A Biharmonic Bernoulli Problem

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History and Applications

Named after **Daniel Bernoulli** (1700-1782), the Bernoulli Problem was initially the study of the **stream line** that a fluid makes around an obstacle. A detailed qualitative description was first found by H. W. Alt and L. Caffarelli, hence the more contemporary name **Alt-Caffarelli Problem**:

$$\min_{\substack{u \in W^{1,2}(\Omega) \\ u_{|\partial\Omega}=1}} \int_\Omega |\nabla u|^2 \, \mathrm{d}x + \lambda |\{u > 0\}|. \tag{1}$$

Figure 1: If Ω corresponds to the river without the inside stone, the stream lines are positive level sets of minimizers of (1).

Nowadays, there exist extensive applications in **shape optimization**. One seeks to find shapes of fixed volume that are most **thermally insulating**. This yields minimization problems like

$$\min_{K \subset\subset \Omega} \left(\mathrm{Cap}(K, \Omega) - \lambda |K| \right), \tag{2}$$

where $\mathrm{Cap}(\cdot, \Omega)$ denotes the **capacity** of a heat conductor in Ω. Minimizers of (2) correspond to **nodal sets** of minimizers of (1).

The Problem and its Free Boundary

Let $\Omega \subset \mathbb{R}^2$ be a domain with smooth boundary, $g \in C^\infty(\overline{\Omega})$ a positive function and $\lambda > 0$. We seek to minimize

$$\mathcal{E}(u) := \int_\Omega (\Delta u)^2 \, \mathrm{d}x + \lambda |\{u > 0\}| \tag{3}$$

among all $u \in W^{2,2}(\Omega)$ such that $u_{|\partial\Omega} = g$.

The first summand measures the *bending* of u. Minimizers thus have to find a **balance** between not **bending** too much but at the same time being **nonpositive** in a large region. Nonpositivity near the boundary requires a lot of bending!

Minimizers exist but are in general **non-unique**. Moreover minimizers are smooth except on their nodal sets $\{u = 0\}$, where **regularity breaks down**. Because of this property, we also call $\{u = 0\}$ the **free boundary** of the problem. One is interested in the structure of the free boundary and the global regularity of minimizers.

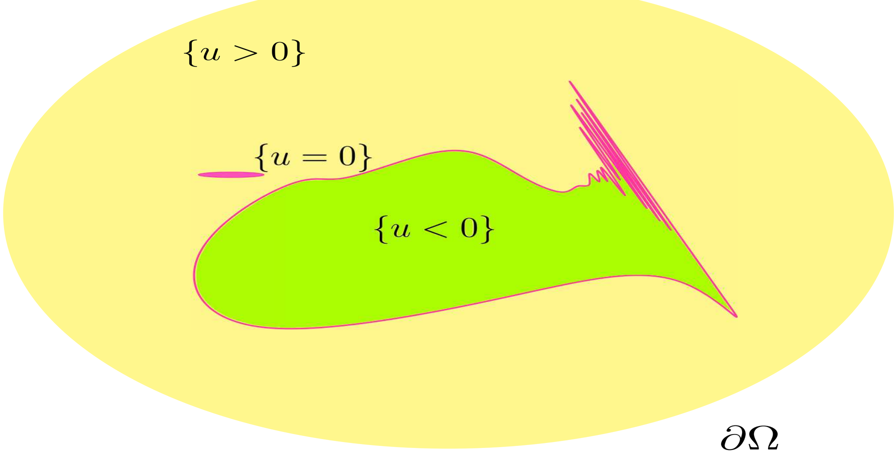

Figure 2: Minimizers divide Ω into three regions. The free boundary $\{u = 0\}$ may a priori look very fuzzy.

Main Result

Theorem: [M. 2020] *Let $u \in W^{2,2}(\Omega)$ be a minimizer. Then $u \in C^2(\overline{\Omega})$ and ∇u does not vanish on $\{u = 0\}$. The nodal set $\{u = 0\}$ is a union of finitely many disjoint and simply closed C^2-curves. Moreover u is a solution to*

$$2 \int_\Omega \Delta u \Delta \phi \, \mathrm{d}x = \lambda \int_{\{u=0\}} \frac{\phi}{|\nabla u|} \, \mathrm{d}\mathcal{H}^1 \quad \forall \phi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega).$$

Discussion of the Theorem

- The **nonvanishing gradient** of minimizers u on $\{u = 0\}$ makes $\{u = 0\}$ a manifold at least as regular as u. It is also needed to write down the equation for the minimizer.
- The equation for the minimizer is not an **Euler-Lagrange equation**, since the measure term is not necessarily differentiable.
- Further Sobolev regularity can be obtained using theory on Poisson equations with **measure-valued** right hand side.
- The explicit equation for u also implies **Navier boundary conditions**, i.e. $\Delta u = 0$ on $\partial\Omega$.
- **Two-dimensionality** is only needed at one point in the proof to obtain **homogeneity of blow-up profiles** when zooming in at singular points.

