} u \\ \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \end{pmatrix} \in W^{1,p}(0,1) \times \mathbb{C}^2 : u(0) = x_0,\ u(1) = x_1 \right\}.$$

Set

$$X := L^p(0,1), \qquad Y := W^{1,p}(0,1), \qquad \partial X := \mathbb{C}^2.$$

We define the operators

$$Au(x) := u''(x) + q(x)u'(x) + r(x)u(x) \qquad u \in D(A) := W^{2,p}(0,1),\ x \in (0,1),$$

$$Lu := \begin{pmatrix} u(0) \\ u(1) \end{pmatrix} \qquad D(L) := Y,$$

$$Bu := \begin{pmatrix} \alpha_0 u'(0) \\ \alpha_1 u'(1) \end{pmatrix}, \qquad D(B) := D(A),$$

$$\tilde{B} := \begin{pmatrix} \beta_0 & 0 \\ 0 & \beta_1 \end{pmatrix}.$$

Therefore, we obtain $V = \ker(L) = W_0^{1,p}(0,1)$.

In the following, it will be convenient to write A as the sum

$$A := A_1 + A_2 := \frac{d^2}{dx^2} + \left(q\frac{d}{dx} + rI \right),$$

and to define A_{1_0} and A_{2_0} as the restrictions of A_1 and A_2, respectively, to

$$D(A) \cap \ker(L) = W^{2,p}(0,1) \cap W_0^{1,p}(0,1).$$

It will be proven soon that $A_{1_0} + A_{2_0}$ generates a cosine operator function, hence Assumption 4.1.1.5 is satisfied. The second derivative on $W^{2,p}(0,1)$ is closed, hence also $\binom{A}{L}$ is closed as an operator from X to $X \times \partial X$, and this checks

Assumption 4.1.1.6. Further, it follows from the embedding $W^{1,p}(0,1) \hookrightarrow C[0,1]$ that we can find suitable constants $\xi, \xi_1, \xi_2, \xi', \tilde{\xi}$, such that

$$\begin{aligned} \|Bu\| &= \left(|u'(0)|^p + |u'(1)|^p\right)^{\frac{1}{p}} \le |u'(0)| + |u'(1)| \le 2\|u'\|_{C[0,1]} \\ &\le \xi\left(\|u''\|_{L^p(0,1)} + \|u'\|_{L^p(0,1)}\right) \le \xi_1\|u''\|_{L^p(0,1)} + \xi_2\|u\|_{L^p(0,1)} \\ &\le \xi'\|u\|_{A_{1_0}} \le \tilde{\xi}\|u\|_{A_0}, \end{aligned}$$

for all $u \in D(A)$, where we have exploited the closed graph theorem and the fact that the second derivative is relatively bounded by the second derivative with relative bound 0. Thus, Assumption 4.1.1.7 holds. The other Assumptions 4.1.1, as well as the Assumptions 4.3.1 are clearly satisfied as well.

In order to prove that $A_{1_0} + A_{2_0}$ generates a cosine operator function with associated phase space $V \times X$, observe that since $q, r \in L^\infty(0,1)$, one has $qu' + ru \in L^p(0,1)$ for all $u \in W_0^{1,p}(0,1)$. Thus, A_{2_0} is bounded from V to X and by Lemma B.15 we can neglect such a perturbation. On the other hand, the operator A_{1_0} is the second derivative with Dirichlet boundary conditions on $L^p(0,1)$, hence it generates a cosine operator function that, as a consequence of the D'Alembert formula, is given by

$$(4.10) \qquad \left(C(t, A_{1_0})f\right)(x) = \frac{\tilde{f}(x+t) + \tilde{f}(x-t)}{2}, \qquad t \in \mathbb{R},\ x \in (0,1),$$

where $\tilde{f}$ is the function obtained by extending $f \in L^p(0,1)$ first by by oddity to $[-1,1]$, and then by 2-periodicity to $\mathbb{R}$ (see [FGGR01, § 2]). One can check that the space of strong differentiability of $(C(t, A_{1_0}))_{t\in\mathbb{R}}$ – that is, the Kisyński space associated to A_{1_0} – is $V = W_0^{1,p}(0,1)$.

Thus, we can apply Corollary 4.3.6 and obtain that the operator matrix $\mathcal{A}$ generates a cosine operator function with associated phase space $\mathcal{V} \times \mathcal{X}$ if the conditions (4.6) and (4.7) are satisfied. It is known that

$$W^{2,p}(0,1) \hookrightarrow W^{\alpha,p}(0,1) = [W^{2,p}(0,1) \cap W_0^{1,p}(0,1), L^p(0,1)]_\alpha, \qquad 0 < \alpha < \frac{1}{2p},$$

hence $[D(A)]_L \hookrightarrow [D(A_0), X]_\alpha$, for suitable α, and by Remark 4.3.7 the condition (4.6) is satisfied.

To check condition (4.7), observe that integrating (4.10) yields that the sine operator function generated by A_{1_0} is given by

$$S(t, A_{1_0})f = \frac{1}{2}\int_{\cdot-t}^{\cdot+t} \tilde{f}(s)ds, \qquad t \in \mathbb{R}.$$

Thus,

$$BS(t, A_{1_0})f = \frac{1}{2}\begin{pmatrix} \alpha_0\left(\tilde{f}(t) - \tilde{f}(-t)\right) \\ \alpha_1\left(\tilde{f}(1+t) - \tilde{f}(1-t)\right) \end{pmatrix}, \qquad t \ge 0,\ f \in D(A_{1_0}).$$

Since $\tilde{f}$ is by definition the odd, 2-periodic extension of f, we conclude that

$$BS(t, A_{1_0})f = \begin{pmatrix} \alpha_0\tilde{f}(t) \\ \alpha_1\tilde{f}(t) \end{pmatrix}, \qquad t \ge 0,\ f \in D(A_{1_0}),$$

and in particular

$$|BS(t, A_{1_0})f| = M|f(t)|, \qquad t \in [0,1],\ f \in D(A_{1_0}),$$

where $M := |\alpha_0| + |\alpha_1|$ (here we have endowed $\mathbb{C}^2$ with the l^1-norm). Thus,

$$\int_0^1 |BS(s, A_0)f|\, ds \ \leq\ M \int_0^1 |f(s)|\, ds = M\|f\|_{L^1(0,1)} \leq M\|f\|_{L^p(0,1)}.$$

This concludes the proof.

Remark 4.3.9. Observe that, as a consequence of Example 4.3.8, we also obtain that the operator matrix $\mathcal{A}$ defined in (4.8)–(4.9) is the generator of an analytic semigroup of angle $\frac{\pi}{2}$ on $L^p(0,1) \times \mathbb{C}^2$, $1 \leq p < \infty$. This improves [KMN03b, Thm. 9.4 and Rem. 9.11], where no angle of analyticity has been obtained, the L^1-setting was not considered, and the assumptions on the parameters q, r, α_0, α_1 were stronger.

An analogous operator matrix $\mathcal{A}$ in an $L^p(\Omega)$-context, $1 < p < \infty$, has also been considered in [FGGR02] (where no analyticity result has been obtained), but with A an elliptic operator in divergence form, and by different means in [AMPR03] and [En04], where $A = \Delta$.

4.4 Regularity and representation formulae

Throughout this section we only impose the Assumptions 4.1.1. Hence, the results below hold in the framework of both Sections 4.2 and 4.3, unless otherwise stated.

Define the class

$$\mathcal{D}_0^\infty := \bigcap_{h=0}^{\infty} \left\{ u \in D^\infty(A) : LA^h u = BA^h u = 0 \right\}.$$

Then by Lemma 2.3.3 and Corollary B.21 we obtain the following.

Proposition 4.4.1. *Let B map $D(A^{k+1})$ into $D(\tilde{B}^k)$, $k \in \mathbb{N}$. Assume $\mathcal{A}$ to generate a cosine operator function on $\mathcal{X}$. Then $C(t, \mathcal{A})$ and $S(t, \mathcal{A})$ map $\mathcal{D}_0^\infty \times \{0\}$ into $D^\infty(A) \times D(\tilde{B}^\infty)$, for all $t \in \mathbb{R}$.*

The following representation formula holds, in analogy to Proposition 2.2.1.

Proposition 4.4.2. *Let A_0, $\tilde{B}$, and $\mathcal{A}$ generate cosine operator functions. If $B = 0$, then*

$$C(t, \mathcal{A} - \lambda) = \begin{pmatrix} C(t, A_0 - \lambda) & (\lambda - A_0) \int_0^t S(t-s, A_0 - \lambda) D_\lambda^{A,L} C(s, \tilde{B} - \lambda) ds \\ 0 & C(t, \tilde{B} - \lambda) \end{pmatrix}$$

for all $t \in \mathbb{R}$ and $\lambda \in \rho(A_0)$.

Proof. Take $\lambda \in \rho(A_0)$. Then by Proposition 2.1.3 there holds

$$\mathcal{A} - \lambda = \mathcal{M}_\lambda^{-1} \tilde{\mathcal{A}}_\lambda \mathcal{M}_\lambda = \mathcal{M}_\lambda^{-1} \begin{pmatrix} A_0 - \lambda & -D_\lambda^{A,L}(\tilde{B} - \lambda) \\ 0 & \tilde{B} - \lambda \end{pmatrix} \mathcal{M}_\lambda,$$

hence by Lemma B.14

$$C(t, \mathcal{A} - \lambda) = \mathcal{M}_\lambda^{-1} C(t, \tilde{\mathcal{A}}_\lambda) \mathcal{M}_\lambda, \qquad t \in \mathbb{R}.$$

Apply Proposition B.28 and obtain

$$C(t, \tilde{\mathcal{A}}_\lambda) = \begin{pmatrix} C(t, A_0 - \lambda) & -\int_0^t C(t-s, A_0 - \lambda) D_\lambda^{A,L} (\tilde{B} - \lambda) S(s, \tilde{B} - \lambda) ds \\ 0 & C(t, \tilde{B} - \lambda) \end{pmatrix},$$

for all $t \in \mathbb{R}$. Integrating by parts we see that

$$\begin{aligned} &\int_0^t C(t-s, A_0 - \lambda) D_\lambda^{A,L} (\tilde{B} - \lambda) S(s, \tilde{B} - \lambda) ds \\ &= \Big[C(t-s, A_0 - \lambda) D_\lambda^{A,L} C(s, \tilde{B} - \lambda) \Big]_0^t \\ &\qquad + \int_0^t (A_0 - \lambda) S(t-s, A_0 - \lambda) D_\lambda^{A,L} C(s, \tilde{B} - \lambda) ds \\ &= D_\lambda^{A,L} C(t, \tilde{B} - \lambda) - C(t, A_0 - \lambda) D_\lambda^{A,L} \\ &\qquad + \int_0^t (A_0 - \lambda) S(t-s, A_0 - \lambda) D_\lambda^{A,L} C(s, \tilde{B} - \lambda) ds, \end{aligned}$$

for all $t \in \mathbb{R}$, and a straightforward matrix computation yields the claimed formula. □

Corollary 4.4.3. *Let the assumptions of Proposition 4.4.2 hold. Then for all* $\lambda \in \rho(A_0)$ *a necessary condition for* $(C(t, \mathcal{A} - \lambda))_{t \in \mathbb{R}}$ *to be bounded is that both* $(C(t, A_0 - \lambda))_{t \in \mathbb{R}}$ *and* $(C(t, \tilde{B} - \lambda))_{t \in \mathbb{R}}$ *be bounded as well.*

Remarks 4.4.4. (a) Under the assumptions of Proposition 4.4.2, let further A_0 be invertible and $\tilde{B} = 0$. Then we obtain that

$$C(t, \mathcal{A}) = \begin{pmatrix} C(t, A_0) & D_0^{A,L} - C(t, A_0) D_0^{A,L} \\ 0 & I_{\partial X} \end{pmatrix}, \qquad t \in \mathbb{R}.$$

Thus, in this very special case $(C(t, \mathcal{A}))_{t \in \mathbb{R}}$ is bounded if and only if $(C(t, A_0))_{t \in \mathbb{R}}$ is bounded. Integrating this formula one sees that the associated sine operator function is

$$S(t, \mathcal{A}) = \begin{pmatrix} S(t, A_0) & tD_0^{A,L} - S(t, A_0) D_0^{A,L} \\ 0 & tI_{\partial X} \end{pmatrix}, \qquad t \in \mathbb{R}.$$

This shows that, under these assumptions, $(S(t, \mathcal{A}))_{t \in \mathbb{R}}$ is *never* bounded, be $(S(t, A_0))_{t \in \mathbb{R}}$ bounded or not.

(b) Consider the abstract second order problem with inhomogeneous boundary conditions

$$\left\{ \begin{array}{rcll} \ddot{v}(t) & = & Av(t), & t \in \mathbb{R}, \\ Lv(t) & = & jt + h, & t \in \mathbb{R}, \\ v(0) & = & f \in X, & \dot{v}(0) = g \in X, \end{array} \right. \tag{4.11}$$

with $h, j \in \partial X$. Differentiating the second equation twice, one can rewrite (4.11) as (dAIBVP$^2_{f,g,h,j}$), with $B = \tilde{B} = 0$. Theorems 1.4.3 and 4.2.2 yield that (4.11) has a unique mild solution for all $f \in Y$, $g \in X$, $h \in \partial Y$, and $j \in \partial X$ if and only if A_0 generates a cosine operator function with associated phase space $Y \times X$.

If we additionally assume A_0 to be invertible, then by (a) such a solution is given by

$$\begin{aligned} v(t) \;=\; & C(t, A_0)f + S(t, A_0)g + D_0^{A,L}h - C(t, A_0)D_0^{A,L}h \\ & +tD_0^{A,L}j - S(t, A_0)D_0^{A,L}j, \qquad t \in \mathbb{R}, \end{aligned}$$

and it is classical if $f \in D(A)$, $g \in Y$, $h \in D(\tilde{B})$, $j \in \partial Y$, and if the compatibility condition $Lf = g$ is satisfied.

(c) It is worth to remark that if $B \in \mathcal{L}(Y, \partial X)$ and $\tilde{B} \in \mathcal{L}(\partial X)$, then by Corollary B.16 the estimate

$$\|u(t) - v(t)\|_X \leq tM, \qquad 0 \leq t \leq 1,$$

holds for the mild solution u to (dAIBVP$^2_{f,0,h,0}$), where $M > 0$ is a suitable constant, and v is the mild solution to (2.11) obtained in (b).

4.5 Asymptotic behavior

As in the previous one, throughout this section we only impose the Assumptions 4.1.1.

We first obtain a result that complements [GGG03, § 5].

Lemma 4.5.1. *The following assertions hold.*

(1) Let the Assumptions 4.1.1 be satisfied. Then the reduction matrix associated to $\mathcal{A}$ has compact resolvent if and only if the embeddings $[D(A_0)] \hookrightarrow Y \hookrightarrow X$ and $[D(\tilde{B})] \hookrightarrow \partial Y \hookrightarrow \partial X$ are all compact.

(2) Let the Assumptions 4.3.1 be satisfied. Then the reduction matrix associated to $\mathcal{A}$ has compact resolvent if and only if ∂X is finite dimensional and the embeddings $[D(A_0)] \hookrightarrow V \hookrightarrow X$ are both compact.

Given two Banach spaces E, F such that $E \hookrightarrow F$, we denote in the following by $i_{E,F}$ the continuous embedding of E in F.

Proof. In the trivial case of finite dimensional spaces Y, X the claim holds at once: from now on we therefore assume that dim $Y =$ dim $X = \infty$. Our aim is to exploit Lemma B.22 in both cases.

(1) Under the Assumption 4.2.1, the domain of the reduction matrix associated to $\mathcal{A}$ is $D(\mathcal{A}) \times \mathcal{Y}$, where $\mathcal{Y} = Y \times \partial Y$. Take $\lambda \in \rho(A_0)$ and recall that the (restriction of the) operator $\mathcal{M}_\lambda$ defined in (2.2) is an isomorphism on $\mathcal{Y}$, but is not compact (because dim $Y = \infty$). Moreover, $\mathcal{M}_\lambda$ maps $D(\mathcal{A})$ into $\mathcal{M}_\lambda D(\mathcal{A}) = D(A_0) \times D(\tilde{B})$. Since we can decompose

$$i_{[D(\mathcal{A})],\mathcal{Y}} = i_{D(A_0)\times D(\tilde{B}),\mathcal{Y}} \circ \mathcal{M}_\lambda,$$

the claim follows.

(2) Under the Assumptions 4.3.1, the domain of the reduction matrix associated to $\mathcal{A}$ is $D(\mathcal{A}) \times \mathcal{V}$, where

$$\mathcal{V} = \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in Y \times \partial X : Lu = x \right\}.$$

Take $\lambda \in \rho(A_0)$ and observe that by Lemma 4.3.3 the (restriction of the) operator $\mathcal{M}_\lambda$ is an isomoprhism from $\mathcal{V}$ to $\mathcal{W} = V \times \partial X$. Thus, we can decompose

$$i_{[D(\mathcal{A})],\mathcal{V}} = \mathcal{M}_\lambda^{-1} \circ i_{[D(A_0)] \times \partial X, \mathcal{W}} \circ \mathcal{M}_\lambda.$$

Likewise we obtain

$$i_{\mathcal{V},\mathcal{X}} = i_{\mathcal{W},\mathcal{X}} \circ \mathcal{M}_\lambda.$$

Since $\mathcal{M}_\lambda$ is not compact (because dim $X = \infty$, we obtain that $i_{[D(\mathcal{A})],\mathcal{V}}$ and $i_{\mathcal{V},\mathcal{X}}$ are both compact if and only if $i_{[D(A_0)] \times \partial X, \mathcal{W}}$ and $i_{\mathcal{W},\mathcal{X}}$ are both compact, and the claim follows. ☐

Taking into account Lemma B.7.(4) and Corollary 2.2.3 we obtain the following.

Corollary 4.5.2. *Let $\mathcal{A}$ generate a cosine operator function. Then $(S(t,\mathcal{A}))_{t\in\mathbb{R}}$ is compact if and only if the embeddings $[D(A_0)] \hookrightarrow X$ and $[D(\tilde{B})] \hookrightarrow \partial X$ are both compact.*

Thus, under the Assumptions 4.3.1 a necessary condition for $(S(t,\mathcal{A}))_{t\in\mathbb{R}}$ to be compact is that ∂X be finite dimensional.

Proposition 4.5.3. *Let $\mathcal{A}$ generate a cosine operator function. Assume the embeddings of $[D(\mathcal{A})]$ into the Kisyński space and of the Kisyński space into $\mathcal{X}$ to be both compact. If $(C(t,\mathcal{A}))_{t\in\mathbb{R}}$ is bounded, then it is also almost periodic. If further $\mathcal{A}$ is invertible, then $(S(t,\mathcal{A}))_{t\in\mathbb{R}}$ is almost periodic as well. If moreover the inclusion*

$$P\sigma(A_0) \cup \{\lambda \in \rho(A_0) : \lambda \in P\sigma(\tilde{B}_\lambda)\} \subset -4\pi^2\alpha^2\mathbb{N}^2 \tag{4.12}$$

holds for some $\alpha > 0$, then $(C(t,\mathcal{A}))_{t\in\mathbb{R}}$ and $(S(t,\mathcal{A}))_{t\in\mathbb{R}}$ are periodic.

Proof. The claims concerning almost periodicity hold by Lemma B.25 and B.27. To check periodicity, by Lemma B.25 it suffices to prove that under our assumptions $P\sigma(\mathcal{A}) \subset -4\pi^2\alpha^2\mathbb{N}^2$ for some suitable $\alpha > 0$. This follows by Lemma 2.2.5. ☐

Remarks 4.5.4. (a) Taking into account [EN00, Cor. V.2.15], one obtains that $P\sigma(A_0) \subset -4\pi^2\alpha^2\mathbb{N}^2$ for some $\alpha > 0$ if $(C(t,A_0))_{t\in\mathbb{R}}$ and $(S(t,A_0))_{t\in\mathbb{R}}$ are periodic. Thus, condition (4.12) holds in particular if $(C(t,A_0))_{t\in\mathbb{R}}$ and $(S(t,A_0))_{t\in\mathbb{R}}$ are periodic and further $\lambda \notin P\sigma(\tilde{B}_\lambda)$ for all $\lambda \in \rho(A_0)$.

