Ulm University
Faculty of Mathematics and Economics

The Prime Number Theorem

Bachelor Thesis

Business Mathematics

submitted by
Wiesel, Johannes
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Referee

Professor Wolfgang Arendt

Contents

1. A Historical Introduction

The Prime Number Theorem looks back on a remarkable history. It should take more than 100 years from the first assumption of the theorem to its complete proof by analytic means. Before we give a detailed description of the historical events, let us first state what it is all about: The Prime Number Theorem says, that the asymptotic behaviour of the number of primes, which are smaller than some value x, is roughly $x/\log(x)$ for $x \to \infty$. This was assumed by 15-year old Carl Friedrich Gauß[1] in 1793 and by Adrien-Marie Legendre[2] in 1798, but was not proven until 1896, when Jacques Salomon Hadamard[3] and Charles-Jean de la Vallée Poussin[4] independently of each other found a way to approach this problem. The proof was later simplified by many famous mathematicians. Amongst others Wiener, Landau and D.J. Newmann could make some important improvements.

Figure 1. Jacques Hadmard

Figure 2. Charles-Jean de la Vallée Poussin

In 1798 Adrien-Marie Legendre published the Prime Number Theorem as an assumption in his work “Théorie des nombres”, while Gauss mentioned his thoughts in 1849 in a letter to J.F. Encke. From this letter it becomes evident, that his occupation with the topic dated back as far as 1793. Both mathematicians scrutinized prime number tables in order to arrive at their assumptions.

A few years later, Bernhard Riemann[5] could find a connection between the distribution of prime numbers and the properties of the so-called Riemann Zeta Function, which was first studied by Euler[6]. He published his results in his famous work of 1859 “Über die Anzahl der Primzahlen unter einer gegebenen Größe”. On only nine pages, Rieman stated a “programme” of ideas, which were to be proven. In his paper he also proposed the study of the Zeta Function by means of complex analysis.

Another important discovery on the way to the proof of the Prime Number Theorem was made by Hans von Mangoldt[7], a German mathematician: He managed to prove the main result of Riemann’s paper, namely that the Prime Number Theorem is equivalent to the fact, that the Rieman Zeta Function has no zeros with a real part of 1. One year later, Hadamard and de la Vallée Poussion used methods of Complex Analysis to show this property of the Zeta Function designing a proof, which was quite long and complicated.

[1]Carl Friedrich Gauß (1777-1855)
[2]Adrien-Marie Legendre (1752-1833)
[3]Jacques Salomon Hadamard (1865-1963)
[4]Charles-Jean Gustave Nicolas Baron de La Vallée Poussin (1866-1962)
[5]Bernhard Riemann (1826–1866)
[6]Leonhard Euler (1707–1783)
[7]Hans von Mangoldt (1854–1925)

FIGURE 3. Bernhard Riemann

FIGURE 4. Don Zagier

For a long time, mathematicians also tried to find elementary proofs (i.e. proofs, which do not use Complex Analysis). In the time between 1851 and 1854, Pafnuti Tschebyscheff[8] worked on a proof of the Prime Number Theorem and could make important findings, which we will partially discuss here. Tschebyscheff also found lower and upper bounds for the ratio of the prime counting function $\pi(x)$ and $x/\log(x)$ for sufficiently large values of x. However, it should still take some time, until the Prime Number Theorem could finally be proven by elementary means: Roughly 100 years later, in 1949, the mathematicians Atle Selberg[9] und Paul Erdős[10] managed to solve this problem. Although the word elementary makes one suggest differently, their proof is quite complicated. This discovery helped elementary methods of Number Theory regain a good reputation in comparison to analytic methods, as the German mathematician Carl Ludwig Siegel stated: "This shows, that one cannot say anything about the real difficulties of a problem, before one has solved it."[11]

The ideas and steps of the proof given here were stated by Don Bernard Zagier[12] in his work "Newman's Short Proof of the Prime Number Theorem" of 1997. Don Zagier follows the work of Donald J. Newman and Jacob Korevaar with a few simplifications.
For the full understanding a basic knowledge of Complex Analysis is assumed.

We now introduce some basic definitions.

Definition 1.1. We define $\mathbb{P}$ as the set of all prime numbers and $\pi(x)$ as the number of primes smaller or equal to x, i.e.

$$\pi(x) = \left|\{p \in \mathbb{P} : p \leq x\}\right|.$$

Definition 1.2. Let us define the *integral logarithm*

$$\mathrm{li}(x) = \int_2^x \frac{1}{\log(t)}\,dt.$$

Definition 1.3. We say two functions $f, g : \mathbb{R} \to \mathbb{R}$ are asymptotically equal, if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$$

and write

$$f \sim g \qquad (x \to \infty).$$

[8]Pafnuti Lwowitsch Tschebyscheff (1821-1894)
[9]Atle Selberg (1917–2007)
[10]Paul Erdős (1913–1996)
[11]see [B08], p. 322.
[12]Don Bernard Zagier (1951-)

Definition 1.4. We write for two functions $f, g : \mathbb{R} \to \mathbb{R}$

$$f = \mathcal{O}(g) \qquad (x \to \infty),$$

if

$$\exists C \in \mathbb{R} \; \exists x_0 > 0 \; \forall x > x_0 : \qquad |f(x)| \leq C|g(x)|.$$

We write for a real number $a < \infty$

$$f = \mathcal{O}(g) \qquad (x \to a),$$

if

$$\exists C \in \mathbb{R} \; \exists \varepsilon > 0 \; \forall |x - a| < \varepsilon : \quad |f(x)| \leq C|g(x)|.$$

With these notations in mind, we state the Prime Number Theorem in the following way:

Theorem 1.5 (Prime Number Theorem)**.** *The prime counting function* $\pi(x)$ *is asysmptotically equal to the ratio* $x/\log x$*, i.e.*

$$\pi(x) \sim \frac{x}{\log x} \qquad (x \to \infty).$$

2. The Riemann Zeta Function and the Tschebyscheff Functions

In this section we introduce the Riemann Zeta Function and the Tschebyscheff Functions. Besides, we take a first look at their properties. The Zeta Function was first examined by Euler in the 18th century, before Riemann made some important discoveries on its properties.

Definition 2.1. [13] We define the following functions, which are useful for the proof of the Prime Number Theorem. The p under the sigma sign means, that we sum over all $p \in \mathbb{P}$.

Riemann Zeta Function:	$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$	$(\mathrm{Re}\,(s) > 1)$
Tschebyscheff Functions:	$\phi(s) = \sum_{p} \frac{\log p}{p^s}$	$(\mathrm{Re}\,(s) > 1)$
	$\vartheta(x) = \sum_{p \leq x} \log p$	$(x \in \mathbb{R})$

Let us now take a closer look at the first two functions.

Lemma 2.2. [14] *For* $\mathrm{Re}\,(s) > 1$, $\zeta(s)$ *and* $\phi(s)$ *are normally convergent and therefore define holomorphic functions in that domain.*

Proof. For $\delta > 0$, $n \in \mathbb{N}$ and $\mathrm{Re}\,(s) \geq 1 + \delta$, we obtain

$$|n^{-s}| = |e^{-s \log n}| = n^{-\mathrm{Re}\,(s)} \leq n^{-(1+\delta)}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}}$ is convergent, so the Zeta Function converges normally and is hence holomorphic on $\{s \in \mathbb{C} : \mathrm{Re}\,(s) > 1\}$. For $\phi(s)$ we use a similar argument. By the slow increase of the logarithm we can find $C \in \mathbb{R}$ such that $\ln(x) \leq Cx^{\frac{\delta}{2}}$ for $x \geq 1$. We conclude for $\mathrm{Re}\,(s) \geq 1 + \delta$

$$\left| \frac{\log p}{p^s} \right| = \frac{\log p}{p^{\mathrm{Re}\,(s)}} \leq \frac{Cp^{\frac{\delta}{2}}}{p^{1+\delta}} = \frac{C}{p^{1+\frac{\delta}{2}}}$$

and hence we obtain the normal convergence and holomorphy of $\phi(s)$ on $\{s \in \mathbb{C} : \mathrm{Re}\,(s) > 1\}$. □

Lemma 2.3. [15] *If* $s \in \mathbb{C}$, $\mathrm{Re}\,(s) > 1$, *then*

$$\zeta(s) = \prod_{p} \frac{1}{1 - p^{-s}}$$

and this is known as the Euler Product.

Proof. We say a product of numbers $a_1, a_2, \dots \neq 0$ is convergent, if $\lim_{n\to\infty} \prod_{i=1}^{n} a_i \neq 0$. Hence we first check $\frac{1}{1-p^{-s}} \neq 0$:
For $\mathrm{Re}\,(s) > 1$ and $p \in \mathbb{P}$ it holds $|p^{-s}| = p^{-\mathrm{Re}\,(s)} < 1$. Hence we state $\mathrm{Re}\,(\overline{1 - p^{-s}}) = 1 - \mathrm{Re}\,(p^{-s}) > 0$ and $|1 - p^{-s}| \geq 1 - |p^{-s}| > 0$. This indicates, that

$$\frac{1}{1 - p^{-s}} = \frac{\overline{1 - p^{-s}}}{|1 - p^{-s}|^2} \in \mathbb{C} \backslash (-\infty, 0].$$

[13] see [Za97], p. 705.
[14] see [SS03], p.169 and [We06], p. 182.
[15] see [We06], pp.108-109.

We show the equality

$$\zeta(s)\prod_p\left(1-\frac{1}{p^s}\right)=1,$$

which is equivalent to Lemma 2.3. Let $\varepsilon>0$ and choose $N\in\mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty}\frac{1}{n^{\mathrm{Re}\,(s)}}<\varepsilon.$$

Since

$$\zeta(s)=1+\frac{1}{2^s}+\frac{1}{3^s}+\ldots,$$

we conclude

$$\left(1-\frac{1}{2^s}\right)\zeta(s)=1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{9^s}+\ldots,$$

$$\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s)=1+\frac{1}{5^s}+\frac{1}{6^s}+\frac{1}{11^s}+\frac{1}{13^s}+\ldots$$

etc. For the first m prime numbers we obtain

$$\left(1-\frac{1}{p_m^s}\right)\left(1-\frac{1}{p_{m-1}^s}\right)\cdots\left(1-\frac{1}{2^s}\right)\zeta(s)=1+\frac{1}{p_{m+1}^s}+\cdots,$$

so it holds

$$\left|\prod_{j=1}^{m}\left(1-\frac{1}{p_j^s}\right)\zeta(s)-1\right|\le\left|\frac{1}{p_{m+1}^s}\right|+\cdots\le\sum_{n=p_{m+1}}^{\infty}\frac{1}{n^{\mathrm{Re}\,(s)}}\le\sum_{n=m+1}^{\infty}\frac{1}{n^{\mathrm{Re}\,(s)}}<\varepsilon$$

for $m\ge N$. This concludes the proof. □

The following lemma was proven by Tschebyscheff, who used similar elementary arguments to find lower and upper bounds for the quotient $\frac{\pi(x)}{x}$ when $x\to\infty$.

Lemma 2.4. [16] *If $x\ge 2$, then*

$$\vartheta(x)\le 4x,$$

which implies

$$\vartheta(x)=\mathcal{O}(x)\qquad(x\to\infty).$$

Proof. First we recognize that $\binom{2n}{n}=\frac{(2n)!}{(n!)^2}$ contains all prime numbers in the interval of integers $[n+1,2n]$ as factors and is an integer. By use of the Binomial Theorem we have for $n\in\mathbb{N}$

$$2^{2n}=(1+1)^{2n}=\sum_{k=0}^{2n}\binom{2n}{k}\ge\binom{2n}{n}\ge\prod_{n<p\le 2n}p=e^{\left(\sum\limits_{n<p\le 2n}\log p\right)}=e^{\vartheta(2n)-\vartheta(n)}.$$

This is equivalent to

$$\vartheta(2n)-\vartheta(n)\le 2n\log 2.$$

[16] see [We06], pp.110-111 and [Za97], p. 706.

Let $x \geq 2$ and choose $k \in \mathbb{N}$ such that $2^k \leq x < 2^{k+1}$. We obtain

$$\begin{aligned}\vartheta(x) \leq \vartheta(2^{k+1}) &= \sum_{l=1}^{k} (\vartheta(2^{l+1}) - \vartheta(2^l)) + \vartheta(2) \\ &\leq \sum_{l=1}^{k} 2^{l+1} \log 2 + \vartheta(2) \\ &\leq 2^{k+2} \log 2 + \vartheta(2) \\ &\leq 4x \log 2 + \log 2 \\ &\leq 5x \log 2 \leq 4x,\end{aligned}$$

which concludes the proof. □

3. Equivalences of the Prime Number Theorem

Let us now take a closer look at the function $\pi(x)$. It is evident by its definition, that $\pi(x) = 0$ for $x < 2$ and that $\pi(x)$ is a step function with steps of height 1 at all prime numbers. Considering some values[17] of the functions $\pi(x)$, $x/\log x$ and $\mathrm{li}(x)$,

x	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
$\pi(x)$	4	25	168	1.229	9.592	78.498	664.579	5.761.455	50.847.534
$\frac{x}{\log x}$	4	22	145	1.086	8.686	72.382	620.421	5.428.681	48.254.942
$\mathrm{li}(x)$	6	30	178	1.246	9.630	78.628	664.918	5.762.209	50.849.235

one could suspect a similar asymptotic behaviour.
Gauß assumed, that the density of prime numbers of scale x is approximately the same as $\frac{1}{\log(x)}$, which means that for the integral logarithm $\mathrm{li}(x)$ it holds

$$\pi(x) \sim \mathrm{li}(x).$$

As a first approach to the proof of the Prime Number Theorem, we therefore note some equivalences.

Theorem 3.1 (Tschebyscheff)**.** [18] *The following are equivalent:*

(i) $\pi(x) \sim li(x)$.
(ii) $\pi(x) \sim \frac{x}{\log(x)}$.
(iii) $\vartheta(x) \sim x$.

