TU Wien
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WS 2016

8. Exercise sheet for the course "Modeling with PDEs"
(Dimensional analysis, homogenization of PDEs)

Exercise 1. Consider the incompressible Navier-Stokes equations:

$$\rho(\partial_t u + \nabla \cdot (u \otimes u)) + \nabla p = \mu \Delta u + \rho f, \qquad \nabla \cdot u = 0, \tag{1}$$

for some given constant density $\rho > 0$.

(i) Take $f = 0$. Apply a scaling in (1) is such a way that the scaled equation only contains the Reynolds number as a-dimensional parameter. Write the dimension matrix A of the system and relate A with the Reynolds number.

(ii) Assume f is given by the gravitational force density, i.e. $f = -g(1 + z/R)^{-2}\,\mathbf{k}$, where g is the gravitational acceleration on the surface of Earth, R is the radius of Earth, and $\mathbf{k}$ indicates the vertical direction. Apply a scaling in (1) is such a way that the scaled equation contains three a-dimensional parameters, one of such parameters being the Reynolds number. Relate these parameters to the dimension matrix A of the system. By estimating the value of some of the parameters, formulate physically reasonable assumptions under which the force f can be replaced by $f' = -g\mathbf{k}$.

Exercise 2. Consider the one-dimensional homogenization problem of the Poisson equation with Dirichlet boundary conditions. Solve the cell problem and calculate the effective coefficient $\overline{A}$. It is a mean value in a certain sense. In which sense? Furthermore, show that $\alpha \le \overline{A} \le \beta$ and $\overline{A} \le \int_0^1 A(y)dy$, i.e., the effective coefficient is smaller than or equal to the arithmetic mean of the original coefficient.

Exercise 3. Consider the two-dimensional homogenization problem of the Poisson equation with Dirichlet boundary conditions and a layered material. This means that the material on $\Omega \subset \mathbb{R}^2$ consists of layers and hence the entries in the coefficient matrix A are functions of (w.l.o.g.) y_1, i.e., $a_{ij} = a_{ij}(y_1)$, $\;i, j = 1, 2$. Solve the cell problem and calculate the four entries of $\overline{A}$.

The exercises will be reviewed in class on January 11, 2017.