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Parabolic systems and evolution equations on networks

(Parabolische Systeme und Evolutionsgleichungen auf Netzwerken)

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1. Historical background

Following earlier intuitions – culminating in tight binding models proposed among others by E. Hückel and L. Pauling – K. Ruedenberg and C. Scherr developed in 1953 a new technique with the aim of studying electronic properties of conjugated bond systems, and in particular of aromatic molecules. Their idea was to set up a Schrödinger equation acting on a quasi-1-dimensional domain that can be schematised as a network of atoms, the network's edges being the chemical bonds. Their results aroused broad interest in the community of quantum chemists and marked the birth of the so-called "free-electron network model".

Over two decades later, when most solid state physicists had already turned back to the original, computationally more feasible discrete tight binding approximations, the free-electron model began to be studied by analysts and theoretical physicists. They developed a sound mathematical theory of differential operators on thin manifolds and were often able to explain observations using a formalism based on the notion of *metric graphs*. In particular, it was G. Lumer who first noticed in 1979 that network models boast discrete spectrum, in accordance with experimental data. Further interesting results followed soon: among others, F. Ali Mehmeti, J. von Below, P. Exner, S. Nicaise and J.-P. Roth extended Lumer's results considering more and more general node conditions, providing interesting descriptions of the spectrum, discussing nonlinear and/or higher dimensional problems and establishing an interplay with quantum physics and theoretical mechanics.

Their investigations, and in particular Roth's work, have established the rule of thumb that several common formulae and principles, which generally only hold approximately, do apply *exactly* in the context of networks. In fact, already back in 1970 E. Montroll and coauthors had observed that Ruedenberg–Scherr's free electron network theory allows for computations of eigenvalues and eigenfunctions of the Hamiltonian (i.e., of energy levels and wavefunctions of the system) without approximations based on perturbation theory.

The final breakthrough has come in the late 1990s, when U. Smilansky and coauthors have shown that models based on differential operators on metric graphs can play a fundamental role in the theory of quantum chaos: ever since, network-based differential models for Schrödinger and, more recently, Dirac and Pauli equations have been commonly referred to as "quantum graphs" in the literature by mathematicians and theoretical physicists. The great attention such models have received in the mathematical and physical communities is reflected by hundreds of articles that have been published in the field of quantum graphs over the last 10 years. Two concise but excellent overviews on this topic have been provided in [PB04, Kuc08].

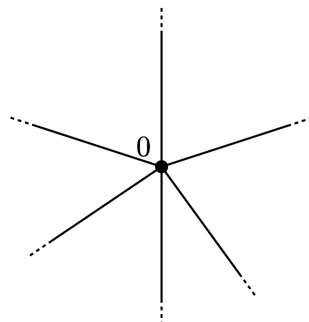
2. Synopsis of my results

The present résumé has only a summarising purpose. I will not recall basic notions of operator and graph theory – like metric graphs, sesquilinear forms and strongly-continuous semigroups of bounded linear operators.

2.1. **Networks and symmetries.** (This section is based on the results obtained in (Mug6, Mug7, Mug11, Mug 13, Mug22, Mug30).)

Throughout, by "network" we mean a metric graph – which in typical applications is embedded in $\mathbb{R}^3$ and represents a (more or less adequate) approximation of an actual three-dimensional object: a molecule, a pipeline, an ensemble of neurons, a fuel cell...

To fix the ideas, consider a metric star G with (finite) edge set E, where each edge is parametrised as a $[0,\infty)$-interval and 0 is identified with the center of the star:



We discuss a diffusion problem

$$\text{(St)} \qquad \begin{cases} \dot{u}_{\mathsf{e}}(t,x) &= u''_{\mathsf{e}}(t,x), \quad t \geq 0,\ x \in (0,\infty),\ \mathsf{e} \in \mathsf{E}, \\ u_{\mathsf{e}}(t,0) &= u_{\mathsf{f}}(t,0), \quad t \geq 0,\ \mathsf{e},\mathsf{f} \in \mathsf{E}, \\ \sum_{\mathsf{e}\in\mathsf{E}} u'_{\mathsf{e}}(t,0) &= 0, \qquad\qquad t \geq 0, \end{cases}$$

on (the edges of) G, imposing on its center a Kirchhoff-type node condition (the heat flowing along the edges through the node must sum up to 0) along with a continuity assumption[1]. In order to solve the problem uniquely we also have to assign an initial data, in form of a function defined edgewise and pointwise. Therefore, the unknown u_{e} is a function that describes the diffusion on the edge e and maps $[0,\infty) \times [0,\infty)$ to $\mathbb{C}$. One can think of such a problem as a model for heat diffusion along equal metal rods, each of them being thin (i.e., almost 1-dimensional), homogeneous and infinitely long. Similar settings naturally arise if we are instead interested in a scattering problem or in wave propagation, in which case the dynamics of the problem may be described by the Schrödinger or the wave equation instead.

It is natural to expect that the evolution of the system reflects the obvious symmetry of the geometric structure we are considering. More specifically, one may conjecture that if the initial data $(u_e(0,\cdot))_{e\in\mathsf{E}}$ are radial, then so is the solution $(u_e(t,\cdot))_{e\in\mathsf{E}}$ for all $t > 0$. In this section I will discuss this and similar question.

One should mention that T. Kato's and J.-L. Lions's theory of sesquilinear form can be used in order to show well-posedness of the above evolution equation. My main contribution to the topic of this section lies in recognising that more modern results on forms (due among other to E.M. Ouhabaz) permit to discuss the interplay of partial differential equations and discrete mathematics in an elegant and efficient way.

The above case of a metric star is particularly simple since only one transmission condition is needed: the network contains only one node. In order to introduce a more interesting setting we consider finitely many intervals of finite length that are connected to realise a simple finite metric graph G with node set V and edge set E. One of the main challenges of the theory of evolution on networks is to relate qualitative properties of a diffusion equation with selected features of the

[1] In the following we will refer to this kind of node conditions as "continuity/Kirchhoff" ones.

underlying simple graph G. We can assume each interval to have length 1, up to modifying the coefficients of the elliptic operator. (In this exposition we mostly restrict ourselves to the case of the Laplace operator, for the general case we refer to (Mug11)). We identify each edge e with its parametrisation and write $\mathsf{e} : [0,1] \to \mathbb{R}^3$. Observe that the parametrisation fixes an orientation of each edge, say $\mathsf{e} = \overrightarrow{(\mathsf{v},\mathsf{w})}$, and in this case we write $\mathsf{e}(0) = \mathsf{v}$, $\mathsf{e}(1) = \mathsf{w}$.

The structure of the network is given by the $|\mathsf{V}| \times |\mathsf{E}|$-*outgoing* and *ingoing incidence matrices* $\mathcal{I}^+ := (\iota^+_{\mathsf{ve}})$ and $\mathcal{I}^- := (\iota^-_{\mathsf{ve}})$ defined by

$$\iota^+_{\mathsf{ve}} : \begin{cases} 1, & \text{if } \mathsf{e}(0) = \mathsf{v}, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \iota^-_{\mathsf{ve}} : \begin{cases} 1, & \text{if } \mathsf{e}(1) = \mathsf{v}. \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The matrix $\mathcal{I} := (\iota_{\mathsf{ve}})$ defined by $\mathcal{I} := \mathcal{I}^+ - \mathcal{I}^-$ is the incidence matrix of G. Furthermore, let $\Gamma(\mathsf{v})$ be the set of all edges incident in v, i.e.,

$$\Gamma(\mathsf{v}) := \{\mathsf{e} \in \mathsf{E} : \mathsf{e}(0) = \mathsf{v} \text{ or } \mathsf{e}(1) = \mathsf{v}\}.$$

For the sake of notational simplicity, if $\mathsf{e} = \overrightarrow{(\mathsf{v},\mathsf{w})}$ we denote the value of a function $f_{\mathsf{e}} : [0,1] \to \mathbb{C}$ on e at 0 and 1 by $f_{\mathsf{e}}(\mathsf{v})$ and $f_{\mathsf{e}}(\mathsf{w})$, respectively. With an abuse of notation, we also set $f_{\mathsf{e}}(\mathsf{v}) = 0$ whenever $\mathsf{e} \notin \Gamma(\mathsf{v})$. When convenient, we shall also write the functions in vector form, i.e., $f = (f_{e_1}, \dots, f_{e_{|\mathsf{E}|}})^T : [0,1] \to \mathbb{C}^{|\mathsf{E}|}$.

The above mentioned early investigations of the 1980s were usually performed in the Hilbert space

$$L^2(\mathsf{G}) := \prod_{\mathsf{e}\in\mathsf{E}} L^2(0,1;\mathbb{C}) \cong L^2(0,1;\mathbb{C}^{|\mathsf{E}|}).$$

It has been proved in [Lum80] that the Cauchy problem associated with the heat equation with continuity/Kirchhoff node conditions

$$\left\{ \begin{array}{rcll} \dot u_{\mathsf{e}}(t,x) &=& u''_{\mathsf{e}}(t,x), & t \geq 0,\ x \in (0,1),\ \mathsf{e} \in \mathsf{E}, \\ u_{\mathsf{e}}(t,\mathsf{v}) &=& u_{\mathsf{f}}(t,\mathsf{v}), & t \geq 0,\ \mathsf{e},\mathsf{f} \in \Gamma(\mathsf{v}),\ \mathsf{v} \in \mathsf{V}, \\ \sum_{\mathsf{e}\in\mathsf{E}} \iota_{\mathsf{ve}} u'_{\mathsf{e}}(t,\mathsf{v}) &=& 0, & t \geq 0,\ \mathsf{v} \in \mathsf{V}, \\ u_{\mathsf{e}}(0,x) &=& u_{0\mathsf{e}}(x), & x \in (0,1),\ \mathsf{e} \in \mathsf{E}, \end{array} \right. \tag{Ki}$$

is governed by a strongly continuous semigroup on $L^2(\mathsf{G})$. Analyticity of this semigroup has been first obtained in [Bel85]; further properties (including a parabolic maximum principle) have been obtained directly in [Bel88, Bel91], but in our approach they will simply follow as special cases of Theorem 12 below. (Observe that the node conditions in (Ki) are a natural generalisation of those in (St).)

