Elasticae in the Hyperbolic Plane

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Hyperbolic Geometry

We endow the **hyperbolic half plane** $\mathbb{H}^2 := \mathbb{R} \times (0, \infty)$ with the Riemannian metric $g = \frac{1}{y^2}\, dx\, dy$. All Möbius transformations of $\mathbb{H}^2$ are now **isometries** - even **scaling**! This makes **geometric variational problems** in $\mathbb{H}^2$ interesting – they have many **invariances**.

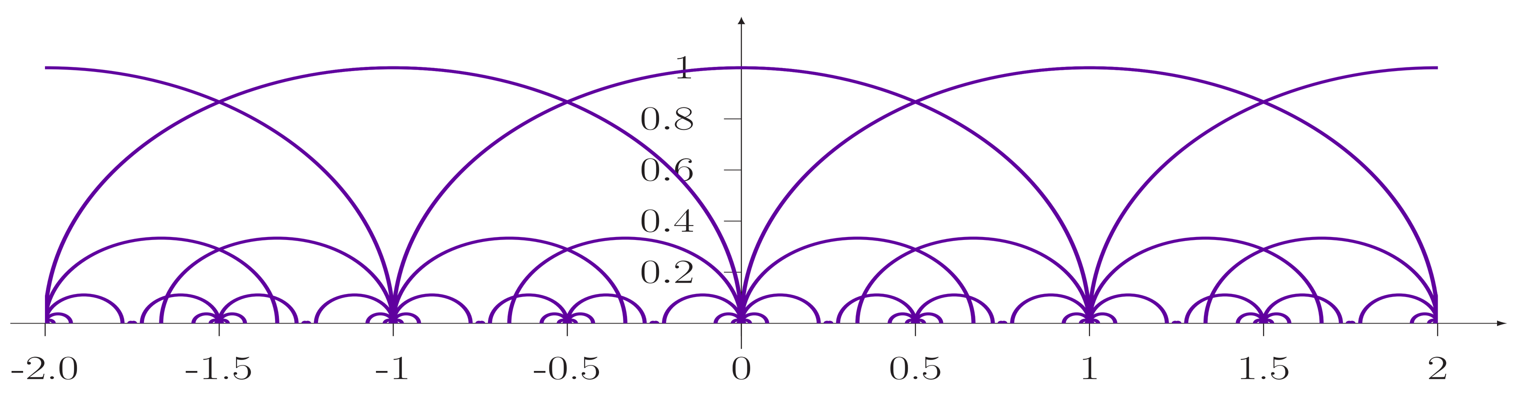

Figure 1: Geodesics in $\mathbb{H}^2$

Euler's Elastica Problem

Let (M, g) be a smooth two-dimensional Riemannian manifold and $\gamma : (a, b) \to M$ be a smooth curve. The **elastic energy** of γ is given by

$$\mathcal{E}_M(\gamma) := \int_\gamma \kappa^2 \, \mathrm{d}s,$$

where $\mathrm{d}s$ denotes the arclength parameter and κ denotes the (signed) curvature of γ in M. The energy was introduced by Euler in 1744 for $M = \mathbb{R}^2$. He was already able to discuss its **critical points**, so-called **Euler elasticae**.

There exists an astounding connection to the **Willmore Energy** of surfaces in case that $M = \mathbb{H}^2$ is the **hyperbolic plane**: Let $\mathcal{S}(\gamma) \subset \mathbb{R}^3$ be the surface that arises from revolution of a closed curve $\gamma : \mathbb{S}^1 \to \mathbb{H}^2$ around the $x-$axis. Then the Willmore energy of $\mathcal{S}(\gamma)$ differs only by a constant factor from $\mathcal{E}_{\mathbb{H}^2}(\gamma)$.

Goal: Understanding the Gradient Flow

We study *smoothly closed elasticae* in $\mathbb{H}^2$ to discuss the convergence of the elastic flow, i.e. the L^2-gradient flow of the elastic energy. Its long-time-existence follows from [1], but one can say more:

MAIN RESULT: We can provide a **complete classification of the asymptotic behavior** of the elastic flow: If we start the flow with a curve of energy **below 16**, the elastic flow **converges**. Moreover, one can find **non-convergent evolutions** with initial data of energy **just slightly above 16**.