Proof. For the equivalence of (i) and (ii), we prove that

$$\text{(3.1)}\quad \mathrm{li}(x) = \int_2^x \frac{1}{\log(t)}dt = \left[\frac{t}{\log(t)}\right]_{t=2}^x + \int_2^x \frac{1}{\log^2(t)}dt = \frac{x}{\log(x)} + \mathcal{O}\left(\frac{x}{\log^2(x)}\right)$$

holds for every real number $x \geq 4$.
By the equation

$$\lim_{x\to\infty} \frac{\mathrm{li}(x)}{\frac{x}{\log(x)}} = \lim_{x\to\infty} \left(\frac{\frac{x}{\log(x)}}{\frac{x}{\log(x)}} + \frac{f(x)}{\frac{x}{\log(x)}}\right) = 1, \qquad \text{where } f(x) = \mathcal{O}\left(\frac{x}{\log^2(x)}\right),$$

it then clearly follows, that

$$\mathrm{li(x)} \sim \frac{x}{\log(x)} \qquad (x \to \infty).$$

The first equality in (3.1) is immediately obtained by integration by parts:

$$\int_2^x \frac{1}{\log(t)}dt = \left[\frac{t}{\log(t)}\right]_{t=2}^x - \int_2^x -\frac{t}{t\log^2(t)}dt = \left[\frac{t}{\log(t)}\right]_{t=2}^x + \int_2^x \frac{1}{\log^2(t)}dt.$$

Thus we show, that

$$\int_2^x \frac{1}{\log^2(t)}dt = \mathcal{O}\left(\frac{x}{\log^2(x)}\right).$$

For this purpose, we split up the integration path into the intervalls $[2, \sqrt{x}]$ and $[\sqrt{x}, x]$ for $x \geq 4$. We estimate

$$\int_2^x \frac{1}{\log^2(t)}dt = \int_2^{\sqrt{x}} \frac{1}{(\log t)^2}dt + \int_{\sqrt{x}}^x \frac{1}{(\log t)^2}dt \leq \frac{\sqrt{x}}{(\log 2)^2} + \frac{x}{(\log\sqrt{x})^2}$$
$$= \mathcal{O}(\sqrt{x}) + \frac{x}{(\frac{1}{2}\log(x))^2} = \mathcal{O}(\sqrt{x}) + \frac{2^2 x}{(\log x)^2} = \mathcal{O}\left(\frac{x}{(\log x)^2}\right),$$

[17] see `http://de.wikipedia.org/wiki/Primzahlsatz`.
[18] see [Fo11], pp.5.1-5,3; [Za97], p.707; [We06], p.110.

since $\log(x)$ is increasing and we can find $C > 0$ such that

$$\log(x) \leq Cx^{\frac{1}{4}} \quad \Leftrightarrow \quad \log^2 x \leq C^2 x^{\frac{1}{2}} \quad \Leftrightarrow \quad x^{-\frac{1}{2}} \leq C^2 \log^{-2} x \qquad \text{for } x > 1.$$

For the equivalence of (ii) and (iii), we establish an easy inequality

$$\vartheta(x) = \sum_{p \leq x} \log p \leq \sum_{p \leq x} \log x = \pi(x) \log x.$$

On the other hand, for $\varepsilon \in (0,1)$ and $x \geq 1$ we have

$$\begin{aligned} \vartheta(x) &= \sum_{p \leq x} \log p \geq \sum_{x^{1-\varepsilon} < p \leq x} \log p \geq \sum_{x^{1-\varepsilon} < p \leq x} \log\left(x^{1-\varepsilon}\right) \\ &= (1-\varepsilon) \sum_{x^{1-\varepsilon} < p \leq x} \log x = (1-\varepsilon) \log x \big(\pi(x) - \pi(x^{1-\varepsilon})\big). \end{aligned}$$

The logarithm is strictly increasing and $\log(x) = 1$ for $x = e$. According to Lemma 2.4 it holds $\vartheta(x^{1-\varepsilon}) \leq Cx^{1-\varepsilon}$ for $x^{1-\varepsilon} \geq 2$ and some $C \in \mathbb{R}$. Hence we obtain for $x^{1-\varepsilon} \geq 5$

$$\pi(x^{1-\varepsilon}) = \sum_{p \leq x^{1-\varepsilon}} 1 \leq \sum_{p \leq x^{1-\varepsilon}} \log p = \vartheta(x^{1-\varepsilon}) \leq Cx^{1-\varepsilon}.$$

Combining these equations we obtain

$$\begin{aligned} \frac{\vartheta(x)}{x} &\leq \frac{\pi(x)\log x}{x} \leq \frac{\vartheta(x)}{x(1-\varepsilon)} + \frac{\log x \ \pi(x^{1-\varepsilon})}{x} \leq \frac{\vartheta(x)}{x(1-\varepsilon)} + \frac{C \log x}{x^{\varepsilon}} \\ &\leq \frac{\pi(x)\log(x)}{x(1-\varepsilon)} + \frac{C\log x}{x^{\varepsilon}}. \end{aligned} \tag{3.2}$$

We know, that

$$\frac{C \log x}{x^{\varepsilon}} \to 0 \qquad \text{for } x \to \infty \text{ and all } \varepsilon > 0.$$

If (ii) holds, we get by subtraction of $\frac{C\log x}{x^{\varepsilon}}$ in (3.2) and taking the limits $x \to \infty$ and then $\varepsilon \to 0$

$$\lim_{\varepsilon \to 0} \lim_{x \to \infty} \frac{\vartheta(x)}{x(1-\varepsilon)} = 1,$$

so (iii) is true.
If (iii) holds, we have

$$\lim_{\varepsilon \to 0} \lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1,$$

so (ii) is true. Hence we proved the equivalence.

□

4. Partial Sums and some elementary results

In this section we discuss some elementary achievements, which were made in an effort to prove the Prime Number Theorem. We have a closer look at a lemma on partial sums, which we also use in section 5. Besides, its application enables us to state results on the asymptotic behaviour of two series.

Lemma 4.1 (Partial Sums)**.** [19] *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers, $(t_n)_{n\in\mathbb{N}}$ a strictly increasing sequence of real numbers, which is not bounded and $A(t)$ the sum over the a_n, for which the indices n fulfill the condition $t_n \le t$. If $g : [t_1, \infty) \to \mathbb{C}$ is continuously differentiable, then the equality*

$$\sum_{\substack{n\in\mathbb{N}\\ t_n\le x}} a_n g(t_n) = A(x)g(x) - \int_{t_1}^{x} A(t)g'(t)dt$$

is true for all real $x \ge t_1$.

Proof. Choose $N \in \mathbb{N}$ such that $t_N \le x < t_{N+1}$. It applies $A(t) = A(t_n)$ for $t_n \le t < t_{n+1}$ and $A(t_n) - A(t_{n-1}) = a_n$ for $n \ge 2$ as well as $A(t_1) = a_1$. We get

$$\begin{aligned}
\int_{t_1}^{x} A(t)g'(t)dt &= \left(\sum_{n=1}^{N-1}\int_{t_n}^{t_{n+1}} + \int_{t_N}^{x}\right) A(t)g'(t)dt \\
&= \sum_{n=1}^{N-1}\Big[A(t_n)g(t)\Big]_{t=t_n}^{t_{n+1}} + \Big[A(t_N)g(t)\Big]_{t=t_N}^{x} \\
&= \sum_{n=1}^{N-1} A(t_n)(g(t_{n+1}) - g(t_n)) + A(t_N)(g(x) - g(t_N)) \\
&= \sum_{n=2}^{N} A(t_{n-1})g(t_n) - \sum_{n=1}^{N} A(t_n)g(t_n) + A(t_N)g(x) \\
&= -\sum_{n=2}^{N}(A(t_n) - A(t_{n-1}))g(t_n) - A(t_1)g(t_1) + A(x)g(x) \\
&= -\sum_{n=1}^{N} a_n g(t_n) + A(x)g(x).
\end{aligned}$$

□

The following theorem was found by Legendre. It will later also be used in section 7 to prove Betrand's Postulate.

Theorem 4.2. [20] *If*

$$r_p(n) = \sum_{k\ge1}\left\lfloor \frac{n}{p^k}\right\rfloor \quad \textit{and} \quad \lfloor x\rfloor = \max\{k \in \mathbb{Z} : k \le x\},$$

then

$$n! = \prod_{p\le n} p^{r_p(n)} \qquad \textit{for all } n \in \mathbb{N}.$$

Proof. We see immediately, that $r_p(n)$ is finite for every $n \in \mathbb{N}$, since $\lfloor n/p^k\rfloor = 0$ for $p^k > n$. Considering the integers in the range $\{1, 2, ..., n\}$ divisble by $p \in \mathbb{P}$, we recognize, that these are the multiples of p, which are smaller than n, so there are $\lfloor n/p\rfloor$. This gives us $\lfloor n/p\rfloor$ p-factors. By the same idea we obtain, that $\lfloor n/p^2\rfloor$

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[19] see [B08], p. 297.

[20] see [B08], pp.295-296 and [AZ04], p.9

numbers are divisble by p^2 and get an extra $\lfloor n/p^2 \rfloor$ p-factors. We continue in the same manner for all powers of p. Adding these numbers gives exactly the number of powers of p, which are contained in $n!$. Since every integer has a prime number factorization, this completes the proof. □

As an application of the lemma on partial sums, we prove two aysmptotic results achieved by Mertens[21] in 1874.

Theorem 4.3. [22] *We have*

$$\sum_{p \le x} \frac{\log p}{p} = \log x + \mathcal{O}(1) \qquad (x \to \infty).$$

Proof. Let $x \in \mathbb{N}$. By $\int \log x \; dx = x(\log x - 1)$ we obtain

$$\sum_{n \le x} \log x \le \int_1^x \log y \; dy + \log x = x(\log x - 1) + 1 + \log x,$$

thus

$$\sum_{n \le x} \log x = x \log x + \mathcal{O}(x) \qquad (x \to \infty). \tag{4.1}$$

By Theorem 4.2 it follows

$$\begin{aligned} \sum_{n \le x} \log n = \log x! = \log \left(\prod_{p \le x} p^{r_p(x)} \right) &= \sum_{p \le x} \sum_{k \ge 1} \left\lfloor \frac{x}{p^k} \right\rfloor \log p \\ &= \sum_{p \le x} \left\lfloor \frac{x}{p} \right\rfloor \log p + \sum_{p \le x} \sum_{k \ge 2} \left\lfloor \frac{x}{p^k} \right\rfloor \log p. \end{aligned} \tag{4.2}$$

Since

$$\sum_{p \le x} \left\lfloor \frac{x}{p} \right\rfloor \log p = \sum_{p \le x} \left(\frac{x}{p} - \left(\frac{x}{p} - \left\lfloor \frac{x}{p} \right\rfloor \right) \right) \log p \le x \sum_{p \le x} \frac{\log p}{p} + \vartheta(x),$$

we obtain

$$\sum_{p \le x} \left\lfloor \frac{x}{p} \right\rfloor \log p = x \sum_{p \le x} \frac{\log p}{p} + \mathcal{O}(x),$$

because $\left(\frac{x}{p} - \left\lfloor \frac{x}{p} \right\rfloor \right) \in [0,1)$ and $\vartheta(x) = \mathcal{O}(x)$.
Furthermore,

$$\begin{aligned} 0 \le \sum_{p \le x} \sum_{k \ge 2} \left\lfloor \frac{x}{p^k} \right\rfloor \log p \le x \sum_{p} \sum_{k \ge 2} \frac{\log p}{p^k} &= x \sum_{p} \log p \left(\frac{1}{1 - \frac{1}{p}} - \frac{1}{p} - 1 \right) \\ = x \sum_{p} \log p \left(\frac{p^2 - (p-1) - p(p-1)}{p(p-1)} \right) &= x \sum_{p} \frac{\log p}{p(p-1)} = \mathcal{O}(x) \end{aligned}$$

for $x \to \infty$, because the last series converges. This can be seen as follows:
Since there exists $C \in \mathbb{R}$ such that $\log x \le C x^{\delta}$ for $x > 1$ and $\delta \in (0,1)$, we state

$$\frac{\log n}{n(n+1)} \le \frac{\log n}{n^2} \le \frac{C n^{\delta}}{n^2} \qquad \text{for } n \ge 2$$

and the series

$$\sum_{n=2}^{\infty} \frac{C}{n^{2-\delta}}$$

[21] Franz Mertens (1840–1927)
[22] see [B08], p.299 and [Fo11], p.2.4.

converges. Hence the series

$$\sum_{n=2}^{\infty} \frac{\log n}{n(n+1)}$$

converges by comparison test.
Plugging the last equations into (4.2) we get

$$\sum_{n \le x} \log n = x \sum_{p \le x} \frac{\log p}{p} + \mathcal{O}(x),$$

which gives by division by $x > 0$

$$\frac{1}{x} \sum_{n \le x} \log n = \sum_{p \le x} \frac{\log p}{p} + \mathcal{O}(1).$$

Using (4.1) we have

$$\log x + f(x) = \frac{1}{x} \sum_{n \le x} \log n = \sum_{p \le x} \frac{\log p}{p} + g(x), \qquad \text{where } f(x), g(x) = \mathcal{O}(1),$$

which proves Theorem 4.3. □

Theorem 4.4. [23] *There is a real constant $B \in \mathbb{R}$ such that*

$$\sum_{p \le x} \frac{1}{p} = \log\log x + B + \mathcal{O}\left(\frac{1}{\log x}\right) \qquad (x \to \infty).$$

Proof. By Theorem 4.3 we have

$$\sum_{p \le x} \frac{\log p}{p} = \log x + r(x), \qquad \text{where } r(x) = \mathcal{O}(1).$$

We use Lemma 4.1 on partial sums, where we set

$$t_n = p_n, \quad a_n = \frac{\log p_n}{p_n}, \quad g(t) = \frac{1}{\log t}.$$

The function $g(t)$ is continuously differentiable for $t \in [2, \infty)$ and $(p_n)_{n \in \mathbb{N}}$ is the sequence of prime numbers in increasing order.
It follows for $x \ge 2$

$$\begin{aligned} \sum_{p \le x} \frac{1}{p} &= \sum_{p \le x} \frac{\log p}{p} \frac{1}{\log p} = \frac{\log x + r(x)}{\log x} + \int_2^x \frac{\log t + r(t)}{t \, \log^2 t} dt \\ &= \frac{\log x + r(x)}{\log x} + \int_2^x \frac{1}{t \, \log t} dt + \int_2^x \frac{r(t)}{t \, \log^2 t} dt \qquad (4.3) \\ &= 1 + \mathcal{O}\left(\frac{1}{\log x}\right) + \log\log x - \log\log 2 + \int_2^x \frac{r(t)}{t \log^2 t} dt, \end{aligned}$$

since

$$\left(\frac{1}{\log t}\right)' = \frac{-1}{t \, \log^2 t} \quad \text{and} \quad (\log\log t)' = \frac{1}{t \, \log t}.$$

Clearly the function $\frac{1}{t \log^2(t)}$ is integrable over $[2, \infty)$, because

$$\lim_{T \to \infty} \int_2^T \frac{1}{t \log^2 t} dt = \lim_{T \to \infty} \left(-\frac{1}{\log T} + \frac{1}{\log 2}\right) = \frac{1}{\log 2}.$$

Since $r(t)$ stays bounded, we state

$$\int_2^x \frac{r(t)}{t \, \log^2 t} dt = \int_2^\infty \frac{r(t)}{t \, \log^2 t} dt - \int_x^\infty \frac{r(t)}{t \, \log^2 t} dt.$$

[23] see [B08] p.300 and [Fo11], p.2.5.

We obtain

$$\int_2^\infty \frac{r(t)}{t\ \log^2 t}dt - \int_x^\infty \frac{r(t)}{t\ \log^2 t}dt = \int_2^\infty \frac{r(t)}{t\ \log^2 t}dt + \mathcal{O}\left(\int_x^\infty \frac{1}{t\ \log^2 t}dt\right)$$
$$= \int_2^\infty \frac{r(t)}{t\ \log^2 t}dt + \mathcal{O}\left(\frac{1}{\log x}\right).$$

Plugging this in (4.3) we get

$$\sum_{p\leq x} \frac{1}{p} = \log\log x + B + \mathcal{O}\left(\frac{1}{\log x}\right) \qquad (x \to \infty),$$

where

$$B = 1 - \log\log 2 + \int_2^\infty \frac{r(t)}{t\ \log^2 t}dt.$$

□

Combining the last two theorems, we are able to prove a theorem by Tschebyscheff concerning the limit of a familiar function.

Theorem 4.5 (Tschebyscheff)**.** [24] *If*

$$\frac{\pi(x)\log(x)}{x}$$

converges for $x \to \infty$, then it converges to 1.

Proof. Let

$$C := \lim_{x\to\infty} \frac{\pi(x)\log(x)}{x},$$

which is equivalent to

$$\pi(x) = \frac{x}{\log x}(C + \varepsilon(x)),$$

where $\varepsilon(x) \to 0$, if $x \to \infty$. We use Lemma 4.1 setting

$$t_n = p_n, \qquad a_n = 1, \qquad g(t) = \frac{1}{t}$$

and hence obtain for $x \geq 2$

$$\sum_{p\leq x} \frac{1}{p} = \sum_{p\leq x} 1\ \frac{1}{p} = \frac{\pi(x)}{x} + \int_2^x \frac{\pi(t)}{t^2}dt = \frac{C+\varepsilon(x)}{\log x} + \int_2^x \frac{C+\varepsilon(t)}{t\ \log t}dt$$
$$= \frac{C+\varepsilon(x)}{\log x} + (C+\delta'(x))(\log\log x - \log\log 2) \tag{4.4}$$
$$= \left(C + \delta'(x) + \frac{C+\varepsilon(x)}{\log x \cdot \log\log x} - \frac{\log\log 2}{\log\log x}(C+\delta'(x))\right)\log\log x$$
$$= (C + \delta(x))\log\log x,$$

where we define

$$\delta(x) = \delta'(x) + \frac{C+\varepsilon(x)}{\log x \log\log x} - \frac{\log\log 2}{\log\log x}(C+\delta'(x)).$$

The second equality of (4.4) can be justified by the following argumentation: We define

$$\delta'(x) = \frac{\int_2^x \frac{\varepsilon(t)}{t\log t}dt}{\int_2^x \frac{1}{t\log t}dt}.$$

[24] see [B08], p.301.

