Ulm University
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Sobolev Spaces of Vector-Valued Functions

Master Thesis

in Mathematics

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1 Introduction

Spaces of *weakly differentiable functions*, so called *Sobolev spaces*, play an important role in modern Analysis. Since their discovery by Sergei Sobolev in the 1930's they have become the base for the study of many subjects such as partial differential equations and calculus of variatons. One Example for their usefulness are Cauchy problems like the d-dimensional heat equation

$$u' = \Delta u \quad u(0) = u_0,$$

where $u : [0, T] \to D(\Delta)$ and $\Delta := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator. If we only considered classical derivatives, we would let $D(\Delta) := C^2(\mathbb{R}^d, \mathbb{R})$. In this case, the operator Δ would not be closed and hence we could not apply the theory of semigroups to this example. If we instead let $D(\Delta) := H^2(\mathbb{R}^d, \mathbb{R})$ – the second Sobolev space consisting of all L^2-functions that are twice weakly differentiable – then Δ generates a semigroup $(e^{t\Delta})_{t\geq 0}$ and $e^{t\Delta}u_0$ solves the above problem in the classical way.

The fact that Sobolev spaces are in some way the "right" domain for differential operators is a reason, but only one reason, for their importance to the theory of partial differential equations. We now extend the example above to see another application of weakly differentiable functions. This example will show why *Sobolev spaces of vector-valued functions* are important as well. We perturb the above Cauchy problem with a vector-valued function $f : [0, T] \to L^2(\mathbb{R}^d, \mathbb{R})$ i.e.

$$u' = \Delta u + f \quad u(0) = u_0.$$

Looking at linear ordinary differential equations one might try to give the solution to this problem via the variation of constants formula

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} f(s)\, ds.$$

Straightforward it is not possible to show that u is a solution to the problem in the classical way. However, if we assume that $f \in W^{1,1}([0, T], L^2(\mathbb{R}^d, \mathbb{R}))$ – the first Sobolev space consisting of all vector-valued L^1-functions that are once weakly differentiable – then the variation of constants formula indeed produces a classical solution.

The introductory example shows that Sobolev spaces of vector-valued functions need to be investigated and this thesis is dedicated to this subject. Rather than looking at examples as the one above we want to give an introduction to the spaces themselves.

In the first chapter we will look at *integration of vector-valued functions*. Measure and integration play a crucial role in the development of scalar-valued Sobolev spaces. It is not possible to define these spaces without the use of the integral type found by Henri Lebesgue in the beginning of the 20^{th} century. We will introduce the *Bochner integral*, a straightforward extension of the Lebesgue integral to vector-valued functions. We will work out similarities and the connection between the two integrals and prove that many classical theorems for the Lebesgue integral hold for vector-valued functions as well. But we will also work out the deviations that occur in the vector-valued case. For example one can bring the dual space of the codomain into play, which obviously is not useful for scalar-valued functions. The geometry of the codomain also plays an important role. For example the dual space of the vector-valued L^p-space is not the same as in the scalar-valued case for any Banach space. We will have a look at the *Radon-Nikodym property*, a geometric property of a Banach space which will play an important role throughout the thesis. With this property it is possible to find the mentioned dual.

In the second chapter we will introduce *Sobolev spaces in one dimension*. As in the scalar valued case, these spaces have special properties which distinguish them from the d-dimensional case. The most important one is that a version of the *Fundamental Theorem of Calculus* holds for weakly differentiable functions in one dimension. With this theorem we are able to accurately describe the spaces. After that we will focus on *criteria for weak differentiability*. Of course it is necessary to give criteria to tell whether a given function is weakly differentiable or not (e.g. in the introductory example one would like to check this for f). Our focus lies on the generalization of a theorem for the scalar-valued case telling us that a function is weakly differentiable if and only if the difference quotient is uniformly bounded. We will prove this theorem and establish its connection to the Radon-Nikodym property.

The next chapter is dedicated to *Sobolev spaces in higher dimensions*. After introducing these spaces, we will survey their structure. The above mentioned Fundamental Theorem does not hold in this case. Instead, we will establish a *regularization process* which helps us to determine weakly differentiable functions in d-dimensional spaces. We will use this process to prove the *Meyers-Serrin Theorem* as well as an alternative to the Fundamental Theorem. These theorems will help us to extend the criterion found in the second chapter to this case. These considerations will also help us to prove the *Sobolev Embedding Theorems*, a collection of theorems which are frequently used in partial differential equations.

In the final chapter, we will have a look at weakly differentiable functions with values in *Banach lattices*, special Banach spaces which are endowed with a partial ordering. In these spaces it makes sense to look at functions such as $u^+ = u1_{\{u \geq 0\}}$. We will use our knowledge from the preceding chapters to investigate whether such functions are still weakly differentiable and what their weak derivatives look like.

We assume that the reader is familiar with the standard results of measure and integration - such as L^p-spaces, their dense subspaces and classical theorems like the Dominated Convergence Theorem and the Lebesgue Differentiation Theorem - as well as those of functional analysis - such as the Hahn-Banach Theorem, the Closed Graph Theorem and reflexive spaces.

2 Integration of Vector-Valued Functions

The results of measure theory are crucial for the introduction of Sobolev spaces. In this chapter we will generalize the fundamental definitions of measurability, integrability and L^p-spaces to the case of vector-valued functions.

2.1 The Bochner Integral

Let (Ω, Σ, μ) be a σ-finite, complete measure space and X be a Banach space. A function $s : \Omega \to X$ is said to be a *simple function* if it can be written as

$$s = \sum_{i=1}^{n} x_i \cdot 1_{E_i}$$

with $x_i \in X$ and pairwise disjoint $E_i \in \Sigma$, $\mu(E_i) < \infty$ $(i = 1, \dots, n)$. A function $f : \Omega \to X$ is said to be *measurable* if there exists a sequence (s_n) of simple functions which converges to f in norm a.e. A function $f : \Omega \to X$ is said to be *weakly measurable* if the function $\langle x', f \rangle$ is measurable for all $x' \in X'$. We say that f is *almost separably valued* if there exists a set N with $\mu(N) = 0$ such that $f(\Omega \backslash N)$ is separable. The most common way to check vector-valued functions for measurability is Pettis' theorem which links the notions of measurability and weak measurability.

Theorem 2.1 (Pettis' Measurability Theorem). *A function $f : \Omega \to X$ is measurable if and only if f is weakly measurable and almost separably valued.*

Proof. If f is measurable then there exist simple functions s_n converging to f a.e. For every $x' \in X'$ the simple functions $\langle x', s_n \rangle$ converge to $\langle x', f \rangle$ pointwise on the same set, thus f is weakly measurable. Apart from a set of measure zero, f takes its values in the closure of the values taken by the functions s_n, hence f is almost separably valued.

Now assume that f is weakly measurable and almost separably valued. Let $N \subset \Omega$ be a null set such that $f(\Omega \backslash N)$ is separable. Let (x_n) be a dense sequence in $f(\Omega \backslash N)$ and use the Hahn-Banach Theorem to choose a sequence (x'_n) of normed elements in X' such that $\langle x'_n, x_n \rangle = \|x_n\|$. Let $\omega \in \Omega \backslash N$ and let $x_{n_k} \to f(\omega)$, then for every $\varepsilon > 0$ there exists a k large enough such that

$$\begin{aligned}\langle x'_{n_k}, f(\omega)\rangle &\leq \|f(\omega)\| \leq \|x_{n_k}\| + \varepsilon \\ &= \langle x'_{n_k}, x_{n_k}\rangle + \varepsilon \\ &= \langle x'_{n_k}, x_{n_k} - f(\omega)\rangle + \langle x'_{n_k}, f(\omega)\rangle + \varepsilon \\ &\leq \langle x'_{n_k}, f(\omega)\rangle + 2\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain $\|f(\omega)\| = \sup_{n\in\mathbb{N}}\langle x'_n, f(\omega)\rangle$, i.e. $\|f\|$ is a.e. the pointwise supremum of countably many measurable functions and hence measurable itself. Let $f_n(\cdot) := \|f(\cdot) - x_n\|$ then f_n is measurable by the same argument as before. Let $\varepsilon > 0$ and $E_n := \{\omega \in \Omega,\ f_n(\omega) \leq \varepsilon\}$, then E_n is measurable as the measure space is complete. Define $g : \Omega \to X$ via

$$g(\omega) := \begin{cases} x_n, & \text{if } \omega \in E_n \backslash \bigcup_{m<n} E_m \\ 0, & \text{otherwise}\end{cases}$$

then $\|f - g\| \leq \varepsilon$ a.e. and g is countably valued. Letting $\varepsilon = 2^{-n}$ we construct a sequence $g_n = \sum_{i=1}^{\infty} x_{i,n} 1_{E_{i,n}}$ ($x_{i,n} \in X$, $\dot{\bigcup}_i E_{i,n} = \Omega$) of countably valued functions converging to f a.e. As (Ω, Σ, μ) is σ-finite we can choose an increasing sequence of measurable sets Ω_n such that $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$ and $\mu(\Omega_n) < \infty$. For each $n \in \mathbb{N}$ let $F_n := \Omega_n \cap \bigcup_{i=1}^{k_n} E_{i,n}$ where k_n is chosen so large that $\mu(\Omega_n \backslash F_n) \leq 2^{-n}$. Let $s_n := g_n 1_{F_n}$, then this defines a sequence of simple functions which also converges to f a.e. To see this let $\omega \in \bigcap_{n=k}^{\infty} F_n$ for some $k \in \mathbb{N}$, then for all $n \geq k$ we have $s_n(\omega) = g_n(\omega)$ and hence $\|f(\omega) - s_n(\omega)\| \leq 2^{-n}$. Thus $s_n(\omega) \to f(\omega)$ for all $\omega \in \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} F_n$. For each j and each $k > j$ we have

$$\mu\left(\Omega_j \backslash \bigcap_{n=k}^{\infty} F_n\right) \leq \sum_{n=k}^{\infty} \mu(\Omega_n \backslash F_n) < 2^{-k+1},$$

thus $\Omega_j \backslash \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} F_n$ is a null set. Hence $s_n \to f$ outside of a null set as claimed. □

Corollary 2.2. *Let $E' \subset X'$ be a norming subspace. A function $f : \Omega \to X$ is measurable if and only if $\langle x', f\rangle$ is measurable for every $x' \in E'$ and f is almost separably valued.*

Proof. As E' is norming we can find a normed sequence x'_n such that $\langle x'_n, x_n\rangle \leq (1 - \varepsilon_n)\|x_n\|$ where $\varepsilon_n \to 0$ and the x_n are chosen as in the proof of Theorem 2.1. Again it follows that $\|f(\omega)\| = \sup_{n\in\mathbb{N}}\langle x'_n, f(\omega)\rangle$ hence the same proof works in this case. □

Corollary 2.3. *Let $\Omega \subset \mathbb{R}^d$ open and let $f : \Omega \to X$ be continuous. Then f is measurable.*

Proof. The continuity implies that f is separably valued. For any $x' \in X'$ the scalar-valued function $\langle x', f\rangle$ is continuous and hence measurable. The claim now follows from Pettis' Theorem. □

Corollary 2.4. *Let f_n be a sequence of measurable functions such that $f_n(\omega) \rightharpoonup f(\omega)$ for almost all $\omega \in \Omega$, then f is measurable.*

Proof. For any $x' \in X'$ and all ω apart from a nullset F we have $\langle x', f_n(\omega)\rangle \to \langle x', f(\omega)\rangle$ and thus $\langle x', f(\omega)\rangle$ is measurable as it is a.e. the pointwise limit of measurable functions. We need to show that f is almost separably valued. For every n choose a null set E_n such that $f_n(\Omega\backslash E_n)$ lies in a separable subspace X_n and let $E := \bigcup_{n\in\mathbb{N}} E_n \cup F$. We have that $\mu(E) = 0$ and that $f_{|\Omega\backslash E}$ takes values in the weak closure of $\operatorname{span}\big(\bigcup_{n\in\mathbb{N}} X_n\big)$. But as this set is convex, its closure and weak closure conincide and thus $f(\Omega\backslash E)$ is separable. Now the assertion follows from Pettis' Theorem. $\square$

For a simple function s the integral $\int_\Omega s\,d\mu$ can be defined in the obvious way

$$\int_\Omega s\,d\mu = \sum_{i=1}^{n} x_i\,\mu(E_i),$$

where $s = \sum_{i=1}^{n} x_i \cdot 1_{E_i}$. It is obvious, that the integral acts linear on simple functions and by the triangular inequality for the norm, we gain the fundamental estimate for the integral

$$\left\|\int_\Omega s\,d\mu\right\| \leq \int_\Omega \|s\|\;d\mu.$$

As for scalar-valued functions, we now extend this integral to certain functions using a limit process.