(b) Let in particular the Assumptions 4.3.1 be satisfied. Since under the assumptions of Proposition 4.5.3 A_0 has compact resolvent and the operator $\tilde{B}_\lambda$, $\lambda \in \rho(A_0)$, is a scalar matrix (because ∂X is necessarily finite dimensional), one sees that $\lambda \notin P\sigma(\tilde{B}_\lambda)$ for all $\lambda \in \rho(A_0)$ reduces to check that a suitable scalar *characteristic equation* has no solution outside a set of countably many points of the real negative halfline. We consider a concrete example of such a characteristic equation in Example 4.5.5 below.

Example 4.5.5. Let us revisit the problem considered in Example 4.3.8, and consider in particular the case $p = 2$, which will be useful in Chapter 5. Then $L^2(0,1) \times \mathbb{C}^2$ equipped with the l^2-norm becomes a Hilbert space.

Fix $q \equiv 0$, $r \le 0$, $\alpha_0 = 1$, $\alpha_1 = -1$, $\beta_0, \beta_1 \le 0$ such that $(\beta_0, \beta_1) \neq (0,0)$. Integrating by parts one sees that for all $u, v \in H^2(0,1)$

$$\left\langle \begin{pmatrix} u'' + ru & 0 \\ \binom{u'(0)}{-u'(1)} & \binom{\beta_0 u(0)}{\beta_1 u(1)} \end{pmatrix}, \begin{pmatrix} v \\ \binom{v(0)}{v(1)} \end{pmatrix} \right\rangle$$

$$= -\int_0^1 u'(x)\overline{v'(x)}dx + \int_0^1 r(x)u(x)\overline{v(x)}dx + \beta_0 u(0)\overline{v(0)} + \beta_1 u(1)\overline{v(1)},$$

hence the operator matrix $\mathcal{A}$ defined in (4.8)–(4.9) is self-adjoint and dissipative. Likewise, one sees that 0 is not an eigenvalue of $\mathcal{A}$, and since as shown below $P\sigma(\mathcal{A}) = \sigma(\mathcal{A})$, it follows that $\mathcal{A}$ is strictly negative definite.

Hence, if $r \le 0$ and $(\beta_0, \beta_1) \in \mathbb{R}^2_- \setminus \{0,0\}$, then by Remark B.18 the cosine operator function generated by the operator matrix

$$\mathcal{A} = \begin{pmatrix} \frac{d^2}{dx^2} + rI & 0 \\ \begin{pmatrix} \delta_0' \\ -\delta_1' \end{pmatrix} & \begin{pmatrix} \beta_0 & 0 \\ 0 & \beta_1 \end{pmatrix} \end{pmatrix}$$

with coupled domain

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} u \\ \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \end{pmatrix} \in H^2(0,1) \times \mathbb{C}^2 : u(0) = x_0,\ u(1) = x_1 \right\}$$

on the Hilbert space $L^2(0,1) \times \mathbb{C}^2$ is contractive and consists of self-adjoint operators, and moreover the associated sine operator function is bounded. Moreover, again by Remark B.18, the associated Kisyński space is isomorphic to $[D((-\mathcal{A})^{\frac{1}{2}})]$. We have already seen that in fact the Kisyński space is given by

$$\mathcal{V} := \left\{ \begin{pmatrix} u \\ \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \end{pmatrix} \in H^1(0,1) \times \mathbb{C}^2 : u(0) = x_0,\ u(1) = x_1 \right\}, \tag{4.13}$$

but it is interesting to prove this in another way.

By [Ka95, Thm. VI.2.23] $D((-\mathcal{A})^{\frac{1}{2}})$ agrees with the form domain of $\mathcal{A}$. The sesquilinear form associated to $\mathcal{A}$ is

$$a(u,v) := -\int_0^1 u'(s)\overline{v'(s)}ds + \int_0^1 r(s)u(s)\overline{v(s)}ds + \beta_0 u(0)\overline{v(0)} + \beta_1 u(1)\overline{v(1)},$$

whose form domain is exactly $\mathcal{V}$. Moreover, the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{V}} := -a(\cdot, \cdot)$, makes the reduction matrix associated to $\mathcal{A}$ dissipative – in other words, we obtain conservation of the energy of solutions to (4.1). This norm is actually equivalent to the product norm as a consequence of the generalized Poincaré inequality, cf. [Ma85, § 1.1.11 and § 3.6.3].

The compactness of the embeddings $H^2(0,1) \cap H^1_0(0,1) \hookrightarrow H^1_0(0,1) \hookrightarrow L^2(0,1)$ implies that the embeddings $[D(A_0)] \hookrightarrow V \hookrightarrow X$ are both compact, hence Lemma 4.5.1, Corollary 4.5.2, and the first part of Proposition 4.5.3 apply. Therefore, $(S(t,\mathcal{A}))_{t\in\mathbb{R}}$ is compact, and further $(C(t,\mathcal{A}))_{t\in\mathbb{R}}$ and $(S(t,\mathcal{A}))_{t\in\mathbb{R}}$ are almost periodic.

Further, it is known that the spectrum of the second derivative on $(0,1)$ with Dirichlet boundary conditions is $\sigma(A_0) = \{-(\pi k)^2 : k = 1,2,\ldots\}$. Hence, to obtain periodicity of $(C(t,\mathcal{A}))_{t\in\mathbb{R}}$ and $(S(t,\mathcal{A}))_{t\in\mathbb{R}}$, by Remark 4.5.4.(b) it is sufficient that no $\lambda \in \rho(A_0)$ is an eigenvalue of the 2×2 matrix $\tilde{B} + BD_\lambda^{A,L}$. It has been computed in [KMN03b, § 9] that a given $\lambda \in \rho(A_0)$ is an eigenvalue of $\tilde{B} + BD_\lambda^{A,L}$ if and only if it is a root of the characteristic equation

$$\lambda^2 + \lambda\left(1 + \frac{2\sqrt{\lambda}}{\tanh\sqrt{\lambda}} - (\beta_0+\beta_1)\right) - \frac{(\beta_0+\beta_1)\sqrt{\lambda}}{\tanh\sqrt{\lambda}} + \beta_0\beta_1 = 0. \tag{4.14}$$

Thus, if for given $(\beta_0,\beta_1) \in \mathbb{R}^2_- \setminus \{0\}$ the roots of (4.14) are contained in the set $\{-(\pi k)^2,\ k = 1,2,\ldots\}$, then the solution to (4.1) is periodic.

Example 4.5.6. We revisit the setting introduced in Example 4.2.3. We have seen that A_0 is the Neumann Laplacian Δ_N, which generates a cosine operator function with associated phase space $H^1(\Omega)\times L^2(\Omega)$, and $\tilde{B}$ is the Laplace–Beltrami operator, which generates a cosine operator function with associated phase space $H^1(\partial\Omega)\times L^2(\partial\Omega)$. By the compactness of the Sobolev embeddings $H^2(\Omega) \hookrightarrow H^1(\Omega) \hookrightarrow L^2(\Omega)$ and $H^2(\partial\Omega) \hookrightarrow H^1(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ it follows that Lemma 4.5.1 and Corollary 4.5.2 apply. We conclude that the reduction matrix associated to $\mathcal{A}$ defined in (4.2) has compact resolvent, and moreover $(S(t,\mathcal{A}))_{t\in\mathbb{R}}$ is compact.

Finally, if B is defined as in (4.3), then $\mathcal{A}$ becomes dissipative and hence it generates a contractive semigroup. Thus, it follows by Proposition A.8 that such a semigroup is asymptotically almost periodic.

Example 4.5.7. We revisit the setting introduced in Example 4.3.8 and define the operator matrix $\mathcal{A}$ as in (4.8)–(4.9). Observe that the Sobolev embeddings $W^{2,p}(0,1) \cap W_0^{1,p}(0,1) \hookrightarrow W_0^{1,p}(0,1) \hookrightarrow L^p(0,1)$ are both compact for all $1 \le p < \infty$. Hence, we can apply Corollary 4.5.2 and conclude that $(S(t,\mathcal{A}))_{t\in\mathbb{R}}$ is compact. For the same reason, the analytic semigroup generated by $\mathcal{A}$ is compact.

Moreover, in the special case of $q = r \equiv 0$, $(C(t,A_0))_{t\in\mathbb{R}} = (C(t,A_{1_0}))_{t\in\mathbb{R}}$ is given by (4.10), and one can check that $\|C(t,A_{1_0})\|_{\mathcal{L}(L^p(0,1))} = 1$ for all $t \in \mathbb{R}$ and $1 \le p < \infty$. Hence if $\alpha_0 = \alpha_1 = \beta_0 = \beta_1 = 0$, then $(C(t,\mathcal{A}))_{t\in\mathbb{R}}$ is bounded, by Remark 4.4.4.(a), and almost periodic, by Proposition 4.5.3.

Chapter 5

Complete second order abstract problems with dynamic boundary conditions

On a Banach space X let us consider a complete abstract second order problem

$$\ddot{u}(t) = Au(t) + C\dot{u}(t), \qquad t \geq 0.$$

We equip such a problem with abstract second order dynamic boundary conditions represented by an equation

$$\ddot{x}(t) = B_1 u(t) + B_2 \dot{u}(t) + B_3 x(t) + B_4 \dot{x}(t), \qquad t \geq 0,$$

on another Banach space ∂X. Here the relation between the variables u and x is expressed by

$$x(t) = Lu(t), \qquad t \geq 0,$$

or else by

$$\dot{x}(t) = L\dot{u}(t), \qquad t \geq 0.$$

Thus, our framework is different from that considered in Section 3.2. We want to investigate well-posedness of such a system. The case of purely boundary damping (i.e., $C = 0$) has already been considered in Remark 4.2.4. Hence, in this chapter we will focus on the case $C \neq 0$.

5.1 General setting

We impose the following throughout this chapter.

Assumptions 5.1.1.

1. *X, Y, and ∂X are Banach spaces such that $Y \hookrightarrow X$.*
2. *$A : D(A) \to X$ and $C : D(C) \to X$ are linear.*
3. *$L : D(A) \cap D(C) \to \partial X$ is linear and surjective.*

4. $B_1 : D(A) \to \partial X$ *and* $B_2 : D(C) \to \partial X$ *are linear.*

5. $B_3 : D(B_3) \subset \partial X \to \partial X$ *and* $B_4 : D(B_4) \subset \partial X \to \partial X$ *are linear and closed.*

To tackle the above problem we investigate the complete second order abstract Cauchy problem

$$(\mathrm{c}\mathcal{ACP}^2) \qquad \begin{cases} \ddot{\mathfrak{u}}(t) &= \mathcal{A}\mathfrak{u}(t) + \mathcal{C}\dot{\mathfrak{u}}(t), \qquad t \geq 0, \\ \mathfrak{u}(0) &= \mathfrak{f} \in \mathcal{X}, \qquad \dot{\mathfrak{u}}(0) = \mathfrak{g} \in \mathcal{X}, \end{cases}$$

on the product space $\mathcal{X} := X \times \partial X$. Here

$$(5.1) \qquad \mathcal{A} := \begin{pmatrix} A & 0 \\ B_1 & B_3 \end{pmatrix} \qquad \text{and} \qquad \mathcal{C} := \begin{pmatrix} C & 0 \\ B_2 & B_4 \end{pmatrix}$$

are operator matrices on $\mathcal{X}$, and their domains will depend on "how unbounded is the damping term C with respect to the elastic term A", as we see next.

Hence, our aim in this chapter is to characterize the generator property of (some part of) the reduction matrix

$$(5.2) \qquad \mathbb{A} := \begin{pmatrix} 0 & I_{D(\mathcal{C})} \\ \mathcal{A} & \mathcal{C} \end{pmatrix}, \qquad D(\mathbb{A}) := D(\mathcal{A}) \times D(\mathcal{C}).$$

Example 5.1.2. The system

$$\begin{cases} \ddot{u}(t,x) &= -u''''(t,x) + \dot{u}''(t,x), & t \geq 0,\ x \in (0,1), \\ u''(t,j) &= (-1)^j u'(t,j) - u(t,j), & t \geq 0,\ j = 0,1, \\ \ddot{u}(t,j) &= (-1)^{j+1} u'''(t,j) + (-1)^j u'(t,j) & \\ & \qquad +(-1)^j \dot{u}'(t,j) - u(t,j) - \dot{u}(t,j), & t \geq 0,\ j = 0,1, \\ u(0,x) &= f(x), & x \in [0,1], \\ \dot{u}(0,x) &= g(x), & x \in (0,1), \\ \dot{u}(0,0) &= x_0, \qquad \dot{u}(0,1) = x_1, & \end{cases}$$

is obtained by equipping a one-dimensional damped plate-like equation with dynamic boundary conditions. We show that such a problem is governed by an analytic, uniformly exponentially stable semigroup that acts on the phase space

$$\left\{ \begin{pmatrix} u \\ \binom{x_0}{x_1} \end{pmatrix} \in H^2(0,1) \times \mathbb{C}^2 : u(0) = x_0,\ u(1) = x_1 \right\} \times \left(L^2(0,1) \times \mathbb{C}^2 \right).$$

In particular, for all $f \in H^2(0,1)$, $g \in L^2(0,1)$, and $x_0, x_1 \in \mathbb{C}^2$ such a system admits a unique mild solution.

Since such a problem involves a fourth-order differential operator, we need two boundary conditions: the first one (given by the second equation of the system) can be looked at as a statical, generalized Wentzell boundary condition (see, e.g., [FGGR01]) on the damping operator, while the second one (given by the third equation of the system) is a dynamic, damped boundary condition on the elastic operator. Then, we can reformulate the problems as (c$\mathcal{ACP}_2$) on the Hilbert space $\mathcal{X} = L^2(0,1) \times \mathbb{C}^2$, where

$$\mathcal{A} = \begin{pmatrix} -\frac{d^4}{dx^4} & 0 \\ \begin{pmatrix} -\delta_0''' + \delta_0' \\ \delta_1''' - \delta_1' \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} \qquad \text{and} \qquad \mathcal{C} = \begin{pmatrix} \frac{d^2}{dx^2} & 0 \\ \begin{pmatrix} \delta_0' \\ -\delta_1' \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix}$$

with coupled domains

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} u \\ \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \end{pmatrix} \in H^4(0,1) \times \mathbb{C}^2 : u(0) = x_0,\ u(1) = x_1, \right.$$

$$\left. u''(0) - u'(0) + u(0) = 0,\ u''(1) + u'(1) + u(1) = 0 \right\},$$

and

$$D(\mathcal{C}) := \left\{ \begin{pmatrix} u \\ \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \end{pmatrix} \in H^2(0,1) \times \mathbb{C}^2 : u(0) = x_0,\ u(1) = x_1 \right\}.$$

We have shown in Example 4.5.5 that $\mathcal{C}$ is self-adjoint and strictly negative definite. Moreover, taking into account Lemma 2.3.1 one can check that $\mathcal{A} = -\mathcal{C}^2$. By Lemma B.33.(2) we can now conclude that the reduction matrix $\mathbb{A}$ (with domain $D(\mathbb{A}) = D(\mathcal{A}) \times D(\mathcal{C})$) defined in (5.2) generates an analytic, compact, uniformly exponentially stable semigroup on the product space $\mathbb{X} = [D(\mathcal{C})] \times \mathcal{X}$.

The crucial point in the above discussion is that a certain operator matrix is self-adjoint and strictly negative definite. Checking that an operator matrix enjoys such properties is usually an application of the Gauss–Green formulae. Our goal is to develop a more abstract theory that permits to characterize the generator property of (some part of) $\mathbb{A}$ by means of its entries $A, C, L, B_1, B_2, B_3, B_4$.

As in Chapter 4, we need to distinguish the cases $L \notin \mathcal{L}(Y, X)$ and $L \in \mathcal{L}(Y, X)$. The final section is devoted to the overdamped case, i.e., the case of a damping operator C that is "more unbounded" than the elastic operator A.

5.2 The damped case: $L \notin \mathcal{L}(Y, X)$

Of concern in this section are complete second order abstract initial-boundary value problems with dynamic boundary conditions of the form

$$(\text{uAIBPV}^2) \qquad \begin{cases} \ddot{u}(t) = Au(t) + C\dot{u}(t), & t \geq 0, \\ \ddot{x}(t) = B_1 u(t) + B_2 \dot{u}(t) + B_3 x(t) + B_4 \dot{x}(t), & t \geq 0, \\ x(t) = Lu(t), & t \geq 0, \\ u(0) = f \in X, \quad \dot{u}(0) = g \in X, & \\ x(0) = h \in \partial X, \quad \dot{x}(0) = j \in \partial X, & \end{cases}$$

on X and ∂X.

Assumptions 5.2.1. *We complement the Assumptions 5.1.1 by the following.*

1. *C is closed, $D(A) \subset D(C)$, and $[D(C)]$ is isomorphic to Y.*

2. $\begin{pmatrix} 0 & I_Y \\ A_0 & C \end{pmatrix}$ *with domain $D(A_0) \times D(C)$, where $A_0 := A|_{\ker(L)}$, generates a C_0-semigroup on $Y \times X$.*

3. $\begin{pmatrix} A \\ L \end{pmatrix} : D(A) \subset Y \to X \times \partial X$ *is closed.*

4. *B_3 is bounded on ∂X.*

5. *B_4 generates a C_0-semigroup on ∂X.*

We denote by $[D(A)]_L^Y$ the Banach space obtained by endowing $D(A)$ with the graph norm of the closed (from Y to $X \times \partial X$) operator $\binom{A}{L}$. It is clear that if $\binom{A}{L}$ is closed as an operator from X to $X \times \partial X$, then it is also closed as an operator from Y to $X \times \partial X$, and $[D(A)]_L^Y \hookrightarrow [D(A)]_L$.

In order to reformulate (dAIBVP^2) as (c$\mathcal{ACP}^2$), we consider the operator matrices $\mathcal{A}$ and $\mathcal{C}$ introduced in (5.1). The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in D(A) \times \partial X : Lu = x \right\},$$

throughout this section, while the domain of $\mathcal{C}$ is

$$D(\mathcal{C}) := D(C) \times D(B_4).$$

A direct matrix computation yields the following.

Lemma 5.2.2. *The part in*

$$\mathbb{X} := Y \times \partial X \times X \times \partial X$$

of the operator matrix $\mathbb{A}$ defined in (5.2) *is similar to*

$$\mathbb{G} := \begin{pmatrix} 0 & I_Y & 0 & 0 \\ A & C & 0 & 0 \\ 0 & 0 & 0 & I_{\partial X} \\ B_1 & B_2 & B_3 & B_4 \end{pmatrix}$$

with domain

$$D(\mathbb{G}) := \left\{ \begin{pmatrix} u \\ x \\ v \\ y \end{pmatrix} \in D(A) \times D(C) \times \partial X \times D(B_4) : Lu = x \right\}$$

on the Banach space

$$\mathbb{Y} := Y \times X \times \partial X \times \partial X.$$

The similarity transformation is performed by the operator matrix

$$\mathbb{U} := \begin{pmatrix} I_Y & 0 & 0 & 0 \\ 0 & 0 & I_X & 0 \\ 0 & I_{\partial X} & 0 & 0 \\ 0 & 0 & 0 & I_{\partial X} \end{pmatrix},$$

which is an isomorphism from $\mathbb{X}$ *onto* $\mathbb{Y}$.

In order to apply our abstract theory, we consider $\mathbb{G}$ as a 2×2 operator matrix with diagonal domain. More precisely,

$$\mathbb{G} = \begin{pmatrix} \mathbf{A} & \mathbf{I} \\ \mathbf{B} & \tilde{\mathbf{B}} \end{pmatrix}, \qquad D(\mathbb{G}) = D(\mathbf{A}) \times D(\tilde{\mathbf{B}}),$$

where the block-entry $\mathbf{A}$ is an operator matrix with coupled domain defined by

$$\mathbf{A} := \begin{pmatrix} 0 & I_Y & 0 \\ A & C & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad D(\mathbf{A}) := \left\{ \begin{pmatrix} u \\ v \\ x \end{pmatrix} \in D(A) \times D(C) \times \partial X : Lu = x \right\},$$

on

$$\mathbf{X} := Y \times X \times \partial X.$$

Further,

$$\mathbf{I} := \begin{pmatrix} 0 \\ 0 \\ I_{\partial X} \end{pmatrix}, \qquad D(\mathbf{I}) := \partial \mathbf{X},$$

is an operator from $\partial \mathbf{X} := \partial X$ to $\mathbf{X}$, while

$$\mathbf{B} := \begin{pmatrix} B_1 & B_2 & B_3 \end{pmatrix}, \qquad D(\mathbf{B}) := D(\mathbf{A}),$$

is an operator from $\mathbf{X}$ to $\partial \mathbf{X}$. Finally,

$$\tilde{\mathbf{B}} := B_4, \qquad D(\tilde{\mathbf{B}}) := D(B_4),$$

is an operator on $\partial \mathbf{X}$.