For $\tilde{\varepsilon} > 0$ there exists an $M > 0$ such that

$$|\varepsilon(t)| \leq \tilde{\varepsilon} \qquad \text{for all } t > M.$$

Since $\varepsilon(x)$ is convergent, it is bounded by a constant $C \in \mathbb{R}$. We state

$$\begin{aligned} |\delta'(x)| &\leq \frac{\int_2^M \frac{C}{t\log t}dt + \tilde{\varepsilon}\int_M^x \frac{1}{t\log t}dt}{\int_2^M \frac{1}{t\log t}dt + \int_M^x \frac{1}{t\log t}dt} \\ &= \frac{C(\log\log M - \log\log 2) + \tilde{\varepsilon}(\log\log x - \log\log M)}{\log\log M - \log\log 2 + \log\log x - \log\log M} \to \tilde{\varepsilon} \qquad \text{for } x \to \infty. \end{aligned}$$

Since $\tilde{\varepsilon}$ is arbitrary, we derive by definition of $\delta(x)$

$$\delta'(x) \to 0 \quad \Rightarrow \quad \delta(x) \to 0 \qquad \text{for } x \to \infty.$$

By Theorem 4.4 we know

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + \mathcal{O}(1) \qquad (x \to \infty).$$

If we combine these two equations and compare the coefficient of $\log\log x$, then $C = 1$ follows. □

5. An Auxiliary Tauberian Theorem

The following theorem was first proven by Ingham[25] in 1935. D.J Newmann[26] could simplify it essentially in 1980. It is an important step on the way to the Prime Number Theorem.
In contrast to the Tauberian Theorems by Wiener[27] and his student Ikehara[28] from 1930, this theorem only uses finite integration paths and does not depend on Fourier Analysis, which makes it particularly handy. The expression "Tauberian Theorem" goes back to Alfred Tauber and his work "Ein Satz aus der Theorie der unendlichen Reihen" from 1897.

Theorem 5.1. [29] *Let $f : [0, \infty) \to \mathbb{R}$ be a bounded function, which is integrable over every finite subinterval. If the Laplace transform of f*

$$\begin{aligned} g \ : \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\} &\to \mathbb{C} \\ z &\mapsto \int_0^\infty f(t)e^{-zt}dt \end{aligned}$$

extends holomorphically to an open superset G of $\{z \in \mathbb{C} : \operatorname{Re}(z) \geq 0\}$, then $\lim_{T\to\infty} \int_0^T f(t)dt$ exists and equals $g(0)$.

Proof. There is an $M \in \mathbb{R}$ such that $|f(t)| \leq M$ for $t \geq 0$. It is clear, that g is well-defined, since f is bounded. For $T > 0$ let us set $g_T : \mathbb{C} \to \mathbb{C}$, where $g_T(z) = \int_0^T f(t)e^{-zt}dt$. Clearly g_T is integrable. We show, that it is holomorphic on $\mathbb{C}$. Therefore, we suspect the derivative to be $\int_0^T -tf(t)e^{-zt}dt$. Thus we show

$$\lim_{h\to 0} \left| \frac{g_T(z+h) - g_T(z)}{h} + \int_0^T tf(t)e^{-zt}dt \right| = 0.$$

We start by stating the inequality

$$\begin{aligned} \left| \frac{g_T(z+h) - g_T(z)}{h} + \int_0^T tf(t)e^{-zt}dt \right| &= \left| \int_0^T \frac{1}{h} \left(f(t)e^{-(z+h)t} - f(t)e^{-zt} + htf(t)e^{-zt} \right) dt \right| \\ &\leq \int_0^T |f(t)e^{-zt}| \left| \frac{e^{-ht} - 1 + ht}{h} \right| dt. \end{aligned} \tag{5.1}$$

Using $F(x) := e^{-xht}$ for $t \in [0, T]$ we get $F'(x) = -hte^{-xht}$ and

$$e^{-ht} - 1 + ht = F(1) - F(0) - F'(0) = \int_0^1 F'(x) - F'(0)dx = \int_0^1 \int_0^x F''(y)dydx,$$

which leads to

$$\left| \frac{e^{-ht} - 1 + ht}{h} \right| \leq \int_0^1 \int_0^x |ht^2 e^{-yht}| dydx \leq |h|T^2 e^{|h|T} \int_0^1 xdx = |h|e^{|h|T} \frac{T^2}{2}$$

[25] Albert Ingham (1900–1967)
[26] Donald J. Newman (1930–2007)
[27] Norbert Wiener (1894–1964)
[28] Shikao Ikehara (1904 – 1984)
[29] see [Ko82], pp. 109,113-115; [We06], pp.114-117; [Za97], pp.707-708; [Be11],pp.2-3.

by $\operatorname{Re}(h) \le |h|$. Together with (5.1) we obtain

$$\left| \frac{g_T(z+h) - g_T(z)}{h} + \int_0^T tf(t)e^{-zt}dt \right| \le \int_0^T |f(t)e^{-zt}||h|e^{|h|T}\frac{T^2}{2}dt$$
$$\le \frac{T^2}{2}e^{|h|T}|h| \int_0^T |f(t)e^{-zt}|dt$$
$$\le \frac{T^2}{2}e^{|h|T}|h|M \int_0^T |e^{-zt}|dt \to 0$$

for $|h| \to 0$, because the integrand is countinuous and thus bounded over a compact intervall. Hence we get the holomorphy of g_T.
Next we show, that

$$\lim_{T\to\infty} (g(0) - g_T(0)) = 0$$

by Cauchy's integral formula. Hence we need to find a suitable integration path around 0. As Korevaar states, the simplest choice would be a circle, but we do not know anything about the holomorphy of the Laplace transform g, if we go too far into the left half plane.[30]
Thus for $R > 0$ fixed, we take a semicircle in the right half plane and a segment of the vertical line $\operatorname{Re}(z) = \delta$ instead. In order to find such a path, which is contained in the open superset G (where g is still holomorphic), we use a compactness argument: For every point z on the line segment $L := \{z \in \mathbb{C} : \operatorname{Re}(z) = 0, -2R \le \operatorname{Im}(z) \le 2R\}$, there exists an open disk of positive radius such that this disk is still contained in G. Since L is compact, we apply the Heine-Borel Theorem to find finitely many such disks, which still cover L. Since we only have finitely many disks to consider, it is evident, that we find a $\delta = \delta(R) > 0$ small enough such that

$$D := \{z \in \mathbb{C} : |z| < 2R, \operatorname{Re}(z) > -2\delta\} \quad \text{and}$$
$$C := \partial\{z \in \mathbb{C} : |z| \le R, \operatorname{Re}(z) \ge -\delta\}$$

are contained in G.

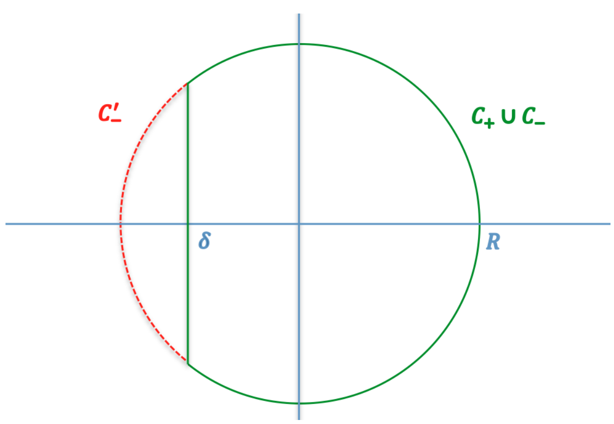

FIGURE 5. Construction

[30]see [Ko82], p.113.

Besides, C defines a simple, piecewise smooth curve such that $C \cup \text{int}(C)$ is contained in D. We assume that C is positively oriented. The function $g(z)$ is holomorphic in D. Hence we apply Cauchy's Integral Formula

$$g(0) - g_T(0) = \frac{1}{2\pi i} \int_C \frac{g(z) - g_T(z)}{z} dz. \tag{5.2}$$

Now we make use of some tricks to obtain nice estimates for the integral:
Let us first observe, that the function $(g(z) - g_T(z))e^{zT}$ is holomorphic on D and for $z = 0$, it has the same value as $g(z) - g_T(z)$. Furthermore, the function

$$\frac{(g(z) - g_T(z))e^{zT} z}{R^2}$$

is holomorphic on D and thus the value of the integral over C is zero. Thus we rewrite (5.2) as follows:

$$g(0) - g_T(0) = \frac{1}{2\pi i} \int_C \frac{(g(z) - g_T(z))\, e^{zT} \left(1 + \frac{z^2}{R^2}\right)}{z} dz.$$

We now split up C to the semicircle $C_+ = C \cap \{z \in \mathbb{C} : \text{Re}\,(z) > 0\}$ and $C_- = C \cap \{z \in \mathbb{C} : \text{Re}\,(z) < 0\}$ as can be seen in Figure 5. For $z \in C_+$ it holds

$$|g(z) - g_T(z)| = \left| \int_T^\infty f(t) e^{-zt} dt \right| \leq M \int_T^\infty |e^{-zt}| dt = \frac{M e^{-\text{Re}\,(z) T}}{\text{Re}\,(z)}$$

and this inequality justifies the multiplication of the integrand in (5.2) with e^{zT}, which eliminates the factor $e^{-\text{Re}\,(z)T}$. To compensate $\text{Re}\,(z)$ in the denominator, we multiply by $\left(1 + \frac{z^2}{R^2}\right)$, which is called Carleman's formula[31].
For $|z| = R$ we obtain

$$\left| e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{1}{z} \right| = e^{\text{Re}\,(z)T} \left| \frac{1}{z} + \frac{z}{R^2} \right| = e^{\text{Re}\,(z)T} \left| \frac{\overline{z}}{|z|^2} + \frac{z}{R^2} \right| = e^{\text{Re}\,(z)T} \frac{2|\text{Re}\,(z)|}{R^2}. \tag{5.3}$$

Thus for the whole integrand we conclude

$$\left| \frac{(g(z) - g_T(z))\, e^{zT} \left(1 + \frac{z^2}{R^2}\right)}{z} \right| \leq \frac{M e^{\text{Re}\,(z)T}}{\text{Re}\,(z)} \frac{2|\text{Re}\,(z)| e^{-\text{Re}\,(z)T}}{R^2} = \frac{2M}{R^2}$$

on C_+ and the integral can be estimated by

$$\left| \frac{1}{2\pi i} \int_{C_+} \frac{(g(z) - g_T(z))\, e^{zT} \left(1 + \frac{z^2}{R^2}\right)}{z} dz \right| \leq \frac{1}{2\pi} \int_{C_+} \frac{2M}{R^2} |dz| \leq \frac{2\pi R M}{2\pi R^2} = \frac{M}{R}.$$

Now we estimate

$$\frac{1}{2\pi i} \int_{C_-} \frac{g(z) e^{zT} \left(1 + \frac{z^2}{R^2}\right)}{z} dz - \frac{1}{2\pi i} \int_{C_-} \frac{g_T(z) e^{zT} \left(1 + \frac{z^2}{R^2}\right)}{z} dz.$$

We start with the second integral. As we showed g_T is entire, so the Fundamental Theorem of Calculus states, that we only need to worry about the starting and ending point of the integration path around 0. Hence instead of C_- it is possible

[31] see [Ko82], p.113.

to integrate over the semicircle $C'_- = \{z \in \mathbb{C} : |z| = R, \mathrm{Re}\,(z) < 0\}$. We obtain for $\mathrm{Re}\,(z) < 0$

$$|g_T(z)| = \left| \int_0^T f(t) e^{-zt} dt \right| \leq M \int_0^T |e^{-zt}| dt \leq M \int_{-\infty}^T |e^{-zt}| dt$$
$$\leq M \int_{-\infty}^T e^{-\mathrm{Re}\,(z) t} dt = \frac{M e^{-\mathrm{Re}\,(z) T}}{|\mathrm{Re}\,(z)|}.$$

Using (5.3) we get

$$\left| \frac{1}{2\pi i} \int_{C_-} \frac{g_T(z) e^{zT} \left(1 + \frac{z^2}{R^2}\right)}{z} dz \right| \leq \frac{\pi R}{2\pi} \frac{M e^{-\mathrm{Re}\,(z) T}}{|\mathrm{Re}\,(z)|} \frac{2 e^{\mathrm{Re}\,(z) T} |\mathrm{Re}\,(z)|}{R^2} = \frac{M}{R}.$$

Since g is holomorphic on $\overline{C_-}$, there exists a $B = B(R, \delta)$ such that

$$\left| g(z) \left(1 + \frac{z^2}{R^2}\right) \frac{1}{z} \right| \leq B$$

for $z \in \mathbb{C}_-$.
The only term of

$$\frac{1}{2\pi i} \int_{C_-} \frac{g(z) e^{zT} \left(1 + \frac{z^2}{R^2}\right)}{z} dz$$

depending on T is e^{zT}, which is holomorphic in z. This implies for $z \in C_-$

$$\frac{g(z) e^{zT} \left(1 + \frac{z^2}{R^2}\right)}{z} \to 0 \quad \text{for } T \to \infty$$

and

$$\left| \frac{g(z) e^{zT} \left(1 + \frac{z^2}{R^2}\right)}{z} \right| \leq B.$$

Therefore, we use Lebesgue's Dominated Convergence Theorem to interchange the limit and integral

$$\lim_{T \to \infty} \frac{1}{2\pi i} \int_{C_-} \frac{g(z) e^{zT} \left(1 + \frac{z^2}{R^2}\right)}{z} dz = \frac{1}{2\pi i} \int_{C_-} \lim_{T \to \infty} \frac{g(z) e^{zT} \left(1 + \frac{z^2}{R^2}\right)}{z} dz = 0.$$

Let $\varepsilon > 0$ and choose R such that $M/R \leq \varepsilon/3$. In addtion choose $T_0(R)$ such that

$$\frac{1}{2\pi i} \int_{C_-} \frac{g(z) e^{zT} \left(1 + \frac{z^2}{R^2}\right)}{z} dz \leq \frac{\varepsilon}{3} \qquad \text{for all } T \geq T_0.$$

This leads to

$$|g(0) - g_T(0)| \leq \varepsilon \qquad \text{for all } T \geq T_0$$

and thus proves Theorem 5.1. □

6. The Proof of the Prime Number Theorem

Now we have done sufficient preparation to start to comprehend the steps of the proof of the Prime Number Theorem given in [Za97]. We begin with the analytic continuation of the Riemann Zeta Function.

Lemma 6.1. [32] *The function*

$$\zeta(s) - \frac{1}{s-1}$$

extends holomorphically to $\{s \in \mathbb{C} : \mathrm{Re}\,(s) > 0\}$.