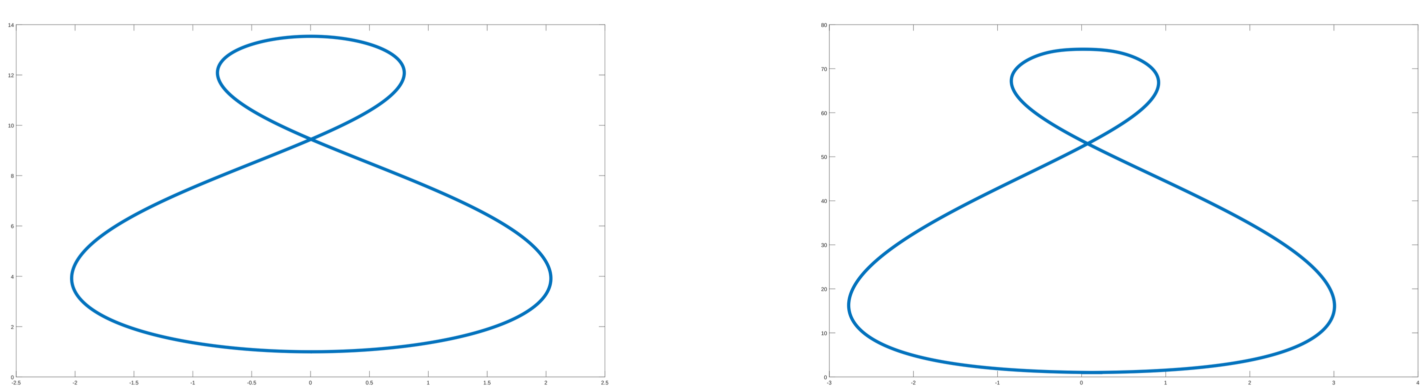

Figure 2: Initial data with non-convergent hyperbolic elastic flow

Construction of Non-Convergent Evolutions

From our explicit parametrization of hyperbolic elasticae, we can deduce the following

Theorem 1. There is no closed elastica in $\mathbb{H}^2$ with vanishing **Euclidean total curvature**.

The euclidean total curvature is a **flow invariant**.

Starting the flow with a closed curve of vanishing Euclidean total curvature (cf. Figure 2) must thus lead to a non-convergent evolution, since each **limit** of such an evolution **has to be an elastica** with the same Euclidean total curvature.

Convergence below an Energy Threshold

The following **Reilly-type inequality** relates hyperbolic length and elastic energy:

Theorem 2. For each $\epsilon > 0$ there exists a constant $c(\epsilon) > 0$ such that for each smoothly closed curve $\gamma : \mathbb{S}^1 \to \mathbb{H}^2$ such that $\mathcal{E}_{\mathbb{H}^2}(\gamma) < 16 - \epsilon$ one has $\mathcal{L}_{\mathbb{H}^2}(\gamma) \leq c(\epsilon)\mathcal{E}_{\mathbb{H}^2}(\gamma)$.

The proof works by examination of critical points of $\mathcal{E}_{\mathbb{H}^2}/\mathcal{L}_{\mathbb{H}^2}$, which we can also parametrize explicitly. If the **hyperbolic arclength** is **controlled along the flow**, results in [1] imply convergence.

Methods and Parametrizations

A first study of closed elasticae in $\mathbb{H}^2$ was conducted in [2], in particular the **Euler-Lagrange equation** $\kappa'' + \frac{1}{2}\kappa^3 - \kappa = 0$ was found. Given this one can easily retrieve the curvature κ, but it remains difficult to determine whether elasticae **close smoothly**.

With **order reduction techniques** from [2], we obtain **explicit solutions**, which allow us to look at hyperbolic elasticae with our **"Euclidean eyes"**.