A measurable function $f : \Omega \to X$ is said to be *Bochner-integrable* or simply *integrable* if there exists a sequence (s_n) of simple functions converging to f a.e. such that

$$\lim_{n\to\infty} \int_\Omega \|f - s_n\|\,d\mu = 0.$$

For an integrable function, the integral is defined via

$$\int_\Omega f\,d\mu = \lim_{n\to\infty} \int_\Omega s_n\,d\mu.$$

Note that this limit exists as by the linearity of the integral and the fundamental estimate we have

$$\begin{aligned}\left\|\int_\Omega s_n\,d\mu - \int_\Omega s_m\,d\mu\right\| &\leq \int_\Omega \|s_n - s_m\|\,d\mu \\ &\leq \int_\Omega \|s_n - f\|\,d\mu + \int_\Omega \|f - s_m\|\,d\mu \to 0\ (n, m \to \infty)\end{aligned}$$

i.e. $\int s_n$ is a Cauchy sequence and thus convergent in X. If (t_n) is another sequence of simple functions converging to f as above, we can define a new sequence by alternating the elements of (s_n) and (t_n). This new sequence will satisfy all of the above criteria and thus its integral will converge. Now all subsequences and in particular $(\int s_n)$ and $(\int t_n)$ will converge to the same limit. This means that the definition of $\int f$ is independent of the choice of (s_n).

The linearity of the integral carries over to the limit of simple functions, hence the integral can be interpreted as an operator. For this reason, we sometimes omit the $d\mu$ when no confusion arises.

The following theorem links the Bochner integral to the Lebesgue integral. This will also be helpful to quickly carry over classical theorems for the Lebesgue integral to the vector-valued case.

Theorem 2.5 (Bochner's Theorem)**.** *Let $f : \Omega \to X$ be a measurable function, then f is Bochner-integrable if and only if $\|f\|$ is Lebesgue-integrable. Further we have the fundamental estimate*

$$\left\| \int_\Omega f d\mu \right\| \leq \int_\Omega \|f\| \, d\mu.$$

Proof. Let f be Bochner integrable and s_n as in the definition. As $\|f\|$ is the a.e.-limit of the simple functions $\|s_n\|$ we obtain that $\|f\|$ is measurable. For any n we have

$$\int_\Omega \|f\| \, d\mu \leq \int_\Omega \|f - s_n\| \, d\mu + \int_\Omega \|s_n\| \, d\mu.$$

The second integral is finite by the definition of a simple function and the first one becomes finite if n is large enough. Further we have

$$\left\| \int_\Omega f \, d\mu \right\| = \lim_{n\to\infty} \left\| \int_\Omega s_n \, d\mu \right\| \leq \lim_{n\to\infty} \int_\Omega \|s_n\| \, d\mu.$$

Now by definition we have that $\int_\Omega \|f - s_n\| \, d\mu \to 0$ and thus the last limit converges to $\int_\Omega \|f\| \, d\mu$, yielding the fundamental estimate.

Now suppose that $\int_\Omega \|f\| \, d\mu < \infty$ and let (s_n) be a sequence of simple functions converging to f a.e. We define new simple functions via

$$t_n(x) := \begin{cases} s_n(x), & \text{if } \|s_n(x)\| \leq 2\|f(x)\| \\ 0, & \text{else} \end{cases}$$

then $\|t_n(x) - f(x)\| \to 0$ a.e. as well and $\|t_n(x) - f(x)\|$ is measurable. We have that $\|t_n(x) - f(x)\| \leq \|t_n(x)\| + \|f(x)\| \leq 3\|f(x)\|$ and thus by the integrability of $\|f\|$ and the Dominated Convergence Theorem for the Lebesgue integral we have that

$$\int_\Omega \|t_n - f\| \, d\mu \to 0.$$

This shows that f is Bochner integrable by definition. $\square$

Corollary 2.6 (Dominated Convergence Theorem)**.** *Let* (f_n) *be a sequence of integrable functions and let* f *be a measurable function such that* $f_n \to f$ *a.e. Further let* $g \in L^1(\Omega, \mathbb{R})$ *such that* $\|f_n\| \leq g$ *a.e. and for all* $n \in \mathbb{N}$. *Then* f *is integrable and we have*

$$\int_\Omega f\, d\mu = \lim_{n\to\infty} \int_\Omega f_n\, d\mu.$$

Proof. By the Dominated Convergence Theorem for the Lebesgue integral we conclude that $\|f\|$ is integrable and thus f is integrable by Bochner's Theorem. Now the real valued function $\|f - f_n\|$ is bounded by the integrable function $\|f\| + g$ and thus we can apply the Dominated Convergence Theorem to this function and compute

$$\left\| \int_\Omega f\, d\mu - \int_\Omega f_n\, d\mu \right\| \leq \int_\Omega \|f - f_n\|\, d\mu \to 0 \quad (n \to \infty)$$

i.e. $\int_\Omega f\, d\mu = \lim_{n\to\infty} \int_\Omega f_n\, d\mu$ □

From the proof of Bochner's Theorem we infer a simple but useful fact.

Corollary 2.7. *Let* f *be integrable, then the sequence of simple functions* s_n *converging to* f *as in the definition of the Bochner integral can be chosen such that* $\|s_n(\omega)\| \leq 2\|f(\omega)\|$ *holds for almost all* $\omega \in \Omega$.

Proposition 2.8. *Let* $x' \in X'$ *and* f *be integrable, then* $\int \langle x', f \rangle = \left\langle x', \int f \right\rangle$.

Proof. By the definition of the integral it holds that $\int \langle x', s \rangle = \left\langle x', \int s \right\rangle$ for any simple function s. Now let (s_n) be a sequence of simple functions as in the definition of $\int f$ such that $\|s_n(\omega)\| \leq 2\|f(\omega)\|$ a.e. Then $\langle x', s_n \rangle \to \langle x', f \rangle$ a.e. and $|\langle x', s_n(\omega) \rangle| \leq 2\|x'\|\|f(\omega)\|$. Thus by the Dominated Convergence Theorem we have

$$\begin{aligned} \int \langle x', f \rangle &= \lim_{n\to\infty} \int \langle x', s_n \rangle \\ &= \lim_{n\to\infty} \left\langle x', \int s_n \right\rangle = \left\langle x', \int f \right\rangle, \end{aligned}$$

where the last equality follows from the continuity of x' and the definition of the integral. □

Theorem 2.9 (Fubini-Tonelli)**.** *Let* $\Omega = \Omega_1 \times \Omega_2$ *be a product measure space with respect to the measure* $\mu_1 \otimes \mu_2$ *and let* $f : \Omega \to X$ *be measurable. Suppose that the integral*

$$\int_{\Omega_1} \int_{\Omega_2} \|f\|\, d\mu_2\, d\mu_1$$

exists, then f *is Bochner integrable and we have that*

$$\int_{\Omega_1 \times \Omega_2} f\, d\mu_1 \otimes \mu_2 = \int_{\Omega_1} \int_{\Omega_2} f\, d\mu_2\, d\mu_1 = \int_{\Omega_2} \int_{\Omega_1} f\, d\mu_1\, d\mu_2.$$

Conversely if f *is Bochner integrable, then the above integrals exist and the equation holds.*

Proof. Throughout this proof let $\{i, j\} = \{1, 2\}$. If the double integral exists, then the Fubini-Tonelli Theorem for the real-valued case implies that $\|f\|$ is integrable, hence f is integrable by Bochner's Theorem. Also the integrals $\int_{\Omega_i} \|f(\omega_1, \omega_2)\| \, d\mu_i(\omega_i)$ exist a.e. on Ω_j, hence the same holds for $\int_{\Omega_i} f(\omega_1, \omega_2) \, d\mu_i(\omega_i)$. As f is almost separably valued, the functions $\omega_j \mapsto \int_{\Omega_i} f(\omega_1, \omega_2) \, d\mu_i(\omega_i)$ are almost separably valued as well. For any $x' \in X'$ we have that $\langle x', f\rangle$ is measurable and the estimate $\langle x', f\rangle \leq \|x'\| \, \|f\|$ shows that it is integrable. Using Fubini's Theorem for real-valued functions we deduce that the functions

$$\omega_j \mapsto \int_{\Omega_i} \langle x', f(\omega_1, \omega_2)\rangle \, d\mu_i(\omega_i)$$

are measurable and integrable. By Proposition 2.8 the values of these functions are equal to $\left\langle x', \int_{\Omega_i} f(\omega_1, \omega_2) \, d\mu_i(\omega_i)\right\rangle$. Now Pettis' Theorem implies that the functions $\omega_j \mapsto \int_{\Omega_i} f(\omega_1, \omega_2) \, d\mu_i(\omega_i)$ are measurable and by the fundamental estimate and Bochner's Theorem they are also integrable. Let $x' \in X'$, then using Fubini's Theorem for real-valued functions we deduce that

$$\int_{\Omega_1 \times \Omega_2} \langle x', f\rangle \, d\mu_1 \otimes \mu_2 = \int_{\Omega_1} \int_{\Omega_2} \langle x', f\rangle \, d\mu_2 \, d\mu_1 = \int_{\Omega_2} \int_{\Omega_1} \langle x', f\rangle \, d\mu_1 \, d\mu_2.$$

By Proposition 2.8 we can interchange integration and the pairing with x' in the above computation. As x' was chosen arbitrary the Hahn-Banach Theorem implies that the claim holds. If conversely f is Bochner-integrable, then $\|f\|$ is integrable by Bochner's Theorem. By Fubini's Theorem for the real-valued case it holds that the iterated integral

$$\int_{\Omega_1} \int_{\Omega_2} \|f\| \, d\mu_2 d\mu_1$$

exists, hence the claim follows from the first part of the theorem. $\square$

2.2 The Spaces $L^p(\Omega, X)$

According to Bochner's Theorem a function is in $L^1(\Omega, X)$ if and only if its norm is in $L^1(\Omega, \mathbb{R})$. We generalize this to the case $p \neq 1$ to define L^p-spaces analogously to the scalar-valued case. For all $1 \leq p \leq \infty$ the space $L^p(\Omega, X)$ is defined as the space of all measurable functions such that $\|f\| \in L^p(\Omega, \mathbb{R})$ and the norm on this space will be defined via $\|f\|_{L^p(\Omega,X)} := \| \, \|f\| \, \|_{L^p(\Omega,\mathbb{R})}$. As in the scalar-valued case we view a measurable function f as an equivalence class of functions that are equal a.e. It now follows as in the scalar-valued case that $L^p(\Omega, X)$ is a Banach space. If X is a Hilbert space, then $L^2(\Omega, X)$ is a Hilbert space as well with respect to the inner product $(f|g) := \int (f|g)$. Occasionally we will write $L^p(\Omega, X, \mu)$ to indicate the chosen measure if confusion might occur otherwise.

Many properties from the scalar-valued case carry over to the vector-valued case. The first example of this is Hölder's inequality. Note that for two X-valued functions the product $f \cdot g$ is not defined in general. But there are two types of measurable functions for which we can make sense of this product: scalar-valued functions, for which the product is defined via the multiplication of scalars and vectors, and functions with values in X', for which the multiplication is defined via the action of X' on X. The Hölder inequality is true in both cases.