We define the $2|\mathsf{E}| \times |\mathsf{V}|$-matrix

$$\tilde{\mathcal{I}} := \begin{pmatrix} (\mathcal{I}^+)^T \\ (\mathcal{I}^-)^T \end{pmatrix} \tag{2}$$

and remark that the two node conditions in (Ki) can be re-written as

$$\begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix} \in \mathrm{Range}\tilde{\mathcal{I}} \quad \text{and} \quad \begin{pmatrix} -u'(t,0) \\ u'(t,1) \end{pmatrix} \in \mathrm{Ker}\tilde{\mathcal{I}}^T, \tag{3}$$

respectively. It is noteworthy that $\mathrm{Range}\tilde{\mathcal{I}}$ and $\mathrm{Ker}\tilde{\mathcal{I}}^T$ are mutually orthogonal subspaces of $\mathbb{C}^{2|\mathsf{E}|}$.

A careful analysis of the Laplace operator associated with (Ki) – i.e., of the second derivative defined edgewise with node conditions given in (3) – is in order. Integration by parts leads to introducing the sesquilinear form with domain

$$H^1(\mathsf{G}) := \left\{ f \in H^1(0,1;\mathbb{C}^{|\mathsf{E}|}) : \begin{pmatrix} f(0) \\ f(1) \end{pmatrix} \in \mathrm{Range}\tilde{\mathcal{I}} \right\}$$

defined by

$$a(f,g) := \sum_{\mathsf{e}\in\mathsf{E}} \int_0^1 f'_{\mathsf{e}}(x)\overline{g'_{\mathsf{e}}(x)}dx, \qquad f,g \in H^1(\mathsf{G}).$$

Up to minor modifications, this is the basic approach upon which the investigations in (Mug6,Mug7,Mug11) rely.

It will be shown in § 2.2 that the node conditions in (3) can be generalised. However, (Ki) is theoretically interesting in its own right, as it displays an interplay between differential and graph theoretical structures.

Let $\mathcal{V}, \mathcal{H}$ be Hilbert spaces such that $\mathcal{V}$ is densely and continuously embedded in $\mathcal{H}$. If $\mathfrak{a} : \mathcal{V}\times\mathcal{V} \to \mathbb{C}$ is an $\mathcal{H}$-elliptic, continuous, symmetric sesquilinear form with associated operator A, then a unitary operator O on $\mathcal{H}$ is called a *global symmetry* of the abstract Cauchy problem

$$\text{(ACP)} \qquad \begin{cases} \dot{u}(t) &= Au(t), \qquad t \geq 0, \\ u(0) &= u_0 \in \mathcal{H} \end{cases}$$

associated with A if $Of \in \mathcal{V}$ and $\mathfrak{a}(Of) = \mathfrak{a}(f)$ for all $f \in \mathcal{V}$, or equivalently if O commutes with the strongly-continuous semigroup generated by A for all $t \geq 0$. While each unitary operator on $\mathcal{H}$ can be embedded in a group (by the spectral theorem), such a group need not consist of global symmetries. It may happen that a set of global symmetries form a Lie group: such a Lie group $\mathcal{O}$ is then called a *global symmetry group*[2]. The set of all global symmetries of (Ki) forms a group (in general not a Lie group!) that we denote by $\mathfrak{A}(\mathsf{G})$. The following result has been proved along the lines of the results in (Mug11, §3), using a known criterion for invariance of closed convex subsets under the action of semigroups due to E.M. Ouhabaz, see [Ouh05, Chapt. 2] that generalizes the classical characterisation of the (sub)Markov property due to A. Beurling and J. Deny.

Theorem 1. *Define a unitary operator $\Sigma_{\mathfrak{S}}$ on $L^2(0,1;\mathbb{C}^{|\mathsf{E}|})$ by*

$$\Sigma_{\mathfrak{S}} f(x) := \mathfrak{S}(f(x)), \qquad x \in (0,1),$$

for any unitary $|\mathsf{E}| \times |\mathsf{E}|$-matrix $\mathfrak{S}$. Then $\Sigma_{\mathfrak{S}}$ is a global symmetry of (Ki) *if and only if*

$$\begin{pmatrix} \mathfrak{S} & 0 \\ 0 & \mathfrak{S} \end{pmatrix} \mathrm{Range}\tilde{\mathcal{I}} \subset \mathrm{Range}\tilde{\mathcal{I}},$$

where $\tilde{\mathcal{I}}$ is the matrix defined in (2).

Consider a closed subspace $\mathcal{Z}$ of $L^2(\mathsf{G})$ and the orthogonal projection $P_{\mathcal{Z}}$ onto $\mathcal{Z}$. It turns out that in order for $(e^{isP_{\mathcal{Z}}})_{s\in\mathbb{R}}$ to be a global symmetry group of (Ki), a certain compatibility condition has to be satisfied by $\mathcal{Z}$ and $\tilde{\mathcal{I}}$. It is particularly instructive to consider the case where $\mathcal{Z}$ has a local, uniform structure, i.e., if there exists a subspace Z of $\mathbb{C}^{|\mathsf{E}|}$ such that the relevant space $\mathcal{Z}$ satisfies

$$\text{(4)} \qquad \mathcal{Z} = \{f \in L^2(0,1;\mathbb{C}^{|\mathsf{E}|}) : f(x) \in Z \text{ for a.e. } x \in (0,1)\}.$$

[2] Global symmetry groups play an important role in mathematical physics, ever since S. Lie and E. Noether have shown how to use them in order to reduce complexity of a differential equation. E.g., by the celebrated Noether's theorem the existence of an r-parameter global symmetry group imply the existence of r conserved quantities of the system.

If in particular (4) holds, then by Theorem 1 the unitary group $(e^{isP_{\mathcal{Z}}})_{s\in\mathbb{R}}$ is a global symmetry group of (Ki) – i.e., it is a Lie subgroup of $\mathfrak{A}(\mathsf{G})$ – if and only if

$$\begin{pmatrix} P_Z & 0 \\ 0 & P_Z \end{pmatrix} \mathrm{Range}\tilde{\mathcal{I}} \subset \mathrm{Range}\tilde{\mathcal{I}},$$

where P_Z denotes the orthogonal projection of H onto Z.

An interesting consequence is the possibility of describing the underlying graph G by only knowing the global symmetries of the network parabolic problem: *One can mirror the shape of a graph*, in some special cases. We mention the following prototypical result, obtained in (Mug11).

Corollary 2. *Let* (4) *hold for some subspace Z of $\mathbb{C}^{|\mathsf{E}|}$.*

(1) Let $Z = \langle \mathbf{1} \rangle := \{(c, c, \ldots, c) : c \in \mathbb{C}\}$. Then $(e^{isP_{\mathcal{Z}}})_{s\in\mathbb{R}}$ is a subgroup of $\mathfrak{A}(\mathsf{G})$ if and only if the underlying graph is bipartite or Eulerian.

(2) Let G be connected. Then G is a star if and only if $(e^{isP_{\mathcal{Z}}})_{s\in\mathbb{R}}$ is a subgroup of $\mathfrak{A}(\mathsf{G})$ for all orthogonal projections P_Z with eigenvector $\mathbf{1}$.

Example 3. *The Petersen graph P is neither bipartite nor Eulerian (since it is 3-regular), hence $\langle \mathbf{1} \rangle$ does not induce a Lie subgroup of $\mathfrak{A}(\mathsf{P})$.*

On the other hand, Tutte's 8-cage TC is not Eulerian (since it is 3-regular) but is indeed bipartite, hence $\langle \mathbf{1} \rangle$ does induce a Lie subgroup of $\mathfrak{A}(\mathsf{TC})$.

For the same reason, $\langle \mathbf{1} \rangle$ induces a Lie subgroup of $\mathfrak{A}(\mathsf{T})$ for any tree T.

A similar approach has been recently used in [PB10] in order to generalize T. Sunada's construction of isospectral, non-isomorphic domains (or graphs). As a direct consequence of Theorem 1 and a known result due to L. Babai, existence of infinitely many such isospectral, non-isomorphic graphs has been shown in (Mug30). Another relevant application of Theorem 1 arises in the cases of matrices $\mathfrak{S}$ associated with permutations π of the set E induced by graph symmetries. A *symmetry* (or *automorphism*) of a graph G is a permutation matrix on $\mathbb{C}^{|\mathsf{V}|}$ that commutes with the adjacency matrix. By the celebrated Frucht's theorem, if Γ is a (possibly infinite) group, then there exist infinitely many graphs G such that Γ is isomorphic to the group of all symmetries of G.

The following analogue of Frucht's theorem has been obtained in (Mug30).

