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WS 2016

5. Exercise sheet for the course "Modeling with PDEs"
(Incompressible Euler equations)

Exercise 1. Consider smooth functions $\rho : \mathbb{R}^d \times [0,T] \to [0,\infty)$, $u : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$, satisfying the continuity equation

$$\partial_t \rho + \operatorname{div}(\rho u) \; = \; 0 \quad \text{for} \quad x \in \mathbb{R}^d \,, t \in [0,T] \,. \tag{1}$$

Let the function $X : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ describe the particles trajectories in the considered fluid, that is, for any $y \in \mathbb{R}^d$, $t \in [0,T]$, $x = X(y,t)$ is the position at time t of the particle that occupies point y at initial time ($t = 0$). The function X satisfies the family of ODEs

$$\begin{aligned} \partial_t X(y,t) &= u(X(y,t),t) \quad \text{for } (y,t) \in \mathbb{R}^d \times [0,T] \,, \\ X(y,0) &= y \quad \text{for } x \in \mathbb{R}^d \,. \end{aligned}$$

(i) Prove that the Jacobi determinant $J(y,t) := \det \left(\frac{\partial X_i}{\partial y_j}(y,t) \right)_{i,j}$ satisfies

$$\partial_t J(y,t) = J(y,t) \operatorname{div}(u)|_{x=X(y,t)} \quad \text{for } (y,t) \in \mathbb{R}^d \times [0,T] \,.$$

(ii) Prove that, given a bounded domain $G \subset \mathbb{R}^d$ and a smooth function $f : \mathbb{R}^d \times [0,T] \to \mathbb{R}$, it holds

$$\frac{d}{dt} \int_{R(t)} \rho f \; \mathrm{d}V \; = \; \int_{R(t)} \rho \frac{Df}{Dt} \; \mathrm{d}V \quad \text{and} \quad \frac{d}{dt} \int_{R(t)} f \; \mathrm{d}V \; = \; \int_{R(t)} \partial_t f + \operatorname{div}(fu) \; \mathrm{d}V$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + (u \cdot \nabla_x)$ is the material derivative.

Exercise 2. Referring to the situation described in Exercise 1, show that the following properties are equivalent.

(i) The fluid is incompressible, that is, for any bounded domain $R_0 \subset \mathbb{R}^d$, the volume of the set

$$R(t) \equiv \{ x \in \mathbb{R}^d \; : \; x = X(y,t) \text{ for some } y \in R_0 \}$$

is constant in time.

(ii) $\operatorname{div} u \equiv 0$.

(iii) The Jacobi determinant J of X is constant (equal to 1) in space and time.

Exercise 3. Prove that

(i) any classical solution of the incompressible Euler equation

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) \;=\; 0\,, \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p \;=\; 0\,, \\ \operatorname{div}(u) \;=\; 0\,, \end{cases} \tag{2}$$

satisfies the energy equation

$$\partial_t\big(\rho \tfrac{|u|^2}{2}\big) \;+\; \operatorname{div}\big(\rho u \tfrac{|u|^2}{2} + pu\big) \;=\; 0\,; \tag{3}$$

(ii) any classical solution of the compressible Euler equation for an ideal fluid with no viscosity and zero internal energy

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) \;=\; 0\,, \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p \;=\; 0\,, \\ \partial_t\big(\rho \tfrac{|u|^2}{2}\big) \;+\; \operatorname{div}\big(\rho u \tfrac{|u|^2}{2} + pu\big) \;=\; 0\,, \end{cases} \tag{4}$$

describes an incompressible fluid, i.e. $\operatorname{div}(u(x,t)) = 0$, under the assumption that $p(x,t) \neq 0$.

The exercises will be reviewed in class on Wednesday, December 7th, 2016.