Proposition 2.10 (Hölder's inequality, scalar-valued case)**.** *Let $1 \leq p \leq \infty$ and let $f \in L^p(\Omega, X)$ and $g \in L^q(\Omega, \mathbb{R})$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then the function fg is in $L^1(\Omega, X)$ and we have that $\|fg\|_{L^1(\Omega,X)} \leq \|f\|_{L^p(\Omega,X)} \|g\|_{L^q(\Omega,\mathbb{R})}$.*

Proof. It is clear that fg is measurable as both functions are pointwise limits of simple functions whose product is again a sequence of simple functions converging to fg a.e. Now by Hölder's inequality applied to the scalar-valued functions $\|f\|$ and g we have that

$$\int_\Omega \|fg\| = \int_\Omega \|f\| \, |g| \leq \|f\|_{L^p(\Omega,X)} \|g\|_{L^q(\Omega,\mathbb{R})}.$$

Thus by Bochner's theorem $fg \in L^1(\Omega, X)$ and the estimate holds. □

Proposition 2.11 (Hölder's inequality, dual-valued case)**.** *Let $1 \leq p \leq \infty$ and let $f \in L^p(\Omega, X)$ and $g \in L^q(\Omega, X')$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then the function $\langle f, g \rangle$ is in $L^1(\Omega, \mathbb{R})$ and we have that $\| \langle f, g \rangle \|_{L^1(\Omega,\mathbb{R})} \leq \|f\|_{L^p(\Omega,X)} \|g\|_{L^q(\Omega,X')}$.*

Proof. The proof is the same as before. $\langle f, g \rangle$ is measurable as the pairing $\langle s_1, s_2 \rangle$ of two simple functions is a simple function with values in $\mathbb{R}$. The absolute value of the product $\langle f, g \rangle$ can be estimated by $\|f\| \, \|g\|$ and thus the estimate for the L^1-norm holds again by applying Hölder's inequality. □

Proposition 2.12. *If Ω is a finite measure space, then $L^q(\Omega, X) \subset L^p(\Omega, X)$ for $1 \leq p \leq q \leq \infty$. In particular for an arbitrary measure space (Ω, Σ, μ) we define the space $L^1_{loc}(\Omega, X)$ to be the space of all X-valued functions f such that for any subset $B \in \Sigma$ with $\mu(B) < \infty$ we have that $f_{|B} \in L^1(B, X)$. Then the first statement implies that $L^p(\Omega, X) \subset L^1_{loc}(\Omega, X)$.*

Proof. The case $q = \infty$ is clear as any bounded measurable function on a set of finite measure is integrable. Now let $q < \infty$. For a function $f \in L^q(\Omega, X)$ we apply Hölder's inequality to the functions $\|f\|^p \in L^{\frac{q}{p}}(\Omega, \mathbb{R})$ and $1_\Omega \in L^{\frac{q}{q-p}}(\Omega, \mathbb{R})$ to obtain

$$\|f\|^p_{L^p(\Omega,X)} \leq \|f\|^p_{L^q(\Omega,X)} \cdot \mu(\Omega)^{1-\frac{p}{q}},$$

which implies that $f \in L^p(\Omega, X)$. □

Another similarity to the scalar-valued case are convergence and density results.

Proposition 2.13. *Let $1 \leq p \leq \infty$ and let $f_n \to f$ in $L^p(\Omega, X)$. Then there exists a subsequence (f_{n_k}) which converges to f pointwise a.e.*

Proof. The sequence $\|f_n - f\|$ converges to 0 in $L^p(\Omega, \mathbb{R})$, thus there exists a subsequence $\|f_{n_k} - f\|$ which converges to 0 pointwise a.e., i.e. $f_{n_k} \to f$ pointwise a.e. $\square$

Proposition 2.14. *Let $f \in L^p(\Omega, X)$. If $1 \le p < \infty$ then there exists a sequence (s_n) of simple functions converging to f in $L^p(\Omega, X)$. If $p = \infty$ then there exists a sequence (s_n) of measurable, countably valued functions converging to f in $L^\infty(\Omega, X)$*

Proof. First let $p < \infty$. As f is measurable there exists a sequnce (s_n) of simple functions converging to f a.e. such that $\|s_n\| \le 2\|f\|$ a.e. Then $\|s_n - f\|^p \le 3^p \|f\|^p$ a.e. and thus $\|s_n - f\| \to 0$ in $L^p(\Omega, X)$ by the Dominated Convergence Theorem.

Now let $p = \infty$ and let $\varepsilon > 0$. As f is measurable we find a null set N and a sequence $(x_n)_{n \in \mathbb{N}}$ that is dense in $f(X \backslash N)$. Let $\Omega_n := \left\{ \omega \in \Omega \backslash \bigcup_{k=1}^{n-1} \Omega_k,\ \|f(\omega) - x_n\| < \varepsilon \right\}$ and define $s := \sum_{n \in \mathbb{N}} x_n 1_{\Omega_n}$. Then s is countably valued, measurable and $\|s - f\|_{L^\infty(\Omega, X)} \le \varepsilon$ from which we infer the result. $\square$

Proposition 2.15. *Let $\Omega \subset \mathbb{R}^d$ open and consider integrations with respect to the Lebesgue measure λ. If $1 \le p < \infty$ and $f \in L^p(\Omega, X)$, then there exists a sequence (φ_n) of functions in $C_c^\infty(\Omega, X)$ converging to f in $L^p(\Omega, X)$.*

Proof. Let E be a measurable set and let $(\varphi_n) \subset C_c^\infty(\Omega, \mathbb{R})$ be a sequence converging to 1_E in $L^p(\Omega, \mathbb{R})$. For a vector $x \in X$ the sequence $(\varphi_n \cdot x) \subset C_c^\infty(\Omega, X)$ converges to $1_E \cdot x$ in $L^p(\Omega, X)$. By linearity we can approximate any simple function and thus any L^p-function by Proposition 2.14. $\square$

A classical result from measure theory is Lebesgue's Differentiation Theorem. Its first part says that the primitive of an integrable function is differentiable a.e. and the derivative is the initial function. This is true for vector-valued functions as well.

Theorem 2.16 (Lebesgue's Differentiation Theorem)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and denote by $B(x, r)$ the ball with radius r centered at $x \in \mathbb{R}^d$. Let $f \in L^1_{loc}(\Omega, X)$, then*

$$\lambda\left(B(x,r)\right)^{-1} \int_{B(x,r)} \|f(y) - f(x)\| \, d\lambda(y) \to 0 \quad (r \to 0)$$

for almost all $x \in \Omega$. In particular we have that

$$f(x) = \lim_{r \to 0} \lambda\left(B(x,r)\right)^{-1} \int_{B(x,r)} f(y) \, d\lambda(y)$$

almost everywhere. Further if $d = 1$ we have

$$f(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) \, d\lambda(t)$$

for almost all $x \in \Omega$.

Proof. As f is measurable, we may assume that X is separable without impact on the claim. Let $(x_n)_{n\in\mathbb{N}}$ be dense in X and consider the scalar functions $\|f(x)-x_n\|$. By Lebesgue's Differentiation Theorem for the scalar case there exists a null set $N_n \subset \mathbb{R}^d$ for every $n \in \mathbb{N}$ such that

$$\|f(x)-x_n\| = \lim_{r\to 0} \lambda\left(B(x,r)\right)^{-1} \int_{B(x,r)} \|f(y)-x_n\|\, d\lambda(y)$$

for all $x \notin N_n$. If we let $N := \bigcup_{n\in\mathbb{N}} N_n$ then the above holds for all $x \notin N$ and all $n \in \mathbb{N}$. Let $\varepsilon > 0$, $x \notin N$ and $n \in \mathbb{N}$ such that $\|f(x)-x_n\| < \frac{\varepsilon}{2}$. Using the above we compute

$$\begin{aligned}
0 &\le \limsup_{r\to 0} \lambda\left(B(x,r)\right)^{-1} \int_{B(x,r)} \|f(y)-f(x)\|\, d\lambda(y) \\
&\le \limsup_{r\to 0} \lambda\left(B(x,r)\right)^{-1} \int_{B(x,r)} \|f(y)-x_n\| + \|x_n - f(x)\|\, d\lambda(y) \\
&= 2\|f(x)-x_n\| < \varepsilon
\end{aligned}$$

from which we infer the assertions. The last claim is proven analogously using the real-valued case. $\square$

We will conclude this section with an inspection of a weak limit of L^p-functions. In the scalar-valued case, this would not yield any further insight.

Proposition 2.17. *Let $(f_n) \subset L^p(\Omega, X)$ such that $\|f_n\|_{L^p(\Omega,X)} \le C < \infty$ for all $n \in \mathbb{N}$ and let $f : \Omega \to X$ such that for almost all $\omega \in \Omega$ we have $f_n(\omega) \rightharpoonup f(\omega)$. Then $f \in L^p(\Omega, X)$ and $\|f\|_{L^p(\Omega,X)} \le C$.*

Proof. By Corollary 2.4 f is measurable. For every $\omega \in \Omega$ choose a normed $x'(\omega) \in X'$ such that $\|f(\omega)\| = \langle x'(\omega), f(\omega)\rangle$. Note that we implicitly assume that $\langle x'(\omega), f(\omega)\rangle \in \mathbb{R}^+$. This is always possible by multiplying $x'(\omega)$ with an element of the unit circle. Now Fatou's Lemma implies that

$$\begin{aligned}
\|f\|^p_{L^p(\Omega,X)} &= \int_\Omega \|f\|^p \, d\mu = \int_\Omega \langle x'(\omega), f(\omega)\rangle^p \, d\mu(\omega) \\
&= \int_\Omega \lim_{n\to\infty} |\langle x'(\omega), f_n(\omega)\rangle|^p \, d\mu(\omega) \\
&\le \liminf_{n\to\infty} \int_\Omega |\langle x'(\omega), f_n(\omega)\rangle|^p \, d\mu(\omega) \\
&\le \liminf_{n\to\infty} \int_\Omega \|f_n(\omega)\|^p \, d\mu(\omega) \le C^p
\end{aligned}$$

hence $f \in L^p(\Omega, X)$ and the estimate holds. $\square$

2.3 The Radon-Nikodym Property and the Dual of $L^p(\Omega, X)$

As we have seen in the previous sections, many properties of the Lebesgue integral carry over to the Bochner integral. But of course we cannot expect every property to work like this. One example is the duality of L^p and L^q. From the scalar-valued case and Hölder's inequality one might expect that for $1 \leq p < \infty$, the dual of $L^p(\Omega, X)$ is given by $L^q(\Omega, X')$, where $\frac{1}{p} + \frac{1}{q} = 1$, but this is not true in general. Some Banach spaces do not behave well enough for this property. Of course Hölder's inequality shows that for all Banach spaces X we have the embedding $L^q(\Omega, X') \hookrightarrow L^p(\Omega, X)'$. Further for all $g \in L^q(\Omega, X')$ it holds that $\|g\|_{L^p(\Omega,X)'} \leq \|g\|_{L^q(\Omega,X')}$ hence this embedding is continuous. We can extend this result:

Proposition 2.18. *For $1 \leq p < \infty$ the inclusion mapping $L^q(\Omega, X') \hookrightarrow L^p(\Omega, X)'$ is an isometry.*

Proof. First let the measure space be finite and let $g = \sum_{i=1}^{\infty} x_i' 1_{E_i} \in L^q(\Omega, X')$ be a countably valued function with $x_i' \in X'$ and pairwise disjoint measurable sets E_i such that $\bigcup_i E_i = \Omega$. We have that $\|g\| \in L^q(\Omega, \mathbb{R})$ so for any $\varepsilon > 0$ there exists a nonnegative function $h \in L^p(\Omega, \mathbb{R})$ with $\|h\|_{L^p(\Omega,\mathbb{R})} = 1$ such that

$$\|g\|_{L^q(\Omega,X')} = \| \, \|g\| \, \|_{L^q(\Omega,\mathbb{R})} \leq \int_\Omega h \, \|g\| \, d\mu + \frac{\varepsilon}{2}.$$

Next choose $x_i \in X$ such that $\|x_i\| = 1$ and $\|x_i'\| \leq \langle x_i, x_i' \rangle + \frac{\varepsilon}{2\|h\|_{L^1(\Omega,\mathbb{R})}}$. Define $f := \sum_{i=1}^{\infty} x_i \, h \, 1_{E_i}$. We have

$$\|f\|_{L^p(\Omega,X')}^p = \sum_{i=1}^{\infty} \int_{E_i} \|x_i \, h(t)\|^p \, d\mu(t) = \int_\Omega |h(t)|^p \, d\mu(t) = \|h\|_{L^p(\Omega,\mathbb{R})}^p = 1$$

and for f seen as a functional we compute

$$\begin{aligned}
\int_\Omega \langle f, g \rangle \, d\mu &= \int_\Omega \left\langle \sum_{i=1}^{\infty} h(t) \, x_i \, 1_{E_i}, g \right\rangle \, d\mu(t) \\
&= \int_\Omega h(t) \sum_{i=1}^{\infty} \langle x_i, x_i' \rangle \; 1_{E_i} \, d\mu(t) \\
&\geq \int_\Omega h(t) \sum_{i=1}^{\infty} \left(\|x_i'\| - \frac{\varepsilon}{2\|h\|_{L^1(\Omega,\mathbb{R})}} \right) 1_{E_i} \, d\mu(t) \\
&= \int_\Omega h(t) \left\| \sum_{i=1}^{\infty} x_i' \, 1_{E_i} \right\| \, d\mu(t) - \int_\Omega h(t) \frac{\varepsilon}{2\|h\|_{L^1(\Omega,\mathbb{R})}} \, d\mu(t) \\
&= \int_\Omega h \|g\| \, d\mu - \frac{\varepsilon}{2} \\
&\geq \|g\|_{L^q(\Omega,X')} - \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain $\|g\|_{L^p(\Omega,X)'} \geq \|g\|_{L^q(\Omega,X')}$ and thus $\|g\|_{L^p(\Omega,X)'} = \|g\|_{L^q(\Omega,X')}$ by Hölder's inequality. Now let $g \in L^q(\Omega, X')$ arbitrary and choose a sequence (g_n) of countably valued functions which converges to g in L^q-norm. As $\|g_n - g\|_{L^p(\Omega,X)'} \leq \|g_n - g\|_{L^q(\Omega,X')}$, the convergence also holds in $L^p(\Omega, X)'$ and thus we compute

$$\|g\|_{L^p(\Omega,X)'} = \lim_{n\to\infty} \|g_n\|_{L^p(\Omega,X)'} = \lim_{n\to\infty} \|g_n\|_{L^q(\Omega,X')} = \|g\|_{L^q(\Omega,X')}.$$