Lemma 5.2.3. *The operator matrix* $\mathbf{A}$ *generates a* C_0*-semigroup on* $\mathbf{X}$. *Such a semigroup is analytic (resp., bounded) if and only if the semigroup considered in the Assumption 5.2.1.2 is analytic (resp., bounded). Finally,* $\mathbf{A}$ *has compact resolvent if and only if the embeddings* $[D(A_0)] \hookrightarrow Y \hookrightarrow X$ *are both compact and* $\dim \partial X < \infty$.

Proof. Consider $\mathbf{A}$ as

$$\mathbf{A} = \left(\begin{array}{cc|c} 0 & I_Y & 0 \\ A & C & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

with coupled domain

$$D(\mathbf{A}) = \left\{ \begin{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ x \end{pmatrix} \in \big(D(A) \times D(C)\big) \times \partial X : \begin{pmatrix} L & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = x \right\}.$$

The only non-zero block-entry

$$\begin{pmatrix} 0 & I_Y \\ A & C \end{pmatrix}$$

of $\mathbf{A}$, restricted to

$$\ker\big(\begin{pmatrix} L & 0 \end{pmatrix}\big) = D(A_0) \times D(C),$$

generates by Assumption 5.2.1.2 a C_0-semigroup on $Y \times X$. Moreover,

$$\begin{pmatrix} L & 0 \end{pmatrix} : D(A) \times D(C) \to \partial X$$

is surjective by Assumption 5.1.1.3. Finally, the operator matrix

$$\begin{pmatrix} \begin{pmatrix} 0 & I_Y \\ A & C \end{pmatrix} \\ \begin{pmatrix} L & 0 \end{pmatrix} \end{pmatrix} : D(A) \times D(C) \subset Y \times X \to Y \times X \times \partial X$$

is closed. To prove this let

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_{n \in \mathbb{N}} \subset D(A) \times D(C), \quad \lim_{n\to\infty} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \qquad \text{in } Y \times X,$$

$$\text{and} \quad \lim_{n\to\infty} \begin{pmatrix} 0 & I_Y \\ A & C \\ L & 0 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \lim_{n\to\infty} \begin{pmatrix} v_n \\ Au_n + Cv_n \\ Lu_n \end{pmatrix} = \begin{pmatrix} v \\ w \\ z \end{pmatrix} \qquad \text{in } Y \times X \times \partial X.$$

Observe that $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} v_n = v$ hold with respect to the norm of Y. Hence, it follows by Assumption 5.2.1.1 that $\lim_{n\to\infty} Cv_n = Cv$. Moreover, since $\lim_{n\to\infty} Au_n = w - Cv$ and $\lim_{n\to\infty} Lu_n = z$, wo obtain by Assumption 5.1.1.3 that $u \in D(A)$, $Au + Cv = w$, and $Lu = z$.

Hence, we have shown that the Assumptions 2.1.1 are satisfied and the operator matrix $\mathbf{A}$ with coupled domain generates a C_0-semigroup on $\mathbf{X}$ by Theorem 2.2.8.(1). The analytic case follows by Theorem 2.2.8.(2). Further, $(e^{t\mathbf{A}})_{t\geq 0}$ is block-triangular and its lower-right entry is $I_{\partial X}$: hence, it is bounded if and only if its upper-left block-entry (i.e., the semigroup considered in the Assumption 5.2.1.2) is bounded. The assertion about the resolvent compactness can be proven likewise. □

We are now in the position to prove the main result of this section.

Theorem 5.2.4. *Under the Assumptions 5.1.1 and 5.2.1 the following assertions hold.*

(1) If $B_1 \in \mathcal{L}([D(A)]_L^Y, [D(B_4)])$, $B_2 \in \mathcal{L}(Y, [D(B_4)])$, and $B_3 \in \mathcal{L}(\partial X, [D(B_4)])$, then the part of $\mathbb{A}$ in $\mathbb{X}$ generates a C_0-semigroup.

(2) *Let $B_1 \in \mathcal{L}([D(A)]^Y_L, \partial X)$, $B_2 \in \mathcal{L}(Y, \partial X)$. If the two semigroups considered in the Assumptions 5.2.1.2–5 are both analytic, then $\mathbb{A}$ generates an analytic semigroup on $\mathbb{X}$.*

(3) *Let $B_1 = B_2 = B_3 = 0$, $B_4 \in \mathcal{L}(\partial X)$. If the two semigroups considered in the Assumptions 5.2.1.2–5 are bounded and uniformly exponentially stable, respectively, then the semigroup generated by the part of $\mathbb{A}$ in $\mathbb{X}$ is bounded.*

(4) *The part of $\mathbb{A}$ in $\mathbb{X}$ has compact resolvent if and only if the embeddings $[D(A_0)] \hookrightarrow Y \hookrightarrow X$ are both compact and $\dim \partial X < \infty$. If this is the case and (2) (resp., (3)) applies, then the semigroup generated by the part of $\mathbb{A}$ in $\mathbb{X}$ is compact (resp., asymptotically almost periodic).*

Proof. By Lemma 5.2.2 it suffices to investigate the similar operator matrix $\mathbb{G}$ on $\mathbb{Y}$, instead of $\mathbb{A}$ on $\mathbb{X}$.We decompose

$$\mathbb{G} := \mathbb{G}_0 + \mathbb{G}_1 := \begin{pmatrix} \mathbf{A} & 0 \\ \mathbf{B} & \tilde{\mathbf{B}} \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{I} \\ 0 & 0 \end{pmatrix},$$

where the operator matrix $\mathbb{G}_1$ is bounded on $\mathbb{Y}$. Hence, the part of $\mathbb{A}$ in $\mathbb{X}$ is a generator if and only if the lower triangular operator matrix $\mathbb{G}_0$ is a generator on $\mathbb{Y}$. Observe that the diagonal block-entries of $\mathbb{G}_0$ both generate a C_0-semigroup, by Lemma 5.2.3 and Assumption 5.2.1.1)

(1) By the closed graph theorem $[D(\mathbf{A})] \hookrightarrow [D(A)]^Y_L \times Y \times \partial X$. Then the off-diagonal entry $\mathbf{B}$ is bounded from $[D(\mathbf{A})]$ to $[D(\tilde{\mathbf{B}})]$. Now it follows by [Na89, Cor. 3.2] that $\mathbb{G}_0$ generates a C_0-semigroup on $\mathbb{Y}$.

(2) The diagonal entries of $\mathbb{G}_0$ both generate analytic semigroups, on $\mathbf{X}$ and $\partial\mathbf{X}$ respectively. Moreover, the off-diagonal entry $\mathbf{B}$ is bounded from $[D(\mathbf{A})]$ to $\partial\mathbf{X}$. Now it follows by [Na89, Cor. 3.3] that $\mathbb{G}_0$ generates an analytic semigroup on $\mathbb{Y}$.

(3) If $\mathbf{B} = 0$, then $\mathbb{G}$ is an upper triangular matrix that generates a C_0-semigroup by (1). The diagonal entries $(e^{t\mathbf{A}})_{t\geq 0}$ and $(e^{t\tilde{\mathbf{B}}})_{t\geq 0}$ of such a semigroup matrix are by assumption bounded and uniformly exponentially stable, respectively. Then the boundedness of $(e^{t\mathbb{G}})_{t\geq 0}$, and hence of $(e^{t\mathbb{A}})_{t\geq 0}$ can be proven mimicking the proof of Corollary 2.2.2.(2), taking into account [Na89, Prop. 3.1].

(4) The block-diagonal operator matrix $\mathbb{G}$ has compact resolvent if and only its diagonal block-entries $\mathbf{A}$ and $\tilde{\mathbf{B}}$ on $\mathbf{X}$ and $\partial\mathbf{X}$, respectively, have compact resolvent. Then, we just need to apply Lemma 5.2.3. The claim about asymptotical almost periodicity holds by Lemma A.8.(1). □

5.3 The damped case: $L \in \mathcal{L}(Y, X)$

Assumptions 5.3.1. *We complement the Assumptions 5.1.1 by the following.*

1. *V is a Banach space such that $V \hookrightarrow Y$.*
2. *L can be extended to an operator that is bounded from Y to ∂X, which we denote again by L, and such that $\ker(L) = V$.*

3. *C is closed, $D(A) \subset D(C)$, and $[D(C)]$ is isomorphic to Y.*

4. *$A_0 := A|_{D(A)\cap\ker(L)}$ is invertible.*

5. *$\begin{pmatrix} 0 & I_V \\ A_0 & C_0 \end{pmatrix}$ with domain $D(A_0) \times D(C_0)$, where $C_0 := C|_{\ker(L)}$, generates a C_0-semigroup on $V \times X$.*

6. *$\begin{pmatrix} A \\ L \end{pmatrix} : D(A) \subset Y \to X \times \partial X$ is closed.*

7. *B_3 is bounded on ∂X.*

8. *B_4 generates a C_0-semigroup on ∂X.*

As in the previous section, we denote by $[D(A)]_L^Y$ the Banach space obtained by endowing $D(A)$ with the graph norm of the closed (from Y to $X \times \partial X$) operator $\binom{A}{L}$.

Consider the non-diagonal Banach space $\underline{\mathbb{X}}$ defined by

$$\underline{\mathbb{X}} := \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in Y \times \partial X : Lu = x \right\} \times X \times \partial X.$$

Motivated by the results of Section 4.3, we investigate the part in $\underline{\mathbb{X}}$ (rather than in

$$\mathbb{X} := Y \times \partial X \times X \times \partial X,$$

as in the previous section) of the reduction matrix $\mathbb{A}$ defined in (5.2).

The initial value problem associated to

$$\dot{\mathbb{u}}(t) = \mathbb{A}\mathbb{u}(t), \qquad t \geq 0, \tag{5.3}$$

on $\underline{\mathbb{X}}$ is formally equivalent to (dAIBVP^2) on X and ∂X, if we identify

$$\mathbb{u}(t) \equiv \begin{pmatrix} u(t) \\ Lu(t) \\ v(t) \\ y(t) \end{pmatrix}, \qquad t \geq 0.$$

The first coordinate of (5.3) reads

$$\frac{du}{dt}(\cdot) = v(\cdot),$$

where the limit is to be understood with respect to the norm of Y. Hence, taking into account the coupling incorporated in the definition of the Banach space $\underline{\mathbb{X}}$ and reasoning as in (1.9), we obtain that for a function

$$\mathbb{u}(\cdot) \equiv \begin{pmatrix} u(\cdot) \\ Lu(\cdot) \\ v(\cdot) \\ y(\cdot) \end{pmatrix} \in C^1(\mathbb{R}, \underline{\mathbb{X}})$$

(hence in particular for a solution to (5.3)) there holds

$$y(\cdot) = \frac{d(Lu)}{dt}(\cdot) = L\frac{du}{dt}(\cdot) = Lv(\cdot), \tag{5.4}$$

where we have used the assumption $L \in \mathcal{L}(Y, \partial X)$.

Summing up, the systems we are concerned with in this section are of the form

$$(\text{mAIBPV}^2) \qquad \begin{cases} \ddot{u}(t) &= Au(t) + C\dot{u}(t), & t\geq 0,\\ \ddot{x}(t) &= B_1u(t) + B_2\dot{u}(t) + B_3x(t) + B_4\dot{x}(t), & t\geq 0,\\ x(t) &= Lu(t), \qquad \dot{x}(t) = L\dot{u}(t), & t\geq 0,\\ u(0) &= f \in X, \qquad \dot{u}(0) = g \in X, & \\ x(0) &= h \in \partial X, \qquad \dot{x}(0) = j \in \partial X, & \end{cases}$$

on X and ∂X. Taking into account (5.4), observe that (mAIBVP^2) can therefore be reformulated as the initial value problem associated to

$$\dot{\mathbb{u}}(t) = \underline{\mathbb{A}}\mathbb{u}(t), \qquad t\geq 0,$$

again on the Banach space $\underline{\mathbb{X}}$. Here the operator matrix $\underline{\mathbb{A}}$ is given by

$$\underline{\mathbb{A}} := \begin{pmatrix} 0 & 0 & I_Y & 0\\ 0 & 0 & L & 0\\ A & 0 & C & 0\\ B_1 & B_3 & B_2 & B_4 \end{pmatrix}.$$

with domain

$$D(\underline{\mathbb{A}}) := \left\{ \begin{pmatrix} u\\ v\\ x\\ y \end{pmatrix} \in D(A) \times \partial X \times D(C) \times D(B_4) : Lu = x,\ Lv = y \right\}.$$

Our aim in the remainder of this section is hence to characterize the generator property of (some part of) $\underline{\mathbb{A}}$.

A direct matrix computation yields the following.

Lemma 5.3.2. *The part in $\underline{\mathbb{X}}$ of the operator matrix $\underline{\mathbb{A}}$ is similar to*

$$\underline{\mathbb{G}} := \begin{pmatrix} 0 & I_Y & 0 & 0\\ A & C & 0 & 0\\ 0 & L & 0 & 0\\ B_1 & B_2 & B_3 & B_4 \end{pmatrix}$$

with domain

$$D(\underline{\mathbb{G}}) := \left\{ \begin{pmatrix} u\\ v\\ x\\ y \end{pmatrix} \in D(A_0) \times D(C) \times \partial X \times D(B_4) : Lu = x,\ Lv = y \right\}$$

on the Banach space

$$\underline{\mathbb{Y}} := \left\{ \begin{pmatrix} u \\ v \\ x \end{pmatrix} \in Y \times X \times \partial X : Lu = x \right\} \times \partial X.$$

The similarity transformation is performed by the operator matrix

$$\underline{\mathbb{U}} := \begin{pmatrix} I_Y & 0 & 0 & 0 \\ 0 & 0 & I_X & 0 \\ 0 & I_{\partial X} & 0 & 0 \\ 0 & 0 & 0 & I_{\partial X} \end{pmatrix},$$

which is an isomorphism from $\underline{\mathbb{X}}$ *onto* $\underline{\mathbb{Y}}$.

To investigate the generator property of $\underline{\mathbb{G}}$, we consider it as a 2×2 operator matrix with coupled domain, i.e.,

$$\underline{\mathbb{G}} = \begin{pmatrix} \underline{\mathbf{A}} & 0 \\ \underline{\mathbf{B}} & \underline{\tilde{\mathbf{B}}} \end{pmatrix}, \qquad D(\underline{\mathbb{G}}) = \left\{ \begin{pmatrix} \mathbf{u} \\ \mathbf{x} \end{pmatrix} \in D(\underline{\mathbf{A}}) \times D(\underline{\tilde{\mathbf{B}}}) : \underline{\mathbf{L}}\mathbf{u} = \mathbf{x} \right\}.$$

Here $\underline{\mathbf{A}}$ is an operator matrix with coupled domain defined by

$$\underline{\mathbf{A}} := \begin{pmatrix} 0 & I_Y & 0 \\ A & C & 0 \\ 0 & L & 0 \end{pmatrix}, \qquad D(\underline{\mathbf{A}}) := \left\{ \begin{pmatrix} u \\ v \\ x \end{pmatrix} \in D(A) \times D(C) \times \partial X : Lu = x \right\},$$

on the Banach space

$$\underline{\mathbf{X}} := \left\{ \begin{pmatrix} u \\ v \\ x \end{pmatrix} \in Y \times X \times \partial X : Lu = x \right\}.$$

Further,

$$\underline{\mathbf{L}} := \begin{pmatrix} 0 & L & 0 \end{pmatrix}, \qquad D(\underline{\mathbf{L}}) := D(\underline{\mathbf{A}}), \qquad \text{and}$$
$$\underline{\mathbf{B}} := \begin{pmatrix} B_1 & B_2 & B_3 \end{pmatrix}, \qquad D(\underline{\mathbf{B}}) := D(\underline{\mathbf{A}}),$$

are operators from $\underline{\mathbf{X}}$ to $\partial\underline{\mathbf{X}} := \partial X$. Finally,

$$\underline{\tilde{\mathbf{B}}} := B_4, \qquad D(\underline{\tilde{\mathbf{B}}}) := D(B_4),$$

is an operator on $\partial\underline{\mathbf{X}}$.

Reasoning as in Lemma C.1 and Lemma C.4 one can see that under the Assumptions 5.1.1 and 5.3.1 the Dirichlet operator $D_0^{A,L}$ associated to (A, L) exists as a bounded operator from ∂X to Y. Using such an operator to "decouple" the non-diagonal Banach space $\underline{\mathbf{X}}$ (similarly to what we did in Section 4.3), we obtain the following.

Lemma 5.3.3. *The restriction* $\underline{\mathbf{A}}_0$ *of* $\underline{\mathbf{A}}$ *to* $\ker(\underline{\mathbf{L}})$ *generates a* C_0*-semigroup on* $\underline{\mathbf{X}}$. *Such a semigroup is analytic (resp., bounded) if and only if the semigroup considered in the Assumption 5.3.1.5 is analytic (resp., bounded). Finally,* $\underline{\mathbf{A}}_0$ *has compact resolvent if and only if the embeddings* $[D(A_0)] \hookrightarrow V \hookrightarrow X$ *are both compact and* $\dim \partial X < \infty$.

Proof. Define the product Banach space

$$\hat{\mathbf{X}} := V \times X \times \partial X$$

and the operator

$$\mathbf{S} := \begin{pmatrix} I_Y & 0 & -D_0^{A,L} \\ 0 & I_X & 0 \\ 0 & 0 & I_{\partial X} \end{pmatrix}. \tag{5.5}$$

Then it can be shown (essentially like in the proof of Lemma 4.3.3) that $\mathbf{S}$ is an isomorphism from $\underline{\mathbf{X}}$ onto $\hat{\mathbf{X}}$.

A direct matrix computation yields that the restriction $\underline{\mathbf{A}}_0$ of the operator matrix $\underline{\mathbf{A}}$ to $\ker(\underline{\mathbf{L}})$ is similar via $\mathbf{S}$ to

$$\hat{\mathbf{A}} := \begin{pmatrix} 0 & I_V & 0 \\ A_0 & C_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad D(\hat{\mathbf{A}}) := D(A_0) \times D(C_0) \times \partial X,$$

on the product Banach space $\hat{\mathbf{X}}$. Such an operator matrix with diagonal domain generates a C_0-semigroup by Assumption 5.3.1.5. The analytic case follows by Theorem 2.2.8.(2). Further, $(e^{t\hat{\mathbf{A}}})_{t\geq 0}$ is block-diagonal and its lower-right entry is $I_{\partial X}$: hence, it is bounded if and only if its upper-left block-entry (i.e., the semigroup considered in the Assumption 5.3.1.2) is bounded. The assertion about the resolvent compactness follows by Corollary 2.2.3. □

The following can be proven similarly to Lemma 1.3.3.