Proof. For $\mathrm{Re}\,(s) > 1$ we have

$$\begin{aligned}\zeta(s) - \frac{1}{s-1} &= \sum_{n=1}^{\infty} \frac{1}{n^s} + \left[\frac{1}{x^{s-1}(1-s)}\right]_{x=1}^{\infty} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_1^{\infty} \frac{1}{x^s} dx \\ &= \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx.\end{aligned}$$

Every integral

$$\int_n^{n+1} \frac{1}{n^s} - \frac{1}{x^s} dx$$

for $n \geq 1$ is holomorphic in s, because the function

$$\begin{aligned}F : \{s \in \mathbb{C} : \mathrm{Re}\,(s) > 0\} \times [n, n+1] &\to \mathbb{C} \\ (s, x) &\mapsto \frac{1}{n^s} - \frac{1}{x^s}\end{aligned}$$

is continuous and for fixed $x \in [n, n+1]$, the function $s \to \frac{1}{n^s} - \frac{1}{x^s}$ is holomorphic on $\{s \in \mathbb{C} : \mathrm{Re}\,(s) > 0\}$. In order to show the holomorphy of the last series above, it thus suffices to show, that the series converges normally for $\mathrm{Re}\,(s) > 0$. For this purpose let us take $\delta > 0$ and $\mathrm{Re}\,(s) \geq \delta$. We calculate:

$$\begin{aligned}\left|\int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx\right| &= \left|s \int_n^{n+1} \left[-\frac{1}{su^s}\right]_{u=n}^{x} dx\right| = \left|s \int_n^{n+1} \int_n^{x} \frac{du}{u^{s+1}} dx\right| \\ &\leq \max_{n \leq v \leq n+1} \left|\frac{s}{v^{s+1}}\right| = \max_{n \leq v \leq n+1} \frac{|s|}{v^{\mathrm{Re}\,(s)+1}} = \frac{|s|}{n^{\delta+1}}\end{aligned}$$

This shows, that $\zeta(s) - \frac{1}{s-1}$ is uniformly convergent for $\mathrm{Re}\,(s) \geq \delta\ \forall \delta > 0$. By properties of normal convence it follows, that it is holomorphic for $\mathrm{Re}\,(s) > 0$. Hence $\zeta(s)$ extends meromorphically to the set $\{s \in \mathbb{C} : \mathrm{Re}\,(s) > 0\}$ with a simple pole at $s = 1$. Since $\{s \in \mathbb{C} : \mathrm{Re}\,(s) > 0\}$ is a domain, this extension is unique. In this text we also denote the analytic extension of a function by the function itself. □

Lemma 6.2. [33] *For the Principal Branch of the logarithm the following Taylor expansion holds:*

$$\log(1+z) = -\sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n} \qquad \text{for } |z| < 1. \tag{6.1}$$

[32] see [Za97], pp.705-706 and [We06], p.108.

[33] see [SS03], p.100.

Proof. We immediately see, that the series on the right converges normally in $D_1(0)$ and hence is holomorphic in $D_1(0)$. Differentiating by summands yields

$$\left(-\sum_{n=1}^{\infty}(-1)^n\frac{z^n}{n}\right)' = -\sum_{n=1}^{\infty}(-1)^n z^{n-1} = \sum_{n=0}^{\infty}(-1)^n z^n = \frac{1}{1+z} = (\log(1+z))'$$

and for $z = 0$ we get 0 on both sides of (6.1). The Identity Theorem of complex analysis completes the proof of Lemma 6.2. □

Before we examine the distribution of zeros of $\zeta(s)$, we first state the following lemma.

Lemma 6.3. [34] *The series*

$$\sum_p \log\left(\frac{1}{1-p^{-s}}\right)$$

converges normally for $\mathrm{Re}\,(s) > 1$.

Proof. As we have seen in Lemma 2.3, $\frac{1}{1-p^{-s}} \in \mathbb{C}\backslash(-\infty, 0]$. Hence $\log\left(\frac{1}{1-p^{-s}}\right)$ is well defined for all $p \in \mathbb{P}$ and $\mathrm{Re}\,(s) > 1$.
For $z \in \mathbb{C}$ and $|z| \geq 2$

$$|z-1| \geq |z| - 1 \geq |z| - \frac{1}{2}|z| = \frac{1}{2}|z| \tag{6.2}$$

holds and for $\mathrm{Re}\,(s) > 1$ it follows

$$|p^s| = p^{\mathrm{Re}\,(s)} \geq 2. \tag{6.3}$$

By (6.2) for $z = p^s$, which is applicable because of (6.3), we obtain

$$\left|\frac{1}{1-p^{-s}} - 1\right| = \left|\frac{1-(1-p^{-s})}{1-p^{-s}}\right| = \left|\frac{1}{p^s(1-p^{-s})}\right| = \frac{1}{|p^s-1|} \leq \frac{2}{|p^s|} \leq \frac{1}{2}, \tag{6.4}$$

where the last inequality is valid for $p \geq 5$.
The function

$$z \mapsto \frac{\log(z)}{z-1},$$

where we use the Principal Branch of the logarithm, has a removable singularity in $z = 1$, since $\log(1) = 0$. Thus it extends holomorphically to $B_1(1)$. In particular the extension is continuous, so we arive at the conclusion, that

$$C := \sup\left\{\left|\frac{\log(z)}{z-1}\right| : z \in \overline{B}_{\frac{1}{2}}(1)\backslash\{1\}\right\}$$

is finite and

$$|\log(z)| \leq C|z-1| \qquad \text{for all } z \in \overline{B}_{\frac{1}{2}}(1). \tag{6.5}$$

We use (6.5) for $\frac{1}{1-p^{-s}} \in \overline{B}_{\frac{1}{2}}(1)$, which is applicable because of (6.4), and we finally have for all $p \geq 5$ and $\mathrm{Re}\,(s) \geq 1 + \delta$

$$\left|\log\left(\frac{1}{1-p^{-s}}\right)\right| \leq C\left|\frac{1}{1-p^{-s}} - 1\right| = C\frac{1}{|p^s-1|} \leq \frac{2C}{p^{1+\delta}},$$

where the last inequality is valid because of (6.4).
We know, that $\sum_p \frac{1}{p^{1+\delta}}$ converges, so we proved the normal convergence of

$$\sum_p \log\left(\frac{1}{1-p^{-s}}\right).$$

□

[34] see [Be11], p.4.

The following result was first proven by de la Vallée Poussin in an article of 25 pages. The proof was later improved by Mertens and by von Mangoldt.

Lemma 6.4. [35] *If* $\mathrm{Re}\,(s) \geq 1$*, then* $\zeta(s) \neq 0$*.*

Proof. Since $\frac{1}{1-p^{-s}} \in \mathbb{C}\backslash(-\infty, 0]$ as we remarked before, it is evident by the convergent Euler Product $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$ from Lemma 2.3, that $\zeta(s) \neq 0$ for $\mathrm{Re}\,(s) > 1$.[36]
We show $\zeta(s) \neq 0$ for $\mathrm{Re}\,(s) = 1$. For this purpose we use the inequality

$$3 + 4\cos(t) + \cos(2t) \geq 0 \qquad \text{for all } t \in \mathbb{R},$$

which can be proven as follows:
We calculate with the Addition Theorem for the Cosine Function

$$\cos(2t) = \cos^2(t) - \sin^2(t) = \cos^2(t) - (1 - \cos^2(t)) = 2\cos^2(t) - 1$$

and obtain

$$\begin{aligned} 3 + 4\cos(t) + \cos(2t) &= 3 + 4\cos(t) + 2\cos^2(t) - 1 \\ &= 2(1 + 2\cos(t) + \cos^2(t)) = 2(1 + \cos(t))^2 \geq 0. \end{aligned} \tag{6.6}$$

Suppose $s > 1$. We apply the logarithm to the Euler Product $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$ and use the Taylor expansion for the logarithm seen in Lemma 6.2. Since $p^{-s} < 1$ and the logarithm is continuous on $\mathbb{R}^+$, we obtain

$$\begin{aligned} \log(\zeta(s)) &= \sum_p \log \frac{1}{1-p^{-s}} = \sum_p -\log(1-p^{-s}) = \sum_p \sum_{k=1}^{\infty} \frac{(p^{-s})^k}{k} \\ &= \sum_p \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{p^{ks}} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \end{aligned} \tag{6.7}$$

where

$$a_n = \begin{cases} \frac{1}{k} & \text{if } n = p^k \text{ with } p \in \mathbb{P} \text{ and } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

To show equation (6.7) for the complex case, we state the following: Let $s = a + it$ be a complex number with $\mathrm{Re}\,(s) = a > 1$. The Riemann Zeta Function $\zeta(s)$ is non-vanishing for $\mathrm{Re}\,(s) > 1$ as we stated above, so we can find a holomorphic "branch" of the complex logarithm, such that $\log(\zeta(s))$ is well defined. It is evident, that this branch conincides with the real logarithm for $s > 1$. Besides, the series $\sum_p \sum_{k=1}^{\infty} \frac{(p^{-s})^k}{k}$ converges normally in the domain $\{s \in \mathbb{C} : \mathrm{Re}\,(s) > 1\}$. Hence by a corollary of the Uniqueness Theorem, (6.7) is also valid for $\mathrm{Re}\,(s) > 1$.
Since $\log|z| = \mathrm{Re}\,(\log(z))$ for all $z \in \mathbb{C}^*$ and $a_n \geq 0$ for all $n \in \mathbb{N}$, it follows

$$\begin{aligned} \log|\zeta(s)| &= \sum_{n=1}^{\infty} a_n \mathrm{Re}\,(n^{-s}) = \sum_{n=1}^{\infty} a_n \mathrm{Re}\,(e^{-s\log n}) = \\ &= \sum_{n=1}^{\infty} a_n \mathrm{Re}\,\left(e^{-a\log n}(\cos(-t\log n) + i\sin(-t\log n))\right) \\ &= \sum_{n=1}^{\infty} a_n e^{-a\log n}\cos(t\log n) = \sum_{n=1}^{\infty} \frac{a_n}{n^a}\cos(t\log n). \end{aligned}$$

Now we use a trick of Hans von Mangoldt: We conclude

$$\log(|\zeta(a)|^3|\zeta(a+it)|^4|\zeta(a+2it)|) = \sum_{n=1}^{\infty} \frac{a_n}{n^a}(3 + 4\cos(t\log n) + \cos(2t\log n) \geq 0,$$

[35] see [Fo11], pp.6.1-6.2; [SS03], pp.185-187; [Be11], p.4.
[36] see [We06], p.109.

because of (6.6). Since e^x is monotone, this is equivalent to

$$|\zeta(a)|^3|\zeta(a+it)|^4|\zeta(a+2it)| \geq 1 \qquad \text{for all } a > 1 \text{ and } t \in \mathbb{R}. \tag{6.8}$$

We assume there exists a $t \neq 0$ such that $\zeta(1+it) = 0$. It is clear from Lemma 6.1, that $\zeta(s)$ has a pole of order 1 in $s = 1$. Thus the function

$$s \mapsto \zeta(s)^3\zeta(s+it)^4$$

has a zero in $s = 1$. To justify this claim, we argue as follows: By properties of poles and zeros, we can WLOG find a neighbourhood $U(1)$ of 1 and holomorphic, non-vanishing functions $f, g : U(1) \to \mathbb{C}$ such that

$$\zeta(s) = \frac{f(s)}{s-1} \quad \text{for all } s \in U(1) \text{ and}$$
$$\zeta(s+it) = g(s)(s-1)^n \quad \text{for all } s \in U^*(1), \text{ where } n \geq 1.$$

This implies

$$\zeta(s)^3\zeta(s+it)^4 = f(s)g(s)(s-1)^{4n-3},$$

where the right side of the equation is holomorphic on $U(1)$. Hence we obtain

$$\lim_{a \to 1} |\zeta(a)^3\zeta(a+it)^4\zeta(a+2it)| = 0,$$

which contradicts (6.8) and completes the proof of the lemma. □

Now we are able to state a meromorphic extension of $\phi(s)$ in the following corollary.

Corollary 6.5. [37] *The function $\phi(s)$ extends meromorphically to the set $\{s \in \mathbb{C} : \mathrm{Re}\,(s) > 1/2\}$ with poles at the zeros of $\zeta(s)$ and in $s = 1$. In particular, if $\mathrm{Re}\,(s) \geq 1$, then $\phi(s) - \frac{1}{s-1}$ is holomorphic.*

Proof. We use the results from the proof of Lemma 6.4, where we had

$$\log(\zeta(s)) = \sum_p \log\left(\frac{1}{1-p^{-s}}\right)$$

for $s > 1$. Since the series on the right converges normally for $\mathrm{Re}\,(s) > 1$ as we saw in Lemma 6.3, this equality is also valid for $\mathrm{Re}\,(s) > 1$ by the Uniqueness Theorem. Furthermore, we can differentiate it by summands. Using the definition of $\phi(s)$, we obtain

$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} &= -\sum_p \frac{d}{ds}\log\left(\frac{1}{1-p^{-s}}\right) = \sum_p \frac{d}{ds}\left(\log(1-p^{-s})\right) \\ &= \sum_p \frac{\log(p)p^{-s}}{1-p^{-s}} = \sum_p \frac{\log(p)}{p^s-1} = \sum_p \frac{\log(p)(p^s-1+1)}{(p^s-1)p^s} \\ &= \sum_p \frac{\log(p)}{p^s} + \sum_p \frac{\log(p)}{(p^s-1)p^s} = \phi(s) + \sum_p \frac{\log(p)}{(p^s-1)p^s}. \end{aligned} \tag{6.9}$$

The last series is normally convergent for $\mathrm{Re}\,(s) > \frac{1}{2}$. This can be seen as follows: Let $\mathrm{Re}\,(s) \geq \frac{1}{2} + \delta$ for some $\delta > 0$. For $n \in \mathbb{N}$ we obtain

$$|n^s| = n^{\mathrm{Re}\,(s)} \geq n^{\frac{1}{2}+\delta}.$$

Since $x \mapsto x^{\mathrm{Re}\,(s)}$ is monotone, we can find $P_0 = P_0(\delta)$ such that

$$\frac{1}{2}|p^s| = \frac{p^{\mathrm{Re}\,(s)}}{2} \geq 1 \qquad \text{for all } p \geq P_0.$$

[37] see [Za97], p.706 and [Be11], p.5.

Hence for $p \geq P_0$ it holds

$$|p^s - 1| \geq |p^s| - 1 \geq |p^s| - \frac{1}{2}|p^s| = \frac{1}{2}|p^s| \geq \frac{1}{2}p^{\frac{1}{2}+\delta}.$$

There is a constant $C \in \mathbb{R}$ such that $\log x \leq Cx^\delta$ for $x > 1$, so we get

$$\left|\frac{\log p}{p^s(p^s-1)}\right| \leq \frac{Cp^\delta}{\frac{1}{2}p^{\frac{1}{2}+\delta}p^{\frac{1}{2}+\delta}} = 2C\frac{1}{p^{1+\delta}}.$$

Thus the series $\sum_p \frac{\log(p)}{(p^s-1)p^s}$ is normally convergent and holomorphic for $\mathrm{Re}\,(s) > 1/2$.

By (6.9) and Lemma 6.1 we obtain, that $\phi(s)$ extends meromorphically to $\mathrm{Re}\,(s) > 1/2$ and only has poles at the zeros of $\zeta(s)$ and at $s = 1$. We take a closer look at the point $s = 1$. It follows

$$\phi(s) - \frac{1}{s-1} = -\sum_p \frac{\log(p)}{(p^s-1)p^s} - \left(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{1-s}\right).$$

Since $\zeta(s)$ has a pole of order 1 in $s = 1$ and is apart from that holomorphic for $\mathrm{Re}\,(s) > 0$, we can find a punctured neighbourhood $U^*(1)$ and a holomorphic, non-vanishing function $h : U(1) \to \mathbb{C}$ such that $\zeta(s) = \frac{h(s)}{s-1}$ for all $s \in U^*(1)$. We get

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{-(s-1)^{-2}h(s) + h'(s)(s-1)^{-1}}{(s-1)^{-1}h(s)} = -\frac{1}{s-1} + \frac{h'(s)}{h(s)}.$$

The last summand on the right is holomorphic on $U(1)$, so we obtain the holomorphic extension of $\phi(s) - \frac{1}{s-1}$ in $s = 1$.