Theorem 4. *Let Γ be a (possibly infinite) group. Then there exist infinitely many graphs G such that Γ is isomorphic to a subgroup of $\mathfrak{A}(\mathsf{G})$.*

A result analogous to Theorem 1 has been applied to the case of a Schrödinger operator with magnetic and electric potentials in (Mug22), where the relation with the gauge group $U(1)$ of electrodynamics has also been pointed out. In fact, this kind of symmetries are analogous to spin rotations for usual (scalar-valued) Schrödinger equations. On the other hand, it has been shown in (Mug13) that $(e^{isP_{\mathcal{Z}}})_{s\in\mathbb{R}}$ is not a global symmetry group of (Ki) whenever $\mathcal{Z}$ has a more general structure that does not satisfy (4).

Proposition 5. *Let $\mathcal{P}_{\mathcal{Z}} = (P_{Z_x})_{x\in(0,1)}$ be the orthogonal projection onto a closed subspace*

$$\mathcal{Z} = \{f \in L^2(0,1;\mathbb{C}^{|\mathsf{E}|}) : f(x) \in Z_x \text{ for a.e. } x \in (0,1)\}. \tag{5}$$

of $L^2(\mathsf{G})$, for a family of closed subspaces $(Z_x)_{x\in(0,1)}$. Then the quadratic form a introduced above satisfies

$$a(e^{is\mathcal{P}_{\mathcal{Z}}} f) = \sum_{\mathsf{e}\in\mathsf{E}} \int_0^1 \|(e^{is}-1)e^{-isP_{Z_x}}(\nabla P_{Z_x}) f_{\mathsf{e}}(x) + f'_{\mathsf{e}}(x)\|^2_{\mathbb{C}^{|\mathsf{E}|}} dx, \qquad f \in H^1(\mathsf{G}),$$

provided that $P_{\mathcal{Z}} H^1(\mathsf{G}) \subset H^1(\mathsf{G})$.

In particular $(e^{is\mathcal{P}_{\mathcal{Z}}})_{s\in\mathbb{R}}$ *is a global symmetry group if and only if* $\nabla P_{\cdot} \equiv 0$ *a.e.*

In physical language, this framework can be seen as a kind of gauge theory of local symmetries where each P_{Z_x} is an orthogonal projection in a different fibre of a bundle. In other words, plugging into the form a unitary group associated with a general orthogonal projection is equivalent to considering a covariant derivative defined by

$$\nabla_s f := \nabla f + (e^{is} - 1)e^{-isd(P_{\cdot})}(\nabla P_{\cdot})f, \qquad f \in H^1(\mathsf{G}),$$

where the role of the gauge field is played by the multiplier

$$(e^{is} - 1)e^{-isd(P_{\cdot})}(\nabla P_{\cdot}) = (e^{isd(P_{\cdot}^{\perp})} - e^{-isd(P_{\cdot})})(\nabla P_{\cdot}).$$

2.2. **Generalized boundary conditions of Kuchment-type and parabolic systems.** (This section is based on the results obtained in (Mug10, Mug11, Mug12, Mug16, Mug17, Mug20).)

So far, we have only considered continuity/Kirchhoff conditions. Also relevant are the so-called *anti-continuity/anti-Kirchhoff* conditions

$$\begin{cases} \iota_{\mathsf{ve}} u'_{\mathsf{e}}(t,\mathsf{v}) &= \iota_{\mathsf{vf}} u'_{\mathsf{f}}(t,\mathsf{v}), \quad t \ge 0,\ \mathsf{e},\mathsf{f} \in \Gamma(\mathsf{v}),\ \mathsf{v} \in \mathsf{V}, \\ \sum_{\mathsf{e}\in\mathsf{E}} u_{\mathsf{e}}(t,\mathsf{v}) &= 0, \qquad\qquad t \ge 0,\ \mathsf{v} \in \mathsf{V}, \end{cases}$$

that arise in approximation schemes, cf. [Kuc04, FKW07, Pos07, ACF07], but many more – in fact, infinitely many – can be considered. V. Kostrykin and R. Schrader have been the first ones to discuss in [KS99] most general (non-dynamic) node conditions in the spirit of the well-known approach of S. Agmon, A. Douglis and L. Nirenberg, and to characterise their self-adjoint character following Friedrichs' extension theory. Soon after, P. Kuchment has proposed in [Kuc04] a parametrisation of self-adjoint (non-dynamic) node conditions: Kuchment's formulation has the great advantage to fit the variational setting neatly. It seems that (Mug12) has been one of the first articles where this feature has been extensively exploited. Kuchment's idea is to consider the reference "boundary space" $\mathbb{C}^{2|\mathsf{E}|}$ and a subspace Y of it, and then to impose conditions

(6) $\quad (u(0,t), u(1,t)) \in Y \qquad$ and $\qquad (-u'(0,t), u'(1,t)) + R(u(0,t), u(1,t)) \in Y^{\perp}, \qquad t \ge 0,$

for some Hermitian $2|\mathsf{E}| \times 2|\mathsf{E}|$-matrix R.[3] It turns out that this particular formulation of boundary conditions allows for an elegant application of the theory of sesquilinear forms.

In fact, there is no particular reason for restricting oneself to the one-dimensional setting, i.e., to abstract Cauchy problems in the function spaces $L^2(0,1;\mathbb{C}^{|\mathsf{E}|})$. It is natural to replace the interval $(0,1)$ by an open domain Ω of $\mathbb{R}^n$ and $\mathbb{C}^{|\mathsf{E}|}$ by a general Hilbert space H, so that the relevant state space is now the Bochner space $L^2(\Omega;H)$ – or more generally $L^p(\Omega;H)$. The boundary conditions are generally defined by a subspace $\mathcal{Y}$ of the "boundary space" $L^p(\partial\Omega;H)$. In the following we will always assume that

- Ω is an open bounded domain of $\mathbb{R}^n$ with boundary $\partial\Omega$ smooth enough (a Lipschitz boundary will do in most cases, but a C^2-boundary is needed for a few technical arguments);
- H is a separable Hilbert space.

[3] Observe that only the part of R in Y – i.e., $R_{|Y} := P_Y R P_Y$ – effectively imposes boundary conditions. Hence, (6) is equivalent to

$$P_{Y^\perp}(u(0,t), u(1,t)) = 0 \qquad \text{and} \qquad P_Y(-u'(0,t), u'(1,t)) + R_{|Y}(u(0,t), u(1,t)) = 0, \qquad t \ge 0,$$

which is actually Kuchment's original formulation.

To the best of our knowledge, this extension of Kuchment's theory to the case of higher-dimensional domains Ω and of possibly infinite dimensional target Hilbert spaces H is novel.

The case of a finite-dimensional Hilbert space H corresponds to the case of finite networks, which is common in applications and is the only one that has been considered in [Kuc04, FKW07, KKVW09]. However, the above formalism is flexible enough to accommodate problems that are not apparently related to networks, like reaction-diffusion systems or population models, and to allow for characterisation of their symmetries and further invariance properties. The setting of network diffusion problems can also be extended to a more general non-local problem, admitting some kind of interaction between edges that do not touch. Theorems 6 has been proved in (Mug12).

Theorem 6. *Let $\mathcal{Y}$ be a closed subspace of $L^2(\partial\Omega;H)$. Let $C \in L^\infty(\Omega;\mathcal{L}(H))$ and $D \in C^1(\overline{\Omega};\mathcal{L}(H^n))$ such that*

$$\mathrm{Re}(D(x)\theta|\theta)_{H^n} \geq \gamma\|\theta\|^2_{H^n} \qquad \text{for all } \theta \in H^n \text{ and a.e. } x \in \Omega$$

is satisfied for some $\gamma > 0$. Let finally $R \in \mathcal{L}(H^1(\Omega;H),\mathcal{Y})$. Then the abstract Cauchy problem

$$\text{(VV)} \qquad \begin{cases} \dot{u}(t,x) &= \nabla\cdot(D\nabla u)(t,x) + C(x)u(t,x), & t \geq 0,\ x \in \Omega, \\ u(t)_{|\partial\Omega} &\in \mathcal{Y}, & t \geq 0, \\ \frac{\partial_D u(t)}{\partial\nu} + Ru(t) &\in \mathcal{Y}^\perp, & t \geq 0, \\ u(0,x) &= u_0(x), & x \in \Omega, \end{cases}$$

is governed by a strongly-continuous contractive analytic semigroup on $L^2(\Omega;H)$.

(Here $\frac{\partial_D}{\partial\nu}$ denotes the conormal derivative associated with the operator matrix D.)

In the special case of networks (i.e., $\Omega = (0,1)$) the above setting has been specifically considered in (Mug12), generalising the setting in [Ali89]. Well-posedness of even more general L^2-problems has been proved in [AN93], in whose setting even structures consisting of bulding blocks of different dimensions can be treated.

Example 7. *Taking into account* (3) *one sees that the class of Kuchment's conditions includes continuity/Kirchhoff ones. As observed in* (3)*, continuity/Kirchhoff node conditions for networks are realised setting*

$$\mathcal{Y} := \mathrm{Range}\tilde{\mathcal{I}},$$

where $\tilde{\mathcal{I}}$ is defined in (2)*, if $H = \mathbb{C}^{|\mathsf{E}|}$. Observe that the associated anti-continuity/anti-Kirchhoff conditions are expressed by means of the orthogonal subspace $\mathcal{Y}^\perp$: for general domains of $\mathbb{R}^n$ anti-continuity/anti-Kirchhoff boundary conditions formally arise whenever coupled parabolic-hyperbolic problems are considered, see e.g.* [DL92, § XVIII.7.5.1]*. Letting $\mathcal{Y} = \mathbb{C}^{2|\mathsf{E}|}$ or $\mathcal{Y} = \{0\}$ leads to considering decoupled intervals with Neumann or Dirichlet boundary conditions, respectively. More generally, node conditions of Kuchment's type always come in pairs and are in general of non-local type.*

Apart from some technicalities related to the vector-valued setting, the proof of Theorem 6 is not very different from that originally provided in [Bel85] for the case of (Ki), i.e., of $\Omega = (0,1)$ and $\mathcal{Y} = \mathrm{Range}\tilde{\mathcal{I}}$. In the general case of $R \neq 0$, the main ingredient of its extension is the following lemma along with a Gagliardo–Nirenberg-type inequality.

