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WS 2016

**6 . Exercise sheet for the course "Modeling with PDEs"**
(incompressible Navier-Stokes equations)

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**Exercise 1.** Consider the stationary flow of a homogeneous fluid in an infinite cylinder with a constant radius $R$. This flow is described by the incompressible Navier-Stokes equations without external forces. Moreover, make the following assumptions (which can be shown to be reasonable by considering the equations in cilindrical coordinates, with the $z-$axis represented by the cylinder's axis, and exploiting the problem's symmetries):

(i) the current has no circular nor radial components, i.e. $u_r = u_\theta = 0$;

(ii) the current is stationary, i.e. $u_z = u_z(r)$;

(iii) the pressure gradient $\frac{\partial p}{\partial z}$ in the cylinder is constant.

Find a bounded solution of the incompressible Navier-Stokes equations satisfying the boundary condition $u(r = R) = 0$. Furthermore, given an arbitrary flat transversal section $S$ of the cylinder, compute the mass flow rate $Q$ through $S$, which is given by the following surface integral

$$Q = \int_S \rho \cdot u_z \, dA.$$

**Exercise 2.** Consider a stationary flow between two infinite, coaxial, slowly rotating cylinders, where the homogeneous fluid is described by the incompressible Navier-Stokes equations with no-flux boundary conditions.

(i) Find a radially symmetric, stationary solution under the assumptions $f = 0$, $\frac{\partial p}{\partial z} = 0$ und $u_z = u_z(r)$.

(ii) Find the limit values of $u_\theta$ and $p$ when the radius of the external cylinder tends to infinity while the internal cylinder has constant radius $R_1$ and angular speed $\omega_1$.

**Exercise 3.** Consider the incompressible Navier-Stokes equations for an homogeneous fluid without external forces

$$u_t \; + \; \nabla \cdot (u \otimes u) \; + \; \frac{1}{\rho_0} \nabla p = \nu_0 \, \Delta u, \qquad \operatorname{div} u = 0, \tag{1}$$

with $\rho_0 > 0$ and $\nu_0 := \frac{\mu}{\rho_0}$. Given a characteristic length $L > 0$ and a characteristic velocity $U > 0$ we can define a-dimensional variables in the following way:

$$\tilde{x} := \frac{x}{L}\,, \quad \tilde{t} := \frac{t}{T}\,, \quad v(\tilde{x}, \tilde{t}) := \frac{u(L\tilde{x}, T\tilde{t})}{U}\,, \quad \tilde{p}(\tilde{x}, \tilde{t}) := \frac{p(L\tilde{x}, T\tilde{t})}{p_0}\,, \tag{2}$$

where $p_0 := \rho_0 U^2$ and $T := \frac{L}{U}$.

(i) Write the PDEs in the scaled variables (2).

(ii) Consider two "flat Couette flows" as described in Example 2.3 (pp. 42-43) in the lecture notes, with parameters $U = U_i$, $d = d_i$, $\rho_0 = \rho_i$, $\mu = \mu_i$ and Reynolds numbers given by $\mathrm{Re}_i := \frac{d_i U_i \rho_i}{\mu_i}$, for $i = 1, 2$. How do these solutions relate together?

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The exercises will be reviewed in class on December 14, 2016.