Now let (Ω, Σ, μ) be σ-finite and choose a sequence $(A_n) \subset \Sigma$ of sets of finite measure that exhaust Ω. Choosing appropriate function values on A_n we can construct a positive countably valued function f such that $\|f\|_{L^1(\Omega,\mathbb{R})} = 1$. This function induces an isometric isomorphism $L^p(\Omega, X, \mu) \cong L^p(\Omega, X, f\,d\mu)$ via $g \mapsto gf^{-\frac{1}{p}}$ and analogously $L^q(\Omega, X', \mu) \cong L^q(\Omega, X', f\,d\mu)$, where formally $\frac{1}{\infty} := 0$. Now let $g \in L^q(\Omega, X', \mu)$, then the finite case above we yields

$$\begin{aligned}
\|g\|_{L^q(\Omega,X',\mu)} &= \left\|gf^{-\frac{1}{q}}\right\|_{L^q(\Omega,X',f\,d\mu)} \\
&= \sup\left\{\int_\Omega \left\langle hf^{-\frac{1}{p}}, gf^{-\frac{1}{q}}\right\rangle f\,d\mu \ , \ \left\|hf^{-\frac{1}{p}}\right\|_{L^p(\Omega,X,f\,d\mu)} = 1\right\} \\
&= \sup\left\{\int_\Omega \langle h, g\rangle \ d\mu \ , \ \left\|hf^{-\frac{1}{p}}\right\|_{L^p(\Omega,X,f\,d\mu)} = 1\right\} \\
&= \sup\left\{\int_\Omega \langle h, g\rangle \ d\mu \ , \ \|h\|_{L^p(\Omega,X,\mu)} = 1\right\} \\
&= \|g\|_{L^p(\Omega,X,\mu)'}
\end{aligned}$$

and thus we obtain the desired equality. □

Now we want to give a sufficient criterion for the equality of $L^q(\Omega, X')$ and $L^p(\Omega, X)'$. We need a few further definitions to do so. A function $\nu : \Sigma \to X$ such that for any sequence of pairwise disjoint measurable sets we have

$$\nu\left(\bigcup_{i=1}^\infty E_i\right) = \sum_{i=1}^\infty \nu(E_i),$$

is called a *vector measure*. As in the scalar-valued case we define its *variation* on a measureable set A to be

$$|\nu|(A) := \sup_\pi \sum_{B\in\pi} \|\nu(B)\|,$$

where the supremum is taken over all finite, measurable partitions π of A. If $|\nu|(\Omega) < \infty$ we will say that ν is of *bounded variation*. If for any measurable set A such that $\mu(A) = 0$ it follows that $\nu(A) = 0$ we will say that ν is *absolutely continuous with respect to* μ and denote this by $\nu \ll \mu$.

Let M and N be metric spaces. A function $f : M \to N$ is said to be *Lipschitz continuous* if there exists a constant L such that $d_N(f(x), f(y)) \leq L\, d_M(x, y)$ for all $x, y \in M$.

Theorem 2.19. *Let X be a Banach space and I be an interval. The following are equivalent*

(i) For any σ-finite, complete measure space (Ω, Σ, μ) the following holds: For any vector measure $\nu : \Sigma \to X$ with bounded variation that is absolutely continuous with respect to μ there exists a function $f \in L^1(\Omega, X, \mu)$ such that $\nu(A) = \int_A f\, d\mu$ for all $A \in \Sigma$.

(ii) Every Lipschitz continuous function $f : I \to X$ is differentiable a.e.

Proof. For a very precise and relatively short proof of this fact see [BL00]. A more circumlocutory treatise of related properties which also covers this proof can be found in [DU77]. We want to mention that these two sources prove the theorem for finite measure spaces, but with the same process as in the previous proposition one can extend the results from finite to σ-finite measure spaces. □

A Banach space X that satisfies the two equivalent criteria from Theorem 2.19 is said to have the *Radon-Nikodym property.* In view of the case $X = \mathbb{R}$ we will refer to (i) as the Radon-Nikodym characterization and to (ii) as the Rademacher characterization.

Before we come back to the duality theory of the Bochner L^p-spaces, we will show that the Radon-Nikodym property is not an unusual property. We will give two large classes of spaces that have this property.

Proposition 2.20 (Dunford-Pettis)**.** *Let Y be a Banach space and let $X := Y'$ be separable, then X has the Radon-Nikodym property.*

Proof. We will use the Rademacher characterization for this proof. Let $F : I \to X$ be Lipschitz continuous with Lipschitz constant L and let $a \in I$. By looking at the function $G := \frac{F - F(a)}{L}$ we find that we may w.l.o.g. assume that $F(a) = 0$ and that $L = 1$. It follows that for any $y \in Y$ the function $\langle y, F(\cdot)\rangle$ is Lipschitz continuous with Lipschitz constant $\|y\|$. Recall that the second part of the scalar-valued Differentiation Theorem of Lebesgue implies that there exists a function g_y, unique up to sets of measure zero, with $\|g_y\|_{L^\infty(I,\mathbb{R})} \leq \|y\|$ such that

$$\langle y, F(t)\rangle = \int_a^t g_y(s)\, ds \quad a.e.$$

As X is separable it follows that Y is separable as well. Let $D \subset Y$ be a countable dense subset and consider all y of the form $y = \sum_{i=1}^n \alpha_i y_i$ for some $n \in \mathbb{N}$, $y_i \in D$ and

$\alpha_i \in \mathbb{Q} + i\mathbb{Q}$. For these y we have

$$\begin{aligned}\langle y, F(t)\rangle &= \left\langle \sum_{i=1}^n \alpha_i y_i, F(t)\right\rangle = \sum_{i=1}^n \alpha_i \langle y_i, F(t)\rangle \\ &= \sum_{i=1}^n \alpha_i \int_a^t g_{y_i}(s)\,ds = \int_a^t \sum_{i=1}^n \alpha_i g_{y_i}(s)\,ds\end{aligned}$$

and thus $g_y = \sum_{i=1}^n \alpha_i g_{y_i}$ a.e. From this we obtain

$$\left|\sum_{i=1}^n \alpha_i g_{y_i}(s)\right| = |g_y(s)| \le \|g_y\|_{L^\infty(I,\mathbb{R})} \le \|y\| = \left\|\sum_{i=1}^n \alpha_i y_i\right\|$$

for almost all $s \in I$. Note that the choice of $(n, (\alpha_i), (y_i))$ is countable so we can choose a null set E such that the above estimate holds for all $s \in I\backslash E$ and all choices of $(n, (\alpha_i), (y_i))$. As $\mathbb{Q} + i\mathbb{Q}$ is dense in $\mathbb{C}$, the estimate carries over to all choices $(n, (\alpha_i), (y_i))$ where $\alpha_i \in \mathbb{C}$. Thus $y \mapsto g_y(s)$ is a linear mapping from $\operatorname{span}(D)$ to $\mathbb{C}$ whose norm is bounded by 1. By the density of D we can uniquely extend this mapping to Y. We obtain an element $f(s) \in Y' = X$ for which $\|f(s)\| \le 1$. For all $y \in D$ and almost all $s \in I$ we have that $\langle f(s), y\rangle = g_y(s)$ which is measurable and bounded. Now let $y \in Y$ and let $(y_n) \subset D$ such that $y_n \to y$. Then on the interval $[a, t]$ the functions $\langle f(\cdot), y_n\rangle = g_{y_n}(\cdot)$ are bounded by $\|y_n\|$ which itself is bounded as (y_n) converges. Thus the Dominated Convergence Theorem yields that $\langle f(\cdot), y\rangle$ is measurable and that

$$\begin{aligned}\langle F(t), y\rangle &= \lim_{n\to\infty} \langle F(t), y_n\rangle = \lim_{n\to\infty} \int_a^t g_{y_n}(s)\,ds \\ &= \lim_{n\to\infty} \int_a^t \langle f(s), y_n\rangle\,ds = \int_a^t \langle f(s), y\rangle\,ds.\end{aligned}$$

As Y is norming for X Corollary 2.2 implies that f is measurable and thus locally integrable as it is bounded. From the above computation we obtain

$$\langle F(t), y\rangle = \left\langle \int_a^t f(s)\,ds, y\right\rangle$$

and by the separating property of Y it follows that

$$F(t) = \int_a^t f(s)\,ds.$$

Lebesgue's Differentiation Theorem implies that F is differentiable a.e. and thus X has the Radon-Nikodym property. □

Let $f : I \to X$ be Lipschitz continuous, then in particular f is continuous and thus $f(I)$ is separable. Thus we may restrict our investigations to the smallest closed subspace of X containing $f(I)$. It follows that X has the Radon-Nikodym property if and only if every closed separable subspace of X does.

Corollary 2.21. *Every reflexive space has the Radon-Nikodym property.*

Proof. By the remark above and the fact that any closed subspace of a reflexive space is reflexive as well we may assume that X is separable. X is the dual space of X', thus by Proposition 2.20 X has the Radon-Nikodym property. $\square$

In the next chapter, we will give examples of spaces that do not have the Radon-Nikodym property. We now come back to the duality theory.

Theorem 2.22. *Let X be a Banach space such that X' has the Radon-Nikodym property, then $L^p(\Omega, X)' \cong L^q(\Omega, X')$ for $1 \leq p < \infty$.*

Proof. We will use the Radon-Nikodym characterization of the Radon-Nikodym property for this proof. First let Ω be a finite measure space. Let $l \in L^p(\Omega, X)'$ and define

$$\nu : \Sigma \to X'$$

via $\langle \nu(E), x \rangle = \langle l, x1_E \rangle$ for any $E \in \Sigma$ and $x \in X$. For any such x and E we have

$$|\langle l, x1_E \rangle| \leq \|l\|_{L^p(\Omega,X)'} \|x1_E\|_{L^p(\Omega,X)} = \|l\|_{L^p(\Omega,X)'} \|x\| \mu(E)^{\frac{1}{p}},$$

thus $\nu(E) \in X'$ with $\|\nu(E)\| \leq \|l\|_{L^p(\Omega,X)'} \mu(E)^{\frac{1}{p}}$, i.e. ν is well defined. Additionally, this directly implies that ν is absolutely continuous with respect to μ. Let (E_n) be pairwise disjoint and measurable and let $x \in X$. Using the continuity of l we compute

$$\begin{aligned} \left\langle \nu\left(\bigcup_{n=1}^{\infty} E_n\right), x \right\rangle &= \left\langle l, x1_{\bigcup_{n=1}^{\infty} E_n} \right\rangle = \left\langle l, \sum_{i=1}^{\infty} x1_{E_n} \right\rangle \\ &= \sum_{i=1}^{\infty} \langle l, x1_{E_n} \rangle = \sum_{i=1}^{\infty} \langle \nu(E_n), x \rangle \end{aligned}$$

where all the series converge as $\left\langle l, x1_{\bigcup_{n=1}^{\infty} E_n} \right\rangle$ exists. Lastly we show that ν is of bounded variation. Note that for a set E_i we have $\|\nu(E_i)\| = \sup_{\|x_i\|=1} \langle x_i, \nu(E_i) \rangle = \sup_{\|x_i\|=1} \langle l, x_i 1_{E_i} \rangle$. This implies that for any partition π we have

$$\sum_{E_i \in \pi} \|\nu(E_i)\| = \sum_{E_i \in \pi} \sup_{\|x_i\|=1} \langle l, x_i 1_{E_i} \rangle = \sup_{\substack{\|x_i\|=1 \\ i=1,\dots,n}} \sum_{E_i \in \pi} \langle l, x_i 1_{E_i} \rangle.$$

