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WS 2016

Exercise sheet n. 1 for "Modelling with PDEs"
(Hyperbolic PDEs, Method of characteristics)

Exercise 1. Consider the following hyperbolic PDE (conservation law):

$$\partial_t u + \partial_x f(u) = 0 \quad \text{for } (t,x) \in (0,\infty) \times \mathbb{R}, \tag{1}$$

where $u : (0,\infty) \times \mathbb{R} \to \mathbb{R}$, $(t,x) \mapsto u(t,x)$, and $f : \mathbb{R} \to \mathbb{R}$.

(i) Write the characteristics equation for (1).

(ii) Solve the characteristic equation for (1) with $f(u) = u^4$ and initial condition $u(0,x) = u_0(x)$ for $x \in \mathbb{R}$.

(iii) Do the characteristics for initial data $u_0(x) = \tanh(x)$ or $u_0(x) = -\tanh(x)$ cross?

Exercise 2. Consider the following hyperbolic balance equation:

$$\partial_t u + \partial_x f(u) = u \quad \text{for } (t,x) \in (0,\infty) \times \mathbb{R}, \tag{2}$$

where $u : (0,\infty) \times \mathbb{R} \to \mathbb{R}$, $(t,x) \mapsto u(t,x)$, and $f : \mathbb{R} \to \mathbb{R}$. Answer the questions of Exercise 1 for equation (2) in place of (1).

Exercise 3. Consider the linear wave equation

$$\partial_t^2 u = c^2\, \partial_x^2 u \quad \text{for } (t,x) \in (0,\infty) \times \mathbb{R}, \tag{3}$$

where $u : (0,\infty) \times \mathbb{R} \to \mathbb{R}$, $(t,x) \mapsto u(t,x)$, and $c > 0$.

(i) Write—under the assumption $u \in C^2((0,\infty) \times \mathbb{R})$—a system of PDEs for $v(t,x) := \partial_t u(t,x)$ and $w(t,x) := \partial_x u(t,x)$.

(ii) Do the same thing as in the previous point for the initial value problem for (3) with initial conditions $u(0,x) = g(x)$ und $\partial_t u(0,x) = h(x)$ where $g, h \in C^1(\mathbb{R})$.

(iii) Solve the initial value problem in (ii) and compare its solution with the D'Alembert form of the solution to the initial value problem for (3).

The exercises will be reviewed in class on Wednesday, October 19th, 2016.