Regarding the zeros of $\zeta(s)$ we can use Lemma 5.3, which states, that $\zeta(s) \neq 0$ for $\mathrm{Re}\,(s) \geq 1$, which proves Corollary 6.5. □

We now use the Auxiliary Tauberian Theorem, which we proved in section 5, in order to show the convergence of the following integral.

Lemma 6.6. [38] *The integral*

$$\int_1^\infty \frac{\vartheta(x) - x}{x^2} dx$$

is convergent.

Proof. By Lemma 4.1 on partial sums

$$\sum_{\substack{n\in\mathbb{N}\\ t_n \leq x}} a_n g(t_n) = A(x)g(x) - \int_{t_1}^x A(t)g'(t)dt$$

holds, where we set $a_n = \log(p_n)$, $t_n = p_n$ and $g(t) = t^{-s}$ for $\mathrm{Re}\,(s) > 1$, which is continuously differentiable for $t \geq 2$. Then $t_n \in \mathbb{R}$ are strictly increasing. Plugging this in the formula above, we get

$$\sum_{p\leq x} \log(p)p^{-s} = \sum_{p\leq x} \log(p)x^{-s} - \int_{p_1}^x \sum_{p\leq t} \log(p)(t^{-s})'dt.$$

Recall, that

$$\vartheta(x) = \sum_{p\leq x} \log p \qquad \text{for all } x \in \mathbb{R}$$

and thus

$$\phi(s) = \lim_{x\to\infty} \sum_{p\leq x} \frac{\log p}{p^s} = \lim_{x\to\infty} \left(\frac{\vartheta(x)}{x^s} - \int_2^x \vartheta(t)(t^{-s})'dt\right) = s\int_1^\infty \frac{\vartheta(t)}{t^{s+1}}dt.$$

[38] see [Za97], p.706; [Ko82], p.110; [Be11], p.7.

By Lemma 2.4 we know, that $\vartheta(x) = \mathcal{O}(x)$ for $x \to \infty$, so $\frac{\vartheta(x)}{x^s} \to 0$ for $x \to \infty$, if $\mathrm{Re}(s) > 1$.
We substitute $t = e^x$ and get

$$\phi(s) = s\int_1^\infty \frac{\vartheta(t)}{t^{s+1}}dt = s\int_0^\infty \frac{\vartheta(e^x)}{e^{x(s+1)}}e^x dx = s\int_0^\infty e^{-sx}\vartheta(e^x)dx.$$

Let

$$f(x) := \vartheta(e^x)e^{-x} - 1 \quad \text{and}$$
$$g(s) := \frac{\phi(s+1)}{s+1} - \frac{1}{s}.$$

Since

$$g(s) = \frac{\phi(s+1)}{s+1} - \frac{1}{s} = \frac{1}{s+1}\left(\phi(s+1) - \frac{s+1}{s}\right) = \frac{1}{s+1}\left(\phi(s+1) - \frac{1}{s} - 1\right),$$

we know from Corollary 6.5, that $g(s)$ is holomorphic for $\mathrm{Re}(s) \geq 0$.
We conclude

$$\begin{aligned} g(s) &= \frac{\phi(s+1)}{s+1} - \frac{1}{s} = \frac{s+1}{s+1}\int_0^\infty e^{-(s+1)x}\vartheta(e^x)dx - \frac{1}{s} \\ &= \int_0^\infty e^{-sx}\vartheta(e^x)e^{-x}dx - \frac{1}{s} = \int_0^\infty e^{-sx}\vartheta(e^x)e^{-x}dx - \left[-\frac{e^{-sx}}{s}\right]_{x=0}^\infty \\ &= \int_0^\infty e^{-sx}(\vartheta(e^x)e^{-x} - 1)dx = \int_0^\infty e^{-sx}f(x)dx \end{aligned}$$

for $\mathrm{Re}(s) > 0$. Since $\vartheta(x) = \mathcal{O}(x)$ $(x \to \infty)$, we know, that $f(x) = \vartheta(x)e^{-x} - 1$ is bounded. Since $\vartheta(x)$ is non-decreasing and e^{-x} is non-vanishing and decreasing, we obtain, that $f(x)$ is measurable as a product and sum of measurable functions. Thus we can apply Theorem 5.1 and we derive by substitution of $e^x = t$, that

$$\lim_{T\to\infty}\int_0^T (\vartheta(e^x)e^{-x} - 1)dx = \lim_{T\to\infty}\int_1^T \left(\frac{\vartheta(t)}{t} - 1\right)\frac{dt}{t} = \lim_{T\to\infty}\int_1^T \frac{\vartheta(t) - t}{t^2}dt$$

exists. □

As we saw in section 3, the Prime Number Theorem is equivalent to $\vartheta(x) \sim x$, which we will show in the final step of the proof.

Theorem 6.7. [39] *The function $\vartheta(x)$ is asymptotically equal to x, i.e.*

$$\vartheta(x) \sim x \qquad (x \to \infty).$$

Proof. We assume towards a contradiction, that $\lim_{x\to\infty}\frac{\vartheta(x)}{x} = 1$ is not true. There are two cases to consider:

(1) Suppose $\limsup_{x\to\infty}\frac{\vartheta(x)}{x} > 1$. Then there exists some real $\lambda > 1$ such that there are arbitrarily large x with $\vartheta(x) \geq \lambda x$. We know, that $\vartheta(t)$ is non-decreasing, so we have

$$\begin{aligned} \int_x^{\lambda x}\frac{\vartheta(t) - t}{t^2}dt &\geq \int_x^{\lambda x}\frac{\vartheta(x) - t}{t^2}dt \geq \int_x^{\lambda x}\frac{\lambda x - t}{t^2}dt \\ &= \int_1^\lambda \frac{\lambda x - ux}{(ux)^2}xdu = \int_1^\lambda \frac{\lambda - u}{u^2}du \\ &= \left[-\frac{\lambda}{u} - \log(u)\right]_{u=1}^\lambda = -1 + \lambda - \log(\lambda) > 0, \end{aligned}$$

where we substituted $u = \frac{t}{x}$.

[39] see [Za97], p.707.

(2) Similarly, suppose $\liminf_{x\to\infty} \frac{\vartheta(x)}{x} < 1$. Then there exists some $\lambda < 1$ such that there are arbitrarily large x with $\vartheta(x) \leq \lambda x$. We conclude

$$\int_{\lambda x}^{x} \frac{\vartheta(t) - t}{t^2} dt \leq \int_{\lambda x}^{x} \frac{\lambda x - t}{t^2} dt = \int_{\lambda}^{1} \frac{\lambda - u}{u^2} du$$
$$= \left[-\frac{\lambda}{u} - \log(u) \right]_{u=\lambda}^{1} = -\lambda + 1 + \log(\lambda) < 0.$$

In both cases, the result of our computation is independent of x. This implies, that the integral diverges, which is a contradiction to Lemma 6.6. □

This concludes the proof of the Prime Number Theorem.

7. Immediate consequences of the Prime Number Theorem and Betrand's Postulate

In this section we state an asymptotic relation for the nth prime number and Betrand's Postulate. In order to prove this asymptotic relation for the nth prime, we state a preliminary lemma.

Lemma 7.1. [40] *If $\lambda > 0$, then*

$$\lim_{n\to\infty} \frac{\pi(\lambda n \log n)}{n} = \lambda.$$

Proof. Since $\log(\lambda n \log n) = \log n + \log \lambda + \log \log n$, we conclude, that $\log(\lambda n \log n) \sim \log n$:

$$\lim_{n\to\infty} \frac{\log(\lambda n \log n)}{\log n} = \lim_{n\to\infty} \left(1 + \frac{\log \lambda + \log \log n}{\log n}\right) = 1.$$

Using the Prime Number Theorem for $x = \lambda n \log n$, we obtain

$$1 = \lim_{n\to\infty} \frac{\pi(\lambda n \log n)}{\lambda n} \frac{\log(\lambda n \log n)}{\log n} = \lim_{n\to\infty} \frac{\pi(\lambda n \log n)}{\lambda n}.$$

Multiplication by λ yields the result. □

Theorem 7.2. [41] *If $(p_n)_{n\in\mathbb{N}}$ is the sequence of all prime numbers in increasing order, then*

$$p_n \sim n \log n.$$

Proof. Theorem 7.2 is equivalent to

$$\lim_{n\to\infty} \frac{p_n}{n \log n} = 1. \tag{7.1}$$

We use an argument similar to the one in the proof of Theorem 6.7. Assume towards a contradiction, that (7.1) is not true. Then there are two possibilities:

(1) Suppose we have

$$\limsup_{n\to\infty} \frac{p_n}{n \log n} > 1.$$

Hence there is a constant $\varepsilon > 0$ such that

$$p_n \geq (1+\varepsilon) n \log n$$

for arbitrarily large n.
For those n it is then true, that

$$\pi\left((1+\varepsilon) n \log n\right) \leq n,$$

which gives

$$\liminf_{n\to\infty} \frac{\pi\left((1+\varepsilon) n \log n\right)}{n} \leq 1.$$

By Lemma 7.1 we obtain, that $\lim_{n\to\infty} \frac{\pi((1+\varepsilon) n \log n)}{n} = (1+\varepsilon)$, which is a contradiction.

(2) Suppose we have

$$\liminf_{n\to\infty} \frac{p_n}{n \log n} < 1.$$

Hence there is a constant $\varepsilon > 0$ such that

$$p_n \leq (1-\varepsilon) n \log n$$

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[40] see [Fo11], pp.6.7-6.8.

[41] see [Fo11], pp.6.6-6.7.

for arbitrarily large n.
For those n it is then true, that

$$\pi\left((1-\varepsilon)n\log n\right) \geq n,$$

which gives us

$$\limsup_{n\to\infty} \frac{\pi\left((1-\varepsilon)n\log n\right)}{n} \geq 1.$$

This is again a contradiction to Lemma 7.1.

□

To motivate Bertrand's Postulate, we prove Theorem 7.3 as a direct corollary of the Prime Number Theorem.

Theorem 7.3. [42] *If $\varepsilon > 0$, then there is a constant $x_0(\varepsilon)$ such that for all $x \geq x_0$, there is at least one prime number in the interval $[x, x(1+\varepsilon)]$.*

Proof. Applying the Prime Number Theorem we have

$$\lim_{x\to\infty} \frac{\pi(x(1+\varepsilon))}{\pi(x)} = \lim_{x\to\infty} \frac{x(1+\varepsilon)}{x} \frac{\log x}{\log((1+\varepsilon)x)} = \lim_{x\to\infty} \frac{(1+\varepsilon)\log x}{\log(1+\varepsilon)+\log x} = 1+\varepsilon.$$

Hence, for every $0 < \delta < \varepsilon$ we can find x_0 such that

$$\pi(x) < \pi(x)(1+(\varepsilon-\delta)) < \pi(x(1+\varepsilon)) \qquad \text{for all } x \geq x_0$$

and since $\pi(x)$ is an integer, this proves the claim. □

As a special case for $\varepsilon = 1$, we will now discuss Betrand's Postulate. Betrand himself could verify this Posulate up to $n = 3000000$. Five years later, it was proven by Tschebyscheff. Here we will restate a proof by Paul Erdös from 1932.

Theorem 7.4 (Betrand)**.** [43] *If $n \geq 1$, there exists a prime number p such that*

$$n < p \leq 2n.$$

Proof. First we show, that Betrand's Postulate is true for $n < 4000$. This is done by "Landau's Trick". It is sufficient to see, that

$$2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 631, 1259, 2503, 4001$$

are prime numbers and every one of them is smaller than twice its predecessor. Hence every interval $\{y : n < y \leq 2n\}$ with $n \leq 4000$ contains one of these prime numbers.
Now we show

$$\prod_{p\leq x} p \leq 4^{x-1} \qquad \text{for all } x \geq 2 \tag{7.2}$$

by induction over the number of primes in the product above. For the largest prime $q \leq x$ we conclude

$$\prod_{p\leq x} p = \prod_{p\leq q} p \quad \text{and} \quad 4^{q-1} \leq 4^{x-1}.$$

This implies, that it is sufficient to prove (7.2) for the case, that x is a prime number.

- Initial step of the induction: For $q = 2$ we get $2 \leq 4$, which seems to be true. All other primes are uneven, so we have a look at a prime number $q = 2m+1$, $m \in \mathbb{N}$.
- Induction hypothesis: (7.2) is true for all integers $2, \ldots, 2m$.

[42] see [Fo11], p.6.8.
[43] see [AZ04], pp.6-13.

- Induction step: Let $q = 2m+1$. Now we state some basic properties:

$$\prod_{p \le m+1} p \le 4^m$$

is true by the induction hypothesis. Besides, in a similar argument as the one in the proof of Lemma 2.4, we obtain

$$\prod_{m+1 < p \le 2m+1} p \le \binom{2m+1}{m}$$

by the equality

$$\binom{2m+1}{m} = \frac{(2m+1)!}{m!(m+1)!}:$$

It is evident, that all prime numbers on the left side of the inequality above divide $(2m+1)!$, but not $m!(m+1)!$, hence the inequality holds.
Since $\binom{2m+1}{m}$ and $\binom{2m+1}{m+1}$ appear in the sum $\sum_{k=0}^{2m+1} \binom{2m+1}{k} = 2^{2m+1}$ and are equal, we obtain

$$2\binom{2m+1}{m} \le 2^{2m+1}.$$

This gives

$$\binom{2m+1}{m} \le 2^{2m}.$$

Combining the equations above it follows

$$\prod_{p \le 2m+1} p = \prod_{p \le m+1} p \prod_{m+1 < p \le 2m+1} p \le 4^m \binom{2m+1}{m} \le 4^m 2^{2m} = 4^{2m}.$$

Next, we use Legendre's Theorem 4.2 to state, that $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$ contains the prime p exactly

$$\sum_{k \ge 1} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \left\lfloor \frac{n}{p^k} \right\rfloor \right)$$

times. Every summand is at most 1, which can be seen by the inequality

$$\left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \left\lfloor \frac{n}{p^k} \right\rfloor < \frac{2n}{p^k} - 2\left(\frac{n}{p^k} - 1 \right) = 2$$

and in addition any such summand is an integer. As we have already seen in Theorem 4.2, the summands are zero for $p^k > 2n$. Hence $\binom{2n}{n}$ contains the factor p

$$\sum_{k \ge 1} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \left\lfloor \frac{n}{p^k} \right\rfloor \right) \le \max\{r : p^r \le 2n\}$$

times. We deduce the following properties :

(1) The largest power of p, which divides $\binom{2n}{n}$, is not larger than $2n$.
(2) In particular, prime numbers p larger than $\sqrt{2n}$ are contained at most once in $\binom{2n}{n}$.
(3) For $3n \ge 3p > 2n$ and $n, p \ge 3$, (1) states, that p and $2p$ are the only multiples of p, which can appear in the numerator of $\frac{(2n)!}{(n!)^2}$. But we also have two p-factors in the denominator. Hence, prime numbers in the region $\frac{2}{3}n < p \le n$ do not appear in $\binom{2n}{n}$ at all.

