TU Wien WS 2016
Institute for Analysis and Scientific Computing
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7. Exercise sheet for the course "Modeling with PDEs"
(Navier-Stokes and Euler equations)

Exercise 1. Consider a flat, incompressible potential flow on an elliptic profile $P := \{\, (x, y) \in \mathbb{R}^2 \,|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \,\}$ for $a > b > 0$, that is parallel to the $x-$axis sufficiently far from P. If ψ is the potential, then such situation is described by the following initial-boundary value problem:

$$\begin{cases} \Delta\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 & \text{in } \mathbb{R}^2\backslash P\,, \\ \psi = 0 & \text{on } \partial P = \{\, (x, y) \in \mathbb{R}^2 \,|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \,\}\,, \\ \psi = y & \text{for } x^2 + y^2 \to \infty\,. \end{cases} \tag{1}$$

(i) Find a solution of (1) such that the flow is symmetric with respect to the $x-$axis.

(ii) Compute the velocity u on the border ∂P of P.

Suggestion: Use elliptic coordinates $(\xi, \eta) \in [0, \infty) \times [0, 2\pi)$, which are related to the cartesian coordinates $(x, y) \in \mathbb{R}^2$ through

$$x = K(\cosh\xi)(\cos\eta) \ \text{ and } \ y = K(\sinh\xi)(\sin\eta)\,,$$

where $K = \sqrt{a^2 - b^2}$ and $\xi_0 = \operatorname{arsinh}\left(\frac{b}{\sqrt{a^2-b^2}}\right)$. The Laplace-Operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ in elliptic coordinates reads as

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{K^2(\sinh^2\xi + \sin^2\eta)}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right).$$

Transform problem (1) in elliptic coordinates and search for a solution with separated variables, i.e. $\psi(\xi, \eta) = f(\xi)g(\eta)$.

Exercise 2. Consider the velocity fields

$$u_1(x, y, z) = \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix} \ \text{ on the domain } \mathbb{R}^3\,,$$

and

$$u_2(x, y, z) = \frac{1}{x^2 + y^2}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \ \text{ on } \Omega_R = \{\, (x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 \geq R^2 \,\} \ \text{ with } R > 0\,.$$

(i) Compute the flow lines and the particle trajectories for both flows. Draw a picture of the result.

(ii) Compute the vorticity $\operatorname{rot} u_i$, for $i = 1, 2$.

(iii) Consider the flow with zero vorticity and find a potential for the velocity field u, i.e. a function φ such that $u = \nabla\varphi$ on the considered domain $\Omega \subset \mathbb{R}^3$. Moreover, prove that a domain $\Omega' \subset \mathbb{R}^3$ exists, such that u admits no potential in Ω'.

Exercise 3. Consider an incompressible and homogeneous fluid (i.e. $\nabla\rho = 0$), described by the Euler equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) \;=\; 0\,, \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p \;=\; 0\,, \\ \operatorname{div}(u) \;=\; 0\,. \end{cases} \tag{2}$$

Furthermore, consider a 2-dimensional canal parametrized by $(x, y) \in [0, L] \times [0, 1]$, which is completely filled with the fluid. Let the pressure p depend only on x, and let the pressure at $x = 0$ be greater than the pressure at $x = L$, that is $p = p(x)$ with $p_1 := p(0) > p(L) =: p_2$.

(i) Find a solution having the form

$$u(x, y, t) = (u_1(x, t), 0) \quad \text{and} \quad p(x, y, t) = p(x)\,.$$

(ii) Discuss the behaviour of u as $t \to \infty$.

Remark: compare the result with Example 2.4 of the lecture notes.

The exercises will be reviewed in class on January 11, 2017.