Lemma 8. *Let $\mathcal{V},\mathcal{H}$ be Hilbert spaces such that $\mathcal{V}$ is continuously and densely embedded in $\mathcal{H}$. Let $\mathfrak{a}: \mathcal{V}\times\mathcal{V} \to \mathbb{C}$ be a sesquilinear mapping. Let $\alpha \in [0,1)$ such that $\mathfrak{b}: \mathcal{V}\times\mathcal{H}_\alpha \to \mathbb{C}$ and $\mathfrak{c}: \mathcal{H}_\alpha\times\mathcal{V} \to \mathbb{C}$ are continuous sesquilinear mappings, where $\mathcal{H}_\alpha$ is some vector space such that $\mathcal{V} \hookrightarrow \mathcal{H}_\alpha \hookrightarrow \mathcal{H}$ and verifying the interpolation inequality*

$$\|f\|_{\mathcal{H}_\alpha} \leq M_\alpha \|f\|^\alpha_{\mathcal{V}} \|f\|^{1-\alpha}_{\mathcal{H}}, \qquad f \in \mathcal{V},$$

for some $M_\alpha > 0$. Then $\mathfrak{a}$ is $\mathcal{H}$-elliptic if and only if $\mathfrak{a}+\mathfrak{b}+\mathfrak{c} : \mathcal{V}\times\mathcal{V}\to\mathbb{C}$ is $\mathcal{H}$-elliptic.

The above lemma is comparable with the results in [Kat95, §VI.1.6] and it has been proved in (Mug20) – where it is also applied to investigate a class of damped wave equations, a setting where usual Desch–Schappacher-type perturbation results are not useful.

Investigating the sesquilinear form associated with the weak formulation of (VV) promptly leads to the following, via a celebrated result due to M. Crouzeix (see e.g. [Cro07]).

Corollary 9. *Under the assumptions of Theorem 6, let additionally $D(x)\in\mathcal{L}(H^n)$ be self-adjoint for all $x\in\overline{\Omega}$. Let furthermore*

$$Ru = S(u_{|\partial\Omega}) \qquad \text{for all } u\in\{v\in H^1(\Omega;H) : v_{|\partial\Omega}\in\mathcal{Y}\} \tag{7}$$

for some $S\in\mathcal{L}(\mathcal{Y})$. Then the (undamped) wave equation associated with (VV)*, i.e.,*

$$\begin{cases} \ddot u(t,x) &=& \nabla\cdot(D\nabla u)(t,x)+C(x)u(t,x), & t\ge 0,\ x\in\Omega,\\ u(t)_{|\partial\Omega} &\in& \mathcal{Y}, & t\ge 0,\\ \frac{\partial_D u(t)}{\partial\nu}+S(u(t)_{|\partial\Omega}) &\in& \mathcal{Y}^\perp, & t\ge 0,\\ u(0,x) &=& u_0(x), & x\in\Omega,\\ \dot u(0,x) &=& u_1(x), & x\in\Omega, \end{cases}$$

has a unique mild solution that continuously depends on the initial data $u_0,u_1\in L^2(\Omega;H)$.

In the special case of finite networks, the above well-posedness result has been observed in [Sch09]. Two different extensions of Corollary 9 to the case of damped wave equations have been obtained in (Mug16, Mug20).

In general, extending an L^2-well-posedness result like Theorem 6 to further L^p-spaces is a difficult task. However, things are much easier if a suitable locality assumption is imposed. Recall that each Hilbert lattice is isometrically lattice isomorphic to $L^2(\Theta)$, for some finite measure space Θ.

Theorems 10 and 11 have been proved in (Mug12).

Theorem 10. *Assume that there exist a closed subspace Y of H such that*

$$\mathcal{Y}=\{f\in L^2(\partial\Omega;H) : f(x)\in Y \text{ for a.e. } x\in\partial\Omega\} \tag{8}$$

and a bounded linear operator S on Y such that

$$(Ru)(z)=S(u(z)) \qquad \text{for all } u\in\{v\in H^1(\Omega;H) : v_{|\partial\Omega}\in\mathcal{Y}\} \text{ and all } z\in\partial\Omega. \tag{9}$$

For $k\in\mathbb{N}$ assume that $D,C\in C^k(\overline{\Omega};\mathcal{L}(H^n))$ and the domain Ω to have C^{2k}-boundary. Then the semigroup that governs (VV) maps $L^2(\Omega;H)$ into $H^{2k}(\Omega;H)$.

In the scalar-valued case, an analogous result is due to H. Brezis and based on L. Nirenberg's technique of incremental quotients for checking properties of Sobolev spaces. We emphasize that the boundary condition (8) is of local nature, in contrast to the case of a boundary condition defined by a general subspace $\mathcal{Y}$ of $L^2(\partial\Omega;H)$. This is essential in allowing to extend Brezis' and Nirenberg's approach to the general case of vector-valued diffusion equations.

Our aim is to discuss invariance of order intervals under the diffusion semigroups that govern (VV). In this way it is possible to characterise their $\|\cdot\|_\infty$-contractivity, which in turn permits to apply Riesz–Thorin's interpolation theorem and to deduce generation of a (contractive) semigroup on all L^p-spaces.

Theorem 11. *Under the assumptions of Theorem 6 let H be a Hilbert lattice, so that $L^2(\Omega; H) \cong L^2(\Omega \times \Theta; \mathbb{C})$, and fix an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of H. Let $D(x)$ be diagonal for all $x \in \Omega$, in the sense that the operator $D(x) = (D_{ij}(x)) \in \mathcal{L}(H^n)$ has to satisfy*

$$(D_{ij}(x)e_k|e_\ell) = 0 \qquad \text{for all } k \neq \ell \text{ and all } x \in \Omega.$$

Assume that

- *there exist a closed subspace Y of H and a bounded linear operator S on Y such that (8) and (9) hold and moreover the orthogonal projection of H onto Y is contractive with respect to the norm of $L^\infty(\Theta; \mathbb{C})$,*
  or else
- *there exist a closed subspace Y of $H \times H$ and a bounded linear operator S on Y such that $\partial\Omega = \partial\Omega_1 \dot\cup \partial\Omega_2$, the identity*

$$\mathcal{Y} = \left\{ f \in L^2(\partial\Omega; H) : \begin{pmatrix} f(z_1) \\ f(z_2) \end{pmatrix} \in Y \text{ for a.e. } z_1 \in \partial\Omega_1,\ z_2 \in \partial\Omega_2 \right\}$$

  and (9) hold and moreover the orthogonal projection of $H \times H$ onto Y is contractive with respect to the norm of $L^\infty(\Theta; \mathbb{C}) \times L^\infty(\Theta; \mathbb{C})$.

Then the Cauchy problem (VV) is governed by a strongly-continuous semigroup $(T_p^{Y,S}(t))_{t\geq 0}$ on $L^p(\Omega; H)$ for all $p \in [1, \infty)$, which is analytic for all $p \in (1, \infty)$. The semigroup is contractive if S and C are contractive with respect to the norms of $L^\infty(\Theta; \mathbb{C})$ and $L^\infty(\Omega \times \Theta; \mathbb{C})$, respectively.

The proof of L^p-contractivity is based on an abstract characterisation of invariance properties, which we present next. It has been obtained in (Mug12, Mug17) by means of the above mentioned invariance result of Ouhabaz. It is an important tool that permits to characterise maximum principles and symmetries of vector-valued equations like (VV) in a very efficient way.

Theorem 12. *Under the assumptions of Theorem 11, assume that there exist a closed subspace Y of H, a closed convex subset C_H of H and operators $B \in \mathcal{L}(H)$, $S \in \mathcal{L}(Y)$ such that*

- *$\mathcal{Y} = \{f \in L^2(\partial\Omega; H) : f(z) \in Y$ for a.e. $z \in \partial\Omega\}$,*
- *$C_{L^2(\Omega;H)} := \{f \in L^2(\Omega; H) : f(x) \in C_H$ for a.e. $x \in \Omega\}$,*
- *$(Cu)(x) = B(u(x))$ for all $u \in H^1(\Omega; H)$ and a.e. $x \in \Omega$,*
- *$(Ru)(z) = S(u(z))$ for all $u \in \{v \in H^1(\Omega; H) : v(z) \in Y$ for a.e. $z \in \partial\Omega\}$ and a.e. $z \in \partial\Omega$.*

Assume that $0 \in C$, or else $\partial\Omega$ to have finite measure.

Then $C_{L^2(\Omega;H)}$ is left invariant under $(T_2^{Y,S}(t))_{t\geq 0}$ if and only if

(i) the inclusion $P_Y C_H \subset C_H$ holds and additionally
(ii) the semigroups $(e^{-tB})_{t\geq 0}$ and $(e^{-tS})_{t\geq 0}$ leave C_H invariant.

Even in the case of uncoupled boundary conditions – i.e., of weakly coupled systems, treated in (Mug10) – the above permits to recover several results previously obtained in [MS95].