We estimate $\binom{2n}{n}$ for $n \geq 2$: By

$$\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}$$

we know, that $\binom{n}{\lfloor n/2 \rfloor}$ is the largest integer in the sequence of n integers

$$\binom{n}{0} + \binom{n}{n}, \binom{n}{1}, \ldots, \binom{n}{n-1},$$

whose sum is 2^n and whose mean is $\frac{2^n}{n}$. So it applies

$$\binom{n}{\lfloor n/2 \rfloor} \geq \frac{2^n}{n} \qquad \text{for } n \geq 2, \tag{7.3}$$

where equality only holds for $n = 2$. By (7.3) we obtain

$$\binom{2n}{n} \geq \frac{4^n}{2n} \qquad \text{for } n \geq 1,$$

which gives

$$\frac{4^n}{2n} \leq \binom{2n}{n} \leq \prod_{p \leq \sqrt{2n}} 2n \cdot \prod_{\sqrt{2n} < p \leq \frac{2}{3}n} p \cdot \prod_{n < p \leq 2n} p \qquad \text{for } n \geq 3,$$

where we used (1),(2),(3) for the three products on the right. Since there are at most $\sqrt{2n}$ prime numbers for $p \leq \sqrt{2n}$, it follows, that

$$4^n \leq (2n)^{1+\sqrt{2n}} \cdot \prod_{\sqrt{2n} < p \leq \frac{2}{3}n} p \cdot \prod_{n < p \leq 2n} p \qquad \text{for } n \geq 3. \tag{7.4}$$

Assume towards a contradiction, that there is no prime number with $n < p \leq 2n$, which means, that the second product in (7.4) is 1. Plugging (7.2) into (7.4) we get

$$4^n \leq (2n)^{1+\sqrt{2n}} 4^{\frac{2}{3}n},$$

which is equivalent to

$$4^{\frac{1}{3}n} \leq (2n)^{1+\sqrt{2n}}. \tag{7.5}$$

If we use the inequality $a + 1 < 2^a$, which is true for all $a \geq 2$, we get

$$2n = (\sqrt[6]{2n})^6 < (\lfloor \sqrt[6]{2n} \rfloor + 1)^6 < 2^{6\lfloor \sqrt[6]{2n} \rfloor} \leq 2^{6\sqrt[6]{2n}} \tag{7.6}$$

and hence for $n \geq 50$ (such that $18 < 2\sqrt{2n}$), we get using (7.5) and (7.6)

$$2^{2n} = 4^{\frac{1}{3}n3} \leq (2n)^{3(1+\sqrt{2n})} < 2^{\sqrt[6]{2n}(18+18\sqrt{2n})} < 2^{\sqrt[6]{2n}20\sqrt{2n}} = 2^{20(2n)^{2/3}}.$$

This is equivalent to

$$2n < 20(2n)^{2/3} \Leftrightarrow (2n)^{1/3} < 20 \Leftrightarrow 2n < 8000 \Leftrightarrow n < 4000.$$

However, we saw by the Landau-Trick, that Theorem 7.4 is true for $n < 4000$, so we obtain the desired contradiction. □

8. Outlook on current developments

As an outlook on current developments in Number Theory regarding prime numbers we give a short overview of a theorem about prime numbers in short intervals and Green-Tao's Theorem. Besides, we mention the Goldbach Conjecture and a new development on the way to the proof of the Twin Prime Conjecture.

8.1. Maier's Theorem: Primes in short intervals.

The Prime Number Theorem states

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1.$$

If we want to know the number of primes "near x" (i.e. in intervals of the type $[x, x+\Phi(x)]$, where $\Phi : \mathbb{R} \to \mathbb{R}$), we could pose the question: For which functions $\Phi(x)$ does the asymptotic equality

$$\pi(x+\Phi(x)) - \pi(x) \sim \frac{\Phi(x)}{\log x} \qquad (x \to \infty) \tag{8.1}$$

hold? In other words: How large do we have to choose $\Phi(x)$ in order to guarantee, that the interval $[x, x+\Phi(x)]$ contains roughly $\Phi(x)/\log x$ prime numbers? As an example we can easily check (8.1) for $\Phi(x) = x$ using the Prime Number Theorem:

$$\frac{\pi(2x)-\pi(x)}{x/\log x} = \underbrace{\frac{\pi(2x)}{2x/\log(2x)}}_{\to 1} \underbrace{\frac{2x}{\log 2+\log x}\frac{\log x}{x}}_{\to 2} - \underbrace{\frac{\pi(x)}{x/\log x}}_{\to 1} \to 1 \quad \text{for } x\to\infty.$$

In order to motivate (8.1) we assume $\Phi(x) \le x$ and regard the following expression, where we use the equivalences of the Prime Number Theorem seen in Theorem 3.1:

$$\pi(x+\Phi(x)) - \pi(x) \sim \int_x^{x+\Phi(x)} \frac{1}{\log t} dt \le \int_x^{x+\Phi(x)} \frac{1}{\log x} dt = \frac{\Phi(x)}{\log x}.$$

On the other hand we state

$$\pi(x+\Phi(x)) - \pi(x) \sim \int_x^{x+\Phi(x)} \frac{1}{\log t} dt \ge \int_x^{x+\Phi(x)} \underbrace{\frac{1}{\log(x+\Phi(x))}}_{\le \log(2x)} dt \ge \frac{\Phi(x)}{\log 2+\log x}.$$

Hence it can be of interest to etablish (8.1) for certain $\Phi(x) \le x$.
Heath-Brown showed (8.1) for $\Phi(x) = x^{7/12-\varepsilon(x)}$ and $\varepsilon(x) \to 0$ for $x \to \infty$. In 1984, Helmut Maier, currently at Ulm University, proved, that it is not sufficient to choose $\Phi(x) = (\log x)^{\lambda_0}$ for $\lambda_0 > 1$. His theorem indicates, that intervals of the type $[x, x+\log(x)^{\lambda_0}]$ for x abritarily large, can contain a number of primes, which is "constantly too high". To be more specific, we state his theorem:

Theorem 8.1 (Maier)**.** [44] *If $\Phi(x) = (\log x)^{\lambda_0}$ and $\lambda_0 > 1$, then*

$$\limsup_{x\to\infty} \frac{\pi(x+\Phi(x)) - \pi(x)}{\Phi(x)/\log x} > 1 \quad \textit{and} \quad \liminf_{x\to\infty} \frac{\pi(x+\Phi(x)) - \pi(x)}{\Phi(x)/\log x} < 1.$$

For the range $1 < \lambda_0 < e^\gamma$ we even have

$$\limsup_{x\to\infty} \frac{\pi(x+\Phi(x)) - \pi(x)}{\Phi(x)/\log x} \ge \frac{e^\gamma}{\lambda_0},$$

where $\gamma = \int_1^\infty \left(\left\lfloor \frac{1}{x} \right\rfloor - \frac{1}{x}\right) dx$ denotes Euler's constant.

[44] see [Ma85], p.1.

8.2. **Green-Tao's Theorem: Primes contain arbitrarily long arithmetic progessions.** A relatively new discovery was made by Ben Green and Terence Tao in 2008. To understand their findings, we introduce so-called arithmetic progressions.

Definition 8.2. [45] An *arithmetic progression* of length $k \in \mathbb{N}$ is a set $Q \subset \mathbb{Z}$ such that there is $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and

$$Q = \{a, a+q, a+2q, \dots, a+(k-1)q\}.$$

Example 8.3. [45] We start with some examples for arithmetic progressions of prime numbers:

(1) $5, 11, 17, 23, 29$
(2) $\{199 + 210n : 0 \leq n \leq 9\} = \{199, 409, \dots, 2089\}$
(3) A record by M. Frind 2003: $\{376859931192959 + 18549279769020k : \ k = 0, 1, \dots, 21\}$

A deep result by Green-Tao is the following theorem:

Theorem 8.4 (Green-Tao)**.** [46] *The prime numbers* $\mathbb{P}$ *contain infinitely many arithmetic progressions of length* k *for all* $k \in \mathbb{N}$*.*

Green-Tao could even prove a stronger result:

Theorem 8.5 (Szemeredi's Theorem in the primes)**.** [46] *If* $A \subset \mathbb{P}$ *is of positive relative upper density, that is*

$$\limsup_{N\to\infty} \frac{|A \cap [1, N]|}{\pi(N)} > 0,$$

then A *contains infinitely many arithmetic progressions of length* k *for all* $k \in \mathbb{N}$*.*

The proof of the last two theorems is not constructive and uses results of many sophisticated areas of mathematics, amongst others Number Theory, Ergodic Theory, Combinatorics and Harmonic Analysis.

8.3. **Helfgott: Minor and Major Arcs for Goldbach's Problem.** In 2013, there were made two major discoveries in the field of prime numbers, which we mention here. The first concerns the so-called Goldbach Conjecture, which is one of the oldest unsolved problems of Number Theory. It states the following:

Conjecture 8.6 (Strong Goldbach Conjecture)**.** *Every even integer greater than 2 can be expressed as the sum of two primes.*

This conjecture has its origin in a correspondence between the German mathematician Christian Goldbach[47] and Leonhard Euler in 1742. In this context, Goldbach also proposed a weaker conjecture:

Conjecture 8.7 (Weak Goldbach Conjecture)**.** *Every odd integer greater than 5 can be written as the sum of three primes.*

While the strong Goldbach Conjecture remains unsolved until today, there have been successful efforts to prove the weak conjecture: In 1923, Hardy[48] and Littlewood[49] proved it for sufficiently large numbers assuming the so-called "Generalized Riemann Hypothesis". This result was improved to be valid without any lower bounds in 1997[50]. A different approach was made by the Soviet mathematician

[45] see [H11], p.2.
[46] see [GT08], p.2.
[47] Christian Goldbach (1690–1764)
[48] Godfrey Harold Hardy (1877–1947)
[49] John Edensor Littlewood (1885–1977)
[50] see [B81].

Vinogradov[51]. He proved the ternary conjecture unconditionally in 1937 for all numbers greater than a constant C. This value C was then lowered a couple of times to $C = e^{3100}$, which was still far too large to make a mechanical verification of the conjecture up to C possible. In fact, Goldbach's ternary conjecture has only been checked by computer for all $n < 10^{29}$. Finally on May 13th 2013 the mathematician Harald Helfgott[52] claimed to have found a proof of the weak Goldbach Conjecture for all numbers $n \geq 10^{29}$, which thus closes the gap between theoretical and mechanical verification of the conjecture. In this context he published a paper on exponential-sum estimates and a paper on the proof itsself[53], where famous methods of Analytic Number Theory like the Circle Method and the Large Sieve play a major role.

8.4. **Zhang: Bounded gaps between primes.** Another famous unsolved problem in the theory of prime numbers is the Twin Prime Conjecture:

Conjecture 8.8 (Twin Prime Conjecture)**.** *There are infinitely many primes p such that $p + 2$ is also prime.*

As in Maier's Theorem, we are interested in the gaps between prime numbers here. In May 2013, Yitang Zhang proved a weaker form of this problem: He showed, that there are infinitely many primes, which differ by at most 70 million. We state the main theorem of his paper[54]:

Theorem 8.9. *It is true, that*

$$\liminf_{n \to \infty} (p_{n+1} - p_n) < 7 \cdot 10^7.$$

While the number 70 million is not chosen optimally as the author himself states, a Polymath project suggested by Terence Tao already reduced the bound to 6712 (unconfirmed) effective July 1st 2013[55]. The result of Yitang Zhang is remarkable because it does not rely on unproven conjectures. It extends already known ideas by Goldston, Pintz and Yildirim, who published two papers on small gaps between prime numbers in 2005 and 2007.

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[51] Ivan Matveevich Vinogradov (1891–1983)

[52] Harald Andrés Helfgott (1977-)

[53] see [He197] and [He297].

[54] see [Y13].

[55] see
`http://michaelnielsen.org/polymath1/index.php?title=Bounded_gaps_between_primes`.

9. The Functional Equation of the Zeta Function

In this section we derive the Functional Equation of the Riemann Zeta Function by examining certain properties of the Theta Series and the Gamma Function. Using the Functional Equation we are able to find the so-called "trivial" zeros of the Zeta Function.

We start with a preceeding lemma, which is needed to prove the Functional Equation of the Theta Series and requires some knowledge of Fourier Series and Fourier Transforms.

Lemma 9.1 (Poisson Summation Formula)**.** [56]

Let $f : \mathbb{R} \to \mathbb{C}$ be a continuously differentiable function such that

$$f(x) = O(|x|^{-2}) \qquad \text{and } f'(x) = O(|x|^{-2}) \qquad \text{for } |x| \to \infty.$$

If $\hat{f} : \mathbb{R} \to \mathbb{C}$ is the Fourier Transform of f, that is

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x t} dx,$$

then

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

Proof. Since f is continuously differentiable and has the above mentioned behaviour for $|x| \to \infty$, the Fourier Transform $\hat{f}(t)$ is well defined. Let

$$F : \mathbb{R} \to \mathbb{C} \qquad F(x) := \sum_{n=-\infty}^{\infty} f(x+n). \tag{9.1}$$

Since $f(x) = O(|x|^{-2})$, the function $F(x)$ as well as its derivative converge uniformly by comparison test with the convergent series $C \sum_{n=n_0}^{\infty} n^{-2}$ for some $n_0 \in \mathbb{N}$ and $C \in \mathbb{R}$. Hence we can exchange the limit and derivative. Since $f(x)$ is continuously differentiable, we conclude, that $F(x)$ is also continuously differentiable. It holds $F(x) = F(x+1)$, so $F(x)$ is periodic with period $T = 1$. Hence the Fourier Series

$$F(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x} \tag{9.2}$$

exists and since F is continuously differentiable, the Fourier Series converges uniformly to F by the Dirichlet Theorem[57]. By

$$c_n = \int_0^1 F(x) e^{-2\pi i n x} dx$$

we obtain

$$\begin{aligned} c_n &= \sum_{k=-\infty}^{\infty} \int_0^1 f(x+k) e^{-2\pi i n x} dx = \sum_{k=-\infty}^{\infty} \int_k^{k+1} f(x+k) e^{-2\pi i n x} dx \\ &= \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x} dx = \hat{f}(n). \end{aligned}$$

Combining the two expressions (9.1) and (9.2) for $F(x)$ we obtain

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x},$$

which concludes the proof of Lemma 9.1 setting $x = 0$. ☐

[56] see [Fo11], p.7.1.

[57] see [Hi05], p.419.

Theorem 9.2 (Functional Equation of the Theta Series)**.** [58] *If the Theta Series is defined by*

$$\Theta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} \qquad \textit{for } x > 0,$$

then

$$\Theta(x) = \frac{1}{\sqrt{x}}\Theta\left(\frac{1}{x}\right) \qquad \textit{holds for all } x > 0.$$

Proof. As one can compute by path integration in the complex plane, the Fourier Transform of f for

$$f : \mathbb{R} \to \mathbb{R} \qquad f(x) = e^{-\pi x^2}$$

is

$$\hat{f}(t) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x t} dx = e^{-\pi t^2} = f(t) \qquad \text{for all } t \in \mathbb{R}. \tag{9.3}$$

Let

$$f_\lambda : \mathbb{R} \to \mathbb{R} \qquad f_\lambda(x) = e^{-\pi\lambda x^2} \qquad \text{for } \lambda > 0.$$

The definition of the Fourier Transform gives

$$\hat{f}_\lambda(t) = \int_{-\infty}^{\infty} e^{-\pi\lambda x^2} e^{-2\pi i x t} dx.$$

Substituting

$$u = \sqrt{\lambda}x \quad \Leftrightarrow \quad \frac{du}{dx} = \sqrt{\lambda} \quad \text{and} \quad v = \frac{t}{\sqrt{\lambda}}$$

we obtain

$$\hat{f}_\lambda(t) = \int_{-\infty}^{\infty} e^{-\pi u^2} e^{-2\pi i v\sqrt{\lambda}x} \frac{du}{\sqrt{\lambda}} = \int_{-\infty}^{\infty} e^{-\pi u^2} e^{-2\pi i u v} \frac{du}{\sqrt{\lambda}},$$

which is the same as (9.3), so

$$\hat{f}_\lambda(t) = \frac{e^{-\pi v^2}}{\sqrt{\lambda}} = \frac{e^{-\pi \frac{t^2}{\lambda}}}{\sqrt{\lambda}}.$$

The function $f_\lambda(t)$ fulfills the requirements of the Poisson Summation Formula 9.1, because it decreases exponentially fast in t and is continuously differentiable. Thus by Lemma 9.1 it follows

$$\sum_{n\in\mathbb{Z}} f_\lambda(n) = \sum_{n\in\mathbb{Z}} \hat{f}_\lambda(n),$$

which is equivalent to

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 \lambda} = \frac{1}{\sqrt{\lambda}} \sum_{n\in\mathbb{Z}} e^{-\pi \frac{n^2}{\lambda}}.$$

If we write x instead of λ, then this concludes the proof of Theorem 9.2. □

[58] see [Fo11], pp. 7.2-7.4 and [SS03], p.169.