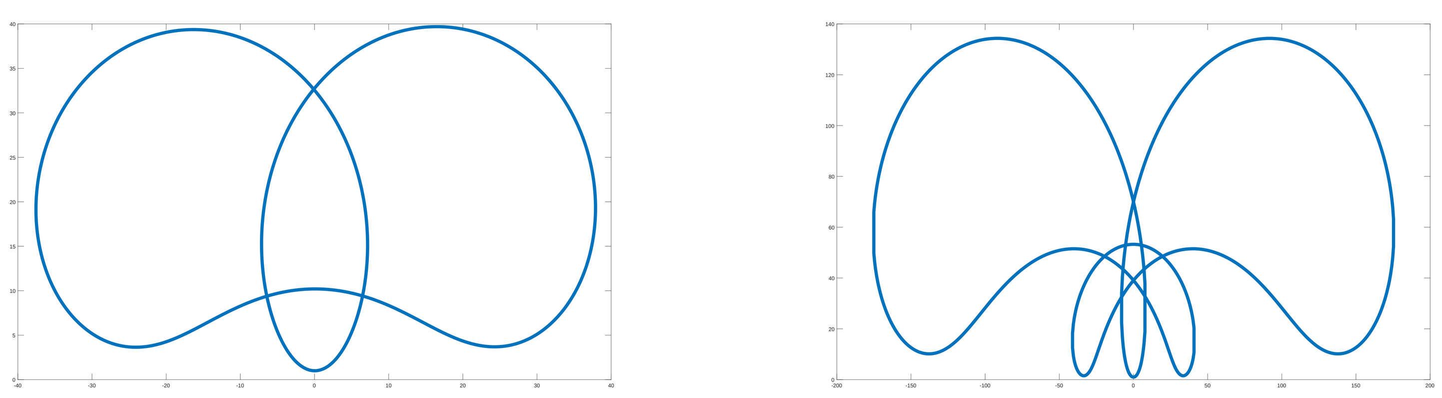

Figure 3: Plots of generic hyperbolic elasticae that are not circles.

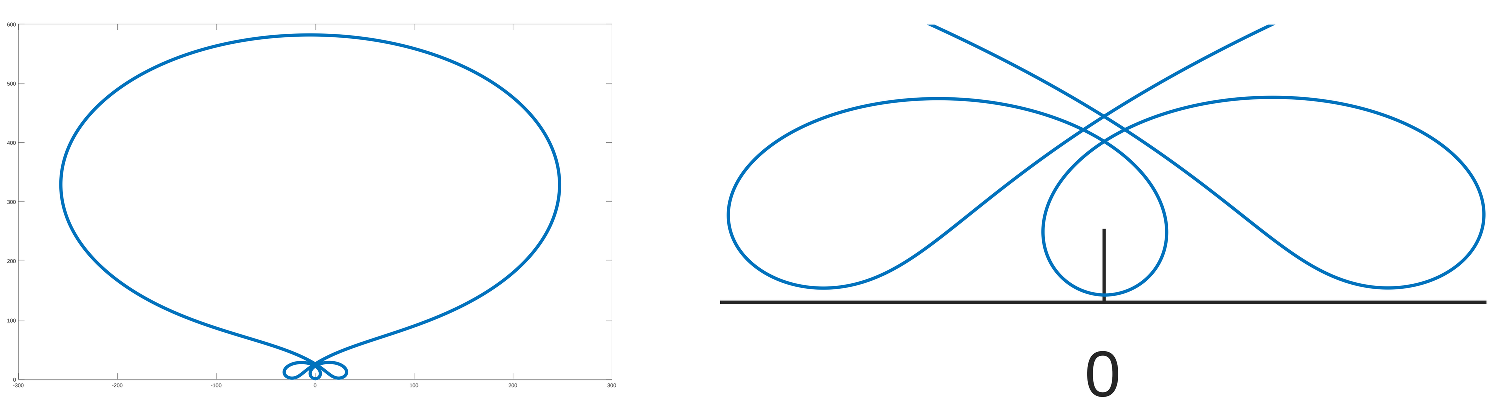

Figure 4: Computing the Euclidean total curvature can be involved and invokes the challenge of excluding shamrock-like looking elasticae.

References: [1]: Dall'Acqua, Spener (2017), *The elastic flow of curves in the hyperbolic plane.*
[2]: Langer, Singer (1984), *The total squared curvature of closed curves.*