Along with the general boundary conditions considered in (VV), it is natural to treat dynamic ones, as a vector-valued pendant of Wentzell–Robin boundary conditions for diffusion on domains. Early results on this kind of diffusion problems are due to J. von Below, S. Nicaise and F. Ali Mehmeti. They have developed an L^2-theory – see e.g. [AN93, BN96] – but only for a very specific choice of couplings since they were mainly interested in discussing networks and interface problems.

Theorems 6, 11 and 12 can be extended to the case of diffusive systems with dynamic boundary conditions, like

$$\text{(VVD)} \qquad \begin{cases} \dot{u}(t,x) &=& \nabla\cdot(D\nabla u)(t,x), & t\geq 0,\ x\in\Omega,\\ u(t)_{|\partial\Omega} &\in& \mathcal{Y}, & t\geq 0,\\ \frac{\partial}{\partial t}u(t)_{|\partial\Omega} &=& -P_{\mathcal{Y}}\frac{\partial_D u(t)}{\partial\nu}+R_1 u(t)_{|\partial\Omega}, & t\geq 0,\\ u(0,x) &=& u_0(x), & x\in\Omega,\\ u(0,z) &=& u_1(z), & z\in\partial\Omega, \end{cases}$$

for some $R\in\mathcal{L}(H^1(\Omega;H);\mathcal{Y})$. This setting has been thoroughly discussed in (Mug17), where in particular its well-posedness in the Hilbert space $L^2(\Omega;H)\times\mathcal{Y}$ has been observed.

The following observation has been obtained in (Mug17). In the special case of $H=\mathbb{C}$, Corollary 13 explains why the same order intervals are left invariant under the diffusion semigroup generated by the Laplacian with Robin and Wentzell–Robin boundary conditions.

Corollary 13. *Under the assumptions of Theorem 12, the closed convex subset*

$$C_{L^2(\Omega;H)}\times\{f\in\mathcal{Y}: f(z)\in C_H \text{ for a.e. } z\in\partial\Omega\}\subset L^2(\Omega;H)\times\mathcal{Y}$$

is left invariant under the strongly-continuous semigroup governing (VVD) *if and only if* $C_{L^2(\Omega;H)}$ *is left invariant under* $(T_2^{Y,S})_{t\geq 0}$.

2.3. **The special case of parabolic network equations.** (This section is based on the results obtained in (Mug7, Mug8, Mug17).)

In the particular case of networks, applying Theorem 12 yields well-posedness of a wide class of parabolic problems in the $L^p(\mathsf{G})$-spaces. Moreover, we can push the analysis further and obtain a kernel representation of the solution operator (i.e., of the semigroup that governs the problem) along with Gaussian-type estimates for this kernel. Gaussian estimates are known to be a fundamental tool in a manifold of contexts, including investigation of analyticity angle of linear semigroups, analysis of semilinear and quasilinear problems, maximal regularity issues, stochastic control theory...

The following results have been established in (Mug7).

Theorem 14. *Let* $p\in[1,\infty)$. *Assume both the* $|\mathsf{V}|\times|\mathsf{V}|$*-matrix* $M=(m_{\mathsf{vw}})$ *and its adjoint to generate* $\|\cdot\|_\infty$*-contractive semigroups, i.e., assume that*

$$\text{(10)} \qquad \mathrm{Re} m_{\mathsf{vv}}+\sum_{\mathsf{w}\neq\mathsf{v}}|m_{\mathsf{vw}}|\leq 0 \quad \text{and} \quad \mathrm{Re} m_{\mathsf{vv}}+\sum_{\mathsf{w}\neq\mathsf{v}}|m_{\mathsf{wv}}|\leq 0 \qquad \text{for all } v\in\mathsf{V}.$$

Then by Theorem 12 the Cauchy problem

$$\text{(DP)} \qquad \begin{cases} \dot{u}_{\mathsf{e}}(t,x) &=& u''_{\mathsf{e}}(t,x), & t\geq 0,\ x\in(0,1),\ \mathsf{e}\in\mathsf{E},\\ u_{\mathsf{e}}(t,\mathsf{v}) &=& u_{\mathsf{f}}(t,\mathsf{v})=:\psi_{\mathsf{v}}(t), & t\geq 0,\ \mathsf{e},\mathsf{f}\in\Gamma(\mathsf{v}),\ \mathsf{v}\in\mathsf{V},\\ \sum_{\mathsf{e}\in\mathsf{E}}\iota_{\mathsf{ve}}u'_{\mathsf{e}}(t,\mathsf{v}) &=& \sum_{\mathsf{w}\in\mathsf{V}}m_{\mathsf{vw}}\psi_{\mathsf{w}}(t), & t\geq 0,\ \mathsf{v}\in\mathsf{V},\\ u_{\mathsf{e}}(0,x) &=& u_{0\mathsf{e}}(x), & x\in(0,1),\ \mathsf{e}\in\mathsf{E}, \end{cases}$$

is governed by a strongly-continuous contractive semigroup $(T_p(t))_{t\geq 0}$ *on* $L^p(\mathsf{G})$ *for all* $p\in[1,\infty)$, *which is analytic for* $p\in(1,\infty)$. *The following assertions hold.*

(1) The semigroup $(T_p(t))_{t\geq 0}$ *on* $L^p(\mathsf{G})$ *is ultracontractive. Moreover,* $T_p(t)$ *is a kernel operator for all* $t>0$*, i.e.,*

$$T_p(t)f(x)=\int_0^1 k_t(x,y)f(y)dy,\qquad x\in(0,1),\ f\in L^p(\mathsf{G}),$$

where the kernel $k_t(\cdot,\cdot)$ *is a function of class* $L^\infty((0,1)\times(0,1);M_{|\mathsf{E}|}(\mathbb{C}))$ *for all* $t>0$*.*

(2) Let additionally M *be diagonal. We define an isometric isomorphism* U *from* $L^p(0,1;\mathbb{C}^{|\mathsf{E}|})$ *onto* $L^p(0,|\mathsf{E}|;\mathbb{C})$ *by*

$$Uf(x):=\tilde f(x):=f_j(x-j+1)\qquad \text{if } x\in(j-1,j),\ j\in\{1,\ldots,|\mathsf{E}|\},\qquad f:[0,1]\to\mathbb{C}^{|\mathsf{E}|},$$

and denote by $\tilde k_t:(0,|\mathsf{E}|)\times(0,|\mathsf{E}|)\to(0,\infty)$ *the kernel associated with the operator.* $UT_p(t)U^{-1}$*. Then the semigroup* $(UT_p(t)U^{-1})_{t\geq 0}$ *on* $L^p(0,|\mathsf{E}|;\mathbb{C})$ *has Gaussian estimates, i.e., there exist constants* $b,c>0$ *such that* $\tilde k_t$ *satisfies*

$$0\leq \tilde k_t(x,y):=Uk_t(x,y)U^{-1}\leq ct^{-\frac12}e^{-\frac{b|x-y|^2}{t}+t},\qquad x,y\in(0,|\mathsf{E}|),\tag{11}$$

uniformly in $t\in(0,1]$*.*

It follows that for any matrix M *that satisfies* (10)*, also* $(T_1(t))_{t\geq 0}$ *is analytic. Furthermore, the angle of analyticity of all semigroups* $(T_p(t))_{t\geq 0}$*,* $p\in[1,\infty)$*, is* $\frac{\pi}{2}$*.*

The above theorem generalises some results obtained by V. Kostrykin, J. Potthoff and R. Schrader. More precisely, sufficient conditions for the Feller property have been obtained in [KPS08]. The same characterisation of the Markov property has been obtained simultaneously by independently by U. Kant, T. Klauss, J. Voigt and M. Weber in [KKVW09].

The short-time kernel estimate in (11) is one of the main results presented in this résumé. It has been later complemented by the long-time estimate established by M. Pang in [Pan09] (in the case of the infinite lattice $\mathbb{Z}_2$ only). Explicit formulae for the heat kernel in the case of $M=0$ have been obtained by S. Nicaise and C. Cattaneo in [Nic87, Cat98].

We mention the following important consequence of Theorem 14.

Corollary 15. *If Theorem 14.(2) applies, then* (DP) *is also governed by a strongly-continuous, positive, analytic semigroup on the space*

$$C(\mathsf{G}):=\left\{f\in C([0,1];\mathbb{C}^{|\mathsf{E}|}):\begin{pmatrix}f(0)\\ f(1)\end{pmatrix}\in \mathrm{Range}\tilde{\mathcal{I}}\right\}$$

of continuous functions over G*. Its analyticity angle is* $\frac{\pi}{2}$*.*

The proof is based on the sub-Markovian property enjoyed by the semigroup in $L^2(\mathsf{G})$, ensuring that the part of the resolvent of Δ in $L^\infty(\mathsf{G})$ is a positive operator that satisfies "good" sectoriality estimates; and on the mentioned Gaussian estimates, which imply that the same properties are enjoyed by part of the resolvent of Δ in $C(\mathsf{G})$. This approach has been inspired by a similar argument in [AMPR03]. While Gaussian estimates are more delicate, one may expect that at least the sub-Markovian property is enjoyed by all diffusion semigroups $(T_2^{Y,S}(t))_{t\geq 0}$ on graphs, at least if $S=0$. Unfortunately, this is not true at all, as shown in (Mug17).