Lemma 5.3.4. *The operator matrices $\underline{\mathbf{A}}$ on $\underline{\mathbf{X}}$ and $\binom{\underline{\mathbf{A}}}{\underline{\mathbf{L}}}$ from $\underline{\mathbf{X}}$ to $\underline{\mathbf{X}} \times \partial\underline{\mathbf{X}}$ are both closed, and their graph norms are equivalent.*

Lemma 5.3.5. *The Dirichlet operators associated to the pair $(\underline{\mathbf{A}}, \underline{\mathbf{L}})$ exist as bounded operators from $\partial\underline{\mathbf{X}}$ to $\underline{\mathbf{Z}}$ for every Banach space $\underline{\mathbf{Z}}$ satisfying $D(\underline{\mathbf{A}}^\infty) \subset \underline{\mathbf{Z}} \hookrightarrow \underline{\mathbf{X}}$. Moreover, the Dirichlet operator $D_\lambda^{\underline{\mathbf{A}},\underline{\mathbf{L}}}$ is given by*

$$D_\lambda^{\underline{\mathbf{A}},\underline{\mathbf{L}}} := \begin{pmatrix} \frac{1}{\lambda} D_\lambda^{\frac{A}{\lambda}+C,L} \\ D_\lambda^{\frac{A}{\lambda}+C,L} \\ \frac{1}{\lambda} I_{\partial X} \end{pmatrix} \qquad \textit{for } \lambda \textit{ large enough.}$$

Proof. It has already been proven in Lemma 5.3.3 and Lemma 5.3.4 that $\underline{\mathbf{A}}_0$ generates a C_0-semigroup on $\underline{\mathbf{X}}$ and that $\binom{\underline{\mathbf{A}}}{\underline{\mathbf{L}}}$ is a closed operator from $\underline{\mathbf{X}}$ to $\underline{\mathbf{X}} \times \partial\underline{\mathbf{X}}$, respectively. The surjectivity of $\underline{\mathbf{L}}$ from $D(\underline{\mathbf{A}})$ to $\partial\underline{\mathbf{X}}$ is a direct consequence of Assumptions 5.1.1.3 and 5.3.1.1. Thus, the existence and the boundedness of the Dirichlet operators associated to the pair $(\underline{\mathbf{A}}, \underline{\mathbf{L}})$ follow by Lemma C.1 and Lemma C.4.

To obtain the claimed representation, take $\mathbf{x} := y \in \partial X = \partial\underline{\mathbf{X}}$. By definition the Dirichlet operator $D_\lambda^{\underline{\mathbf{A}},\underline{\mathbf{L}}}$ maps $\mathbf{x}$ into the unique vector

$$\mathbf{u} := \begin{pmatrix} u \\ v \\ Lu \end{pmatrix} \in D(\underline{\mathbf{A}}) \quad \text{such that} \quad \begin{cases} \underline{\mathbf{A}}\mathbf{u} = \lambda \mathbf{u}, \\ \underline{\mathbf{L}}\mathbf{u} = \mathbf{x}, \end{cases}$$

or rather

$$\begin{cases} v &= \lambda u, \\ Au + Cv &= \lambda v, \\ Lv &= \lambda Lu, \\ Lv &= y. \end{cases}$$

Thus, we see that $\frac{1}{\lambda}Av + Cv = \lambda v$, whence $v = D_\lambda^{\frac{A}{\lambda}+C,L} y$, and the claim follows. $\square$

We are now in the position to prove the main result of this section, which parallels Theorem 5.2.4.

Theorem 5.3.6. *Under the Assumptions 5.1.1 and 5.3.1 the following assertions hold.*

(1) Let $B_1 \in \mathcal{L}([D(A)]_L^Y, \partial X)$, $B_2 \in \mathcal{L}(Y, \partial X)$ and $B_4 \in \mathcal{L}(\partial X)$. Then the part of $\underline{\mathbb{A}}$ in $\underline{\mathbb{X}}$ generates a C_0-semigroup if the reduction matrix

$$\begin{pmatrix} 0 & I_V \\ A_0 - D_\lambda^{\frac{A}{\lambda}+C,L} B_1 & C_0 - D_\lambda^{\frac{A}{\lambda}+C,L} B_2 \end{pmatrix} \qquad \textit{with domain} \quad D(A_0) \times D(C_0)$$

generates a C_0-semigroup on $V \times X$ for some λ large enough.

(2) Let $B_1 \in \mathcal{L}(Y, \partial X)$ and $B_2 \in \mathcal{L}(X, \partial X)$. Then the part of $\underline{\mathbb{A}}$ in $\underline{\mathbb{X}}$ generates an analytic semigroup if and only if the two semigroups considered in the Assumptions 5.3.1.5–8 are both analytic.

(3) Let the semigroups considered in the Assumptions 5.3.1.5–8 be both analytic. Assume that for some $0 < \alpha < 1$ there holds $Y \hookrightarrow [V, X]_{1-\alpha}$. If further $B_1 \in \mathcal{L}([D(A)]_L^Y, [D(B_4), \partial X]_\alpha)$, $B_2 \in \mathcal{L}(Y, \partial X) \cap \mathcal{L}(V, [D(B_4), \partial X]_\alpha)$, and $B_3 \in \mathcal{L}(\partial X, [D(B_4), \partial X]_\alpha)$, then the part of $\mathbb{A}$ in $\underline{\mathbb{X}}$ generates an analytic semigroup.

(4) Let $B_1 = B_2 = B_3 = 0$, $B_4 \in \mathcal{L}(\partial X)$. If the two semigroups considered in the Assumptions 5.3.1.5–8 are bounded and uniformly exponentially stable, respectively, then the the semigroup generated by the part of $\underline{\mathbb{A}}$ in $\underline{\mathbb{X}}$ is bounded.

(5) The part of $\underline{\mathbb{A}}$ in $\underline{\mathbb{X}}$ has compact resolvent if and only if the embeddings $[D(A_0)] \hookrightarrow V \hookrightarrow X$ are both compact and $\dim \partial X < \infty$. *If this is the case and (2) or (3) (resp., (4)) apply, then the semigroup generated by the part of $\underline{\mathbb{A}}$ in $\underline{\mathbb{X}}$ is compact (resp., asymptotically almost periodic).*

Proof. By Lemma 5.3.2 the part of $\underline{\mathbb{A}}$ in $\underline{\mathbb{X}}$ is a generator if and only if the operator matrix $\underline{\mathbb{G}}$ is a generator on $\underline{\mathbb{Y}}$.

It has been shown in the proof of Lemma 5.3.5 that the spaces $\underline{\mathbf{X}}, \partial\underline{\mathbf{X}}$ and the operators $\underline{\mathbf{A}}, \underline{\mathbf{L}}$ satisfy the Assumptions 2.1.1. Hence the operator matrix with coupled domain $\underline{\mathbb{G}}$ can be studied by means of the results of Chapter 2.

(1) By Lemma 5.3.4 the graph norms of $\underline{\mathbf{A}}$ and $\binom{\underline{\mathbf{A}}}{\underline{\mathbf{L}}}$ are equivalent, hence it follows by the closed graph theorem that $[D(\underline{\mathbf{A}})]_{\underline{\mathbf{L}}} \hookrightarrow [D(A)]_L^Y \times Y \times \partial X$. Thus,

by assumption $\underline{\mathbf{B}}$ is bounded from $[D(\underline{\mathbf{A}})]_{\mathbf{L}}$ to $\partial\underline{\mathbf{X}}$. Therefore, the claim follows by Theorem 2.2.8.(1) if we can prove that $\underline{\mathbf{A}}_0 - D_\lambda^{\mathbf{A},\mathbf{L}}\underline{\mathbf{B}}$ on $\underline{\mathbf{X}}$, or equivalently the similar operator $\mathbf{S}\left(\underline{\mathbf{A}}_0 - D_\lambda^{\mathbf{A},\mathbf{L}}\underline{\mathbf{B}}\right)\mathbf{S}^{-1}$ on $\hat{\mathbf{X}} = V \times X \times \partial X$ generate a C_0-semigroup, where $\mathbf{S}$ is defined as in (5.5).

Take into account the proof of Lemma 5.3.3 and compute the operator matrix $\mathbf{S}D_\lambda^{\mathbf{A},\mathbf{L}}\underline{\mathbf{B}}\mathbf{S}^{-1}$ on $\hat{\mathbf{X}}$. By Lemma 5.3.5 we obtain that, if the Dirichlet operator $D_\lambda^{\mathbf{A},\mathbf{L}}$ exists, then

$$\mathbf{S}D_\lambda^{\mathbf{A},\mathbf{L}}\underline{\mathbf{B}}\mathbf{S}^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ D_\lambda^{\frac{A}{\lambda}+C,L}B_1 & D_\lambda^{\frac{A}{\lambda}+C,L}B_2 & D_\lambda^{\frac{A}{\lambda}+C,L}\left(B_3 + B_1 D_0^{A,L}\right) \\ \frac{1}{\lambda}B_1 & \frac{1}{\lambda}B_2 & \frac{1}{\lambda}\left(B_3 + B_1 D_0^{A,L}\right) \end{pmatrix}.$$

Summing up, if $D_\lambda^{\mathbf{A},\mathbf{L}}$ exists, then $\underline{\mathbf{A}}_0 - D_\lambda^{\mathbf{A},\mathbf{L}}\underline{\mathbf{B}}$ is similar to

$$\begin{aligned}\hat{\mathbf{A}} - \mathbf{S}D_\lambda^{\mathbf{A},\mathbf{L}}\underline{\mathbf{B}}\mathbf{S}^{-1} \;&=\; \left(\begin{array}{cc|c} 0 & I_V & 0 \\ A_0 - D_\lambda^{\frac{A}{\lambda}+C,L}B_1 & C_0 - D_\lambda^{\frac{A}{\lambda}+C,L}B_2 & 0 \\ \hline \frac{1}{\lambda}B_1 & \frac{1}{\lambda}B_2 & 0 \end{array}\right) \\ &\quad + \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & D_\lambda^{\frac{A}{\lambda}+C,L}\left(B_3 + B_1 D_0^{A,L}\right) \\ \hline 0 & 0 & \frac{1}{\lambda}\left(B_3 + B_1 D_0^{A,L}\right) \end{array}\right).\end{aligned}$$

Now the second operator on the right-hand side is bounded on $\hat{\mathbf{X}}$, while the first one is a block-diagonal matrix. Since the lower-left block-entry is by assumption bounded from the domain of the upper-left block-entry (i.e., $D(A_0) \times D(C_0)$) to ∂X, by [Na89, Cor. 3.2] we conclude that the claim follows.

(2) Observe that by assumption $\underline{\mathbf{B}} \in \mathcal{L}(\underline{\mathbf{X}}, \partial\underline{\mathbf{X}})$. Hence the claim is a direct consequence of Theorem 2.2.8.(2) and Lemma 5.3.3.

(3) Taking into account Lemma 5.3.3, $\underline{\mathbf{A}}_0$ and $\underline{\tilde{\mathbf{B}}}$ generate analytic semigroups on $\underline{\mathbf{X}}$ and $\partial\underline{\mathbf{X}}$, respectively. Moreover, one sees that the interpolation spaces between $[D(\underline{\mathbf{A}}_0)]$ and $\underline{\mathbf{X}}$ are given by

$$[D(\underline{\mathbf{A}}_0), \underline{\mathbf{X}}]_\epsilon = \left\{ \begin{pmatrix} u \\ v \\ x \end{pmatrix} \in [D(A), Y]_\epsilon \times [V, X]_\epsilon \times \partial X : Lu = x \right\}$$

for $0 < \epsilon < 1$. Hence, by assumption $[D(\underline{\mathbf{A}})]_{\mathbf{L}} \hookrightarrow [D(\underline{\mathbf{A}}_0), \underline{\mathbf{X}}]_\alpha$. Further, $[D(\underline{\mathbf{A}}_0)] \hookrightarrow [D(A)]_L^Y \times V \times \partial X$ and consequently $\underline{\mathbf{B}} \in \mathcal{L}([D(\underline{\mathbf{A}})]_{\mathbf{L}}, \partial\underline{\mathbf{X}}) \cap \mathcal{L}([D(\underline{\mathbf{A}}_0)], [D(\underline{\tilde{\mathbf{B}}}), \partial\underline{\mathbf{X}}]_\alpha)$, and the claim follows by Theorem 2.2.8.(3).

(4) There holds $\underline{\mathbf{B}} = 0$, and the claim follows by Corollary 2.2.2.(2) and Lemma 5.3.3.

(5) By Corollary 2.2.3, the part of the operator matrix $\mathbb{G}$ with coupled domain has compact resolvent if and only if $\underline{\mathbf{A}}_0$ and $\underline{\tilde{\mathbf{B}}}$ have compact resolvent. Hence the claim follows by Lemma 5.3.3 and Lemma A.8.(1). $\square$

5.4 The overdamped case

Of concern in this section are complete second order abstract initial-boundary value problems with dynamic boundary conditions of the form

$$
(\mathrm{oAIBPV}^2) \qquad \begin{cases} \ddot{u}(t) &= Au(t) + C\dot{u}(t), & t \geq 0, \\ \ddot{x}(t) &= B_1 u(t) + B_2 \dot{u}(t) + B_3 x(t) + B_4 \dot{x}(t), & t \geq 0, \\ \dot{x}(t) &= L\dot{u}(t), & t \geq 0, \\ u(0) &= f \in X, \qquad \dot{u}(0) = g \in X, \\ x(0) &= h \in \partial X, \qquad \dot{x}(0) = j \in \partial X, \end{cases}
$$

on X and ∂X. Observe that the coupling relation expressed by the third equation is not the same of (uAIBPV^2) or (mAIBPV^2).

Assumptions 5.4.1. *We complement the Assumptions 5.1.1 by the following.*

1. $D(C) \subset D(A) \subset Y$.
2. $C_0 := C|_{\ker(L)}$ *is densely defined and has nonempty resolvent set.*
3. $\begin{pmatrix} C \\ L \end{pmatrix} : D(C) \subset X \to X \times \partial X$ *is closed.*
4. $B_2 : [D(C)]_L \to \partial X$ *is bounded.*

In order to reformulate (oAIBVP^2) as (c$\mathcal{ACP}^2$), we consider the operator matrices $\mathcal{A}$ and $\mathcal{C}$ introduced in (5.1). The domain of $\mathcal{C}$ will be

$$
D(\mathcal{C}) := \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in D(C) \times D(B_4) : Lu = x \right\}
$$

throughout this section, while the domain of $\mathcal{A}$ is either diagonal or coupled, i.e.,

$$
\left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in D(A) \times D(B_3) : Lu = x \right\}. \tag{5.6}
$$

We show sufficient conditions under which a suitable part of the reduction matrix $\mathbb{A}$ defined in (5.2) is a generator, i.e., under which (c$\mathcal{ACP}^2$) is governed by a C_0-semigroup.

Proposition 5.4.2. *Let $D(A) = D(C)$ and $D(B_3) = D(B_4)$. If $\binom{A}{L}$ is closed and $B_1 \in \mathcal{L}([D(A)]_L, \partial X)$, then the following assertions hold.*

(1) Let $B_2 \in \mathcal{L}(X, \partial X)$ and $B_4 \in \mathcal{L}(\partial X)$. Then $\mathbb{A}$ with domain $D(\mathbb{A}) = D(\mathcal{C}) \times D(\mathcal{C})$ generates a C_0-semigroup on $[D(\mathcal{C})] \times \mathcal{X}$ if and only if C_0 generates a C_0-semigroup on X.

(2) Let $B_2 \in \mathcal{L}(X, \partial X)$. Then $\mathbb{A}$ with domain $D(\mathbb{A}) = D(\mathcal{C}) \times D(\mathcal{C})$ generates an analytic semigroup on $[D(\mathcal{C})] \times \mathcal{X}$ if and only if C_0 and B_4 generate analytic semigroups on X and ∂X, respectively.

(3) Let C_0 and B_4 generate anayltic semigroups on X and ∂X, respectively. If for some $0 < \alpha < 1$ there holds $[D(C)]_L \hookrightarrow [D(C_0), X]_\alpha$ and further $B_2 \in \mathcal{L}([D(C_0)], [D(B_4), \partial X]_\alpha)$, then $\mathbb{A}$ with domain $D(\mathbb{A}) = D(\mathcal{C}) \times D(\mathcal{C})$ generates an analytic semigroup on $[D(\mathcal{C})] \times \mathcal{X}$.

Proof. First of all, observe that by Lemma 1.1.2.(1) the operator matrix $\mathcal{A}$ on $\mathcal{X}$ defined in (5.1) with coupled domain as in (5.6) is closed. Moreover, by assumption $D(\mathcal{A}) = D(\mathcal{C})$. If we can prove that $\mathcal{C}$ is a generator, then the claim follows by Lemma B.31 and the closed graph theorem.

Under the assumptions of (1) (resp., of (2), of (3)), the operator matrix $\mathcal{C}$ on $\mathcal{X}$ generates a C_0-semigroup by Theorem 2.2.8.(1) and Remark 2.2.11 (resp, generates an analytic semigroup by Theorem 2.2.8.(2), generates an analytic semigroup by Theorem 2.2.8.(3)). □

Remark 5.4.3. Look back at the original problem and observe we obtain automatically a regularity result for solutions to (oAIBVP$_2$), if Lemma 5.4.2 applies. More precisely, the part of $\mathbb{A}$ in $D(\mathcal{C}) \times D(\mathcal{C})$ is not exactly associated to (oAIBVP$_2$), but rather to (oAIBVP$_2$) complemented with the extra compatibility condition

$$x(t) = Lu(t), \qquad t \geq 0,$$

(i.e., to (mAIBVP$_2$) as in Section 5.3).

In the following two propositions we show that we can also permit a more unbounded damping operator C, provided it generates a cosine operator function. Recall that a generator of a cosine operator function also generates an analytic semigroup of angle $\frac{\pi}{2}$.

Proposition 5.4.4. *Consider a Banach space ∂Y such that $D(B_4) \subset \partial Y \hookrightarrow \partial X$. Let either $B_2 \in \mathcal{L}([D(C_0)], \partial Y)$ or $B_2 \in \mathcal{L}(Y, \partial X)$. If further*

$$A \in \mathcal{L}(Y, X), \qquad B_1 \in \mathcal{L}(Y, \partial X), \qquad \text{and} \quad B_3 \in \mathcal{L}(\partial Y, \partial X), \tag{5.7}$$

then $\mathbb{A}$ with domain $D(\mathbb{A}) = (Y \times \partial Y) \times D(\mathcal{C})$ generates a cosine operator function on

$$(Y \times \partial Y) \times (X \times \partial X)$$

if and only if C_0 and B_4 generate cosine operator functions with associated phase space $Y \times X$ and $\partial Y \times \partial X$, respectively.

Proof. By Lemma 4.2.2 the operator matrix $\mathcal{C}$ generates a cosine operator function with associated phase space $(Y \times \partial Y) \times (X \times \partial X)$ if and only if C_0 and B_4 generate cosine operator functions with associated phase space $Y \times X$ and $\partial Y \times \partial X$, respectively. Moreover, by assumption $\mathcal{A}$ is bounded from $Y \times \partial Y$ to $X \times \partial X$, hence the claim follows by Lemma B.32. □

Proposition 5.4.5. *Consider a Banach space V such that $V \hookrightarrow Y$. Assume that L can be extended to Y and $\ker(L) = V$. Let finally A, B_1, B_3 satisfy (5.7), and $B_4 \in \mathcal{L}(\partial X)$. Then $\mathbb{A}$ with domain*

$$D(\mathbb{A}) = \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in Y \times \partial X : Lu = x \right\} \times D(\mathcal{C})$$

generates a cosine operator function on

$$\left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in Y \times \partial X : Lu = x \right\} \times (X \times \partial X)$$

if and only if $C_0 - D_\lambda^{C,L} B_2$ *generates a cosine operator function with associated phase space* $V \times X$ *for some* $\lambda \in \rho(C_0)$ *if and only if* $C_0 - D_\lambda^{C,L} B_2$ *generates a cosine operator function with associated phase space* $V \times X$ *for all* $\lambda \in \rho(C_0)$, *where* C_0 *is the restriction of* C *to* $D(C) \cap \ker(L)$.

Proof. By Theorem 4.3.4 the operator matrix $\mathcal{C}$ generates a cosine operator function with associated phase space

$$\left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in Y \times \partial X : Lu = x \right\} \times (X \times \partial X)$$

if and only if $C_0 - D_\lambda^{C,L} B_2$ generates a cosine operator function with associated phase space $V \times X$ for some $\lambda \in \rho(C_0)$ if and only if $C_0 - D_\lambda^{C,L} B_2$ generates a cosine operator function with associated phase space $V \times X$ for all $\lambda \in \rho(C_0)$. Moreover, by assumption $\mathcal{A}$ is bounded from $Y \times \partial X$ to $X \times \partial X$, hence the claim follows by Lemma B.32. □

Remark 5.4.6. As in Remark 5.4.3, we notice that under the assumptions of Proposition 5.4.5 the semigroup generated by the part of $\mathbb{A}$ yields a solution to (oAIBVP$_2$) that moreover satisfies the additional condition

$$x(t) = Lu(t), \qquad t \geq 0.$$

Appendix A

Basic results on semigroups of linear operators

Throughout this Appendix, A will be an operator on a Banach space X. Consider the the abstract Cauchy problem

$$(\mathrm{ACP}_f) \qquad \begin{cases} \dot{u}(t) = Au(t), & t \geq 0, \\ u(0) = f \in X. \end{cases}$$

(We can also replace $t \geq 0$ by $t \in \mathbb{R}$.) If it is clear from the context which initial data f we are considering, we will simply write (ACP) instead of ($\mathrm{ACP}_{f,g}$).

