TU Wien WS 2016
Institute for Analysis and Scientific Computing
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4. Exercise sheet for the course "Modeling with PDEs"
(Numerical mathods/Riemann problem)

Exercise 1. Consider the linear advection equation

$$u_t + a u_x = 0\,, \qquad x \in \mathbb{R}\,, t > 0\,,$$

with a positive constant a. An explicit finite difference scheme of the form

$$u_j^{n+1} \;=\; \sum_{l=-P}^{P} \alpha_l \, u_{j-l}^n \quad \text{with} \quad j \in \mathbb{Z}\,,\; n \in \mathbb{N}_0\,,\; P \in \mathbb{N},$$

with space step h, is called l^2-**stable** if for $h > 0$ and all $u^0 \in l_h^2$, where

$$l_h^2 := \left\{ (v_j)_{j\in\mathbb{Z}} \;:\; \|v\|_{2,h}^2 := h \sum_{j\in\mathbb{Z}} |v_j|^2 < \infty \right\},$$

the inequality $\|u^1\|_{2,h} \leq (1 + ch)\|u^0\|_{2,h}$ holds for some $h-$independent constant $c > 0$. Investigate the l^2-stability for

(i) the central scheme,

(ii) the Lax-Friedrichs scheme,

(iii) the Upwind scheme.

Hint. Multiply the difference scheme times u_j^0 or u_{j-1}^0 and exploit the identity

$$b(a-b) = \frac{1}{2}\big(a^2 - b^2 - (a-b)^2\big)\,.$$

Exercise 2. Consider the Upwind and Lax-Friederichs schemes for the linear advection equation

$$u_t + a u_x = 0\,, \qquad x \in \mathbb{R}\,, t > 0\,,$$

with a positive constant a. Show that, for any choice of k and h such that the CFL constraint $ak \leq h$ is fulfilled, the artificial diffusion of the Upwind scheme is smaller than the artificial diffusion of the Lax-Friederichs scheme.

Exercise 3. For $f \in C^2(\mathbb{R})$ and $u_l \neq u_r$ the function

$$u(x,t) \;=\; \begin{cases} u_l & \text{for } x < st\,, \\ u_r & \text{for } x > st\,, \end{cases} \qquad s = \frac{f(u_r) - f(u_l)}{u_r - u_l}\,,$$

is a weak solution of the Riemann problem

$$u_t \;+\; f(u)_x \;=\; 0\,, \qquad u(x,0) \;=\; \begin{cases} u_l & \text{for } \; x < 0\,, \\ u_r & \text{for } \; x > 0\,. \end{cases} \tag{1}$$

Show that u satisfies the Oleinik entropy condition if and only if u satisfies in weak sense the inequaliy

$$\eta(u)_t \;+\; \psi(u)_x \le 0 \tag{2}$$

for all convex entropies $\eta \in C^2(\mathbb{R})$ and related entropy fluxes $\psi \in C^1(\mathbb{R})$.

Suggestion. Show that u satisfies (2) in weak sense if and only if the inequality

$$-\, s\,\big(\eta(u_r) - \eta(u_l)\big) + \psi(u_r) - \psi(u_l) \le 0 \tag{3}$$

holds. Write the left-hand side of (3) as integral.

The exercises will be reviewed in class on Wednesday, November 23th, 2016.