Example 16. *If* $Y=\mathrm{Range}\tilde{\mathcal{I}}$*, then* $\tilde{\mathcal{I}}$ *is a row-stochastich* $0-1$*-matrix and it can be easily seen that* P_Y *is a Markovian matrix, and hence so is the semigroup that governs* (Ki)*.*

However, for general Y *the Markovian property of* P_Y *is a rare feature. Even in the almost trivial case of* $H=\mathbb{C}^2$ *it follows from Theorem 12 that the semigroup* $(T_2^{Y,0}(t))_{t\geq 0}$ *is* $\|\cdot\|_\infty$*-contractive if and only if* Y *agrees with any of the following subspaces:*

- $\{(0,0)\}$ *(decoupled Dirichlet boundary conditions);*
- $\mathbb{C}^2$ *(decoupled Neumann boundary conditions);*
- $\mathbb{C}\times\{0\}$ *(decoupled Neumann/Dirichlet boundary conditions);*
- $\{0\}\times\mathbb{C}$ *(decoupled Dirichlet/Neumann boundary conditions);*
- $\langle(1,1)\rangle$ *(continuity/Kirchhoff conditions);*
- $\langle(1,-1)\rangle$ *(anti-continuity/anti-Kirchhoff conditions.*

Furthermore, it is Markovian (i.e., positive and $\|\cdot\|_\infty$-contractive) if and only if any of the first five cases holds.

In some sense, continuity/Kirchhoff node conditions represent a singular case among those giving rise to a self-adjoint realisation of the Laplacian. It is still unknown whether $(T_2^{Y,0}(t))_{t\geq 0}$ extends to all (or even just some) $L^p(\mathsf{G})$-spaces whenever it fails to be Markovian.

In the case of networks, dynamic node conditions are particularly motivated by their appearance in biomathematical models. Laplace operators with dynamic boundary conditions on networks also appear as limiting cases of 3D-1D approximation schemes proposed by P. Kuchment, H. Zeng, P. Exner and O. Post, among others in [KZ03, EP05].

The following has been obtained already in (Mug8). It follows from Theorem 6 and Corollary 13.

Theorem 17. *Let $M=(m_{\mathsf{vw}})$ be a $|\mathsf{V}|\times|\mathsf{V}|$-matrix. Consider the abstract Cauchy problem*

$$\text{(DPD)}\qquad \left\{\begin{array}{rcll} \dot u_{\mathsf{e}}(t,x) &=& u''_{\mathsf{e}}(t,x), & t\geq 0,\ x\in(0,1),\ \mathsf{e}\in\mathsf{E},\\ u_{\mathsf{e}}(t,\mathsf{v}) &=& u_{\mathsf{f}}(t,\mathsf{v}) =: \psi_{\mathsf{v}}(t), & t\geq 0,\ \mathsf{e},\mathsf{f}\in\Gamma(\mathsf{v}),\ \mathsf{v}\in\mathsf{V},\\ \dot\psi_{\mathsf{v}}(t) &=& -\sum_{\mathsf{e}\in\mathsf{E}}\iota_{\mathsf{ve}}u'_{\mathsf{e}}(t,\mathsf{v})+\sum_{\mathsf{w}\in\mathsf{V}} m_{\mathsf{vw}}\psi_{\mathsf{w}}(t), & t\geq 0,\ \mathsf{v}\in\mathsf{V},\\ u_{\mathsf{e}}(0,x) &=& u_{0\mathsf{e}}(x), & x\in(0,1),\ \mathsf{e}\in\mathsf{E},\\ \psi_{\mathsf{v}}(0) &=& \psi_{0\mathsf{v}}, & \mathsf{v}\in\mathsf{V}. \end{array}\right.$$

Then the following assertions hold.

(1) (DPD) *is governed by a strongly-continuous, analytic semigroup on $L^2(\mathsf{G})\times\mathbb{C}^{|\mathsf{V}|}$.*

(2) If (10) *holds, then* (DPD) *is governed by a strongly-continuous ultracontractive semigroup $(T_p(t))_{t\geq 0}$ on $L^p(\mathsf{G})\times\mathbb{C}^{|\mathsf{V}|}$ that is analytic for $p\in(1,\infty)$; in this case $T_p(t)$ is a kernel operator for all $t>0$.*

Before concluding our discussion of network problems we mention that the vector-valued setting poses a severe restriction to irreducibility (i.e., invariance of no non-trivial closed ideals) of diffusion processes . In fact, not even the heat semigroup is irreducible if $\dim H>1$: e.g., the strongly-continuous semigroup on $L^2(\mathbb{R};\mathbb{C}^2)$ generated by the Laplacian leaves invariant the closed ideal $L^2(\mathbb{R};\{0\}\times\mathbb{C})$. However, irreducibility can be restored by means of lower order terms or boundary conditions: in particular, $(T_2^{Y,0}(t))_{t\geq 0}$ is irreducible if and only if the orthogonal projection of H^2 onto Y is an irreducible operator. A typical example is given by the orthogonal projection onto $Y=\mathrm{Range}\tilde{\mathcal{I}}$, if $\mathcal{I}$ is the incidence matrix of a connected graph G. Then, the semigroup $(T_2^{Y,0}(t))_{t\geq 0}$ that governs (Ki) converges towards the rank-1 orthogonal projection P onto the constant functions at a rate given by

$$\|T_2^{Y,0}(t)-P\|\leq Me^{(\epsilon+\lambda_2)t}$$

for all $t\geq 0$, all $\epsilon>0$ and some $M>0$, where λ_2 denotes the second eigenvalue of the elliptic operator with continuity/Kirchhoff conditions. In the special case of the Laplacian several known results of spectral graph theory can be applied (see e.g. [Moh97]), since a result due to von Below ([Bel85]) relates λ_2 and the eigenvalues of the graph Laplacian of G.

2.4. **Vectör-valued analysis.** (This section is based on the results obtained in (Mug13, Mug14).)

In typical cases we want to apply the criterion in Theorem 12 to an order interval C_H of H. Although u^+ is formally given by the composition of a Lipschitz continuous mapping on H and a function in $H^1(\Omega;H)$, providing a chain rule yielding an explicit formula for $\nabla(u^+)$ is not trivial. This is due to the fact that Rademacher's theorem fails to hold in infinite dimensional spaces and it is in particular not easy to understand in which sense the orthogonal projection of H onto H_+ is "differentiable a.e.", as one would expect in the finite dimensional case. In order to work out the details, the following result is needed.

Lemma 18. *Let H be a Hilbert lattice, so that $L^2(\Omega;H) \cong L^2(\Omega\times\Theta;\mathbb{C})$. Let $u,v\in H^1(\Omega;H)$. Then also $\max\{u,v\}\in H^1(\Omega;H)$. Furthermore,*

$$\begin{aligned}\nabla\max\{u,v\}(\cdot,\theta) &= \mathbf{1}_{\{u(\cdot,\theta)\geq v(\cdot,\theta)\}}\nabla u(\cdot,\theta)+\mathbf{1}_{\{u(\cdot,\theta)<v(\cdot,\theta)\}}\nabla v(\cdot,\theta),\\ \nabla(u-v)^-(\cdot,\theta) &= \mathbf{1}_{\{u(\cdot,\theta)<v(\cdot,\theta)\}}\nabla v(\cdot,\theta),\end{aligned}$$

for a.e. $\theta\in\Theta$.

The above formulae are meant as identities of functions in $L^2(\Omega;\mathbb{C})$. In particular, each "slice" $u(\cdot,\theta)$ defines a scalar-valued function on Ω: it is the differential of this slice-function that is denoted by $\nabla u(\cdot,\theta)$.

Another possible application of Theorem 12 is given by the characterisation of irreducibility of $(e^{t\Delta_Y})_{t\geq 0}$, i.e., the property of not leaving invariant any nontrivial closed ideals of $L^2(\Omega;H)$. However, to begin with it is necessary to characterise all closed ideals of $L^2(\Omega;H)$. One can see that each family $(P_{Z_x})_{x\in\Omega}$ such that P_{Z_x} is an orthogonal projection onto a closed ideal of H, $x\in\Omega$, defines an orthogonal projection onto a closed ideal of $L^2(\Omega;H)$. In fact, the converse also holds: it is a nontrivial result based on a characterisation (obtained by H. Vogt in [Vog09]) of orthogonal projections onto closed subspaces of $L^2(\Omega;H)$ as those families $(P_{Z_x})_{x\in\Omega}$ such that P_{Z_x} is an orthogonal projection onto a closed subspace of H and satisfying a certain locality assumption. Such a locality assumption can be dropped if one only considers orthogonal projections onto closed ideals, as shown in (Mug13).

Theorem 19. *Let H be a Hilbert lattice. For each orthogonal projection $P_{\mathcal{W}}$ onto a closed ideal $\mathcal{W}$ of $L^2(\Omega;H)$ there exists a family $(P_{Z_x})_{x\in\Omega}$ of orthogonal projections onto closed subspaces of H such that $x\mapsto P_{Z_x}$ is strongly measurable and such that*

$$(P_{\mathcal{W}}f)(x)=P_{Z_x}(f(x))\qquad \text{for a.e. } x\in\Omega \text{ and all } f\in L^2(\Omega;H). \tag{12}$$

The ranges of the orthogonal projections P_{Z_x} appearing in (12) are in fact closed ideals of H for a.e. $x\in\Omega$.

Let Theorem 11 apply. Under the additional regularity assumptions imposed in Theorem 10, $(T_2^{Y,S}(t))_{t\geq 0}$ maps $L^2(\Omega;H)$ into $H^{2k}(\Omega;H)\hookrightarrow L^\infty(\Omega;H)$, provided that $2k\geq\frac{n}{p}$. By a theorem due to L.V. Kantorovitch and B.Z. Vulikh extended to the vector-valued case in (Mug14), each operator $T_2^{Y,S}(t)$ is a kernel operator, i.e., there exists a mapping K_t such that

$$T_2^{Y,S}(t)f(x)=\int_\Omega K_t(x,y)f(y)dy,\qquad t>0,\ x\in\Omega.$$

In order to make precise the properties of this integral kernel, some notations have to be introduced.