In the context of the theory of C_0-semigroups the following is standard, cf. [EN00, § II.6].

Definition A.1. *A* classical solution to (ACP_f) *is a function* $u(\cdot)$ *such that*

- $u(\cdot) \in C^1(\mathbb{R}_+, X)$,
- $u(t) \in D(A)$ *for all* $t \geq 0$, *and*
- (ACP_f) *is satisfied.*

The problem (ACP_f) *is called* well-posed *(in the Banach space X) if*

- $D(A)$ *is dense in* X,
- (ACP_f) *admits a unique classical solution* $u(\cdot, f)$ *for all* $f \in D(A)$, *and*
- *for every sequence of initial data* $(f_n)_{n\in\mathbb{N}} \subset D(A)$ *tending to* 0 *there holds* $\lim_{n\to\infty} u(t, f_n) = 0$ *uniformly for* t *in compact intervals.*

We can also relax the notion of solution and introduce the following.

Definition A.2. *A* mild solution to (ACP_f) *is a function* $u(\cdot)$ *such that*

- $u(\cdot) \in C(\mathbb{R}_+, X)$,
- $\int_0^t u(s)ds \in D(A)$ *for all* $t \geq 0$, *and*

- *$u(\cdot)$ satisfies the integrated problem*

$$u(t) = f + A\int_0^t u(s)ds, \qquad t \geq 0.$$

The following result is well-known, cf. [EN00, Prop. II.6.2, Prop. II.6.4, and Cor. II.6.9] and [ABHN01, Thm. 3.1.12].

Lemma A.3. *Let A be a closed operator on a Banach space X. Then the following are equivalent.*

(i) The operator A generates a C_0-semigroup on X.

(ii) The problem (ACP_f) is well-posed.

(iii) The problem (ACP_f) admits a unique mild solution for all $f \in X$.

If (i) holds, then the unique classical (resp., mild) solution to (ACP_f) is given by

$$u(t) := e^{tA}f, \qquad t \geq 0,$$

for all $f \in D(A)$ (resp., $f \in X$).

Lemma A.4. *Let X, Y be Banach space, and let U be an isomorphism from X onto Y. Then an operator A on X generates a C_0-semigroup on X if and only if UAU^{-1} generates a C_0-semigroup on Y, and in this case*

$$Ue^{tA}U^{-1} = e^{tUAU^{-1}}, \qquad t \geq 0.$$

Lemma A.5. *Let A generate a C_0-semigroup on a Banach space X. Then e^{tA} maps $D(A^n)$ into itself for all $t \geq 0$ and all $n \in \mathbb{N}$. If it is immediately differentiable, then in fact e^{tA} maps X into $D^\infty(A)$ for all $t > 0$.*

Here and in the following, for a sectorial operator A on X we denote by $[D(A), X]_\alpha$, $0 < \alpha < 1$, the associated (complex) interpolation space, cf. [Lu95, Chapt. 1] for the abstract theory and [LM72, Vol. I, Chapt. 1] for concrete spaces.

The following perturbation results are well-known. We refer to [EN00, Chapt. 3] for more results in this field.

Lemma A.6. *Let A generate a C_0-semigroup. Then the following assertions hold.*

(1) If $B \in \mathcal{L}(X)$, then $A + B$ generates a C_0-semigroup as well. Such a semigroup is analytic if and only if the semigroup generated by A is analytic. Moreover, the estimate

$$\|e^{tA} - e^{t(A+B)}\| \leq tM, \qquad 0 \leq t \leq 1,$$

holds, for some $M > 0$.

(2) If $B \in \mathcal{L}([D(A)])$, then $A + B$ generates a C_0-semigroup as well. Such a semigroup is analytic if and only if the semigroup generated by A is analytic.

(3) Let the semigroup generated by A be analytic. If $B \in \mathcal{L}([D(A)], [D(A), X]_\alpha)$, $0 < \alpha < 1$, then $A + B$ generates an analytic semigroup as well.

While considering (incomplete) second order problems, a key asymptotical notion is that of (asymptotical) almost periodicity. We recall this concept, cf. [ABHN01, § 4.5, § 5.4, and references therein].

Definition A.7. *Let an operator A generate a bounded C_0-semigroup on a Banach space X. Then $(e^{tA})_{t\geq 0}$ is called* asymptotically almost periodic *if $X = X_0 \oplus X_{ap}$, where*

$$\begin{aligned} X_0 &:= \left\{x \in X : \lim_{t\to\infty} \|e^{tA}x\| = 0\right\} \qquad \textit{and} \\ X_{ap} &:= \overline{\operatorname{span}}\, \{x \in D(A) : Ax = i\eta x \textit{ for some } \eta \in \mathbb{R}\}. \end{aligned}$$

If A generates a bounded C_0-group, then $(e^{tA})_{t\in\mathbb{R}}$ is called almost periodic *if $X = X_{\mathrm{ap}}$ with X_{ap} defined as above.*

The following holds by [ABHN01, Rem. 4.5.13 and Prop. 5.4.7] and [EN00, Thm. IV.2.26].

Lemma A.8. *Let A be an operator on a Banach space X. If the embedding $[D(A)] \hookrightarrow X$ is compact, then the following assertions hold.*

(1) If A generates a bounded C_0-semigroup, then it is asymptotically almost periodic. Such a C_0-semigroup is periodic if and only if $X_{\mathrm{ap}} = X$ and further

$$P\sigma(A) \subset 2\pi i\alpha\mathbb{Z} \quad \textit{for some } \alpha > 0. \tag{A.1}$$

(2) If A generates a bounded C_0-group, then it is almost periodic. Such a C_0-group is periodic if and only if (A.1) *holds.*

Appendix B

Basic results on cosine operator functions and complete second order problems

Definition B.1. *Let X be a Banach space. A strongly continuous function $C : \mathbb{R} \to \mathcal{L}(X)$ is called a* cosine operator function *if it satisfies the D'Alembert functional relations*

$$\begin{cases} C(t+s) + C(t-s) &= 2C(t)C(s), \qquad t, s \in \mathbb{R}, \\ C(0) &= I_X. \end{cases}$$

As for the case of C_0-semigroups, it is possible to associate to any cosine operator function a unique generator.

Definition B.2. *Consider a cosine operator function $(C(t))_{t\in\mathbb{R}}$ on a Banach space X. Then we call*

$$Ax := \lim_{t\to 0} \frac{2}{t^2}(C(t)x - x), \qquad D(A) := \left\{ x \in X : \lim_{t\to 0} \frac{2}{t^2}(C(t)x - x) \text{ exists} \right\},$$

the generator of $(C(t))_{t\in\mathbb{R}}$, *and denote $C(t) = C(t, A)$, $t \in \mathbb{R}$.*

A real characterisation of generators of cosine operator functions is due to M. Sova, in analogy to the theorem of Hille–Yosida. Moreover, H.O. Fattorini has proven that on UMD-spaces (i.e., Banach spaces on which the Hilbert transform is bounded) generators of cosine operator functions are not far from being squares of C_0-group generators. Summing up, we can state the following, cf. [ABHN01, Thm. 3.15.3 and Cor. 3.16.8].

Proposition B.3. *Let A be a closed, densely defined operator on a Banach space X. Then the following are equivalent.*

(i) A generates a cosine operator function.

(ii) There exist $M \geq 1$ and $\omega \geq 0$ such that $(\omega^2, \infty) \subset \rho(A)$ and

$$\text{(B.1)} \quad \left\| (\lambda - \omega)^{k+1} \frac{d^k}{d\lambda^k} \lambda R(\lambda^2, A) \right\| \leq M k! \qquad \text{for } \lambda > \omega,\ k = 0, 1, 2, \ldots .$$

If (i) and (ii) hold, then

$$\text{(B.2)} \qquad \|C(t, A)\| \leq M e^{\omega |t|}, \qquad t \in \mathbb{R}.$$

If further X is a UMD*-space, then (i)–(ii) are also equivalent to the following.*

(iii) There exists a generator B of a C_0-group on X such that $A = B^2 + \omega'$, for some $\omega' \geq 0$.

(If $\omega = 0$, then we can take $\omega' = 0$.)

Remarks B.4. (a) Observe that if condition (ii) in the above proposition holds with $\omega = 0$, then in particular it follows from (B.1) with $k = 0$ that

$$\left\| \lambda^2 R(\lambda^2, A) \right\| \leq M \qquad \text{for all } \lambda > 0.$$

In other words, a necessary condition for A to generate a bounded cosine operator function is that A be sectorial and moreover that A generate a bounded C_0-semigroup (and in this case the bounds agree).

This condition is not sufficient, since there are examples of generators of cosine operator functions A such that $(C(t, A + \omega))_{t \in \mathbb{R}}$ is not bounded for any $\omega \in \mathbb{R}$, cf. [Go80, § 3].

(b) As an application of Proposition B.3 and the spectral theorem, one obtains that every self-adjoint, dissipative (resp., upper bounded) operator on a Hilbert space generates a contractive (resp., quasi-contractive, i.e., $M = 0$ in (B.2)) cosine operator function of self-adjoint operators, cf. [ABHN01, Exa. 3.14.16].

To every cosine operator function $(C(t, A))_{t \in \mathbb{R}}$ is associated another strongly continuous family of operators.

Definition B.5. *Let $(C(t, A))_{t \in \mathbb{R}}$ be a cosine operator function on a Banach space X. Then we define the associated* sine operator function *$(S(t, A))_{t \in \mathbb{R}}$ by*

$$S(t, A)x := \int_0^t C(s, A) x ds, \qquad t \in \mathbb{R},\ x \in X.$$

Remark B.6. If Proposition B.3.(iii) applies, then $S(\cdot, A) \in C(\mathbb{R}, \mathcal{L}(X, [D(B)]))$, and $D(B)$ agrees with the space of strong differentiability of $(C(t, A))_{t \in \mathbb{R}}$, cf. [Go69, Thm. 2.1]. Moreover,

$$e^{tB} = C(t, A) + BS(t, A), \qquad t \in \mathbb{R}.$$

In the following we collect some basic properties of cosine and sine operator functions, cf. [So66, § 2], [Ki72], [TW77], [Lu82], and [Ba85] (see also [Fa85, Chapt. II and Chapt. V], [Go85, § 8], and [ABHN01, § 3.14-15]).

Lemma B.7. *Let A generate a cosine operator function $(C(t,A))_{t\in\mathbb{R}}$ with associated sine operator function $(S(t,A))_{t\in\mathbb{R}}$ on a Banach space X. Then the following properties hold.*

(1) A generates an analytic semigroup of angle $\frac{\pi}{2}$. Such a semigroup is bounded if also $(C(t,A))_{t\in\mathbb{R}}$ is bounded.

(2) The functions $C(\cdot,A)x:\mathbb{R}\to X$ and $S(\cdot,A)x:\mathbb{R}\to X$ are even and odd, respectively, for all $x\in X$.

(3) $C(\cdot,A)x\in C^2(\mathbb{R},X)$ for all $x\in D(A)$, and one has

$$\begin{array}{rcl} \frac{d}{dt}C(t,A)x &=& AS(t,A)x=S(t,A)Ax, \\ \frac{d^2}{dt^2}C(t,A)x &=& AC(t,A)x=C(t,A)Ax, \end{array} \qquad t\in\mathbb{R},\ x\in D(A).$$

(4) $S(t,A)$ (resp., $C(t,A)$) is a compact operator for t in some interval of non-zero length – or equivalently for all $t\in\mathbb{R}$ – if and only if A has compact resolvent (resp., if and only if $\dim X<\infty$).

(5) Let $x\in X$. If $(C(t,A))_{t\in\mathbb{R}}$ is bounded and $\lim_{t\to\infty}C(t,A)x=0$, then $x=0$.

(6) Assume $(C(t,A))_{t\in\mathbb{R}}$ to be bounded. Then $\sigma(A)\subset(-\infty,0]$ and moreover A is invertible if and only if $(S(t,A))_{t\in\mathbb{R}}$ is bounded.

(7) For $\lambda\in\mathbb{R}$ large enough there holds

$$\lambda R(\lambda^2,A)=\int_0^\infty e^{-\lambda t}C(t,A)dt.$$

The following observation seems to be new.

Lemma B.8. *A necessary condition for an invertible operator A to generate a bounded cosine operator function on a* UMD*-space is that $-A$ has a bounded $H^\infty(\Sigma_\phi)$-calculus for all $\phi>0$.*

Proof. If $(C(t,A))_{t\in\mathbb{R}}$ is bounded, then by Proposition B.3 there exists a generator B of a C_0-group such that $A=B^2$. By Lemma B.7.(6) and Remark B.6 such a C_0-group is bounded. Hence, by [HP98, Cor. 4] $-A$ has bounded $H^\infty(\Sigma_\phi)$-calculus for all $\phi>0$. □

Lemma B.7.(3) shows that a cosine operator function $(C(t,A))_{t\in\mathbb{R}}$ is a natural candidate for the solution to the second order Cauchy problem

$$(\mathrm{ACP}^2_{f,g}) \qquad \begin{cases} \ddot{u}(t)=Au(t), & t\in\mathbb{R}, \\ u(0)=f\in X, & \dot{u}(0)=g\in X, \end{cases}$$

for $g=0$. Before making this intuition precise we need the following.

Definition B.9. *Let V a Banach space such that $[D(A)]\hookrightarrow V\hookrightarrow X$. A* classical solution to (ACP^2) in (V,X) *is a function $u(\cdot)$ such that*

- $u(\cdot) \in C^2(\mathbb{R}, X) \cap C^1(\mathbb{R}, V)$,
- $u(t) \in D(A)$ *for all* $t \in \mathbb{R}$, *and*
- $(\mathrm{ACP}^2_{f,g})$ *is satisfied.*

The problem (ACP^2) *is called* well-posed in (V, X) *if*

- *The embeddings* $[D(A)] \hookrightarrow V \hookrightarrow X$ *are both dense,*
- $(\mathrm{ACP}^2_{f,g})$ *admits a unique classical solution* $u = u(\cdot, f, g)$ *in* (V, X) *for all* $f \in D(A)$, $g \in V$, *and*
- *for every sequence of initial data* $(f_n, g_n)_{n \in \mathbb{N}} \subset D(A) \times V$ *tending to* 0 *there holds* $\lim_{n\to\infty} u(t, f_n, g_n) = 0$ *uniformly for* t *in compact intervals.*

We can also relax the notion of solution and introduce the following, cf. [ABHN01, § 3.14].

Definition B.10. *Let* V *a Banach space such that* $[D(A)] \hookrightarrow V \hookrightarrow X$. *A* mild solution to $(\mathrm{ACP}^2_{f,g})$ in (V, X) *is a function* $u(\cdot)$ *such that*

- $u(\cdot) \in C^1(\mathbb{R}, X) \cap C(\mathbb{R}, V)$,
- $\int_0^t \int_0^s u(r)drds = \int_0^t (t-s)u(s)ds \in D(A)$ *for all* $t \in \mathbb{R}$,
- $u(\cdot)$ *satisfies the integrated problem*

$$u(t) = f + tg + A \int_0^t (t-s)u(s)ds, \qquad t \in \mathbb{R}.$$

Observe that, for any given Banach space V such that $[D(A)] \hookrightarrow V \hookrightarrow X$, $(\mathrm{ACP}^2_{f,g})$ can be formally reduced to a first order abstract Cauchy problem

$$(\mathbf{ACP_f}) \qquad \begin{cases} \dot{\mathbf{u}}(t) = \mathbf{A}\mathbf{u}(t), & t \in \mathbb{R}, \\ \mathbf{u}(0) = \mathbf{f} \in \mathbf{X}, \end{cases}$$

on the product Banach space $\mathbf{X} := V \times X$, where $\mathbf{A}$ is the operator matrix

$$\text{(B.3)} \qquad \mathbf{A} := \begin{pmatrix} 0 & I_V \\ A & 0 \end{pmatrix}, \qquad D(\mathbf{A}) := D(A) \times V,$$

on $\mathbf{X}$. Here

$$\mathbf{u}(t) := \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}, \qquad t \in \mathbb{R}, \qquad \text{and} \quad \mathbf{f} := \begin{pmatrix} f \\ g \end{pmatrix}.$$

The intuitive equivalence of (ACP^2) and $(\mathbf{ACP})$ is precised in the following.

Theorem B.11. *Let* A *be closed. For a Banach space* V *such that* $[D(A)] \hookrightarrow V \hookrightarrow X$ *the following are equivalent.*

(i) The problem (ACP^2) *is well-posed in* (V, X).

(ii) *The problem* $(\mathrm{ACP}^2_{f,g})$ *admits a unique mild solution in* (V,X) *for all* $f \in V$, $g \in X$.

(iii) *The operator* A *generates a cosine operator function on* X *and*

$$V = \left\{x \in X : C(\cdot, A)x \in C^1(\mathbb{R}, X)\right\}.$$

(iv) *The operator matrix* $\mathbf{A}$ *generates a* C_0*-group in* $\mathbf{X}$.

(v) *The operator matrix* $\mathbf{A}$ *generates a* C_0*-semigroup in* $\mathbf{X}$.

If (iii) and (iv) hold, then

$$e^{t\mathbf{A}} = \begin{pmatrix} C(t,A) & S(t,A) \\ AS(t,A) & C(t,A) \end{pmatrix}, \qquad t \in \mathbb{R}. \tag{B.4}$$

Accordingly, the unique classical (resp., mild) solution to $(\mathrm{ACP}^2_{f,g})$ *is given by*

$$u(t) := C(t,A)f + S(t,A)g, \qquad t \in \mathbb{R},$$

for all $f \in D(A)$, $g \in V$ *(resp.,* $f \in V$, $g \in X$*).*

Proof. Take initial data f, g. To begin with, one can check directly that $\mathbf{u}(\cdot)$ is a classical (resp., mild) solution to $(\mathbf{ACP}_{\binom{f}{g}})$ if and only if $\binom{u(\cdot)}{\dot u(\cdot)}$ is a classical solution (resp., $\binom{u(\cdot)}{g + A\int_0^\cdot u(s)ds}$ is a mild solution) to $(\mathrm{ACP}^2_{f,g})$. It follows that (ACP^2) is well-posed in (V,X) if and only if $(\mathbf{ACP})$ is well-posed in the Banach space $\mathbf{X}$, and moreover that $(\mathrm{ACP}^2_{f,g})$ admits a unique mild solution in (V,X) for all $f \in V$, $g \in X$ if and only if $(\mathbf{ACP}_{\binom{f}{g}})$ admits a unique mild solution in $\mathbf{X}$ for all $\in V$, $g \in X$.

Since the operator matrix $\mathbf{A}$ is closed if (and only if) A is closed, by Lemma A.3 the conditions (i) and (ii) are both equivalent to saying that the operator matrix $\mathbf{A}$ generates a C_0-group in $\mathbf{X}$. Thus, the equivalence of (i), (ii), and (iv) is proven.

To see that (v) implies (iv), observe that the reduction matrix $\mathbf{A}$ is similar to $-\mathbf{A}$ via

$$\begin{pmatrix} I_V & 0 \\ 0 & -I_X \end{pmatrix},$$

which is an isomorphism on $\mathbf{X}$.

Finally, the equivalence of (iii) and (iv) is a celebrated result due to Kisyński, cf. [Ki72, § 2], as well as the formula (B.4). □

Remark B.12. Observe that $(e^{t\mathbf{A}})_{t\in\mathbb{R}}$ is bounded if and only if $(C(t,A))_{t\in\mathbb{R}}$ and $(S(t,A))_{t\in\mathbb{R}}$ are bounded. Also, $(e^{t\mathbf{A}})_{t\in\mathbb{R}}$ is periodic if and only if $(C(t,A))_{t\in\mathbb{R}}$ and $(S(t,A))_{t\in\mathbb{R}}$ are periodic (and in this case the periods coincide).