This, together with the fact that

$$\begin{aligned}
\sum_{E_i \in \pi} \langle l, x_i 1_{E_i} \rangle &= \left\langle l, \sum_{E_i \in \pi} x_i 1_{E_i} \right\rangle \\
&\leq \|l\| \left\| \sum_{E_i \in \pi} x_i 1_{E_i} \right\|_{L^p(\Omega, X)} \\
&= \|l\| \left(\sum_{E_i \in \pi} \|x_i\|^p \mu(E_i) \right)^{\frac{1}{p}} \\
&= \|l\| (\mu(\Omega))^{\frac{1}{p}},
\end{aligned}$$

implies that $|\nu|(\Omega) \leq \|l\|(\mu(\Omega))^{\frac{1}{p}} < \infty$ and thus ν is of bounded variation as claimed. As X' has the Radon-Nikodym property, there exists a function $g \in L^1(\Omega, X')$ such that

$$\nu(E) = \int_E g \, d\mu \quad (E \in \Sigma).$$

Let $s := \sum_{i=1}^n x_i 1_{E_i}$ be a simple function, then

$$\begin{aligned}
\langle l, s \rangle &= \sum_{i=1}^n \langle l, x_i 1_{E_i} \rangle = \sum_{i=1}^n \langle \nu(E_i), x_i \rangle \\
&= \sum_{i=1}^n \left\langle \int_{E_i} g \, d\mu, x_i \right\rangle = \sum_{i=1}^n \int_{E_i} \langle g, x_i \rangle \, d\mu \\
&= \sum_{i=1}^n \int_\Omega \langle g, x_i 1_{E_i} \rangle \, d\mu = \int_\Omega \langle g, s \rangle \, d\mu,
\end{aligned}$$

which we need to extend to arbitrary L^p-functions. To do so let $E_n := \{x \in \Omega, \|g(x)\| \leq n\}$ and note that $\bigcup_{n=1}^\infty E_n = \Omega$. For a fixed n we have $g 1_{E_n} \in L^\infty(\Omega, X') \subset L^q(\Omega, X')$, thus by Hölder's inequality

$$f \mapsto \int_{E_n} \langle g, f \rangle \, d\mu$$

is a bounded functional in $L^p(\Omega, X)'$ which is equal to $\langle l, f \rangle$ if f is a simple function supported by E_n. For any $f \in L^p(\Omega, X)$ there exists a sequence (s_k) of simple functions supported by E_n which converges to $f 1_{E_n}$ in L^p-norm. Thus we have

$$\langle l, f 1_{E_n} \rangle = \lim_{k \to \infty} \langle l, s_k \rangle = \lim_{k \to \infty} \int_{E_n} \langle g, s_k \rangle \, d\mu = \int_{E_n} \langle g, f \rangle \, d\mu,$$

where the last equality holds as $g1_{E_n} \in L^q(\Omega, X') \subset L^p(\Omega, X)'$. With this, the norm of $g1_{E_n}$ can be computed to be

$$\begin{aligned}
\|g1_{E_n}\|_{L^q(\Omega,X')} &= \|g1_{E_n}\|_{L^p(\Omega,X)'} \\
&= \sup_{\|f\|_{L^p(\Omega,X)}=1} \int_\Omega \langle f, g1_{E_n}\rangle \, d\mu = \sup_{\|f\|_{L^p(\Omega,X)}=1} \int_{E_n} \langle f, g\rangle \, d\mu \\
&= \sup_{\|f\|_{L^p(\Omega,X)}=1} \langle l, f1_{E_n}\rangle \leq \|l\|_{L^p(\Omega,X)'}.
\end{aligned}$$

Now $\|g(x)1_{E_n}(x)\| \to \|g(x)\|$ for all $x \in \Omega$ and the convergence is monotonically increasing. Thus by Beppo Levi's Monotone Convergence Theorem we have that $g \in L^q(\Omega, X')$, $\|g\|_{L^q(\Omega,X')} \leq \|l\|_{L^p(\Omega,X)'}$ and the convergence of $g1_{E_n}$ happens in L^q-norm. From this we infer

$$\begin{aligned}
\langle l, f\rangle &= \lim_{n\to\infty} \langle l, f1_{E_n}\rangle = \lim_{n\to\infty} \int_{E_n} \langle g, f\rangle \, d\mu \\
&= \lim_{n\to\infty} \int_\Omega \langle g1_{E_n}, f\rangle \, d\mu = \int_\Omega \langle g, f\rangle \, d\mu,
\end{aligned}$$

for all $f \in L^p(\Omega, X)$. Thus we have $L^p(\Omega, X)' = L^q(\Omega, X')$ whenever Ω is finite. For a σ-finite measure space we conclude the result using the same technique as in the beginning of the section. □

We want to remark that the converse of Theorem 2.22 is true as well: $L^p(\Omega, X)' = L^q(\Omega, X')$ implies that X' has the Radon-Nikodym property. Thus we see, that the duality does not carry over to general Banach spaces. For a proof of this we refer to [DU77, IV, Theorem 1].

Corollary 2.23 (Phillips)**.** *If X is reflexive and $1 < p < \infty$ then $L^p(\Omega, X)$ is reflexive.*

Proof. Theorem 2.22 and Corollary 2.21 imply that $L^p(\Omega, X)'' = L^q(\Omega, X')' = L^p(\Omega, X'')$ and the reflexivity of X implies that $L^p(\Omega, X'') = L^p(\Omega, X)$. □

We have already seen that $L^2(\Omega, X)$ is a Hilbert space if X is. Corollary 2.21 implies that $L^2(\Omega, X)$ has the Radon-Nikodym property. This can be extended to more general L^p-spaces. We will need this result, which we state without proof.

Theorem 2.24 (Sundaresan, Turett and Uhl)**.** *If $1 < p < \infty$ then X has the Radon-Nikodym property if and only if $L^p(\Omega, X)$ does.*

Proof. See [Sun77] for the original proof and [TU76] for an alternative proof. □

2.4 Notes

The first two sections are a fairly standard treatise of the Bochner integral and the vector-valued L^p- spaces. There exists a variety of books that contain chapters about these topics. However we did not find a source that contains all results we needed. We have found results, proofs and inspiration in the following books: [DS64], [Edw65], [HP68], [Yos68], [GGZ74], [DU77], [CH98], [BL00] and [ABHN11].

The Radon-Nikodym property was discovered in the 1970's to be a very important property of a Banach space. For many applications it seems to be the natural least property that a Banach space must have in order to yield desired results. Thus there are many different approaches to and many different definitions of this property. A very thorough treatise of the Radon-Nikodym property can be found in [DU77]. Our approach is based on this book and [ABHN11] and covers the Radon-Nikodym property only as far as it is needed for this thesis.

Theorem 2.23 was proven directly by Phillips [Phi43]. The proof of Theorem 2.24 has been omitted here as a much deeper understanding of the Radon-Nikodym property and related topics would be needed. Sundaresan's proof utilizes the Radon-Nikodym characterization but the proof in [Sun77] is not entirely correct and would need to be mended while reading. To understand the proof of Turett and Uhl, one would first need to proof another equivalent definition of the Radon-Nikodym property. This proof is also contained in [DU77]. We are not aware whether there exists a proof of this theorem using the Rademacher characterization. Such a proof would be very interesting as we will use this characterization throughout the rest of the thesis.

3 Sobolev Spaces in One Dimension

Let $I = (a, b) \subset \mathbb{R}$ be an open interval with $a, b \in \mathbb{R} \cup \{\infty, -\infty\}$. The Sobolev spaces of functions $u : I \to \mathbb{R}$ have special properties which distinguish them from the general case where I is replaced by a subset of $\mathbb{R}^d$. For example, these functions are continuous which is not true in general. In this chapter we will have a look at the same case for vector-valued functions. Some results carry over to this case, others require special demands on the vector space we are looking at.

3.1 Vector-Valued Distributions

Let X be a Banach space. The space of *X-valued distributions* or *X-valued generalized functions* is defined as the space $\mathcal{D}'(I, X) := \mathcal{L}(C_c^\infty(I, \mathbb{R}), X)$, where the space $C_c^\infty(I, \mathbb{R})$ is topologized in the following way: a sequence (φ_n) converges to φ if and only if all φ_n and φ are supported in the same compact set and on this set $\varphi_n^{(k)} \to \varphi^{(k)}$ uniformly for any $k \in \mathbb{N}_0$. For a general treatise of the topology of $C_c^\infty(I, \mathbb{R})$ see the chapter about locally convex spaces and inductive limits in [RS81]. For any function $f \in L^1_{loc}(I, X)$ we can define a distribution via $\varphi \mapsto \int_I f\varphi$ which we denote by T_f. For us, this case is the most important one.

We define the derivative of a vector-valued distribution analogously to the scalar-valued case. Let $T \in \mathcal{D}'(I, X)$ then the *derivative* T' of T is the distribution defined via

$$\langle T', \varphi \rangle := - \langle T, \varphi' \rangle \quad (\varphi \in C_c^\infty(I, \mathbb{R})).$$

The above definition is motivated by the case where $X = \mathbb{R}$ and T is given as a differentiable L^p-function. In this case, the definition of the derivative is just the usual integration by parts.

To be able to work with the distribution defined by L^p-functions, we will prove the following result.

Proposition 3.1. *Let $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}, X)$. For each $h > 0$ we define a new function $M_h f$ via*

$$M_h f(t) := \frac{1}{h} \int_t^{t+h} f(s) ds,$$

where "ds" is an abbreviation for $d\lambda(s)$. The function $M_h f$ is in $L^p(\mathbb{R}, X) \cap C(\mathbb{R}, X)$ and we have that $\lim_{h \to 0} M_h f = f$ in L^p and a.e.

Proof. As the interval $I = [t, t+h]$ is bounded, we have that $f \in L^1(I, X)$ and thus the integral exists. Let $t_n \to t$ and define $f_n(s) := f(s) \cdot 1_{[t_n, t_n+h]}$ then f_n is bounded by f and on $[t, t+h]$ we have that $f_n \to f$ a.e. By the Dominated Convergence Theorem we now have that $M_h f(t_n) \to M_h f(t)$ and thus $M_h f$ is continuous.

By Hölder's inequality we obtain that

$$\begin{aligned} \|M_h f(t)\| &\leq \frac{1}{h} \int_t^{t+h} \|f(s)\| ds \\ &\leq \frac{1}{h} h^{\frac{1}{q}} \left(\int_t^{t+h} \|f(s)\|^p \, ds \right)^{\frac{1}{p}} \quad \left(\text{where } \frac{1}{p} + \frac{1}{q} = 1 \right) \end{aligned}$$

and thus $\|M_h f(t)\|^p \leq \frac{1}{h} \int_t^{t+h} \|f(s)\|^p ds$. Using Fubini's Theorem for the scalar-valued case we derive that

$$\begin{aligned} \|M_h f\|_p^p &\leq \frac{1}{h} \int_{\mathbb{R}} \int_t^{t+h} \|f(s)\|^p ds \, dt \\ &= \frac{1}{h} \int_{\mathbb{R}} \int_0^h \|f(s+t)\|^p ds \, dt \\ &= \frac{1}{h} \int_0^h \int_{\mathbb{R}} \|f(s+t)\|^p ds \, dt \\ &= \|f\|^p, \end{aligned}$$

thus $M_h f \in L^p(I, X)$ and $M_h \in \mathcal{L}(L^p(I, X))$ with $\|M_h\| \leq 1$. In Theorem 2.16 we have already shown that $M_h f$ converges to f pointwise a.e. To show the convergence in $L^p(I, X)$ let $A_h := Id - M_h$ and note that $\|A\| \leq 2$. As $p < \infty$ we can find a sequence $(\varphi_n) \subset C_c^\infty(\mathbb{R}, X)$ such that $\varphi_n \to f$ in $L^p(\mathbb{R}, X)$.

$$\|A_h f\|_{L^p(\mathbb{R},X)} \leq 2\|f - \varphi_n\|_{L^p(I,X)} + \|A_h \varphi_n\|_{L^p(I,X)}.$$

The first summand converges to 0 for $n \to \infty$ by the choice of φ_n. The second summand converges to 0 for $h \to 0$ as φ_n is uniformly continuous for every fixed n. This shows the claimed convergence in $L^p(\mathbb{R}, X)$. □

With this result we can prove two important corollaries. The second corollary can be seen as a version of the Fundamental Theorem of Calculus for L^p-functions.