Let F be a Banach space and (Ω,μ) be a measure space. Denote by $\mathcal{L}_{\sigma^*}(\Omega;F')$ the space of $\sigma(F',F)$-measurable [4], $\sigma(F',F)$-bounded functions from Ω to F'.

One of our main results is the following, taken from (Mug14). It deeply relies on the theory of tensor products of Banach spaces and lattices.

Theorem 20. *Let (Ω_1,μ_1) and (Ω_2,μ_2) be complete, strictly localisable measure spaces.*

- *If E,F are Banach spaces, then the mapping $k\mapsto T_k$ defined by*
$$\langle (T_k f)(\omega_2),v\rangle_{F',F} := \int_{\Omega_1} \langle k(\omega_1,\omega_2)f(\omega_1),v\rangle_{F',F}\, d\mu_1(\omega_1)$$
for all $v\in F$, $f\in L^1(\Omega_1;E)$ and μ_2-a.e. $\omega_2\in\Omega_2$ is well-defined. It is an isometric isomorphism from $L^\infty_{\sigma^*}(\Omega_1\times\Omega_2;\mathcal{L}(E,F'))$ to $\mathcal{L}(L^1(\Omega_1;E),L^\infty_{\sigma^*}(\Omega_2;F'))$.
- *If E,F are Banach lattices, then the kernel k is positive if and only if the operator T_k is positive and moreover $k\mapsto T_k$ is an isometric lattice isomorphism from $L^\infty_{\sigma^*}(\Omega_1\times\Omega_2;\mathcal{L}^r(E,F'))$ to $\mathcal{L}^r(L^1(\Omega_1;E),L^\infty_{\sigma^*}(\Omega_2;F'))$, where $\mathcal{L}^r$ denotes the space of regular operators.*
- *Let Ω_1 and Ω_2 be σ-finite measure spaces. If E,F are Hilbert spaces, then the mapping $k\mapsto T_k$ is an isometric isomorphism from $L^2(\Omega_1\times\Omega_2;\mathcal{L}_2(E,F))$ to $\mathcal{L}_2(L^2(\Omega_1;E);L^2(\Omega_2;F))$, where $\mathcal{L}_2$ denotes the space of Hilbert–Schmidt operators.*

2.5. **Biological models, stochastic and nonlinear differential equations.** (This section is based on the results obtained in (Mug9, Mug 10, Mug15).)

It has been known at least since the seminal works of Nobel laureates A.L. Hodgkin and A.F. Huxley that the spread of electric potential in the neurons can be modelled as a diffusive process. Ever since, parabolic network equations have been pervasive in theoretical literature on neuronal networks.

The cable model of a dendritic tree proposed by W. Rall in [Ral59] leads to network diffusion problems analogous to (DPD), cf. [Cam80, Nic85]. A thorough biomathematical investigation of them has been performed in a series of four papers by G. Major et al. beginning with [MEJ93]. This model has been investigated in detail in (Mug8), where several qualitative properties of (DPD) have been obtained. Moreover, its connections with Rall's linear cable theory for passive biological fibres have been discussed in the same paper, confirming the outcome of numerical simulations performed in [MEJ93].

Theorem 17 can also be applied in the study of so-called FitzHugh–Nagumo model of active propagation of electric activity in the neuron. This model is expressed by means of a coupled system consisting of a semilinear diffusion equation and a linear inhomogeneous ordinary differential equation for an auxiliary unknown R. In the case of a network, the FitzHugh–Nagumo model takes the form

$$(13)\quad \begin{cases} \frac{\partial}{\partial t}u_{\mathsf{e}}(t,x) = u''_{\mathsf{e}}(t,x)-u_{\mathsf{e}}(t,x)-\Theta_{\mathsf{e}}(u_{\mathsf{e}}(t,x))-R_{\mathsf{e}}(t,x), & t\ge 0,\ x\in(0,1),\ \mathsf{e}\in\mathsf{E},\\ \frac{\partial}{\partial t}R_{\mathsf{e}}(t,x) = \alpha_{\mathsf{e}}(x)u_{\mathsf{e}}(t,x)-\beta_{\mathsf{e}}(x)R_{\mathsf{e}}(t,x)+\zeta_{\mathsf{e}}(t), & t\ge 0,\ x\in(0,1),\ \mathsf{e}\in\mathsf{E},\end{cases}$$

[4] Here and in the following $f\colon\Omega\to F'$ is called $\sigma(F',F)$-measurable if the scalar function $\omega\mapsto\langle f(\omega),v\rangle$ is measurable for every $v\in F$, and it is called $\sigma(F',F)$-bounded if
$$\|f\|_{\sigma^*} := \sup_{v\in B_F}\operatorname*{ess\,sup}_{\omega\in\Omega}|\langle f(\omega),v\rangle|<\infty,$$
where B_F is the unit ball in F).

Elements f and g of $\mathcal{L}_{\sigma^*}(\Omega;F')$ are considered equivalent, for which we write $f\sim g$, if $\|f-g\|_{\sigma^*}=0$. Finally, we denote by $L^\infty_{\sigma^*}(\Omega;F') := \mathcal{L}_{\sigma^*}(\Omega;F')/\sim$ the space of equivalence classes with respect to this equivalence relation, equipped with the norm $\|[f]_\sim\|_\infty := \|f\|_{\sigma^*}$.

where each Θ_{e} is a polynomial of third degree. Theorem 17 has been exploited in (Mug9) in order to deduce the following global well-posedness result, by means of the theory of semigroups of nonlinear operators.

Theorem 21. *The following assertions hold.*

(1) The abstract Cauchy problem for the boundary-value problem obtained equipping (13) *with continuity and dynamic (resp., non-dynamic) Kirchhoff conditions as in (DP) (resp., as in (DPD)) is globally well-posed in $L^2(\mathsf{G}) \times L^2(\mathsf{G}) \times \mathbb{C}^{|\mathsf{V}|}$ (resp., in $L^2(\mathsf{G}) \times L^2(\mathsf{G})$), i.e., this Cauchy problem is governed by a semigroups of nonlinear operators.*

Moreover, such a semigroup does not leave invariant either the positive cone of L^2 nor the unit ball of L^∞, i.e.,

(2) there exists positive initial condition (u_0, R_0) such that the solution (u, R) satisfies $u_{\tilde{\mathsf{e}}}(t_0, x) < 0$ or $R_{\tilde{\mathsf{e}}}(t_0, x) < 0$ for some time t_0, some edge $\tilde{\mathsf{e}}$, and all x in a subset of non-zero Lebesgue measure of $(0, 1)$;

(3) there exists some initial conditions (u_0, R_0) such that $|u_0| \leq 1$ and $|R_0| \leq 1$ at a.e. point of the network but such that the solution (u, R) satisfies $u_{\tilde{\mathsf{e}}}(t_0, x) > 1$ or $R_{\tilde{\mathsf{e}}}(t_0, x) > 1$ for some time t_0, some edge $\tilde{\mathsf{e}}$, and all x in a set of non-zero Lebesgue measure of $(0, 1)$.

Comparable results have been later obtained in [BCP09, Ven09] in the case of a similar cardiac model, where the state space is instead $L^2(\Omega; \mathbb{C}^3)$. While the proof of global well-posedness for the nonlinear equation is based on a classical argument relying upon a combination of properties of maximal monotone operators and globally Lipschitz perturbations, the non-invariance results seem to be new. Although the results in Theorem 21 hold even in the linear case, i.e., even taking $\Theta_{\mathsf{e}} \equiv 0$, it is in the nonlinear setting that they offer new insights. In fact, it has been observed in (Mug9) that the above analysis is in accordance with experimental observations that corroborate the following description of transmembrane voltage's behaviour in excitable fibres during transmission of an action potential:

- before an action potential initiates, the transmembrane voltage is observed to cross the threshold value of approx. -55 mV (corresponding to ξ_1);
- the voltage quickly rises to approx. +40 mV, corresponding to the asymptotic signal amplitude ξ_2: depolarisation occurs;
- afterwards, voltage suddenly sinks to an undershoot value of approx. -80 mV, i.e. hyperpolarisation occurs;
- finally, voltage reaches its resting value of approx. -70 mV, corresponding to ξ.

Also motivated by biomathematical considerations, the result in Theorem 17 has been applied in (Mug15) to study a network diffusion equation with additive stochastic terms (fractional Brownian motion) in the dynamic as well as in the non-dynamic node conditions. In particular, the following holds.

Theorem 22. *Let $\mathcal{A}$ be a sectorial operator on a Banach space $\mathcal{X}$ and let $\mathcal{C}$ be a bounded linear operator mapping $\mathcal{X}$ into the complex interpolation space $[D(\mathcal{A}), \mathcal{X}]_\alpha$ for some $\alpha \in (0, 1)$. Consider the stochastic Cauchy problem*

$$\begin{cases} \mathrm{d}\mathfrak{v}(t) = \mathcal{A}\mathfrak{v}(t)\,\mathrm{d}t + \mathcal{C}_a\,\mathrm{d}Z^a(t), & t \geq 0, \\ \mathfrak{v}(0) = \mathfrak{v}_0, \end{cases} \tag{14}$$

where $Z^a(\cdot)$ is a fractional Brownian motion with Hurst parameter $h \in (3/4, 1)$ and take the initial condition $\mathfrak{v}_0 \in D(\mathcal{A})$. Then there exists a unique strong solution to equation (14).