Using arguments of uniform convergence, we can derive a useful corollary.

Corollary 9.3. [59] *For the Theta Series* $\Theta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}$ *it holds*

$$\Theta(x) = \mathcal{O}\left(\frac{1}{\sqrt{x}}\right) \qquad \text{for } x \downarrow 0.$$

Proof. For $\varepsilon > 0$ and $x \in [\varepsilon, \infty)$ we conclude

$$e^{-\pi n^2 x} \leq e^{-\pi n^2 \varepsilon},$$

because $e^{-\pi n^2 x}$ is non-increasing for $n \in \mathbb{Z}$. The series $\sum_{n=-\infty}^{\infty} e^{-\pi n^2 \varepsilon}$ converges. Since e^{-x} decreases faster than any polynomial in x, we conclude the following: The series $\Theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}$ as well as its derivatives are uniformly convergent on $[\varepsilon, \infty)$, where $\varepsilon > 0$, so $\Theta(x) \in \mathbb{C}^{\infty}(0, \infty)$.
By Theorem 9.2

$$\Theta(x) = \frac{1}{\sqrt{x}} \sum_{n\in\mathbb{Z}} e^{-\pi \frac{n^2}{x}}$$

holds. Since $\Theta(x)$ is uniformly convergent, we obtain

$$\lim_{x\to\infty} \left(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} \right) = \sum_{n\in\mathbb{Z}} \lim_{x\to\infty} e^{-\pi n^2 x} = 1,$$

which proves Corollary 9.3. □

We now introduce the integral form of the Gamma Function, which was first given by Euler[60] in 1729. Since we make use of some of its properties, we first examine it more closely.

Lemma 9.4. [61] *The Gamma Function, which is defined by*

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt \qquad \text{for } s \in \mathbb{C},\ \mathrm{Re}\,(s) > 0,$$

can be extended meromorphically to the whole complex plane with simple poles at $s = 0, -1, -2 \ldots.$

Proof. It holds $|t^{s-1}e^{-t}| = t^{\mathrm{Re}\,(s)-1}e^{-t}$. The function

$$\int_0^{\infty} t^{\mathrm{Re}\,(s)-1} e^{-t} dt = \Gamma(\mathrm{Re}\,(s))$$

is the real Gamma Function, which we examine now. We show the existence of the improper Riemann integral. If we split the integral up at $t = 1$, we have

$$\int_0^1 t^{\mathrm{Re}\,(s)-1} e^{-t} dt + \int_1^{\infty} t^{\mathrm{Re}\,(s)-1} e^{-t} dt.$$

We show the convergence of both integrals seperately.
Since

$$|t^{\mathrm{Re}\,(s)-1} e^{-t}| \leq t^{\mathrm{Re}\,(s)-1} \qquad \text{for } 0 \leq t \leq 1$$

and

$$\lim_{\varepsilon\to 0} \int_{\varepsilon}^1 t^{\mathrm{Re}\,(s)-1} dt = \lim_{\varepsilon\to 0} \left[\frac{t^{\mathrm{Re}\,(s)}}{\mathrm{Re}\,(s)} \right]_{t=\varepsilon}^1 = \lim_{\varepsilon\to 0} \frac{1}{\mathrm{Re}\,(s)} \left(1 - \varepsilon^{\mathrm{Re}\,(s)}\right) = \frac{1}{\mathrm{Re}\,(s)},$$

[59] see [Fo11], p.7.4
[60] Leonhard Euler (1707-1783)
[61] see [Fo08], p.6.1 and [SS03], pp. 160-162.

the first integral converges for $\mathrm{Re}\,(s) > 0$.
For $\alpha \in \mathbb{R}$ there is a constant $C \in \mathbb{R}$ such that

$$t^\alpha \le Ce^{\frac{t}{2}} \qquad \text{for } t \ge 1.$$

We calculate the integral

$$\begin{aligned}\int_1^\infty t^{\mathrm{Re}\,(s)-1} e^{-t} dt &\le \int_1^\infty C\; e^{\frac{t}{2}} e^{-t} dt \\ &= \int_1^\infty C\; e^{-\frac{t}{2}} dt = \lim_{T\to\infty} C\left[(-2)e^{-\frac{t}{2}}\right]_{t=1}^T = 2Ce^{-\frac{1}{2}},\end{aligned}$$

so the second integral converges as well and we proved the existence of the improper integrals.
By the triangle inequality

$$|\Gamma(s)| \le \int_0^\infty t^{\mathrm{Re}\,(s)-1} e^{-t} dt = \Gamma(\mathrm{Re}\,(s))$$

the existence of the complex integral follows.
We show, that $\Gamma(s)$ is holomorphic for $\mathrm{Re}\,(s) > 0$. For this purpose we only need to show that for $0 < \varepsilon < 1$

$$F_\varepsilon(s) = \int_\varepsilon^{1/\varepsilon} t^{s-1} e^{-t} dt$$

converges uniformly to $\Gamma(s)$ on

$$S_{\delta,M} = \{\delta \le \mathrm{Re}\,(s) \le M\}, \qquad \text{where } 0 < \delta < M < \infty.$$

Then the property follows by normal convergence in addition to the fact, that $t^{s-1}e^{-t}$ is continuous and holomorphic in z for fixed $t \in [\varepsilon, 1/\varepsilon]$, so F_ε is holomorphic. We obtain

$$|\Gamma(s) - F_\varepsilon(s)| \le \int_0^\varepsilon e^{-t} t^{\mathrm{Re}\,(s)-1} dt + \int_{1/\varepsilon}^\infty e^{-t} t^{\mathrm{Re}\,(s)-1} dt.$$

Since $0 < \varepsilon < 1$, we can estimate $|e^{-t}t^{\mathrm{Re}\,(s)-1}| = t^{\mathrm{Re}\,(s)-1}$ for $t \in [0, \varepsilon]$, thus the first integral is

$$\int_0^\varepsilon e^{-t} t^{\mathrm{Re}\,(s)-1} dt \le \int_0^\varepsilon t^{\delta-1} dt = \frac{\varepsilon^\delta}{\delta}.$$

The second integral is bounded from above as follows

$$\int_{1/\varepsilon}^\infty e^{-t} t^{\mathrm{Re}\,(s)-1} dt \le \int_{1/\varepsilon}^\infty e^{-t} t^{M-1} dt \le C \int_{1/\varepsilon}^\infty e^{-t/2} dt = -2Ce^{-\frac{1}{2\varepsilon}},$$

so both integrals converge uniformly to 0, if $\varepsilon \to 0$.
Partial integration yields

$$\begin{aligned}\Gamma(s+1) &= \int_0^\infty t^s e^{-t} dt = \lim_{T\to\infty} \lim_{\delta\to 0} \int_\delta^T t^s e^{-t} dt \\ &= \lim_{T\to\infty} \lim_{\delta\to 0} \Big[- t^s e^{-t} \Big]_{t=\delta}^T + \int_\delta^T s t^{s-1} e^{-t} dt \\ &= 0 - 0 + s\Gamma(s) \qquad\qquad \text{for } \mathrm{Re}\,(s) > 0.\end{aligned}$$

By this equation, we obtain

$$\Gamma(s) = \frac{\Gamma(s+1)}{s} = \frac{\Gamma(s+2)}{s\;(s+1)} = \cdots = \frac{\Gamma(s+n+1)}{s\;(s+1)\cdots(s+n)}.$$

The right-hand side is meromorphic in the right half plane

$$H(-n-1) = \{s \in \mathbb{C} : \mathrm{Re}\,(s) > -n-1\}$$

with simple poles at $s = 0, -1, \dots, -n$. Thus we found the meromorphic continuation of $\Gamma(s)$. $\square$

We use the Gamma Function in order to prove the following lemma.

Lemma 9.5. [62] *If* $s \in \mathbb{C}$ *and* $\mathrm{Re}\,(s) > 1$, *then*

$$\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{\frac{s}{2}} \int_0^\infty t^{\frac{s}{2}} \left(\sum_{n=1}^\infty e^{-\pi n^2 t}\right) \frac{dt}{t}.$$

Proof. We start by examining the integral on the right-hand side of the equality. We define the function

$$\psi(t) = \sum_{n=1}^\infty e^{-\pi n^2 t}$$

and observe, that

$$\Theta(t) = \sum_{n=-\infty}^\infty e^{-\pi n^2 t} = 1 + 2\psi(t) \quad \Leftrightarrow \quad \psi(t) = \frac{1}{2}(\Theta(t) - 1).$$

As we discussed in the proof of Corollary 9.3, the Theta Series $\Theta(t)$ is uniformly convergent on $[\varepsilon, \infty)$ for $\varepsilon > 0$, so the same holds for $\psi(t)$. Hence by the same corollary we obtain

$$\psi(t) = \mathcal{O}\left(\frac{1}{\sqrt{t}}\right) \qquad \text{for } t \downarrow 0.$$

Since

$$\begin{aligned}
\lim_{t\to\infty} \frac{\psi(t)}{e^{-\pi t}} &= \lim_{t\to\infty} \sum_{n=1}^\infty e^{-\pi(n^2-1)t} = \lim_{t\to\infty}\left(1 + \sum_{n=2}^\infty e^{-\pi(n^2-1)t}\right) \\
&= \lim_{t\to\infty}\left(1 + \sum_{n=1}^\infty e^{-\pi(n^2+2n)t}\right) = 1 + \sum_{n=1}^\infty \lim_{t\to\infty} e^{-\pi(n^2+2n)t} = 1,
\end{aligned}$$

we conclude, that $\psi(t)$ converges exponentially fast to 0 for $t \to \infty$, where the last series in the equation above converges uniformly, because it is dominated by $\psi(t)$. Hence for $\varepsilon > 0$ we obtain the properties

$$\begin{aligned}
&\exists t_0 \in (0,1) \quad \exists C \in \mathbb{R}: && \psi(t) \le \tfrac{C}{\sqrt{t}} && \text{for all } t_0 > t > 0, \\
&\exists t_1 > 1: && \tfrac{\psi(t)}{e^{-\pi t}} \le 1 + \varepsilon && \text{for all } t > t_1 \text{ and} \\
&\exists C_1 \in \mathbb{R}: && t^{\frac{\mathrm{Re}\,(s)-2}{2}} \le C_1 e^{\frac{\pi t}{2}} && \text{for all } t > t_1.
\end{aligned}$$

Splitting up the integral, we have

$$\begin{aligned}
&\int_0^{t_0} \left| t^{\frac{s}{2}} \psi(t) \frac{dt}{t} \right| + \int_{t_0}^{t_1} \left| t^{\frac{s}{2}} \psi(t) \frac{dt}{t} \right| + \int_{t_1}^\infty \left| t^{\frac{s}{2}} \psi(t) \frac{dt}{t} \right| \\
&\le C \int_0^{t_0} t^{\frac{\mathrm{Re}\,(s)-3}{2}} dt + \int_{t_0}^{t_1} t^{\frac{\mathrm{Re}\,(s)}{2}} \psi(t) \frac{dt}{t} + \int_{t_1}^\infty t^{\frac{\mathrm{Re}\,(s)-2}{2}} (1+\varepsilon) e^{-\pi t} dt \\
&= \lim_{\delta\to 0} C \left[\frac{2}{\mathrm{Re}\,(s) - 1}\, t^{\frac{\mathrm{Re}\,(s)-1}{2}}\right]_{t=\delta}^{t_0} + M(t_1 - t_0) + C_1 \lim_{T\to\infty} (1+\varepsilon) \int_{t_1}^T e^{-\frac{\pi t}{2}} dt \\
&= C \frac{2}{\mathrm{Re}\,(s) - 1}\, t_0^{\frac{\mathrm{Re}\,(s)-1}{2}} + M(t_1 - t_0) + C_1(1+\varepsilon)\frac{2}{\pi} e^{-\frac{\pi t_1}{2}} < \infty
\end{aligned}$$

[62] see [Fo11], pp.7.4-7.5 and [SS03], p.170.

for $M \in \mathbb{R}$, because the function $t \mapsto t^{\frac{\mathrm{Re}\,(s)}{2}-1}\psi(t)$ is continuously differentiable on $[t_0, t_1]$. This implies, that the integral converges.
By the definition of the Gamma Function

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty t^{\frac{s}{2}-1}e^{-t}dt = \int_0^\infty t^{\frac{s}{2}}e^{-t}\frac{dt}{t} \qquad \text{for } \mathrm{Re}\,(s) > 0$$

and by subsituting $t = \pi n^2 \tilde{t}$, where $n \in \mathbb{N}$ and

$$\frac{dt}{d\tilde{t}} = \pi n^2 = \frac{t}{\tilde{t}} \quad \Leftrightarrow \quad \frac{d\tilde{t}}{\tilde{t}} = \frac{dt}{t}$$

holds, we obtain

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty (\pi n^2 \tilde{t})^{\frac{s}{2}} e^{-\pi n^2 \tilde{t}} \frac{d\tilde{t}}{\tilde{t}} = n^s \pi^{\frac{s}{2}} \int_0^\infty \tilde{t}^{\frac{s}{2}} e^{-\pi n^2 \tilde{t}} \frac{d\tilde{t}}{\tilde{t}}.$$

This gives for $\mathrm{Re}\,(s) > 1$

$$\Gamma\left(\frac{s}{2}\right)\zeta(s) = \sum_{n=1}^\infty \Gamma\left(\frac{s}{2}\right)\frac{1}{n^s} = \sum_{n=1}^\infty \pi^{\frac{s}{2}} \int_0^\infty \tilde{t}^{\frac{s}{2}} e^{-\pi n^2 \tilde{t}} \frac{d\tilde{t}}{\tilde{t}} = \pi^{\frac{s}{2}} \int_0^\infty \tilde{t}^{\frac{s}{2}} \left(\sum_{n=1}^\infty e^{-\pi n^2 \tilde{t}}\right) \frac{d\tilde{t}}{\tilde{t}},$$

where the exchange of the series and the integral is justified by Lebesgue's Dominated Convergence Theorem. The Convergence Theorem is applicable due to the argument, that there is an integrable function

$$g(t) = Ct^{\frac{\mathrm{Re}\,(s)-3}{2}}\mathbb{1}_{[0,t_0)}(t) + \psi(t)t^{\frac{\mathrm{Re}\,(s)-2}{2}}\mathbb{1}_{[t_0,t_1]}(t) + t^{\frac{\mathrm{Re}\,(s)-2}{2}}(1+\varepsilon)e^{-\pi t}\mathbb{1}_{(t_1,\infty)}(t)$$

for $\mathrm{Re}\,(s) > 1$ fixed with

$$|\psi_k(t)| := \left|\frac{t^{\frac{s}{2}}}{t}\sum_{n=1}^k e^{-\pi n^2 t}\right| \le g(t) \qquad \text{for all } k \in \mathbb{N}$$

by the same estimate as above and $\psi_k(t) \to \psi(t)\frac{t^{\frac{s}{2}}}{t}$ for $k \to \infty$. □

Lemma 9.5 enables us to prove two important Functional Equations, which clear the ground for a theorem about the zeros of the Riemann Zeta Function $\zeta(s)$ for $\mathrm{Re}\,(s) < 0$.

Theorem 9.6 (Functional Equations). [63] *We state the following Functional Equations:*

a) Let

$$\xi(s) := \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

This function extends meromorphically to $\mathbb{C}$. It is holomorphic everywhere apart from simple poles at $s = 0$ and $s = 1$. Furthermore, the Functional Equation

$$\xi(s) = \xi(1-s)$$

holds.

b) The Zeta Function extends meromorphically to $\mathbb{C}$ with a simple pole at $s = 1$. Furthermore, the Functional Equation

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

holds.