Corollary 3.2. *Let $f \in L^1_{loc}$ such that $T_f = 0$. Then $f = 0$ a.e.*

Proof. Let $J \subset I$ be a bounded, open subinterval, then $f \cdot 1_J \in L^1(J, X)$. Let $(\varphi_n) \subset C_c^\infty(J)$ such that $\varphi_n \leq 1$ and $\varphi_n \to 1_J$ a.e., then by the Dominated Convergence Theorem we obtain

$$\int_J f = \lim_{n\to\infty} \int_I f \varphi_n = \lim_{n\to\infty} \langle T_f, \varphi_n \rangle = 0.$$

Thus for all $t \in J$ and h small enough we have that $M_h(f \cdot 1_J(t)) = 0$. Proposition 3.1 implies that $f \cdot 1_J = 0$ a.e. As J was chosen arbitrary, we conclude that $f = 0$ a.e. $\square$

Corollary 3.3. *Let $g \in L^1_{loc}(I, X)$, $t_0 \in I$ and f given by*

$$f(t) := \int_{t_0}^{t} g(s)ds,$$

then $f \in C(I, X)$ and we have that

(i) $T_f' = T_g$
(ii) f *is differentiable a.e. with* $f' = g$

Proof. We can extend g to $\mathbb{R}$ by setting $g = 0$ outside of I. Thus we can assume that $I = \mathbb{R}$. We have that $M_h g(t) = \frac{f(t+h)-f(t)}{h}$ thus (ii) is a direct consequence of Proposition 3.1.

Now let $\varphi \in C_c^\infty(I)$, then by the uniform continuity of φ' and the mean value theorem we have that the difference quotient $\frac{\varphi(t+h)-\varphi(t)}{h}$ converges uniformly to $\varphi'(t)$. Using this and we obtain

$$\begin{aligned}
\langle T_f', \varphi \rangle &= - \langle T_f, \varphi' \rangle \\
&= - \int_{\mathbb{R}} f(t) \lim_{h \to 0} \frac{\varphi(t+h) - \varphi(t)}{h} dt \\
&= - \lim_{h \to 0} \int_{\mathbb{R}} f(t) \frac{\varphi(t+h) - \varphi(t)}{h} dt \\
&= - \lim_{h \to 0} \left(\int_{\mathbb{R}} f(t) \frac{\varphi(t+h)}{h} dt - \int_{\mathbb{R}} f(t) \frac{\varphi(t)}{h} dt \right) \\
&= - \lim_{h \to 0} \left(\int_{\mathbb{R}} f(t-h) \frac{\varphi(t)}{h} dt - \int_{\mathbb{R}} f(t) \frac{\varphi(t)}{h} dt \right) \\
&= - \lim_{h \to 0} \int_{\mathbb{R}} \left[M_{-h} g(t) \right] \varphi(t) dt
\end{aligned}$$

By Proposition 3.1 we have that $T_{-h}g(t)$ converges to $g(t)$ pointwise and in L^p thus the last expression is equal to $\langle T_g, \varphi \rangle$. This proves (i). $\square$

The last proposition in this section shows that the distributional derivative behaves analogously to the usual derivative, namely if the derivative is zero the distribution must be constant.

Proposition 3.4. *Let $T \in \mathcal{D}'(I, X)$ such that $T' = 0$. Then there exists $x_0 \in X$ such that $T = x_0$ or more specific $\langle T, \varphi \rangle = x_0 \int \varphi$.*

Proof. Let $\vartheta \in C_c^\infty(I)$ such that $\int_I \vartheta = 1$ and define $x_0 := \langle T, \vartheta \rangle$. Let $\operatorname{supp}(\vartheta) \subset [a, b]$ and let $t_0 < a$. For an arbitrary $\varphi \in C_c^\infty(I)$ we define

$$\psi(t) := \int_{t_0}^{t} \left(\varphi(s) - \vartheta(s) \int_I \varphi(u) du \right) ds.$$

By the assumption on ϑ we have that $\psi \in C_c^\infty(I)$ and thus we can apply $T' = 0$ to ψ. We compute

$$0 = -\langle T', \psi \rangle = \langle T, \psi' \rangle = \left\langle T, \varphi - \vartheta \int_I \varphi \right\rangle = \langle T, \varphi \rangle - x_0 \int_I \varphi$$

and thus $T = x_0$ as claimed. □

3.2 The Spaces $W^{1,p}(I, X)$

We are now ready to define Sobolev spaces analogously to the scalar-valued case. A function $u \in L^p(I, X)$ is called *weakly differentiable* if there exists a function $v \in L^p(I, X)$ such that $T_u' = T_v$ in the sense of Distributions, i.e.

$$\int_I u\varphi = -\int_I v\varphi' \quad (\varphi \in C_c^\infty).$$

In this case, v is called the *weak derivative* of u and we denote $u' := v$. This is well defined as Corollary 3.2 assures that the function v is unique in $L^p(I, X)$. The *first Sobolev space* is defined by

$$W^{1,p}(I, X) := \{u \in L^p(I, X), u \text{ is weakly differentiable}\}$$

and on $W^{1,p}(I, X)$ we define a norm via $\|u\|_{W^{1,p}(I,X)} := \|u\|_{L^p(I,X)} + \|u'\|_{L^p(I,X)}$.

Proposition 3.5. *For $1 \le p \le \infty$ the space $W^{1,p}(I, X)$ is a Banach space. If X is a Hilbert space, then $H^1(I, X) := W^{1,2}(I, X)$ is a Hilbert space with respect to the norm $\|u\|_{H^1(I,X)} := \left(\|u\|_{L^2(I,X)}^2 + \|u'\|_{L^2(I,X)}^2 \right)^{\frac{1}{2}}$ and this norm is equivalent to the norm on $W^{1,2}(I, X)$.*

Proof. Let $(u_n) \subset W^{1,p}(I, X)$ be a Cauchy sequence with respect to the $W^{1,p}$-norm. By the definition of $\|\cdot\|_{W^{1,p}(I,X)}$ and the completeness of $L^p(I, X)$ we have that there exist functions u and v such that $u_n \to u$ and $u_n' \to v$ $L^p(I, X)$. Let $\varphi \in C_c^\infty(I)$, then $\varphi, \varphi' \in L^q(I, X)$ as well, where $\frac{1}{p} + \frac{1}{q} = 1$, and thus using Hölder's inequality we obtain that

$$\int_I u_n \varphi' \to \int_I u\varphi' \quad \text{and} \quad \int_I u_n' \varphi \to \int_I v\varphi.$$

Thus the relation $\int_I u_n \varphi' = -\int_I u_n' \varphi$ carries over to u and v, i.e. u is weakly differentiable and $u' = v$. This shows that $W^{1,p}(I,X)$ is complete and therefore a Banach space.

Now let X be a Hilbertspace , then $L^2(I,X)$ is a Hilbert space as well. Define

$$(u,v)_{H^1(I,X)} := (u,v)_{L^2(I,X)} + (u',v')_{L^2(I,X)},$$

then it is obvious that this defines an inner product on $H^1(I,X)$ and that the norm defined by this product is the one given in the proposition. The two norms are equivalent as all norms on $\mathbb{R}^2$ are equivalent, hence the completeness follows from the first part of the proof. $\square$

Proposition 3.6. *If X is reflexive, then $W^{1,p}(I,X)$ is reflexive for $1 < p < \infty$.*

Proof. By Corollary 2.23 we have that $L^p(I,X)$ and thus $L^p(I,X) \times L^p(I,X)$ is reflexive. We define an isometry $T : W^{1,p}(I,X) \to L^p(I,X) \times L^p(I,X)$ via $u \mapsto (u,u')$. By Proposition 3.5 we have that $T(W^{1,p}(I,X)) \subset L^p(I,X) \times L^p(I,X)$ is closed and therefore reflexive. Thus $W^{1,p}(I,X)$ is reflexive. $\square$

In Section 3.1 we have already seen that one part of the Fundamental Theorem of Calculus holds in the sense of distributions, i.e. in $W^{1,p}(I,X)$. We now want to show that the second part of the Fundamental Theorem in this case is true as well.

Theorem 3.7. *Let $1 \leq p \leq \infty$ and $u \in W^{1,p}(I,X)$. Then there exists a $t_0 \in I$ such that for almost all $t \in I$ we have that*

$$u(t) = u(t_0) + \int_{t_0}^{t} u'(s)ds.$$

Proof. Let $t_1 \in I$ and define $g(t) = \int_{t_1}^{t} u'(s)ds$ and $w(t) = u(t) - g(t)$. By Corollary 3.3 we have that $T_w' = 0$. Proposition 3.4 yields that $T_w = x_0 \in X$ and Corollary 3.2 implies that $u(t) = x_0 + \int_{t_1}^{t} u'(s)ds$ a.e. Choose a $t_0 \in I$ such that this equation holds, then $u(t) - u(t_0) = \int_{t_0}^{t} u'(s)ds$ a.e. which is equivalent to the claim. $\square$

The proof shows even more: There exists a null set $N \subset I$ such that $u(t) = u(t_0) + \int_{t_0}^{t} u'(s)ds$ for all $t, t_0 \notin N$. The Fundamental Theorem yields a useful characterization of the Sobolev space $W^{1,p}(I,X)$. As in the scalar-valued case we say that a function $f : [a,b] \to X$ is *absolutely continuous* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\sum_{i=1}^{n} \|f(b_i) - f(a_i)\| \leq \varepsilon$ for every finite collection $\{[a_i,b_i]\}$ of disjoint intervals with a total length of at most δ. If $g \in L^1([a,b],X)$ and f is given via

$$f(t) = \int_a^t g(s)\, ds$$

then by the integrability of $\|g\|$ we deduce that f is absolutely continuous. Recall that Lebesgue's Differentiation Theorem implies the converse if $X = \mathbb{R}$. If I is an arbitrary interval, we say that f is *locally absolutely continuous* if $f_{|[a,b]}$ is absolutely continuous for all compact intervals $[a, b] \subset I$. All of the above easily translates to this case when we replace $L^1(I, X)$ by $L^1_{\text{loc}}(I, X)$. Let $u \in W^{1,p}(I, X)$, then Theorem 3.7 tells us that there exists a representative of u which is locally absolutely continuous and differentiable a.e. by Corollary 3.3. Hence given u as above we may always implicitly assume that u is the representative having the afore mentioned properties. We summarize and extend this in the following proposition.

Proposition 3.8. *Let $u \in L^p(I, X)$ for $1 \leq p \leq \infty$, then the following are equivalent*

(i) $u \in W^{1,p}(I, X)$

(ii) *u is locally absolutely continuous, differentiable a.e. and $u' \in L^p(I, X)$*

(iii) *there exists an L^p-function u' such that for any functional $x' \in X'$ the function $\psi := \langle x', u\rangle$ is locally absolutely continuous (hence differentiable a.e.) and $\psi' = \langle x', u'\rangle$*

(iv) *there exists an L^p-function u' such that for any functional x' in a seperating subset $E' \subset X'$ the function $\psi := \langle x', u\rangle$ is locally absolutely continuous (hence differentiable a.e.) and $\psi' = \langle x', u'\rangle$*

The functions u' in (ii), (iii) and (iv) are all the same and equal to the weak derivative of u.

Proof. $(i) \Rightarrow (ii)$ From Theorem 3.7 we know that a.e.

$$u(t) = u(t_0) + \int_{t_0}^{t} u'(s)\, ds$$

and as u' is in L^1_{loc} we have that u is locally absolutely continuous. Corollary 3.3 implies that u is differentiable a.e. with derivative $u' \in L^p(I, X)$.
$(ii) \Rightarrow (iii)$ This is an easy consequence of the linearity of x' and Proposition 2.8.
As $(iii) \Rightarrow (iv)$ is trival, we need to show that $(iv) \Rightarrow (i)$. Define the function g via

$$g(t) = u(t_0) + \int_{t_0}^{t} u'(s)\, ds,$$

for some $t_0 \in I$. Then $g \in W^{1,p}(I, X)$ by Corollary 3.3. For any $x' \in E'$ the function ψ as above is locally absolutely continuous, so Lebesgue's Differentiation Theorem yields that

$$\big\langle x', u(t)\big\rangle = \psi(t) = \psi(t_0) + \int_{t_0}^{t} \psi'(s)\, ds = \big\langle x', u(t_0)\big\rangle + \int_{t_0}^{t} \big\langle x', u'(s)\big\rangle\, ds,$$

where we may need to change the choice of t_0 above. Thus, using Proposition 2.8, we gain $\langle x', g\rangle = \langle x', u\rangle$. As E' is separating we conclude that $u = g \in W^{1,p}(I, X)$. □

At this point we want to remark that the existence of the L^p-function u' is crucial in (iii) and (iv). It is not true in general that $\langle x', u\rangle \in W^{1,p}(I, \mathbb{R})$ for all x' implies that $u \in W^{1,p}(I, X)$. We will study some examples for this.