The above theorem extends to the *fractional* Brownian motion case a known result proved by V. Barbu, G. Da Prato and M. Röckner in the case of a simple Wiener noise, see [BDR09]. The proof is nontrivial and is based on a rate for the convergence of the Yosida approximants of a sectorial operator obtained by Blunck in [Blu02]. In (Mug15), this result is the starting point for several general results on long-time behaviour of stochastic differential equations, like the following.

Theorem 23. *Under the assumptions of Theorem 6, let $B \in \mathcal{L}(H)$, $S \in \mathcal{L}(Y)$ be dissipative operators such that*

- $(Cu)(x) = B(u(x))$ *for all* $u \in H^1(\Omega; H)$ *and a.e.* $x \in \Omega$,
- $(Ru)(z) = S(u(z))$ *for all* $u \in \{v \in H^1(\Omega; H) : v(z) \in Y$ *for a.e.* $z \in \partial\Omega\}$ *and a.e.* $z \in \partial\Omega$.

If $B + \epsilon\mathrm{Id}$ or $S + \epsilon\mathrm{Id}$ are dissipative for some $\epsilon > 0$, then $(T_2^{Y,R}(t))_{t\geq 0}$ is uniformly exponentially stable and the stochastic convolution process

$$W(t) := \int_0^t T_2^{Y,R}(t-s)\mathcal{C}_a \, \mathrm{d}Z^a(s), \qquad t \geq 0,$$

that governs (14) *with initial conditions $\mathfrak{v}_0$ converges to a centered Gaussian random variable.*

We conclude with a comment on synchronised systems. Intuitively, a system is unsynchronised if the time evolution tears its subsystems apart.

The notion of synchronisation is relevant in several concrete models, e.g. in the study of ephaptic fiber interactions.

Lemma 24. *Denote by $P_{\mathcal{Z}}$ the orthogonal projection onto a closed subspace $\mathcal{Z}$ of a Hilbert space $\mathcal{H}$, and by $\mathcal{C}_\epsilon^{\mathcal{Z}}$ the closed convex subset of $\mathcal{H}$ defined as the strip around $\mathcal{Z}$ of thickness 2ϵ, i.e.,*

$$\mathcal{C}_\epsilon^{\mathcal{Z}} := \{f \in \mathcal{H} : \|f - P_{\mathcal{Z}} f\|_{\mathcal{H}} \leq \epsilon\}.$$

Let $(T(t))_{t\geq 0}$ be a strongly-continuous semigroup on $\mathcal{H}$. Consider the following assertions.

(a) $\mathcal{C}_\epsilon^{\mathcal{Z}}$ is invariant under $(T(t))_{t\geq 0}$ for all $\epsilon > 0$.
(b) $\mathcal{C}_{\epsilon_0}^{\mathcal{Z}}$ is invariant under $(T(t))_{t\geq 0}$ for some $\epsilon_0 > 0$.
(c) $\mathcal{Z}$ is invariant under $(T(t))_{t\geq 0}$.

Then (a)$\Longleftrightarrow$(b)$\Longrightarrow$(c). If $(T(t))_{t\geq 0}$ is contractive, then also (c) $\Longrightarrow$ (a) holds.

If (4) holds for $Z = \langle \mathbf{1} \rangle$ – the subspace of entrywise constant vectors – then invariance of some $\mathcal{C}_\epsilon^{\mathcal{Z}}$ can be seen as measure of synchronicity of the system, and Lemma 24 (proved in (Mug10)) can be seen as a characterisation of the self-synchronisation property of the system. If the Hilbert space $\mathcal{H}$ is replaced by a finite dimensional L^∞-space and in the particular case of the $Z = \langle \mathbf{1} \rangle$, the property of invariance of $\mathcal{C}_\epsilon^{\mathcal{Z}}$ under the flow governing a dynamical system has been investigated in several recent papers by Atay et al., see e.g. [AJW04].

List of publications

Journals

(Mug1) *A semigroup analogue of the Fonf–Lin–Wojtaszczyk ergodic characterisation of reflexive Banach spaces with a basis*, Studia Math. **164** (2004), 243–251

(Mug2) *Abstract wave equations with acoustic boundary conditions*, Math. Nachr. **279** (2006), 299–318

(Mug3) *Operator matrices as generators of cosine operator functions*, Int. Eq. Oper. Theory **54** (2006), 441–464

(Mug4) *Matrix methods for wave equations*, Math. Z. **253** (2006), 667–680

(Mug5) *Dirichlet forms for general Wentzell boundary conditions, analytic semigroups, and cosine operator functions*, Electr. J. Diff. Eq. **118** (2006), 1–20 (joint with S. Romanelli)

(Mug6) *Variational and semigroup methods for waves and diffusion in networks*, Appl. Math. Optim. **55** (2007), 219–240 (joint with M. Kramar Fijavž, E. Sikolya)

(Mug7) *Gaussian estimates for a heat equation on a network*, Netw. Heter. Media **2** (2007), 55–79

(Mug8) *Dynamic and generalized Wentzell node conditions for network equations*, Math. Meth. Appli. Sci. **30** (2007), 681–706 (joint with S. Romanelli)

(Mug9) *Analysis of a FitzHugh-Nagumo-Rall model of a neuronal network*, Math. Meth. Appl. Sci. **30** (2007), 2281–2308 (joint with S. Cardanobile)

(Mug10) *Qualitative properties of parabolic systems of evolution equations*, Ann. Sc. Norm. Super. Pisa, Cl. Sci., V. Ser. **VII** (2008) 287–312 (joint with S. Cardanobile)

(Mug11) *Well-posedness and symmetries of strongly coupled network equations*, J. Phys. A **41** (2008) 055102 (joint with S. Cardanobile, R. Nittka)

(Mug12) *Parabolic systems with coupled boundary conditions*, J. Differ. Equ. **247** (2009), 1229–1248 (joint with S. Cardanobile)

(Mug13) *Towards a gauge theory for evolution equations on vector-valued spaces*, J. Math. Phys. **50** (2009), 103520 (joint with S. Cardanobile)

(Mug14) *Properties of representations of operators acting between spaces of vector-valued functions*, Positivity, in press (joint with R. Nittka)

(Mug15) *Long-time behavior of stochastically perturbed neuronal networks*, Stoch. Dyn. **10** (2010), 441–464 (joint with S. Bonaccorsi)

(Mug16) *Damped wave equations with dynamic boundary conditions*, J. Appl. Anal., in press

(Mug17) *Vector-valued heat equations with coupled, dynamic boundary conditions*, Adv. Diff. Equ. **15** (2010), 1125-1160

Refereed proceedings

(Mug18) *Semigroups for initial boundary value problems*, in: M. Iannelli, G. Lumer (eds.): Evolution Equations 2000: Applications to Physics, Industry, Life Sciences and Economics (Proceedings Levico Terme 2000), Progress in Nonlinear Differential Equations **55**, Birkhäuser, Basel, 2003, 276–292 (joint with M. Kramar, R. Nagel)

(Mug19) *Theory and applications of one-sided coupled operator matrices*, Conf. Semin. Matem. Univ. Bari **283** (2003), 1–29 (joint with M. Kramar, R. Nagel)

(Mug20) *A variational approach to damped wave equations*, in H. Amann et al. (eds.): Functional Analysis and Evolution Equations, Birkhäuser, Basel, 2008, 503–514

Book contributions

(Mug21) *Milestones of evolution, information and complexity*, in W. Arendt, W. Schleich (eds.): Mathematical Analysis of Evolution, Information and Complexity, Wiley-VCH, Weihnheim, 2009, XXIII–XXIX (joint with W. Arendt, W. Schleich)

(Mug22) *Symmetries in quantum graphs*, in W. Arendt, W. Schleich (eds.): Mathematical Analysis of Evolution, Information and Complexity, Wiley-VCH, Weihnheim, 2009, 181–196 (joint with J. Bolte, S. Cardanobile, R. Nittka)

(Mug23) *Investigation of input-output gain in dynamical systems for neural information processing*, in W. Arendt, W. Schleich (eds.): Mathematical Analysis of Evolution, Information and Complexity, Wiley-VCH, Weihnheim, 2009, 379–393 (joint with S. Cardanobile, M. Cohen, S. Corchs, H. Neumann)

(Mug24) *Relating simulation and modelling of neural networks*, in W. Arendt, W. Schleich (eds.): Mathematical Analysis of Evolution, Information and Complexity, Wiley-VCH, Weihnheim, 2009, 137–155 (joint with S. Cardanobile, H. Markert, G. Palm, F. Schwenkert)

Submitted

(Mug25) *On the domain of a Fleming–Viot type operator on an L^p-space with invariant measure* (joint with A. Rhandi)

(Mug26) *Convergence of sectorial operators on varying Hilbert spaces* (joint with R. Nittka, O. Post)

Preprints

(Mug27) *Spectral asymptotics for diffusive interface problems* (joint with J. von Below)

(Mug28) *Expansions in generalized eigenfunctions for elliptic operators on a star-shaped network with general self-adjoint boundary conditions* (joint with F. Ali Mehmeti, R. Haller-Dintelmann, V. Régnier)

(Mug29) *Asymptotics of semigroups generated by operator matrices* (arXiv/0801.1963)

(Mug30) *A Frucht's theorem for quantum graphs* (arXiv/1003.2529)

(Mug31) *Gradient systems on networks* (joint with R. Pröpper)

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