Take into account Lemma B.7.(7) and the inversion formula for the Laplace transform: Then, it follows from the equivalence of (ii) and (iii) in Proposition B.11 that if (ACP^2) is well-posed in (V_1, X) as well as in (V_2, X), then the spaces V_1 and V_2 coincide. Such a space V, which is unique if it exists, deserves a name.

Definition B.13. *Let A generate a cosine operator function. The (unique) space V introduced in Proposition B.11 is called* Kisyński space associated to $(C(t,A))_{t\in\mathbb{R}}$ *(or* to A*). The product space $\mathbf{X} = V \times X$ is called* phase space associated to $(C(t,A))_{t\in\mathbb{R}}$.

Theorem B.11 shows that the notion of phase space is the key to investigate several properties of cosine operator function, using techniques that are already developed in the context of C_0-groups.

Lemma B.14. *Let V_1, V_2, X_1, X_2 be Banach spaces with $V_1 \hookrightarrow X_1$ and $V_2 \hookrightarrow X_2$, and let U be an isomorphism from V_1 onto V_2 and from X_1 onto X_2. Then an operator A generates a cosine operator function with associated phase space $V_1 \times X_1$ if and only if UAU^{-1} generates a cosine operator function with associated phase space $V_2 \times X_2$. In this case, there holds*

$$UC(t,A)U^{-1} = C(t, UAU^{-1}), \qquad t \in \mathbb{R}.$$

Proof. The operator matrix

$$\mathbf{U} := \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$$

is an isomorphism from $V_1 \times X_1$ onto $V_2 \times X_2$. Thus, by Lemma A.4 and Lemma B.11 it follows that $\mathbf{A}$ defined as in (B.3) (with $V := V_1$) generates a C_0-group on $V_1 \times X_1$ if and only if $\mathbf{UAU}^{-1}$ generates a C_0-group on $V_2 \times X_2$. Now

$$\mathbf{UAU}^{-1} = \begin{pmatrix} 0 & I_{V_1} \\ UAU^{-1} & 0 \end{pmatrix},$$

hence $\mathbf{UAU}^{-1}$ generates a C_0-group on $V_2 \times X_2$ if and only if UAU^{-1} generates a cosine operator function with associated phase space $V_2 \times X_2$. ☐

Phase spaces also allow to obtain the following perturbation result.

Lemma B.15. *Let A generate a cosine operator function with associated phase space $V \times X$. Then $A + B$ generates a cosine operator function with associated phase space $V \times X$ as well, provided B is an operator that is either bounded from $[D(A)]$ to V, or bounded from V to X.*

Proof. The operator matrix $\mathbf{A}$ defined in (B.3) generates a C_0-group on $\mathbf{X} = V \times X$. Consider its perturbation

$$\mathbf{B} := \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}.$$

Since $[D(\mathbf{A})] = [D(A)] \times V$, then by assumption either $\mathbf{B} \in \mathcal{L}([D(\mathbf{A})])$, or $\mathbf{B} \in \mathcal{L}(\mathbf{X})$. Thus, by Lemma A.6.(1),(2) also their sum

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 0 & I_V \\ A + B & 0 \end{pmatrix}$$

generates a C_0-group on $\mathbf{X}$, that is, $A + B$ generates a cosine operator function with associated phase space $\mathbf{X}$. ☐

The following is a direct consequence of Lemma B.15 and Lemma A.6.(1).

Remark B.16. Let A generate a cosine operator function with associated phase space $V \times X$. If $B \in \mathcal{L}(V,X)$, then the estimate

$$\|C(t,A) - C(t,A+B)\| \leq tM, \qquad 0 \leq t \leq 1,$$

holds, for some $M > 0$.

Even when it is known that a given operator A generates a cosine operator function, it is usually impossible to write down an explicit formula for $(C(t,A))_{t\in\mathbb{R}}$, and consequently to compute its space of strong differentiability, i.e., its Kisyński space. Things look better if we work on a UMD-space.

Corollary B.17. *Let A be the generator of a cosine operator function on a* UMD*-space X. If $(C(t,A-\omega))_{t\in\mathbb{R}}$ is bounded for some $\omega \in \rho(A)$, then the associated Kisyński space is isomorphic to $[D(A),X]_{\frac{1}{2}}$.*

Proof. Lemma B.15 ensures that $A - \omega$ generates a cosine operator function as well. By Remark B.4.(a) and Lemma B.8, the operator $\omega - A$ is invertible and sectorial and has bounded H^∞-calculus. In particular, $\omega - A$ has bounded imaginary powers. It follows by [Tr78, § 1.15.3] that $[D((\omega - A)^\alpha)]$ is isomorphic to $[D(A-\omega),X]_{1-\alpha} = [D(A),X]_{1-\alpha}$ for $0 < \alpha < 1$. In particular, $[D(B)]$ is isomorphic to $[D(A),X]_{\frac{1}{2}}$, where B is the square root of $\omega - A$ introduced in the proof of Lemma B.8. On the other hand, by Remark B.6 the Kisyński space associated to $(C(t,A-\omega))_{t\in\mathbb{R}}$ is $[D(B)]$. The proof is concluded by observing that by Lemma B.15 the Kisyński space associated to $(C(t,A-\omega))_{t\in\mathbb{R}}$ agrees with the Kisyński space associated to $(C(t,A))_{t\in\mathbb{R}}$. ☐

Remark B.18. A corollary of Remark B.4.(b) and Corollary B.17 we derive the following known result, cf. [GW03, Prop. 2.1]: Every self-adjoint, strictly negative definite operator A on a Hilbert space H generates a (contractive) cosine operator function (with associated (bounded) sine operator function, cf. [Fa85, § V.6]) whose associated Kisyński space is isomorphic to $[D(A),H]_{\frac{1}{2}}$.

With completely different methods the following has been proven by Rhandi ([Rh92, Thm. 1.2]).

Lemma B.19. *Let A generate a cosine operator function. If B is an operator such that $D(A) \subset D(B)$ and moreover $t_0 > 0$ and $q < 1$ can be chosen such that*

$$\int_0^{t_0} \|BS(s,A)f\| ds \leq q\|f\| \qquad \text{for all } f \in D(A),$$

then also $A + B$ generates a cosine operator function, and the associated phase spaces coincide.

Regularity can also be investigated by means of phase spaces.

Lemma B.20. *Let A be an operator on a Banach space X, V a Banach space such that $[D(A)] \hookrightarrow V \hookrightarrow X$. For the operator $\mathbf{A}$ defined in* (B.3) *there holds*

$$D(\mathbf{A}^{2k-1}) = D(A^k) \times D((A^{k-1})_{|V}) \quad \text{and} \quad D(\mathbf{A}^{2k}) = D((A^k)_{|V}) \times D(A^k)$$

for all $k \in \mathbb{N}$. In particular, $D(\mathbf{A}^\infty) = D^\infty(A) \times D^\infty(A)$.

Proof. The claim can be performed by induction on n, using the fact that

$$D((A^k)_{|Y}) = \{u \in D(A) \, : \, Au \in D((A^{k-1})_{|Y})\}$$

and recalling that $D(A) \subset V$. $\square$

The following is an immediate consequence of Lemma A.5, Proposition B.11, and Lemma B.20.

Corollary B.21. *Let A generate a cosine operator function, and let $f, g \in D(A^k)$. Then the classical solution $u = u(t)$ to* ($\mathrm{ACP}^2_{f,g}$) *belongs to $D(A^k)$ as well, for all $t \in \mathbb{R}$.*

Due to the key role played by the reduction matrix $\mathbf{A}$, it is natural to investigate some of its spectral properties.

Lemma B.22. *Let A be a closed operator on a Banach space X, V a Banach space such that $[D(A)] \hookrightarrow V \hookrightarrow X$. Then the reduction operator matrix $\mathbf{A}$ introduced in* (B.3) *is closed. Moreover, its resolvent set is*

$$\rho(\mathbf{A}) = \left\{\lambda \in \mathbb{C} : \lambda^2 \in \rho(A)\right\},$$

and

$$R(\lambda, \mathbf{A}) = \begin{pmatrix} \lambda R(\lambda^2, A) & R(\lambda^2, A) \\ AR(\lambda^2, A) & \lambda R(\lambda^2, A) \end{pmatrix}, \qquad \lambda \in \rho(\mathbf{A}).$$

If $\rho(A) \neq \emptyset$, then $\mathbf{A}$ has compact resolvent if and only if the embeddings $[D(A)] \hookrightarrow V \hookrightarrow X$ are both compact.

Let us now turn to the asymptotic behavior of solutions to a second order problem. By Proposition B.3 and Lemma B.7.(5), a cosine operator function has growth bound at least 0, and its space of strong stability can only be trivial. Hence, we focus instead on results about (almost) periodicity.

The notion of almost periodicity for a cosine or sine operator function is less obvious than that for a C_0-group, cf. [ABHN01, Def. 4.5.6 and Thm. 4.5.7].

Definition B.23. *Let $(C(t))_{t\in\mathbb{R}}$ be a bounded cosine operator function on a Banach space X. Then $(C(t))_{t\in\mathbb{R}}$ is called* almost periodic *if for every $x \in X$ and every $\epsilon > 0$ there exists a lenght $l > 0$ such that*

$$\{t > 0 : \|C(t+s)x - C(s)x\| \leq \epsilon \text{ for all } s \in \mathbb{R}\} \cap [a, a+l] \neq \emptyset$$

for all $a \in \mathbb{R}$.

We can likewise define then notion of almost periodicity for a bounded sine operator function.

Remark B.24. The two notions of almost periodicity (for C_0-groups and for cosine/sine operator functions) are consistent: that is, if the group $(e^{t\mathbf{A}})_{t\in\mathbb{R}}$ defined in (B.4) is almost periodic in the sense of Definition A.7, then its operator entries are almost periodic in the sense of Definition B.23, cf. [ABHN01, § 4.5].

Lemma B.25. *Let A be the invertible generator of a bounded cosine operator function with associated phase space $V \times X$. If the embeddings $[D(A)] \hookrightarrow V \hookrightarrow X$ are both compact, then the following assertions hold.*

(1) $(C(t,A))_{t\in\mathbb{R}}$ and $(S(t,A))_{t\in\mathbb{R}}$ are almost periodic.

(2) $(C(t,A))_{t\in\mathbb{R}}$ and $(S(t,A))_{t\in\mathbb{R}}$ are periodic (with same period) if and only if $P\sigma(A) \subset -4\pi^2\alpha^2\mathbb{N}^2$ for some $\alpha > 0$.

Proof. The key point of the proof is that by Lemma B.7.(6), Remark B.12 and Lemma B.22 the reduction matrix $\mathbf{A}$ has compact resolvent and generates a bounded C_0-group $(e^{t\mathbf{A}})_{t\in\mathbb{R}}$ on $\mathbf{X} = V \times X$, hence by Lemma A.8.(2) $(e^{t\mathbf{A}})_{t\in\mathbb{R}}$ is almost periodic. The almost periodicity of $(C(t,A))_{t\in\mathbb{R}}$ and $(S(t,A))_{t\in\mathbb{R}}$ then follows by Remark B.24.

To prove (2), observe that by Lemma B.22 one obtains $P\sigma(\mathbf{A}) \subset 2\pi i\alpha\mathbb{Z}$. Hence, the periodicity of $(e^{t\mathbf{A}})_{t\in\mathbb{R}}$ follows by Lemma A.8.(2). □

Remarks B.26. (a) Let A be the invertible generator of a bounded cosine operator function on a Banach space X. Then the conclusion of Lemma B.25.(1) still holds if we only assume $[D(A)] \hookrightarrow X$ to be compact, and moreover X not to contain c_0, cf. [ABHN01, Rem. 5.7.6].

(b) Let A generate a cosine operator function. If $(S(t,A))_{t\in\mathbb{R}}$ is almost periodic, then also $(C(t,A))_{t\in\mathbb{R}}$ is almost periodic, cf. [XL98, Thm. 7.1.7].

If no assumption is made on the invertibility of A, one can still investigate the almost periodicity of $(C(t,A))_{t\in\mathbb{R}}$ only. As a consequence of [AB97, Prop. 4.8] we obtain the following.

Lemma B.27. *Let A generate a cosine operator function with associated phase space $V \times X$. Assume $(C(t,A))_{t\in\mathbb{R}}$ to be bounded and the embeddings $[D(A)] \hookrightarrow V \hookrightarrow X$ to be both compact. Then $(C(t,A))_{t\in\mathbb{R}}$ is almost periodic.*

We will need the following, which is analogous to [Na89, Prop. 3.1].

Lemma B.28. *Let A and D be generators of cosine operator functions with associated phase space $V \times X$ and $W \times Y$, respectively. Consider an operator H that is bounded from $[D(D)]$ to X and the operator matrix*

$$\mathcal{A} := \begin{pmatrix} A & H \\ 0 & D \end{pmatrix}, \qquad D(\mathcal{A}) := D(A) \times D(D).$$

Then the operator matrix $\mathcal{A}$ generates a cosine operator function with associated phase space $(V \times W) \times (X \times Y)$ if and only if

$$\int_0^t C(t-s,A)HS(s,D)ds, \qquad t \geq 0,$$

can be extended to a family of operators from Y to X which is uniformly bounded as $t \to 0^+$. In this case, there holds

$$C(t,\mathcal{A}) = \begin{pmatrix} C(t,A) & \int_0^t C(t-s,A)HS(s,D)ds \\ 0 & C(t,D) \end{pmatrix}, \qquad t \in \mathbb{R}, \tag{B.5}$$

and the associated sine operator function is

$$\text{(B.6)} \qquad S(t,\mathcal{A}) = \begin{pmatrix} S(t,A) & \int_0^t S(t-s,A)HS(s,D)ds \\ 0 & S(t,D) \end{pmatrix}, \qquad t \in \mathbb{R}$$

(up to considering the extensions from Y to X of the upper-right entries in (B.5) *and* (B.6)*).*

Proof. The operator matrix $\mathcal{A}$ generates a cosine operator function with associated phase space $(V \times W) \times (X \times Y)$ if and only if the reduction matrix

$$\mathbb{A} := \begin{pmatrix} 0 & I_{V\times W} \\ \mathcal{A} & 0 \end{pmatrix}, \qquad D(\mathbb{A}) := (D(A) \times D(D)) \times (V \times W),$$

generates a C_0-semigroup on $(V \times W) \times (X \times Y)$. Define the operator matrix

$$\mathsf{U} := \begin{pmatrix} I_V & 0 & 0 & 0 \\ 0 & 0 & I_X & 0 \\ 0 & I_W & 0 & 0 \\ 0 & 0 & 0 & I_Y \end{pmatrix},$$

which is an isomorphism from $(V \times W) \times (X \times Y)$ onto $(V \times X) \times (W \times Y)$ with inverse

$$\mathsf{U}^{-1} := \begin{pmatrix} I_V & 0 & 0 & 0 \\ 0 & 0 & I_W & 0 \\ 0 & I_X & 0 & 0 \\ 0 & 0 & 0 & I_Y \end{pmatrix}.$$

Then the similar operator matrix $\tilde{\mathbb{A}} := \mathsf{U}\mathbb{A}\mathsf{U}^{-1}$ is given by

$$\tilde{\mathbb{A}} = \begin{pmatrix} \mathbf{A} & \mathbf{H} \\ 0 & \mathbf{D} \end{pmatrix}, \qquad D(\tilde{\mathbb{A}}) := D(\mathbf{A}) \times D(\mathbf{D}).$$

Here $\mathbf{A}$ is defined as in (B.3) and

$$\mathbf{D} := \begin{pmatrix} 0 & I_W \\ D & 0 \end{pmatrix}, \qquad \mathbf{H} := \begin{pmatrix} 0 & 0 \\ H & 0 \end{pmatrix}, \qquad D(\mathbf{D}) : D(\mathbf{H}) := D(D) \times W.$$

By assumption $\mathbf{A}$ and $\mathbf{D}$ generate C_0-semigroups on $V \times X$ and $W \times Y$, respectively. Moreover $\mathbf{H} \in \mathcal{L}([D(\mathbf{D})], V \times X)$, and a direct computation shows that

$$e^{(t-s)\mathbf{A}}\mathbf{H}e^{s\mathbf{D}} = \begin{pmatrix} S(t-s,A)HC(s,D) & S(t-s,A)HS(s,D) \\ C(t-s,A)HS(s,D) & C(t-s,A)HS(s,D) \end{pmatrix}, \qquad 0 \le s \le t.$$

By virtue of [Na89, Prop. 3.1] we obtain that $\tilde{\mathbb{A}}$ generates a C_0-semigroup if and only if the family of operators

$$\int_0^t e^{(t-s)\mathbf{A}}\mathbf{H}e^{s\mathbf{D}}ds, \qquad t \ge 0,$$

from $W\times Y$ to $V\times X$ is uniformly bounded as $t\to 0^+$. Hence, if $\tilde{\mathbb{A}}$ generates a C_0-semigroup, then in particular $\int_0^t C(t-s,A)HS(s,D)ds$ is uniformly bounded as $t\to 0^+$.

Again by [Na89, Prop. 3.1]

$$e^{t\tilde{\mathbb{A}}} = \begin{pmatrix} e^{t\mathbf{A}} & \int_0^t e^{(t-s)\mathbf{A}}\mathbf{H}e^{s\mathbf{D}}ds \\ 0 & e^{t\mathbf{D}} \end{pmatrix}, \qquad t\geq 0.$$

By Lemma A.4 $e^{t\mathbb{A}} = e^{t\mathsf{U}^{-1}\tilde{\mathbb{A}}\mathsf{U}} = \mathsf{U}^{-1}e^{t\tilde{\mathbb{A}}}\mathsf{U}$, $t\geq 0$. Thus, a direct computations shows that the semigroup generated by $\mathbb{A}$ is given by

$$\left(\begin{array}{cc|cc} C(t,A) & \int_0^t S(t-s,A)HC(s,D)ds & S(t,A) & \int_0^t S(t-s,A)HS(s,D)ds \\ 0 & C(t,D) & 0 & S(t,D) \\ \hline AS(t,A) & \int_0^t C(t-s,A)HC(s,D)ds & C(t,A) & \int_0^t C(t-s,A)HS(s,D)ds \\ 0 & DS(t,D) & 0 & C(t,D) \end{array}\right),$$

for $t\geq 0$. Since by assumption $\mathbb{A}$ generates a cosine operator function with associated phase space $(V\times W)\times(X\times Y)$, comparing the above formula with (B.4) yields (B.5) and (B.6).

One can also check directly that the lower-right block-entry defines a cosine operator function on $X\times Y$. Further, integrating by parts one sees that the upper-right and lower-right block-entries can be obtained by integrating the upper-left and lower-left block-entries, respectively, and moreover that the diagonal blocks coincide. Hence, by definition of sine operator function, all the blocks are strongly continuous families as soon as the lower-right is strongly continuous. Consequently, if the family $\int_0^t C(t-s,A)HS(s,D)ds$ is uniformly bounded as $t\to 0^+$, then the family $\int_0^t e^{(t-s)\mathbf{A}}\mathbf{H}e^{s\mathbf{D}}ds$ is uniformly bounded as $t\to 0^+$, and the claim follows. □

The following is the analogous of [Na89, Cor. 3.2].

Corollary B.29. *Let A and D be closed operators, V,X,W,Y be Banach spaces such that $[D(A)]\hookrightarrow V\hookrightarrow X$ and $[D(D)]\hookrightarrow W\hookrightarrow Y$. Assume the operator H to be bounded either from $[D(D)]$ to V, or from W to X, and the operator K to be bounded either from $[D(A)]$ to W, or from V to Y. Then the operator matrix*

$$\mathcal{A} := \begin{pmatrix} A & H \\ K & D \end{pmatrix}, \qquad D(\mathcal{A}) := D(A)\times D(D),$$

generates a cosine operator function with associated phase space $(V\times W)\times(X\times Y)$ if and only if A and D generate cosine operator functions with associated phase space $V\times X$ and $W\times Y$, respectively.