Example 3.9 (Counterexample on c_0)**.** Let $X = c_0$, the space of all sequences converging to 0 equipped with the sup-norm, and let

$$\begin{aligned} f : (0,1) &\to c_0 \\ t &\mapsto f(t) := \left(\frac{\sin(nt)}{n}\right). \end{aligned}$$

The function f is nowhere differentiable and thus not in $W^{1,p}(I, c_0)$ by Proposition 3.8. Note that if it were differentiable, then due to the sup-norm on c_0 the differentiation would be done coordinatewise. But $f_n'(t) = \cos(nt)$ which does not define a c_0-sequence for any t. However, we have that $\langle x', f\rangle \in W^{1,p}((0,1), \mathbb{R})$ for any $x' \in X' = l^1$. To see this let

$$s_N(t) := \sum_{k=1}^{N} x_k' \frac{\sin(kt)}{k} \in C^1((0,1), \mathbb{R}).$$

If $m < n$ are big enough we have that

$$\sup_{t\in(0,1)} \left| \sum_{k=m}^{n} x_k' \frac{\sin(kt)}{k} \right| \leq \sup_{t\in(0,1)} \sum_{k=m}^{n} |x_k'| \leq \varepsilon$$

by the absolute convergence of the series defined by (x_k'). Thus the partial sums $s_N(t)$ converge uniformly to $\langle x', f\rangle$. The same argument shows that the derivatives $s_N'(t)$ converge uniformly. It follows that $\langle x', f\rangle \in C^1((0,1), \mathbb{R}) \subset W^{1,p}((0,1), \mathbb{R})$.

Example 3.10 (Counterexample on L^p)**.** Let $1 \leq p \leq \infty$ and $X = L^p((0,1), \mathbb{R})$. Define

$$\begin{aligned} f : (0,1) &\to L^p((0,1), \mathbb{R}) \\ t &\mapsto f(t) := 1_{(0,t)}, \end{aligned}$$

then f is nowhere differentiable and thus not in $W^{1,r}((0,1), X)$ for any $1 \leq r \leq \infty$ by Proposition 3.8. To see this let $t_0 \in (0,1)$ and assume that f is differentiable in t_0. Then for any $g \in L^q((0,1), \mathbb{R}) \subset L^p((0,1), \mathbb{R})'$ the function $t \mapsto \langle g, f(t)\rangle$ is differentiable in t_0 aswell. Let $g := 1_{\{t\leq t_0\}} - 1_{\{t>t_0\}}$ then

$$\langle g, f(t)\rangle = \int_0^t g(s)\, ds = \begin{cases} \int_0^t 1\, ds = t, & \text{if } t \leq t_0 \\ \int_0^{t_0} 1\, ds - \int_{t_0}^t 1\, ds = 2t_0 - t, & \text{if } t > t_0 \end{cases}$$

which is not differentiable in t_0 and thus contradicts the assumption. But for any $g \in L^q((0,1), \mathbb{R}) \subset L^r((0,1), \mathbb{R})$ $(r \leq q)$ we have

$$\langle g, f(t)\rangle = \int_0^t g(s)\, ds$$

thus $\langle g, f(\cdot)\rangle \in W^{1,r}((0,1), \mathbb{R})$ for all $r \leq q$ by Proposition 3.8.

We give another example that shows how delicate the characterizations (*iii*) and (*iv*) have to be handled.

Example 3.11. Let $A \subset \mathbb{R}$ be a non-measurable set and consider the Hilbert space $l^2(A) = \left\{(x_t)_{t\in A},\, x_t \in \mathbb{R},\, \sum_{t\in A} x_t^2 < \infty\right\}$. Note that $(x_t) \in l^2(A)$ implies that $x_t = 0$ for all but at most countably many $t \in A$. The standard orthonormal base for $l^2(A)$ is given by $e_t := (\delta_{ts})_{s\in A}$ $(t \in A)$. We define a function $f : \mathbb{R} \to l^2(A)$ via

$$f(t) := \begin{cases} 0, & t \notin A \\ e_t, & t \in A \end{cases}$$

For any $y \in l^2(A)$ we have the Fourier representation

$$y = \sum_{t\in A} (y|e_t)e_t = \sum_{j=1}^{\infty} y_{t_j} e_{t_j},$$

where the last sum is countable as $y_t = 0$ for all but at most countably many $t_j \in A$. Thus we have that

$$(y|f(t)) = \sum_{j=1}^{\infty} y_{t_j}(e_{t_j}|f(t)) = 0$$

if $t \notin \{t_j\}_{j=1}^{\infty}$, i.e. $(y|f(\cdot)) = 0$ a.e.

Let $f' : \mathbb{R} \to l^2(A)$ be the constant zero function, then we have that for all $y \in l^2(A)' = l^2(A)$ the function $(y|f(\cdot))$ is in $W^{1,p}(\mathbb{R}, \mathbb{R})$ with weak derivative $(y|f(\cdot))' = (y|f'(\cdot))$. So unlike the previous two examples there actually exists a candidate for the weak derivative of f as demanded in (*iii*). But of course, $f \notin W^{1,p}(\mathbb{R}, l^2(A))$ as $\|f\| = 1_A$ is not even measurable.

With the Fundamental Theorem given for weakly differentiable functions, we can prove that the density of C^∞-functions carries over from $L^p(I, X)$ to $W^{1,p}(I, X)$.

Corollary 3.12. *If I is a bounded interval, then $C^\infty\left(\bar{I}, X\right)$ is dense in $W^{1,p}(I, X)$.*

Proof. Let $u \in W^{1,p}(I, X)$, then $u' \in L^p(I, X)$ and thus by Proposition 2.15 there exists a sequence $(\varphi_n) \subset C_c^\infty(I, X)$ such that $\varphi_n \to u'$ in $L^p(I, X)$. By Theorem 3.7 there exists a $t_0 \in I$ such that $u(t) = u(t_0) + \int_{t_0}^t u'(s)\, ds$ a.e. Let $u_n(t) := u(t_0) + \int_{t_0}^t \varphi_n(s)\, ds$ then obviously $(u_n) \subset C^\infty(\bar{I}, X)$ and $u_n' = \varphi_n$, thus it only remains to show that $u_n \to u$ in $L^p(I, X)$. By Hölder's inequality we have that

$$\begin{aligned} \|u_n(t) - u(t)\| &= \left\| \int_{t_0}^t \varphi_n(s) - u'(s)\, ds \right\| \\ &\leq |t - t_0|^{\frac{1}{q}} \left(\int_{t_0}^t \|\varphi_n(s) - u'(s)\|^p\, ds \right)^{\frac{1}{p}} \end{aligned}$$

with $\frac{1}{p} + \frac{1}{q} = 1$. Thus we obtain

$$\begin{aligned}\|u_n - u\|_{L^p(I,X)}^p &\leq \int_I |t-t_0|^{\frac{p}{q}} \int_{t_0}^t \|\varphi_n(s) - u'(s)\|^p \, ds \, dt \\ &\leq \int_I \lambda(I)^{\frac{p}{q}} \int_I \|\varphi_n(s) - u'(s)\|^p \, ds \, dt \\ &= \lambda(I)^{\frac{p}{q}+1} \|\varphi_n - u'\|_{L^p(I,X)}^p\end{aligned}$$

which converges to 0 as $n \to \infty$. □

In the next chapter, we will examine this corollary in a more general way.

3.3 Criteria for Weak Differentiability

Given a function $u \in L^p(I, X)$ it is of interest to give criteria that tell us whether u is weakly differentiable. There are several such criteria for scalar-valued functions, but these criteria do not carry over to the vector-valued case in general. We need to make geometric assumptions on the vector space X such as reflexivity or the Radon-Nikodym property.

Theorem 3.13. *Let X be a reflexive Banach space and $u \in L^p(I, X)$ for some $1 \leq p \leq \infty$. Then $u \in W^{1,p}(I, X)$ if and only if there exists a function $g \in L^p(I, \mathbb{R})$ such that*

$$\|u(t) - u(s)\| \leq \left| \int_s^t g(x) \, dx \right|$$

for almost all $s, t \in I$, i.e. for all s, t outside a common null set.

Proof. For the necessity note that $g = \|u'\|$ satisfies the criterion by Theorem 3.7 or more specifically the note after this theorem. Note that we do not need reflexivity for this implication. Now let the estimate be true. First note that the estimate implies that u is continuous outside a null set N, i.e. if $t_n \to t$ in $I \backslash N$, then $u(t_n) \to u(t)$. We may consider another representative of u by changing the values in N. For each $t \in N$ let $(t_n) \subset I \backslash N$ such that $t_n \to t$. By the estimate $u(t_n)$ is a Cauchy sequence and hence convergent in X. We let $u(t) := \lim_{n\to\infty} u(t_n)$. This choice is unique as for any other sequence $(s_n) \subset I \backslash N$ with $s_n \to t$ the alternating sequence of $(u(t_n))$ and $(u(s_n))$ is a Cauchy sequence. Using the Dominated Convergence Theorem it is now straightforward to show that the chosen representative of u is continuous and the estimate holds for all $t, s \in I$. As u is continuous, it follows that $u(I)$ is separable. Any closed subspace of X is reflexive and thus we may w.l.o.g. assume that X is separable. We show that

$$u_h(t) := \frac{u(t+h) - u(t)}{h}$$

is bounded in $L^p(J, X)$ for any $J \subset\subset I$ – that is $\bar{J} \subset I$ is compact – such that $|h| < \operatorname{dist}(J, \partial I)$. If $p = \infty$ then it is clear that $\|u_h\|_{L^\infty(J,X)} \leq \|g\|_{L^\infty(J,\mathbb{R})} \leq \|g\|_{L^\infty(I,\mathbb{R})}$. For

$p < \infty$ we assume that $h > 0$. The case $h < 0$ is handled analogously. Hölder's inequality implies that for almost all $t \in J$ we have

$$\|u_h(t)\|^p = \frac{1}{|h|^p}\|u(t+h) - u(t)\|^p \le \frac{1}{|h|^p}\left(\int_t^{t+h} |g(s)|\,ds\right)^p \le |h|^{\frac{p}{q}-p}\int_t^{t+h} |g(s)|^p\,ds.$$

Thus using Fubini's Theorem we can estimate the L^p-norm to be

$$\begin{aligned}\|u_h\|^p_{L^p(J,X)} &= \int_J \|u_h(t)\|^p\,dt \le |h|^{\frac{p}{q}-p}\int_J\int_t^{t+h} |g(s)|^p\,ds\,dt \qquad (3.1)\\ &= |h|^{\frac{p}{q}-p}\int_I |g(s)|^p \int_J 1_{(s-h,s)}\,dt\,ds \le |h|^{\frac{p}{q}-p+1}\int_I |g(s)|^p\,ds = \|g\|^p_{L^p(I,\mathbb{R})}\end{aligned}$$

The separability of $X = X''$ implies that X' is separable as well. Let $(x'_n) \subset X'$ be dense, define

$$\psi_n(t) := \langle x'_n, u(t)\rangle$$

and compute that

$$|\psi_n(t) - \psi_n(s)| \le \|x'_n\| \left|\int_s^t g(x)\,dx\right|$$

for all $t, s \in I$. As $g \in L^1_{\text{loc}}(I,\mathbb{R})$ this implies that ψ_n is locally absolutely continuous and thus differentiable a.e. by Lebesgue's Differentiation Theorem. For the same reason there exists a null set F such that for all $t \in I\backslash F$ we have

$$g(t) = \lim_{h\to 0}\frac{1}{|h|}\int_t^{t+h} g(s)\,ds$$

Let E_n be a null set such that ψ_n is differentiable on $I\backslash E_n$ and define $E := \bigcup_{n=1}^\infty E_n \cup F$. Then for all $t \in I\backslash E$ we have that

$$\|u_h(t)\| \le \frac{1}{|h|}\left|\int_t^{t+h} g(s)\,ds\right| \to |g(t)|$$

and thus $\|u_h(t)\|$ is bounded by some constant K_t if $|h|$ is small enough. By the reflexivity of X there exists a sequence $h_n \to 0$ and some element $\omega(t) \in X$ such that $u_{h_n}(t) \rightharpoonup \omega(t)$. In particular we have that

$$\langle x'_m, \omega(t)\rangle = \lim_{n\to\infty}\langle x'_m, u_{h_n}(t)\rangle = \psi'_m(t),$$

thus for any sequence $h_k \to 0$ we have

$$\lim_{k\to\infty}\langle x'_m, u_{h_k}(t)\rangle = \psi'_m(t) = \langle x'_m, \omega(t)\rangle.$$

Now let $x' \in X'$ be arbitrary, $\varepsilon > 0$ and choose an x'_m such that $\|x' - x'_m\| \leq \varepsilon$. For $|h|$ small enough we now obtain

$$\begin{aligned} |\langle x', u_h(t) - \omega(t) \rangle| &\leq |\langle x' - x'_m, u_h(t) - \omega(t) \rangle| + |\langle x'_m, u_h(t) - \omega(t) \rangle| \\ &\leq \varepsilon(K_t + \|\omega(t)\|) + \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ yields that $u_h(t) \rightharpoonup \omega(t)$. By Proposition 2.17 we have that $\omega \in L^p(I, X)$ and thus Proposition 3.8 implies that $u \in W^{1,p}(I, X)$. □

An important application of Theorem 3.13 is the weak differentiability of the composition of a weakly differentiable function and a Lipschitz continuous function. A typical example would be the norm $\| \cdot \| : X \to \mathbb{R}$ which is Lipschitz continuous by the reverse triangle inequality.