[63] see [Fo11], pp.7.5-7.7 and [SS03], pp.170-172.

Proof. a) By Lemma 9.5 we conclude, that for $\mathrm{Re}\,(s) > 1$ and $\psi(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 t}$

$$\xi(s) = \pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty t^{\frac{s}{2}}\psi(t)\frac{dt}{t} = \int_0^1 t^{\frac{s}{2}}\psi(t)\frac{dt}{t} + \int_1^\infty t^{\frac{s}{2}}\psi(t)\frac{dt}{t} \tag{9.4}$$

holds. Recall, that $\psi(t) = \frac{1}{2}(\Theta(t) - 1)$. By Theorem 9.2

$$\begin{aligned}\psi(t) = \frac{1}{2}(\Theta(t) - 1) &= \frac{1}{2}\left(\frac{1}{\sqrt{t}}\ \Theta\left(\frac{1}{t}\right) - 1\right)\\ &= \frac{1}{2\sqrt{t}}\left(\Theta\left(\frac{1}{t}\right) - 1\right) - \frac{1}{2}\left(1 - \frac{1}{\sqrt{t}}\right)\\ &= \frac{1}{\sqrt{t}}\psi\left(\frac{1}{t}\right) - \frac{1}{2}\left(1 - \frac{1}{\sqrt{t}}\right)\end{aligned}$$

applies. We obtain

$$\begin{aligned}\int_0^1 t^{\frac{s}{2}}\psi(t)\frac{dt}{t} &= \int_0^1 t^{\frac{s}{2}}\left(\frac{1}{\sqrt{t}}\psi\left(\frac{1}{t}\right) - \frac{1}{2}\left(1 - \frac{1}{\sqrt{t}}\right)\right)\frac{dt}{t}\\ &= \int_0^1 t^{\frac{s-1}{2}}\psi\left(\frac{1}{t}\right)\frac{dt}{t} + \frac{1}{2}\int_0^1 (t^{\frac{s-1}{2}} - t^{\frac{s}{2}})\frac{dt}{t}.\end{aligned}$$

We conclude for the last integral

$$\frac{1}{2}\int_0^1 (t^{\frac{s-1}{2}} - t^{\frac{s}{2}})\frac{dt}{t} = \left[\frac{1}{s-1}t^{\frac{s-1}{2}} - \frac{1}{s}t^{\frac{s}{2}}\right]_{t=0}^1 = \frac{1}{s-1} - \frac{1}{s}$$

and the other summand

$$\int_0^1 t^{\frac{s-1}{2}}\psi\left(\frac{1}{t}\right)\frac{dt}{t} = \int_1^\infty \tilde{t}^{\frac{1-s}{2}}\psi\left(\tilde{t}\right)\frac{d\tilde{t}}{\tilde{t}},$$

where we substituted

$$\tilde{t} = \frac{1}{t} \quad \Leftrightarrow \quad \frac{d\tilde{t}}{dt} = -\frac{1}{t^2} = -\frac{\tilde{t}}{t} \quad \Leftrightarrow \quad \frac{dt}{t} = -\frac{d\tilde{t}}{\tilde{t}}.$$

Plugging the last equations in (9.4) and writing t instead of $\tilde{t}$ again, we obtain

$$\xi(s) = \int_0^\infty t^{\frac{s}{2}}\psi(t)\frac{dt}{t} = \int_1^\infty \left(t^{\frac{1-s}{2}} + t^{\frac{s}{2}}\right)\psi(t)\frac{dt}{t} + \frac{1}{s-1} - \frac{1}{s}. \tag{9.5}$$

We examine the last integral in (9.5): The function $\psi(t)$ goes to 0 exponentially fast, while the other factors of the integrand only grow polynomially fast for $t \to \infty$, so the integral exists as seen in a similar computation in the proof of Lemma 9.5. Since $\psi(t)$ is uniformly convergent, it is continuous on $[\varepsilon, \infty)$ for $\varepsilon > 0$, so the whole integrant is continuous. For a fixed $t \in [1, \infty)$ the integrand is holomorphic in s, thus the integral converges to a holomorphic function $g(s)$. Hence we found a meromorphic extension of $\xi(s)$ to $\mathbb{C}$ with first order poles at $s = 0$ and $s = 1$.
We also recognize, that

$$\xi(1-s) = \int_1^\infty \left(t^{\frac{s}{2}} + t^{\frac{1-s}{2}}\right)\psi(t)\frac{dt}{t} - \frac{1}{s} + \frac{1}{s-1} = \xi(s)$$

applies.

b) We first provide a second proof for the meromorphic extension of $\zeta(s)$. In Lemma 6.1 we showed the extension to $\{s \in \mathbb{C} : \mathrm{Re}\,(s) > 0\}$, which was sufficient for the proof of the Prime Number Theorem. Here we extend $\zeta(s)$ to $\mathbb{C}$ in order to examine the zeros of $\zeta(s)$ for $\mathrm{Re}\,(s) < 0$. By a) we have

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\xi(s). \tag{9.6}$$

As we saw in Lemma 9.4, the Gamma Function is meromorphic with simple poles at $s = 0, -1, \ldots$, thus the function

$$s \to \frac{1}{\Gamma\left(\frac{s}{2}\right)}$$

is holomorphic on $\mathbb{C}$ with zeros of order 1 at $s = 0, -2, -4 \ldots.$ The zero of order 1 at $s = 0$ removes the pole of order 1 in $s = 0$ of the function $\xi(s)$. Hence $\zeta(s)$ is holomorphic in $\mathbb{C}$ apart from a single pole in $s = 1$.
By (9.6) and the Functional Equation of $\xi(s)$ we obtain

$$\xi(1-s) = \pi^{-\frac{1-s}{2}}\,\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) = \pi^{\frac{-s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \xi(s)$$

and

$$\zeta(1-s) = \pi^{\frac{1}{2}-s}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)^{-1}\zeta(s). \tag{9.7}$$

Two important properties of the Gamma Function are

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = \frac{\sin \pi z}{\pi} \tag{9.8}$$

$$\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{1+z}{2}\right) = 2^{1-z}\sqrt{\pi}\,\Gamma(z), \tag{9.9}$$

which were found by Euler and Legendre.
By (9.8) we obtain with $z = (1+s)/2$, that

$$\Gamma\left(\frac{1+s}{2}\right)^{-1}\Gamma\left(\frac{1-s}{2}\right)^{-1} = \frac{\sin\left(\pi\frac{1+s}{2}\right)}{\pi} = \frac{\cos\left(\frac{\pi s}{2}\right)}{\pi},$$

which leads to

$$\Gamma\left(\frac{1-s}{2}\right)^{-1} = \frac{\cos\left(\frac{\pi s}{2}\right)}{\pi}\,\Gamma\left(\frac{1+s}{2}\right).$$

Hence (9.7) is equivalent to

$$\zeta(1-s) = \pi^{\frac{1}{2}-s}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)^{-1}\zeta(s) = \pi^{-\frac{1}{2}-s}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)\cos\left(\frac{\pi s}{2}\right)\zeta(s).$$

By (9.9) it follows

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\,\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s).$$

□

Theorem 9.7. [64] *If* $k \in \mathbb{N}$, *then*

$$\zeta(-2k) = 0$$

and these are the only zeros of the function $\zeta(s)$ *for* $\mathrm{Re}\,(s) < 0$.

Proof. We conclude

$$\mathrm{Re}\,(1-s) = 1 - \mathrm{Re}\,(s) < 0 \quad \Leftrightarrow \quad \mathrm{Re}\,(s) > 1.$$

Recall Lemma 6.4, which said $\zeta(s) \neq 0$ for $\mathrm{Re}\,(s) \geq 1$. Besides, by (9.8) it follows, that $\Gamma(s)$ is non-vanishing in $\mathbb{C}$.
By the Functional Equation of the Zeta Function

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

from Theorem 9.6 we observe, that on the right side of the equality only the Cosine Function has zeros for $\mathrm{Re}\,(s) > 1$. Since

$$\cos\left(\frac{\pi s}{2}\right) = 0 \quad \Leftrightarrow \quad s = 2k+1 \quad \text{where } k \in \mathbb{Z},$$

we conclude for $s = 2k+1$

$$\zeta(1-(2k+1)) = \zeta(-2k) = 0 \qquad \text{for all } k \in \mathbb{N}.$$

□

[64] see [Fo11], p.7.7 and [B08], p.317.

10. The Riemann Hypothesis

The Riemann Zeta Function has so-called "trivial" zeros at $-2, -4, \ldots$, whose existence can be proven relatively easily as we saw in section 7. By Lemma 6.4 $\zeta(s)$ has no zeros for $\mathrm{Re}\,(s) \geq 1$. It is still unknown, how the zeros of $\zeta(s)$ are distributed in the strip $\{s \in \mathbb{C} : 0 < \mathrm{Re}\,(s) < 1\}$, which is called the "Critical Strip". The Riemann Hypothesis claims, that for any non-trivial zero s, $\mathrm{Re}\,(s) = 1/2$ holds. Today it is also assumed, that all of these zeros are simple. J.B. Conrey showed in 1989, that at least 40% of the non-trivial zeros are on the line $\{z \in \mathbb{C} : \mathrm{Re}\,(z) = 1/2\}$.
The Riemann Hypothesis is considered one of the most important unresolved problems in mathematics and is one of the "Clay Mathematics Institute Millenium Prize Problems".
Even if we do not know yet, if this problem can be solved, we can state its implications on the Prime Number Theorem in the following theorem:

Theorem 10.1. [65] *Let $\frac{1}{2} \leq a < 1$. The following are equivalent:*

i) The Riemann Zeta Function has no zeros with $\mathrm{Re}\,(s) > a$.
ii) $\pi(x) = li(x) + \mathcal{O}(x^{a+\varepsilon})$ holds for all $\varepsilon > 0$ and $x \to \infty$.
iii) $\vartheta(x) = x + \mathcal{O}(x^{a+\varepsilon})$ holds for all $\varepsilon > 0$ and $x \to \infty$.

The proof of this theorem (in particular the step $(i) \to (ii)$) requires some deeper knowledge of Dirichlet Series and the Perron Formula, so we do not give the proof here.
Nevertheless, we state an interesting fact about the distribution of the zeros of $\zeta(s)$ in the Critical Strip, which can be obtained far more easily. Examining the Functional Equation from Theorem 9.6 a)

$$\xi(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

we realise, that $\xi(s)$ and $\zeta(s)$ have the same zeros in the Critical Strip, because $\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)$ is nonzero and holomorphic in this area. We conclude

$$\xi(1-s) = \xi(s) \qquad \text{and} \qquad \xi(\overline{s}) = \overline{\xi(s)}.$$

The second equality follows directly from the integral form (9.5) of $\xi(s)$ (over the real axis), since the function $z \mapsto \overline{z}$ is $\mathbb{R}$-linear and continuous.
If

$$s = \frac{1}{2} + x + it, \qquad \text{where } -\frac{1}{2} < x < \frac{1}{2}$$

and

$$\xi(s) = \xi\left(\frac{1}{2} + x + it\right) = 0,$$

then

$$\xi(1-s) = \xi\left(\frac{1}{2} - x - it\right) = 0,$$
$$\xi(\overline{s}) = \xi\left(\frac{1}{2} + x - it\right) = 0 \text{ and}$$
$$\xi\left(\overline{1-s}\right) = \xi\left(\frac{1}{2} - x + it\right) = 0.$$

The non-trivial zeros of $\zeta(s)$ are therefore symmetric to the real axis and the line $\{s \in \mathbb{C} : \mathrm{Re}\,(s) = 1/2\}$.

[65] see [Fo11], p.10.1, p.7.9

In 1893, Hadamard proved one of Riemann's propositions, which says, that $\zeta(s)$ has infinitely many zeros in the Critical Strip. There are many results about the vertical distribution of the zeros in the Critical Strip. In particular von Mangoldt proved Riemann's proposition

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} + O(\log T) \qquad \text{for } T \to \infty$$

in 1905, where $N(T)$ denotes the number of zeros of $\zeta(s)$ including order in $0 \leq \mathrm{Im}\,(s) \leq T$.
Regarding the horizontal distribution of the zeros, there is not much known. De la Vallée Poussin proved the existence of a positive function $\eta(|t|)$, which converges to 0 for $|t| \to \infty$ such that $\zeta(a + it) \neq 0$ for $a > 1 - \eta(|t|)$ and $|t|$ sufficiently large.

On the next page, one can find two images of the Riemann Zeta Function. In the first image, the colour at a certain point in the complex plane is used to show the value of $\zeta(s)$: If it is close to black, this means, that $\zeta(s)$ is near zero. The hue encodes the argument of the value of $\zeta(s)$ in a point. Values, which have arguments close to zero, are shown in red.
The second image shows the real (red) and imaginary (blue) parts of $\zeta(s)$ on the line $\{s \in \mathbb{C} : \mathrm{Re}\,(s) = 1/2\}$. One can spot the first zeros at $\mathrm{Im}\,(s) = \pm 14.135$, ± 21.022 and ± 25.011.
At the end of this paper we quote Riemann himself, who uttered some thoughts on his Hypothesis:

"It would certainly be desirable to have a rigorous demonstration for this proposition; nevertheless I have for the moment set this aside, after several quick but unsuccessful attempts, because it seemed unneeded for the immediate goal of my study". [66]

[66] see [SS03], p.185.

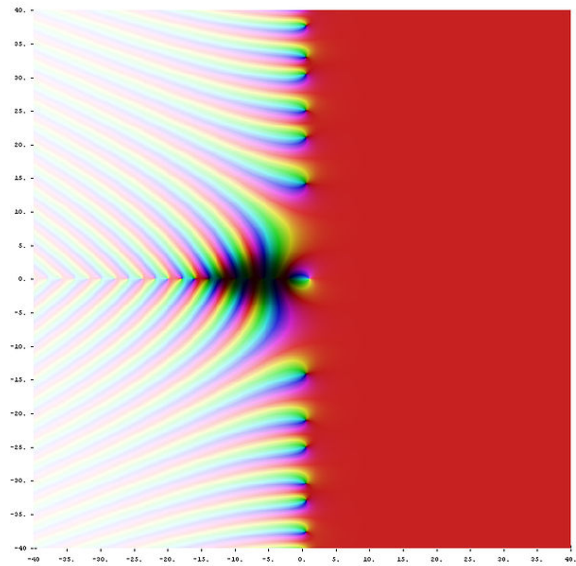

FIGURE 6. Riemann Zeta Function in the complex plane

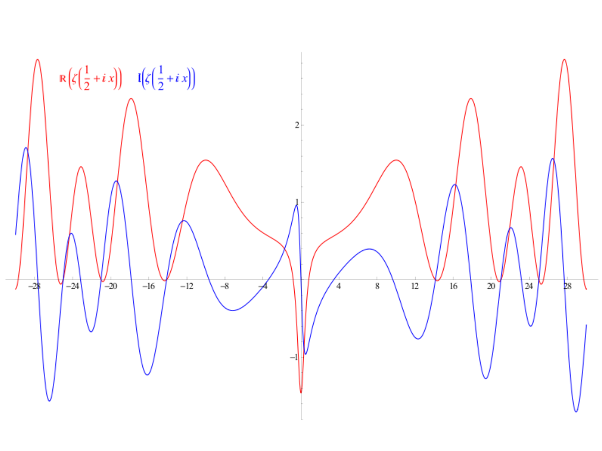

FIGURE 7. Real and imaginary parts of the Riemann Zeta Function on the line $\mathrm{Re}\,(s) = 1/2$

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List of Figures

Declaration

I hereby declare that this thesis was performed and written on my own and that references and resources used within this work have been explicitly indicated.
I am aware that making a false declaration may have serious consequences.

Ulm, 05.07.2013 ________________________________

(Signature)