Proof. It follows by Lemma B.28 that the diagonal matrix

$$\mathcal{A}_0 := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \qquad D(\mathcal{A}_0) := D(\mathcal{A}),$$

generates a cosine operator function with associated phase space $(V\times W)\times(X\times Y)$ if (and only if, since $\mathcal{A}_0$ is diagonal) A and D generate cosine operator functions with associated phase space $V \times X$ and $W \times Y$, respectively. Now consider perturbations of $\mathcal{A}_0$ given by the operator matrices

$$\mathcal{H} := \begin{pmatrix} 0 & H \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathcal{K} := \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}.$$

Observe that both $\mathcal{H}$ and $\mathcal{K}$ are, by assumption, either bounded from $[D(A)] \times [D(D)]$ to $V \times W$, or from $V \times W$ to $X \times Y$. By Lemma B.15 also their sum $\mathcal{A}_0 + \mathcal{H} + \mathcal{K} = \mathcal{A}$ generates a cosine operator function with associated phase space $(V \times W) \times (X \times Y)$. □

Introduce now a closed *damping* operator $C : D(C) \subset X \to X$. Whenever we consider a *complete* second order abstract Cauchy problem

$$(\mathrm{cACP}^2) \qquad \begin{cases} \ddot{u}(t) = Au(t) + C\dot{u}(t), & t \geq 0, \\ u(0) = f, \quad \dot{u}(0) = g, \end{cases}$$

the theory becomes different (for example, we cannot in general expect backward solvability) and, as a rule, more complicated. A natural step is to introduce the reduction matrix

$$\tilde{\mathbf{A}} := \begin{pmatrix} 0 & I_{D(C)} \\ A & C \end{pmatrix}, \qquad D(\tilde{\mathbf{A}}) = D(A) \times D(C),$$

and investigate the generator property of (some suitable part of) $\tilde{\mathbf{A}}$.

The following seems to be new. In particular, it shows that the unboundedness (on X) of the damping term C may not prevent backward solvability of (cACP^2).

Lemma B.30. *If C is bounded either on V or on X, then $\tilde{\mathbf{A}}$ generates a C_0-group on $V \times X$ if and only if A generates a cosine operator function with associated phase space $V \times X$.*

Proof. Reason as in the proof of Lemma B.15 and observe that, under our assumptions, the operator matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}$$

is bounded either on $[D(\tilde{\mathbf{A}})] = [D(A)] \times V$, or on the phase space $\mathbf{X} = V \times X$. The claim follows by Lemma A.6.(1). □

In the overdamped case (i.e., C is "more unbounded" than A) the following can be shown with the same proof of [EN00, Cor. VI.3.3].

Lemma B.31. *Let A be bounded from $[D(C)]$ to X. Then $\tilde{\mathbf{A}}$ (with domain $D(C) \times D(C)$) generates a C_0-semigroup (resp., an analytic semigroup) on $[D(C)] \times X$ if and only if C generates a C_0-semigroup (resp., an analytic semigroup) on X.*

We can allow a damping term even "more unbounded", if this is the generator of a cosine operator function.

Lemma B.32. *Let V be a Banach space such that $[D(C)] \hookrightarrow V \hookrightarrow X$ and $A \in \mathcal{L}(V, X)$. Then $\tilde{\mathbf{A}}$ (with domain $V \times D(C)$) generates a cosine operator function on $V \times X$ if and only if C generates a cosine operator function with associated phase space $V \times X$.*

Proof. The proof mimics that of [EN00, Cor. VI.3.3]. Let first C generate a cosine operator function with associated phase space $V \times X$. Then by Corollary B.29 the operator matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \qquad \text{with domain} \qquad V \times D(C)$$

generates a cosine operator function on $V \times X$ with associated Kisyński space $V \times V$. Now observe that

$$\begin{pmatrix} 0 & I_{D(C)} \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$$

are bounded from $V \times [D(C)]$ to $V \times V$ and from $V \times V$ to $V \times X$, respectively. Hence, $\tilde{\mathbf{A}}$ generates a cosine operator function on $V \times X$ by Lemma B.15.

The converse implication can be proven likewise, and the claim follows. □

We emphasize that, if the above lemma applies, then in particular $\tilde{\mathbf{A}}$ generates an analytic semigroup of angle $\frac{\pi}{2}$.

The following are known results: Lemma B.33.(1) is [EN00, Thm. VI.3.14], while (2) follows by [XL98, Thm. 6.4.3 and Thm. 6.4.4].

Lemma B.33. *Let X be a Hilbert space. Then the following assertions hold.*

(1) Consider a densely defined, invertible operator D on X, and assume that $A = -D^*D$. If $D(D) \subset D(C)$ and C is dissipative, then $\tilde{\mathbf{A}}$ with domain $D(\tilde{\mathbf{A}}) = D(A) \times D(C)$ generates a contraction C_0-semigroup on the product space $[D(D)] \times X$.

(2) Assume A, C to be self-adjoint and strictly negative definite. If $D((-A)^{\frac{1}{2}}) \subset D(C)$, then $\tilde{\mathbf{A}}$ with domain $D(\tilde{\mathbf{A}}) = D(A) \times D(C)$ generates a uniformly exponentially stable C_0-semigroup on the product space $[D((-A)^{\frac{1}{2}})] \times X$. If moreover $C = -(-A)^{\frac{1}{2}}$, then such a semigroup is analytic, and also compact if additionally the embeddings $D(A) \hookrightarrow D((-A)^{\frac{1}{2}}) \hookrightarrow X$ are both compact.

Remarks B.34. (a) The theory of complete second order abstract Cauchy problem with an operator C "subordinated" to A has been started by some papers by S.P. Chen and R. Triggiani, cf. [CT88], [CT89], [CT90], and [CT90b]. Their investigations has been further developed by T.-J. Xiao and J. Liang in [XL98, § 6.4], where functional calculus for self-adjoint operators is also applied. To avoid technicalities, in the above lemma we have stuck to their easier results.

(b) We have not discussed the relation between well-posedness of (cACP^2) and generator property of $\tilde{\mathbf{A}}$. This has been thoroughly investigated in [XL98, Chapt. 2] and [EN00, § VI.3].

Appendix C

The Dirichlet operator

Consider a domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. Then one can study the Dirichlet problem

$$\text{(C.1)} \qquad \begin{cases} \Delta u(x) &= 0, \qquad x \in \Omega, \\ u(z) &= f(z), \quad z \in \partial\Omega, \end{cases}$$

for some given function f on $\partial\Omega$. It is well-known that this problem can be solved whenever there exists a Green function G for the domain Ω, and in this case the solution is given by

$$u(x) = \int_{\partial\Omega} f(z) \frac{\partial G}{\partial \nu}(x,z) d\sigma(z), \qquad x \in \Omega,$$

cf. [Ev98, § 2.2.4] (see also [ABHN01, § 6.1] for a nice operator theoretical approach to such problems).

More generally, consider the abstract setting introduced in Section 1.1, and in particular the spaces $X, \partial X$ and the operators A, L. One can now consider the abstract (eigenvalue) Dirichlet problem

$$\text{(ADP)} \qquad \begin{cases} Au &= \lambda u, \\ Lu &= x, \end{cases}$$

where L models some *boundary* operator (e.g., the trace operator or the normal derivative). In the following a key role will be played by the operators $\binom{A}{L}$ and A_0 defined in (1.3) and (1.4).

A sufficient condition for the solvability of (ADP) is expressed in the following result due to Greiner, cf. [Gr87, Lemma 1.2].

Lemma C.1. *Assume A_0 to have nonempty resolvent set. If L is surjective from $D(A)$ to ∂X, then* (ADP) *admits a unique solution $u := D_\lambda^{A,L} x$ for all $x \in \partial X$ and $\lambda \in \rho(A_0)$.*

We have thus introduced a family of (linear, since (ADP) is linear) operators $D_\lambda^{A,L} : \partial X \to X$, defined for $\lambda \in \rho(A_0)$: we call them *Dirichlet operators associated to the pair* (A, L).

It is sometimes useful to know what is the relation between different Dirichlet operators. Lemma C.2.(1) below is due to Greiner, cf. [Gr87, Lemma 1.3].

Lemma C.2. *Under the assumptions of Lemma C.1, the following assertions hold.*

(1) Let $\lambda, \lambda' \in \rho(A_0)$. Then there holds

$$D_\lambda^{A,L} - D_{\lambda'}^{A,L} = (\lambda' - \lambda) R(\lambda, A_0) D_{\lambda'}^{A,L}, \tag{C.2}$$

and further

$$R(\lambda', A_0) D_\lambda^{A,L} = R(\lambda, A_0) D_{\lambda'}^{A,L}.$$

(2) Let $\lambda \in \rho(A_0)$. Let L' be another operator from X to ∂X with $D(A) \subset D(L')$. If the Dirichlet operator $D_\lambda^{A,L'} : \partial X \to X$ exists, then there holds

$$D_\lambda^{A,L} = D_\lambda^{A,L'} L' D_\lambda^{A,L}.$$

(3) Let $\lambda \in \rho(A_0)$. Let A' be another operator on X, such that $D(A') \subset D(A)$. If the Dirichlet operator $D_\lambda^{A',L} : \partial X \to X$ exists, then there holds

$$D_\lambda^{A,L} - D_\lambda^{A',L} = R(\lambda, A_0)(A - \lambda) D_\lambda^{A',L}.$$

Proof. *(1)* Take $\lambda, \lambda' \in \rho(A_0)$. Then

$$(\lambda - \lambda') D_{\lambda'}^{A,L} = \lambda D_{\lambda'}^{A,L} - \lambda D_\lambda^{A,L} + A D_\lambda^{A,L} - A D_{\lambda'}^{A,L} = (\lambda - A_0)(D_{\lambda'}^{A,L} - D_\lambda^{A,L}),$$

where we have used the fact that $D_\lambda^{A,L}$ and $D_{\lambda'}^{A,L}$ are both right inverses of L, hence $(D_{\lambda'}^{A,L} - D_\lambda^{A,L})(\partial X) \subset D(A_0)$. Thus,

$$(\lambda - \lambda') R(\lambda, A_0) D_{\lambda'}^{A,L} = D_{\lambda'}^{A,L} - D_\lambda^{A,L}.$$

Multiplying both sides by $R(\lambda', A_0)$ and applying the resolvent identity we finally show that the claim holds.

(2) Let $x \in \partial X$ and set $u := D_\lambda^{A,L} x$. Then by definition

$$\begin{cases} Au &= \lambda u, \\ Lu &= x, \end{cases} \quad \text{or equivalently} \quad \begin{cases} Au &= \lambda u, \\ L'u &= x + (L' - L)u. \end{cases}$$

Hence, by definition, $u = D_\lambda^{A,L'}\big(x + (L' - L)u\big)$, and there follows

$$D_\lambda^{A,L} x = u = D_\lambda^{A,L'}\big(x + (L' - L) D_\lambda^{A,L} x\big) = D_\lambda^{A,L'} L' D_\lambda^{A,L} x,$$

because $D_\lambda^{A,L}$ is by definition a right inverse of L.

(3) Take $x \in \partial X$. Set $u := D_\lambda^{A,L} x$ and $v := D_\lambda^{A',L} x$. Observe that by assumption $v \in D(A)$, and moreover $u - v \in D(A_0)$. Then

$$\lambda v - Av = \lambda v - \lambda u + Au - Av = (\lambda - A_0)(v - u),$$

whence $u - v = R(\lambda, A_0)(A - \lambda) v$. This yields the claim. □

The solutions to (C.1) depend continuously on the given boundary value. Lemma C.4 below shows that the same holds for abstract Dirichlet problem.

Lemma C.3. *Assume* $\binom{A}{L}$ *to be closed (as an operator from* X *to* $X \times \partial X$*). Let* Z *be a Banach space such that* $Z \hookrightarrow X$*, and consider the operator matrix*

$$\begin{pmatrix} A & 0 \\ L & 0 \end{pmatrix} : D(A) \times \partial X \to X \times \partial X$$

on $X \times \partial X$*. Then its part in* $Z \times \partial X$ *is closed.*

Proof. Denote by $A_|$ the part of A in Z and let

$$\begin{pmatrix} u_n \\ x_n \end{pmatrix}_{n\in\mathbb{N}} \subset D(A_|) \times \partial X, \qquad \lim_{n\to\infty} \begin{pmatrix} u_n \\ x_n \end{pmatrix} = \begin{pmatrix} u \\ x \end{pmatrix} \qquad \text{in } Z \times \partial X,$$

$$\text{and} \quad \lim_{n\to\infty} \begin{pmatrix} A_| & 0 \\ L & 0 \end{pmatrix} \begin{pmatrix} u_n \\ x_n \end{pmatrix} = \lim_{n\to\infty} \begin{pmatrix} Au_n \\ Lu_n \end{pmatrix} = \begin{pmatrix} w \\ y \end{pmatrix} \qquad \text{in } Z \times \partial X,$$

for some $u, w \in Z$, $x, y \in \partial X$. Since $Z \hookrightarrow X$, we can apply the closedness of $\binom{A}{L}$ and conclude that $u \in D(A)$, $Au = w$, and $Lu = y$. ☐

Lemma C.4. *Under the assumptions of Lemma C.1, take* $\lambda \in \rho(A_0)$ *and consider the Dirichlet operator* $D_\lambda^{A,L}$*. If* $\binom{A}{L}$ *is closed (as an operator from* X *to* $X \times \partial X$*), then* $D_\lambda^{A,L}$ *is bounded from* ∂X *to* Z *for every Banach space* Z *satisfying* $D^\infty(A) \subset Z \hookrightarrow X$*. In particular,* $D_\lambda^{A,L} \in \mathcal{L}(\partial X, [D(A)]_L)$*.*

Proof. Observe that $\ker(\lambda - A) \subset D^\infty(A)$. Therefore the boundedness of $D_\lambda^{A,L}$ from ∂X to some Banach space Z containing $D^\infty(A)$ is implied by the closedness of the operator $L_{|\ker(\lambda-A)} : \ker(\lambda - A) \subset Z \to \partial X$ (as an operator from Z to ∂X).

To show that $L_{|\ker(\lambda-A)}$ is actually closed, let

$$(u_n)_{n\in\mathbb{N}} \subset \ker(\lambda - A), \qquad Z - \lim_{n\to\infty} u_n = u$$
$$\text{and} \quad \lim_{n\to\infty} Lu_n = x,$$

where "$Z - \lim$" stands for the limit with respect to the norm of Z. It follows that $Au_n = \lambda u_n \overset{Z}{\to} \lambda u$, that is

$$\lim_{n\to\infty} \begin{pmatrix} A & 0 \\ L & 0 \end{pmatrix}_| \begin{pmatrix} u_n \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda u \\ x \end{pmatrix} \qquad \text{in } Z \times \partial X.$$

By Lemma C.3 we conclude that $u \in D(A)$ and that $Au = \lambda u$, $Lu = x$. ☐

Remarks C.5. (a) Lemma C.2.(1) implies that $(D_\lambda^{A,L})_{\lambda\in\rho(A_0)}$ is a family of compact operators from ∂X to X if and only if $D_{\lambda_0}^{A,L}$ is a compact operator from ∂X to X for some $\lambda_0 \in \rho(A_0)$. In this case we say that the pair (A, L) *has compact Dirichlet operator*.

(b) If ∂X is finite dimensional, or else if a Banach space Z as in the statement of Lemma C.4 can be compactly embedded in X, then we obtain that the pair (A, L) has compact Dirichlet operator.

(c) If one only assumes L to be surjective from $D(A)$ to some subspace V of ∂X, then the Dirichlet operators can still be defined, but only as bounded operators from V to Z for every Banach space Z satisfying $D^\infty(A) \subset Z \hookrightarrow X$ (see [Gr87]).

To conclude this section, we mention two important results about the decay rate of $\|D_\lambda^{A,L}\|$ for $\lambda \in \rho(A_0) \cap \mathbb{R}$.

Lemma C.6. *Under the assuptions of Lemma C.1, let A_0 generate a C_0-semigroup on X. Then the following assertions hold.*

(1) There exists $K > 0$ such that $\|D_\lambda^{A,L}\|_{\mathcal{L}(\partial X, X)} \leq K$ for all $\lambda \in \rho(A_0) \cap \mathbb{R}$.

(2) Let further the semigroup generated by A_0 be analytic. Then $[D(A)]_L$ is continuously embedded in some interpolation space $[D(A_0), X]_\alpha$, $0 < \alpha < 1$, if and only if

$$\lim_{\substack{|\lambda|\to\infty \\ \mathrm{Re}\lambda>0}} |\lambda|^{\beta-\alpha} \|D_\lambda^{A,L}\|_{\mathcal{L}(\partial X, [D(A_0),X]_\beta)} = 0, \qquad 0 \leq \beta < \alpha.$$

Proof. *(1)* Let the semigroup $(e^{tA_0})_{t\geq 0}$ satisfy

$$\|e^{tA_0}\| \leq Me^{\omega t}, \qquad t \geq 0,$$

for some $M \geq 1$ and $\omega \in \mathbb{R}$. Take $\lambda, \lambda' \in (\omega, \infty)$. Then by virtue of (C.2) there holds

$$\|D_\lambda^{A,L}\|_{\mathcal{L}(\partial X, X)} \leq \big(1 + \|(\lambda' - \lambda)R(\lambda, A_0)\|\big) \|D_{\lambda'}^{A,L}\|_{\mathcal{L}(\partial X, X)}.$$

It follows by the theorem of Hille–Yosida that

$$\|D_\lambda^{A,L}\|_{\mathcal{L}(\partial X, X)} \leq (1 + M)\|D_{\lambda'}^{A,L}\|_{\mathcal{L}(\partial X, X)} =: K,$$

and this yields the claim.

(2) This is [GK91, Lemma. 2.4]. □

In Chapter 3 we consider complicated operators L' from X to ∂X that can be looked at as "good" perturbations of a suitable, more usual operator L. Checking their surjectivity might in general be a tough task, but things can be handled more easily if L is surjective and, additionally, the restriction A_0 of A to $\ker(L)$ has good properties.

Corollary C.7. *Under the assumptions of Lemma C.1, let further A_0 generate an analytic semigroup on X. Assume that $[D(A)]_L \hookrightarrow [D(A_0), X]_\alpha$ for some $0 < \alpha < 1$. Then every operator $L' : D(A) \to \partial X$ such that $(L - L') \in \mathcal{L}([D(A_0), X]_\alpha, \partial X)$ is surjective.*

Proof. Under our assumptions, the operator $(L - L')D_\lambda^{A,L}$ is bounded on ∂X for all $\lambda \in \rho(A_0)$. Moreover, by Lemma C.6.(2) we obtain that

$$\lim_{\substack{|\lambda|\to+\infty \\ \mathrm{Re}\lambda>0}} \|D_\lambda^{A,L}\|_{\mathcal{L}(\partial X, [D(A_0),X]_\alpha)} = 0.$$

Thus, $I_{\partial X} - (L - L')D_{\lambda_0}^{A,L} = L'D_{\lambda_0}^{A,L}$ can be inverted for λ_0 large enough.

To prove the surjectivity of L', take $x \in \partial X$, and observe that for $u := D_\lambda^{A,L}\big(L'D_\lambda^{A,L}\big)^{-1}x$ there holds $L'u = x$. □

Bibliography

[AF97] H. Amann and M. Fila, *A Fujita-type theorem for the Laplace equation with a dynamical boundary condition*, Acta Math. Univ. Comen. **66** (1997), 321–328.

[AB97] W. Arendt and C.J.K. Batty, *Almost periodic solutions of first- and second-order Cauchy problems*, J. Diff. Equations **137** (1997), 363–383.

[ABHN01] W. Arendt, C.J.K. Batty, M. Hieber, and F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics **96**, Birkhäuser 2001.

[AMPR03] W. Arendt, G. Metafune, D. Pallara, and S. Romanelli, *The Laplacian with Wentzell–Robin boundary conditions on spaces of continuous functions*, Semigroup Forum **67** (2003), 247–261.

[Ba85] A.G. Baskakov, *Harmonic analysis of cosine and exponential operator-valued functions*, Math. USSR Sb. **52** (1985), 63–90.

[BE04] A. Bátkai and K.-J. Engel, *Abstract wave equations with generalized Wentzell boundary conditions.* J. Diff. Equations (to appear).

[BLX03] C.J.K. Batty, J. Liang, and T.-J. Xiao, *On the spectral and growth bound of semigroups associated with hyperbolic equations.* As a preprint in: Tübinger Berichte zur Funktionalanalysis **12** (2002/03), 38–48.

