TU Wien
Institute for Analysis and Scientific Computing
Prof. C. Heitzinger, Dr. N. Zamponi WS 2016

3. Exercise sheet for the course "Modeling with PDEs"
(Traffic flow/Riemann problem)

Exercise 1. Show that

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0 \quad \text{and} \quad \left(u^2\right)_t + \left(\tfrac{2}{3}u^3\right)_x = 0$$

have identical classic solutions, but for the Riemann problem with $u_l > u_r \geq 0$ different weak solutions. How do you explain this fact?

Exercise 2.

(i) Show that for the viscous Burgers equation ($\varepsilon > 0$)

$$(u_\varepsilon)_t + \left(\frac{u_\varepsilon^2}{2}\right)_x = \varepsilon(u_\varepsilon)_{xx} \quad \text{for} \quad x \in \mathbb{R}\,, t > 0\,, \tag{1}$$

can be transformed into the linear heat equation

$$(v_\varepsilon)_t = \varepsilon\,(v_\varepsilon)_{xx} \quad \text{für} \quad x \in \mathbb{R}\,, t > 0\,, \tag{2}$$

through the Cole-Hopf transformation:

$$u_\varepsilon = (\phi_\varepsilon)_x\,, \quad v_\varepsilon = e^{-\frac{\phi_\varepsilon}{2\varepsilon}}\,.$$

The solution to the initial value problem for the equation (2) with the initial datum $v_\varepsilon(x,0) = v_{\varepsilon,0}(x)$ (where $v_{\varepsilon,0} \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$) reads as

$$v_\varepsilon(x,t) = \frac{1}{\sqrt{4\pi\varepsilon t}} \int_{\mathbb{R}} v_{\varepsilon,0}(\xi) \cdot e^{-\frac{(x-\xi)^2}{4\varepsilon t}}\, d\xi \quad \text{for} \quad x \in \mathbb{R}\,, t \geq 0\,.$$

(ii) Consider the initial data ($u_l > u_r$)

$$u_{\varepsilon,0}(x) = \begin{cases} u_l & \text{für } x < 0\,, \\ u_r & \text{für } x \geq 0\,. \end{cases}$$

Write the solution u_ε to (1) in the form

$$u_\varepsilon(x,t) = \frac{u_l + u_r\, q(x,t,\varepsilon)}{1 + q(x,t,\varepsilon)}$$

for a suitable function $q(x,t,\varepsilon)$ and compute for $t > 0$ the pointwise limit $u(x,t) := \lim_{\varepsilon \to 0} u_\varepsilon(x,t)$ on the line $x = \frac{u_l + u_r}{2}\, t$.

Hint: for $z \gg 1$ the expansion holds: $\int_z^\infty e^{-y^2} dy = \frac{e^{-z^2}}{2z}\left(1 - \frac{1}{2z^2} + \mathcal{O}(\frac{1}{z^4})\right)$.

Exercise 3. Discuss the traffic light problem for the modified Greenberg model

$$v(\rho) = \frac{v_{max}}{\ln(1+\rho_{\max})} \ln\left(\frac{1+\rho_{\max}}{1+\rho}\right)$$

with $0 \leq \rho \leq \rho_{max}$ and a constant initial condition: $\rho(x, t=0) = \bar{\rho}$.

(i) Show that the function u defined as

$$u := a + b\rho \quad \text{mit} \quad a := \tfrac{\rho_{\max}}{1+\rho_{\max}}\,, \quad b := -\tfrac{1}{1+\rho_{\max}}$$

satisfies an hyperbolic conservation law with a convex flux function.

(ii) The traffic light is placed at $x = 0$. What are the boundary conditions for the traffic light during the red phase $0 \leq t \leq T$? Solve the initial-boundary value problem for the equation satisfied by u.

(iii) At $t = T$ the traffic light becomes green and remains green forever. Compute the first time instant when all the cars are again in motion. For how long it is possible to observe vehicles moving at the maximum speed?

The exercises will be reviewed in class on Wednesday, November 16th, 2016.