Radial Solutions are Explicit

If $\Omega = B_1(0)$ and g is constant, **Talenti's inequality** shows the existence of a radial minimizer u. As minimizers are also **subharmonic**, u must be radially increasing. The free boundary $\{u = 0\}$ is therefore a single circle, whose radius is then the only free parameter.

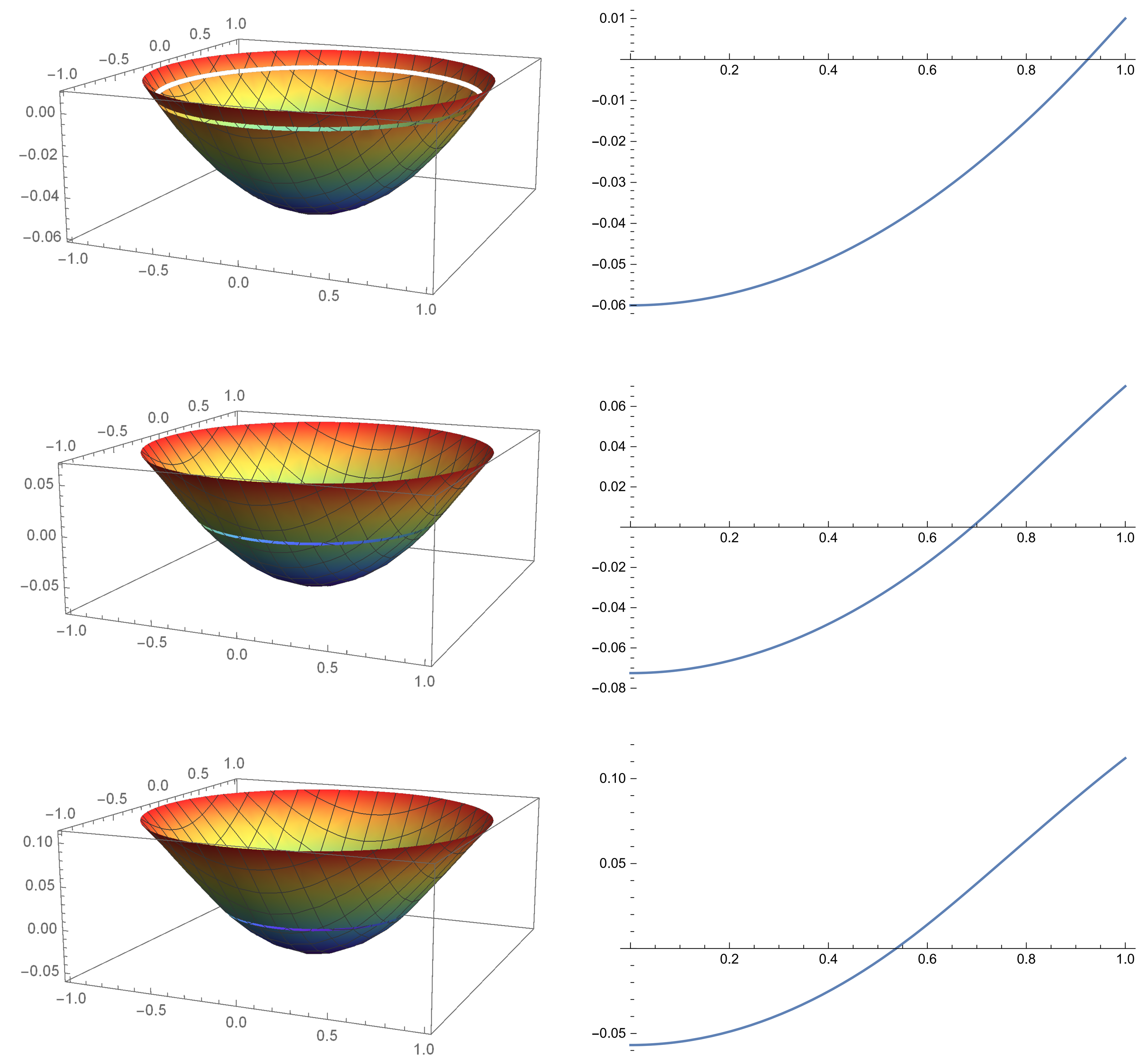

Figure 3: Explicit radial minimizers and their radial profile curves.

References: [1]: Dipierro, Karakhanyan, Valdinoci (2018), *A free boundary problem driven by the biharmonic operator.*
[2]: Müller (2020), *The biharmonic Alt-Caffarelli Problem in 2D.*