[Be76] J.T. Beale, *Spectral properties of an acoustic boundary condition*, Indiana Univ. Math. J. **25** (1976), 895–917.

[Be77] J.T. Beale, *Acoustic scattering from locally reacting surfaces*, Indiana Univ. Math. J. **26** (1977), 199–222.

[BR74] J.T. Beale and S.I. Rosencrans, *Acoustic boundary conditions*, Bull. Amer. Math. Soc. **80** (1974), 1276–1278.

[Be00] B.P. Belinsky, *Wave propagation in the ice-covered ocean wave guide and operator polynomials*, in: H.G.W. Begehr, R.P. Gilbert, and J. Kajiwara (eds.), "Proceedings of the Second ISAAC Congress", Kluwer Academic Publishers 2000, 1319–1333.

[CENN03] V. Casarino, K.-J. Engel, R. Nagel, and G. Nickel, *A semigroup approach to boundary feedback systems*, Integral Equations Oper. Theory **47** (2003), 289–306.

[CENP04] V. Casarino, K.-J. Engel, G. Nickel, and S. Piazzera, *Decoupling techniques for wave equations with dynamic boundary conditions.* Preprint 2004.

[CT88] S.P. Chen and R. Triggiani, *Proof of two conjectures by G. Chen and D. L. Russell on structural damping for elastic systems*, in: A. Gómez and others (eds.), "Approximation and optimization", (Proceedings Havana 1987), Lect. Notes Math. **1354**, Sringer-Verlag 1988, 234–256.

[CT89] S.P. Chen and R. Triggiani, *Proof of extensions of two conjectures on structural damping for elastic systems*, Pacific J. Math. **136** (1989), 15–55.

[CT90] S.P. Chen and R. Triggiani, *Characterization of domains of fractional powers of certain operators arising in elastic systems, and applications*, J. Diff. Equations **88** (1990), 279–293.

[CT90b] S.P. Chen and R. Triggiani, *Gevrey class semigroups arising from elastic systems with gentle dissipation: the case* $0 < \alpha < 1/2$, Proc. Amer. Math. Soc. **110** (1990), 401–415.

[En97] K.-J. Engel, *Positivity and stability for one-sided coupled operator matrices*, Positivity **1** (1997), 103–124.

[En99] K.-J. Engel, *Spectral theory and generator property for one-sided coupled operator matrices*, Semigroup Forum **58** (1999), 267–295.

[En03] K.-J. Engel, *Second order differential operators on* $C[0,1]$ *with Wentzell-Robin boundary conditions*, in: G.R. Goldstein, R. Nagel, and S. Romanelli (eds.), "Evolution Equations: Proceedings in Honor of J.A. Goldstein's 60th Birthday", Lecture Notes in Pure and Applied Mathematics **234**, Marcel Dekker 2003, 159–166.

[En04] K.-J. Engel, *The Laplacian on* $C(\overline{\Omega})$ *with generalized Wentzell boundary conditions*, Arch. Math. **81** (2003), 548–558.

[En04b] K.-J. Engel, *Matrix methods for Wentzell boundary conditions.* As a preprint in: Tübinger Berichte zur Funktionalanalysis **13** (2003/04), 78–99.

[EN00] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics **194**, Springer-Verlag 2000.

[Es93] J. Escher, *Quasilinear parabolic problems with dynamical boundary conditions*, Commun. Partial Differ. Equations **18** (1993), 1309–1364.

[Ev98] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics **19**, American Mathematical Society 1998.

[Fa85] H.O. Fattorini, *Second Order Linear Differential Equations in Banach Spaces*, Mathematics Studies **108**, North-Holland 1985.

[FGGR00] A. Favini, G. Ruiz Goldstein, J.A. Goldstein, and S. Romanelli, *C_0-semigroups generated by second order differential operators with general Wentzell boundary conditions*, Proc. Amer. Math. Soc. **128** (2000), 1981–1989.

[FGGR01] A. Favini, G.R. Goldstein, J.A. Goldstein, and S. Romanelli, *The one dimensional wave equation with Wentzell boundary conditions*, in: S. Aicovici and N. Pavel (eds.), "Differential Equations and Control Theory" (Proceedings Athens 2000), Lecture Notes in Pure and Applied Mathematics **225**, Marcel Dekker 2001, 139–145.

[FGGR02] A. Favini, G.R. Goldstein, J.A. Goldstein, and S. Romanelli, *The heat equation with generalized Wentzell boundary condition*, J. Evol. Equations **2** (2002), 1–19.

[FGG+03] A. Favini, G.R. Goldstein, J.A. Goldstein, E. Obrecht, and S. Romanelli, *The Laplacian with generalized Wentzell boundary conditions*, in: M. Iannelli and G. Lumer (eds.): "Evolution Equations 2000: Applications to Physics, Industry, Life Sciences and Economics" (Proceedings Levico Terme 2000), Progress in Nonlinear Differential Equations, Birkhäuser 2003, 169–180.

[FG00] C.L. Frota and J.A. Goldstein, *Some nonlinear wave equations with acoustic boundary conditions*, J. Differ. Equations Soc. **164** (2000), 92–109.

[Ga04] C. Gal, Ph.D. thesis (in preparation).

[GGG03] C. Gal, G.R. Goldstein, and J.A. Goldstein, *Oscillatory boundary conditions for acoustic wave equations*, J. Evol. Equations **3** (2004), 623–636.

[Go69] J.A. Goldstein, *Semigroups and second-order differential equations*, J. Funct. Anal. **4** (1969), 50–70.

[Go80] J.A. Goldstein, *The universal addability problem for generators of cosine functions and operator groups*, Houston J. Math. **6** (1980), 365–373.

[Go85] J.A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford Mathematical Monographs, Oxford University Press 1985.

[GW03] J.A. Goldstein and M. Wacker, *The energy space and norm growth for abstract wave equations*, Appl. Math. Lett **16** (2003), 767–772.

[Gr87] G. Greiner, *Perturbing the boundary conditions of a generator*, Houston J. Math. **13** (1987), 213–229.

[Gr99] A Grigor'yan, *Estimates of heat kernels on Riemannian manifolds*, in: B. Davies and Yu. Safarov (eds.), "Spectral Theory and Geometry. ICMS Instructional Conference" (Proceedings Edinburgh 1998), London Math. Soc. Lecture Notes **273**, Cambridge Univ. Press 1999, 140–225.

[GK91] G. Greiner and K. Kuhn, *Linear and semilinear boundary conditions: the analytic case*, in: Ph. Clément, E. Mitidieri, and B. de Pagter (eds.), "Semigroup Theory and Evolution Equations" (Proceedings Delft 1989), Lecture Notes in Pure and Appl. Math. **135**, Marcel Dekker 1991, 193–211.

[Gu94] D. Guidetti, *On elliptic boundary value problems with dynamic boundary conditions of parabolic type*, Differ. Integral Equ. **7** (1994), 873–884.

[HP98] M. Hieber and J. Prüss, *Functional calculi for linear operators in vector-valued L^p-spaces via transference principle*, Adv. Diff. Equ. **3** (1998), 847–872.

[Hi89] T. Hintermann, *Evolution equations with dynamic boundary conditions*, Proc. Royal Soc. Edinburgh **113A** (1989), 43–60.

[Ka95] T. Kato, *Perturbation Theory for Linear Operators*, Classics in Mathematics, Springer-Verlag 1995.

[KW03] V. Keyantuo and M. Warma, *On the wave equation in L^p spaces.* As a preprint in: Ulmer Seminare über Funktionalanalysis and Differentialgleichungen **8** (2003), 225–241.

[Ki72] J. Kisyński, *On cosine operator functions and one-parameter groups of operators*, Studia Math. **44** (1972), 93–105.

[KMN03] M. Kramar, D. Mugnolo, and R. Nagel, *Semigroups for initial-boundary value problems*, in: M. Iannelli and G. Lumer (eds.): "Evolution Equations 2000: Applications to Physics, Industry, Life Sciences and Economics" (Proceedings Levico Terme 2000), Progress in Nonlinear Differential Equations, Birkhäuser 2003, 277–297.

[KMN03b] M. Kramar, D. Mugnolo, and R. Nagel, *Theory and applications of one-sided coupled operator matrices*, Conf. Sem. Mat. Univ. Bari **283** (2003).

[Kr61] V.N. Krasil'nikov, *On the solution of some boundary-contact problems of linear hydrodynamics*, J. Appl. Math. Mech. **25** (1961), 1134–1141.

[LSY93] P. Lancaster, A. Shkalikov, Q. Ye, *Strongly definitizable linear pencils in Hilbert space*, Integral Equations Oper. Theory **17** (1993), 338–360.

[Li61] J.L. Lions, *Équations différentielles opérationnelles et problèmes aux limites*, Grundlehren der mathematischen Wissenschaften **111**, Springer-Verlag 1961.

[LM72] J.L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications. Vol. I–II*, Grundlehren der mathematischen Wissenschaften **181–182**, Springer-Verlag 1972.

[Lu95] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Progress in Nonlinear Differential Equations and their Applications **16**, Birkhäuser 1995.

[Lu82] D. Lutz, *Some spectral properties of bounded operator cosine functions*, Math. Repts. Acad. Sci. Canada **4** (1982), 81–85.

[Ma85] V.G. Maz'ja, *Sobolev Spaces*, Springer-Verlag 1985.

[MI68] P.M. Morse and K.U. Ingard, *Theoretical Acoustics*, McGraw-Hill 1968.

[Mu04] D. Mugnolo, *Abstract wave equations with acoustic boundary conditions*, Math. Nachr. (to appear).

[Mu04b] D. Mugnolo, *Operator matrices as generators of cosine operator functions*. As a preprint in: Tübinger Berichte zur Funktionalanalysis **13** (2003/04), 290–303.

[Mu04c] D. Mugnolo, *A semigroup analogue of the Fonf-Lin-Wojtaszczyk ergodic characterization of reflexive Banach spaces with a basis*, Studia Math. (to appear).

[Na89] R. Nagel, *Towards a "matrix theory" for unbounded operator matrices*, Math. Z. **201** (1989), 57–68.

[Na96] R. Nagel, *Characteristic equations for the spectrum of generators*, Ann. Scuola Norm. Pisa Cl. Sci. (4) **XXIV** (1997), 703–717.

[Ni04] G. Nickel, *Semigroup methods for dynamical boundary value problems*, Habilitationsschrift, Tübingen 2004.

[Rh92] A. Rhandi, *Positive perturbations of linear Volterra equations and sine functions of operators*, J. Integral Equations Appl. **4** (1992), 409–420.

[So66] M. Sova, *Cosine operator functions*, Rozpr. Mat. **49** (1966).

[TW77] C.C. Travis and G.F. Webb, *Compactness, regularity, and uniform continuity properties of strongly continuous cosine families*, Houston J. Math. **3** (1977), 555–567.

[Tr78] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, Mathematical Library **18**, North-Holland 1978.

[XL98] T.-J. Xiao and J. Liang, *The Cauchy Problem for Higher-Order Abstract Differential Equations*, Lecture Notes in Mathematics **1701**, Springer-Verlag, 1998.

[XL04] T.-J. Xiao and J. Liang, *A solution to an open problem for wave equations with generalized Wentzell boundary conditions*, Math. Ann. **327** (2003), 351–363.

[XL04b] T.-J. Xiao and J. Liang, *Second order evolution equations with dynamic boundary conditions*. As a preprint in: Tübinger Berichte zur Funktionalanalysis **11** (2001/02), 303–321.

Lebenslauf

11.09.1978	Geboren in Bari (Italien)
1983-1988	Besuch der “Scuola Elementare G. Mazzini” in Bari
1988-1991	Besuch der “Scuola Media G. De Sanctis” in Bari
1991-1996	Besuch der “Liceo Ginnasio Q. Orazio Flacco” in Bari
Juli 1996	Maturità classica (Abitur) mit Note: 60/60
1996-2000	Studium der Mathematik mit Nebenfach Physik and der Università degli Studi di Bari
Oktober 1998 - Juli 1999	Austausch im Rahmen des Socrates-Erasmus-Projektes an der Eberhard-Karls-Universitï¿½ Tbingen
Oktober 2000 - Februar 2001	Wissenschaftliche Hilfskraft an der Mathematischen Fakultï¿½ der Universitï¿½ Tbingen
Dezember 2000	Laurea (Diplom) in Mathematik an der Università degli Studi di Bari mit Note: 110/110 e lode
Februar - September 2001	Wissenschaftlicher Angestellter an der Mathematischen Fakultï¿½ der Eberhard-Karls-Universitï¿½ Tbingen im Rahmen vom ≪Landesforschungsschwerpunkt “Evolutionsgleichungen”≫
ab September 2001	Promotion an der Mathematischen Fakultï¿½ der Eberhard-Karls-Universitï¿½ Tbingen
ab September 2001	Stipendium vom ≪Istituto Nazionale di Alta Matematica “Franceso Severi”≫ (Nationales Institut fr die Hï¿½ere Mathematik)
Mï¿½z - Mai 2002	Auslandsaufenthalt bei Prof. J.A. Goldstein und Prof. G.R. Goldstein an der University of Memphis (USA)
November 2002 - Juli 2003	Dozent an der Berufsakademie Horb
Mï¿½z 2003	DAAD-Stipendium fr einen Auslandsaufenthalt bei Prof. A. Batkai und Prof. Z. Sebestyï¿½ an der Eï¿½vï¿½ Lorï¿½d Tudomï¿½yegyetem (Ungarn)

Meine akademischen Lehrer waren

in Mathematik

F. Altomare, A. Cossu, D. Fortunato, G. Muni, A. Pastore, G. Puglisi, S. Romanelli, A. Salvatore (in Bari)

H. Heyer, G. Huisken, F. Loose, R. Nagel, U. Schlotterbeck, M. Wolff (in Tbingen)

in Physik

M. Maiellaro, L. Palese, V. Picciariello, G. Selvaggi (in Bari)

Zusammenfassung in deutscher Sprache

Von physikalischen Problemen ausgehend, oder als L^p-Pendant zur Theorie der Wentzell'schen Randbedingungen fr Diffussionsprozesse, sind in den letzten Jahren partielle Differentialgleichungungen auf beschrï¿½kten Gebieten von $\mathbb{R}^n$ mit dynamischen Randbedingungen betrachtet worden. In der vorliegenden Arbeit richten wir den Fokus auf Halbgruppenmethoden fr abstrakte Wellengleichungen mit verschiedenen Arten von dynamischen Randbedingungen.

Im ersten Kapitel fhren wir den Begriff Wohlgestelltheit fr abstrakte Probleme *1. Ordnung* der Form

$$\text{(AIBVP)} \qquad \begin{cases} \dot{u}(t) &= Au(t), & t \geq 0, \\ \dot{x}(t) &= Bu(t) + \tilde{B}x(t), & t \geq 0, \\ x(t) &= Lu(t), & t \geq 0, \\ u(0) &= f \in X, & \\ x(0) &= g \in \partial X, & \end{cases}$$

ein und charakterisieren diesen anschlieï¿½nd. Die erste dieser Gleichungen findet in einem Banach-*Zustandsraum* statt: in konkreten Anwendungen ist dieser oft ein Raum von Funktionen auf einem Gebiet $\Omega \subset \mathbb{R}^n$ mit glattem, nichtleeren Rand $\partial\Omega$. Die dritte Gleichung stellt eine Kopplungsbeziehung zwischen der Variablen $u(\cdot)$ in X und $x(\cdot)$ in einem Banach-*Randraum* ∂X dar: in konkreten Anwendungen ist ∂X oft ein Raum von Funktionen auf $\partial\Omega$. Schlieï¿½ich stellt die zweite Gleichung eine Evolutionsgleichung auf dem Rand dar mit einem Feedback durch den Operator B reprï¿½entiert.

Um die Theorie der starkstetigen Halbgruppen von beschrï¿½kten linearen Operatoren (kurz: C_0-Halbgruppen) anzuwenden, ist entscheidend, dass (AIBVP) auf ein passendes abstraktes Cauchyproblem auf dem Produktraum $X \times \partial X$ reduziert wird.

Es wird bewiesen, dass unter passenden Voraussetzungen ein solches Cauchyproblem von einer C_0-Halbgruppe gesteuert wird, deren Generator eine Operatormatrix mit *nicht-diagonalem* Definitionsbereich ist. Spektral- und Generatoreigenschaften dieser Art von Operatormatrizen werden ausfhrlich im zweiten Kapitel diskutiert.

Es ist dann natrlich, Resultate dieser Art zu Problemen 2. Ordnung fortzusetzen, wie

$$\begin{cases} \ddot{u}(t) &= Au(t), & & t \in \mathbb{R}, \\ \ddot{x}(t) &= Bu(t) + \tilde{B}x(t), & & t \in \mathbb{R}, \\ u(0) &= f \in X, & \dot{u}(0) = g \in X, & \\ x(0) &= h \in \partial X, & \dot{x}(0) = j \in \partial X. & \end{cases}$$

Hier braucht man jedoch noch eine Kopplungsbeziehung zwischen den Variablen $u(\cdot)$ und $x(\cdot)$. In der Tat kann ein abstraktes Problem 2. Ordnung mit verschiedenen Arten von dynamischen Randbedingungen ausgestattet werden. Diese unterscheiden sich grundsï¿½zlich in der Kopplungsbeziehung: angesichts der Anwendungen betrachen wir drei Arten. Wir zeigen, dass die Wohlgestelltheit solcher Probleme mit der Theorie der Cosinusfamilien, die in Appendix B erwähnt wird, verwandt ist.

Im dritten Kapitel betrachten wir ein abstraktes Problem 2. Ordnung, in dem die Kopplungsbeziehung durch

$$\dot{x}(\cdot) = Lu(\cdot)$$

gegeben ist. Das ist physikalisch durch sogenannte *Wellengleichungen mit akustischen Randbedingungen* motiviert, die in den letzten 40 Jahren untersucht worden sind. Als Anwendung unserer abstrakten Techniken diskutieren wir verschiedene Systeme dieser Art, beweisen einige bekannte Ergebnisse (hinsichtlich Generatoreigenschaft und Resolventenkompaktheit) erneut, und bringen diese in einen abstrakten Rahmen.

Im vierten Kapitel untersuchen wir abstrakte Probleme 2. Ordnung, die mit dynamischen Randbedingungen 2. Ordnung ausgestattet sind, und in denen die Kopplungsbeziehung durch

$$(*) \qquad x(\cdot) = Lu(\cdot)$$

oder durch

$$(**) \qquad x(\cdot) = Lu(\cdot) \qquad \text{und} \qquad \dot{x}(\cdot) = L\dot{u}(\cdot)$$

gegeben ist. Dynamische Randbedingungen mit solchen Kopplungsbeziehungen stellen recht unterschiedliche konkrete Probleme dar. In konkreten Anwendungen modellieren sie beispielsweise dynamische Neumann'sche (oder Robin'sche) bzw. Dirichlet'sche Randbedingungen der 2. Ordnung. Wir zeigen, dass der zu solchen Problemen assoziierte Phasenraum von der angenommenen Kopplungsbeziehung abhï¿½gt. Genauer, wenn $(*)$ gilt, dann ist der erste Koordinatenraum des zum Problem assoziierten Phasenraumes ein diagonaler Teilraum von $X \times \partial X$. Wenn jedoch $(**)$ gilt, ist der erste Koordinatenraum des Phasenraumes ein gewisser Teilraum von $X \times \partial X$, der in seiner Definition selbst eine Kopplungsbeziehung enthï¿½t.

Im fnften Kapitel betrachten wir *vollstï¿½dige* Probleme 2. Ordnung, d.h., Systeme in denen die erste Gleichung durch

$$\ddot{u}(t) = Au(t) + C\dot{u}(t), \qquad t \in \mathbb{R},$$

gegeben ist. Auch bei dieser Verallgemeinerung mssen wir zwischen Fï¿½len unterscheiden, die konkret dynamischen Dirichlet'schen bzw. Neumann'schen (oder Robin'schen) Randbedingungen entsprechen. Auï¿½rdem betrachten wir den Fall bergedï¿½pfter vollstï¿½diger Probleme, d.h., Probleme, in denen C "unbeschränkter" als A ist.