Corollary 3.14. *Let X and Y be Banach spaces and let $1 \leq p \leq \infty$ such that $u \in W^{1,p}(I, X)$. If $F : X \to Y$ is Lipschitz continuous and Y is reflexive, then $F \circ u \in W^{1,p}(I, Y)$. In particular, if $u \in W^{1,p}(I, X)$, then $\|u\| \in W^{1,p}(I, \mathbb{R})$.*

Proof. If $u \in W^{1,p}(I, X)$, then the proof of Theorem 3.13 shows that there exists a function $g \in L^p(I, \mathbb{R})$ such that the condition of this theorem is satisfied even without X being reflexive. Let L be the Lipschitz constant of F, then $F \circ u$ satisfies the condition with the function $L \cdot g$. As Y is reflexive, we obtain that $F \circ u \in W^{1,p}(I, Y)$ by Proposition 3.13. The last claim follows from the reflexivity of $\mathbb{R}$ and the fact that the norm is Lipschitz continuous. □

As an application of this, we want to prove an easy example of a Sobolev Embedding Theorem.

Theorem 3.15 (Sobolev Embedding Theorem)**.** *Let X be a Banach space and $1 \leq p \leq \infty$, then there exists a contant C such that*

$$\|u\|_{L^\infty(I,X)} \leq C \|u\|_{W^{1,p}(I,X)}$$

for all $u \in W^{1,p}(I, X)$, i.e. $W^{1,p}(I, X) \hookrightarrow L^\infty(I, X)$ and the embedding is continuous. Further $W^{1,p}(I, X) \subset C_b(I, X)$.

Proof. We first show, that every function $u \in W^{1,p}(I, X)$ is indeed an L^∞-function. Corollary 3.14 implies that $\|u\| \in W^{1,p}(I, \mathbb{R})$. By the scalar-valued Sobolev Embedding Theorem (see [Bré10, Theorem 8.8]) we have that $\|u\|$ is bounded a.e. and thus u is bounded a.e.
Now we show that the graph of the injection $W^{1,p}(I, X) \hookrightarrow L^\infty(I, X)$ is closed. Let $u, u_n \in W^{1,p}(I, X)$ and $v \in L^\infty(I, X)$ such that $u_n \to u$ in $W^{1,p}(I, X)$ and $u_n \to v$ in $L^\infty(I, X)$. By Proposition 2.13 there exists a subsequence (u_{n_k}) such that $u_{n_k} \to u$ pointwise a.e. Using the argument again there eyists a subsequence $(u_{n_{k_l}})$ such that $u_{n_{k_l}} \to v$ pointwise a.e. Hence $u = v$ a.e. and thus the graph is closed. The existence of the constant C now follows from the Closed Graph Theorem. The last claim follows from

the above together with the fact that any $u \in W^{1,p}(I,X)$ has a continuous representative as seen in Theorem 3.7. $\square$

Although Theorem 3.13 yielded some important results, it is usually hard to find the function g in the prerequisites. We now want to give a criterion that is easier to compute. The drawback is that we will loose the case $p = 1$. A huge advantage will be that X is no longer required to be reflexive. Instead we will see, that the natural environment for the criterion is a space that has the Radon-Nikodym property.

Recall that if $u \in W^{1,p}(I,X)$ the requirement of Theorem 3.13 is satisfied with the function $g = u'$. Then (3.1) implies that there exists a C such that for any $J \subset\subset I$ and any $h \in \mathbb{R}$ with $|h| < \operatorname{dist}(J, \partial I)$ we have

$$\|\tau_h u - u\|_{L^p(J,X)} \leq C|h|, \tag{3.2}$$

where $\tau_h u := u(\cdot + h)$. Moreover we know that C can be chosen to be $\|u'\|_{L^p(I,X)}$. For the real valued case $X = \mathbb{R}$, it is known that the converse of this is true as well: If a function $u \in L^p(I,\mathbb{R})$ satisfies the above, then $u \in W^{1,p}(I,\mathbb{R})$, see [Bré10, Theorem 9.3]. We now want to identify the Banach spaces X for which this criterion holds.

Example 3.16 (Criterion (3.2) fails for c_0)**.** Let $I = [0,1]$, $X = c_0$ and let $f : I \to c_0$ be given by $f(x) = (f_n(x)) = \left(\frac{\exp(inx)}{n}\right)$. f is continuous and hence in $L^p(I,X)$. Let J and h be as in (3.2). We first assume that $h > 0$ and $p < \infty$. Then for all $n \in \mathbb{N}$ we have

$$|f_n(x+h) - f_n(x)| = \left| \int_x^{x+h} i\cos(nt) - \sin(nt)\, dt \right| \leq \int_x^{x+h} 1\, dt = |h|$$

and thus $\|f(x+h) - f(x)\|_{c_0} \leq |h|$. Integrating over J we can estimate the L^p-norm by

$$\|\tau_h f - f\|_{L^p(J,c_0)} \leq |h|.$$

The cases $h < 0$ and $p = \infty$ can be shown analogously. We conclude that f satisfies (3.2). But analogously to Example 3.9 f is not in $W^{1,p}(I,X)$.

Example 3.17 (Criterion (3.2) fails for $L^1([0,1],\mathbb{R})$)**.** Let $I = (0,1)$ and $f : I \to L^1([0,1],\mathbb{R})$ given by $x \mapsto 1_{[0,x]}$. f is continuous and hence in $L^p(I,X)$. Let J and h as above. Again we first assume that $h > 0$ to compute

$$\|f(x+h) - f(x)\|_{L^1([0,1])} = \|1_{(x,x+h]}\|_{L^1([0,1])} = |h|$$

and thus the L^p-norm can be estimated by

$$\|\tau_h f - f\|_{L^p(J,L^1([0,1]))} \leq |h|,$$

which means that f satisfies (3.2). But in Example 3.10 we have seen that $f \notin W^{1,p}(I, L^1([0,1],\mathbb{R}))$.

The two previous examples are actually special cases of the following example. Note that the computations above show that the utilized functions are Lipschitz continuous but not differentiable a.e. This means that the spaces c_0 and $L^1([0,1],\mathbb{R})$ do not have the Radon-Nikodym property.

Example 3.18 (Criterion (3.2) fails for any space that does not have the Radon-Nikodym property)**.** Let I be an interval and X be a Banach space that does not have the Radon-Nikodym property, then there exists a Lipschitz continuous function $f : I \to X$ with Lipschitz constant L that is not differentiable a.e. Let $N \subset I$ be bunded with $\lambda(N) > 0$ such that f is not differentiable in $t \in N$ and let $\varphi \in C_c^\infty(I)$ with $\operatorname{supp}\varphi = K \supset N$. Note that due to the mean value theorem, φ is Lipschitz continuous with constant $\max_{x\in I} |\varphi'(x)| = \max_{x\in K} |\varphi'(x)|$. The function φf has compact support and is not a.e. differentiable. We show that φf is Lipschitz continuous as well. If $x, y \in K$ then by a simple utilisation of the triangular inequality we compute

$$\|\varphi(x)f(x) - \varphi(y)f(y)\| \leq \max_{x_1,x_2\in K} \|f(x_1)\|\|\varphi'(x_2)\||x-y| + L\max_{\xi\in K}|\varphi(\xi)||x-y|,$$

where the maxima exist due to the compactness of K. If $y \notin K$ we have that $\varphi(y) = 0$ and thus

$$\begin{aligned}\|\varphi(x)f(x) - \varphi(y)f(y)\| &= \|\varphi(x)f(x)\| = \|(\varphi(x) - \varphi(y))f(x)\| \\ &\leq \max_{x_1,x_2\in K} \|f(x_1)\|\|\varphi'(x_2)\||x-y|,\end{aligned}$$

and thus we have the desired Lipschitz estimate. The above computation shows that we can w.l.o.g. assume that f is compactly supported. We let J and h be as above and note that the function $x \mapsto \|f(x+h) - f(x)\|$ takes positive values only on a set of measure at most $2\lambda(\operatorname{supp} f)$. With this in mind we can estimate

$$\int_J \|f(x+h) - f(x)\|^p \, dx \leq 2\lambda(\operatorname{supp}\ f)L^p|h|^p,$$

for $p < \infty$. The case $p = \infty$ can be shown analogously. Hence f satisfies (3.2) but is not weakly differentiable due to Proposition 3.8.

We will now show that the converse is true as well: A space that does have the Radon Nikodym property yields the criterion. We will need the following Lemma.

Lemma 3.19. *Let X be a Banach space and I be an open interval. Let (J_n) be a monotonically increasing sequence of intervals such that $J_n \subset\subset I$ and $\bigcup_n J_n = I$. Assume that $g_n \in L^p(J_n, X)$ with $\|g_n\|_{L^p(J_n,X)} \leq C$ for every $n \in \mathbb{N}$ and that $g_n(x) = g_m(x)$ for almost all $x \in J_n \cap J_m$. Then there exists a function $g \in L^p(I,X)$ with $\|g\|_{L^p(I,X)} \leq C$ such that $g_{|J_n} = g_n$.*

Proof. Let $g(x) := g_n(x)$ for some n such that $x \in J_n$. By assumption the choice of n does not matter in the L^p-sense. Clearly we have $g_{|J_n} = g_n$. Let $\hat{g}_n := g \cdot 1_{J_n} \in L^p(I,X)$, then $\|\hat{g}_n\|_{L^p(I,X)} \leq C$ and $\hat{g}_n \to g$ a.e. By Beppo Levi's Theorem it follows that

$$\int_I \|g\|^p \, d\lambda \leq C^p,$$

thus g satisfies the claimed attributes. $\square$

Theorem 3.20. *Let I be an interval and X be a Banach space that has the Radon-Nikodym property. For $1 < p \leq \infty$, a function $u \in L^p(I,X)$ is in $W^{1,p}(I,X)$ if and only if there exists a constant $C > 0$ such that for all J and h as in (3.2) we have*

$$\|\tau_h u - u\|_{L^p(J,X)} \leq C|h|$$

Proof. First let $1 < p < \infty$ and let $J_n := \left\{x \in I, |x| < n, \operatorname{dist}(x, \partial I) > \frac{3}{n}\right\}$. We define functions

$$f_n : \left(-\frac{1}{n}, \frac{1}{n}\right) \to L^p(J_n, X)$$
$$t \mapsto \tau_t u$$

For all $s, t \in \left(-\frac{1}{n}, \frac{1}{n}\right)$ it holds that $|s-t| < \operatorname{dist}(J_n + s, \partial I)$, thus by assumption we have that

$$\begin{aligned}
\|f_n(t) - f_n(s)\|^p_{L^p(J_n,X)} &= \int_{J_n} \|u(x+t) - u(x+s)\|^p \, dx \\
&= \int_{J_n+s} \|u(x+t-s) - u(x)\|^p \, dx \\
&= \|\tau_{t-s} u - u\|^p_{L^p(J_n+s,X)} \leq C^p |t-s|^p,
\end{aligned}$$

i.e. f_n is Lipschitz continuous. By Theorem 2.24 the space $L^p(J_n, X)$ has the Radon-Nikodym property and thus there exists a function $f_n' : \left(-\frac{1}{n}, \frac{1}{n}\right) \to L^p(J_n, X)$ such that $\frac{f_n(t+h) - f_n(t)}{h} \to f_n'(t)$ $(h \to 0)$ for almost all $t \in \left(-\frac{1}{n}, \frac{1}{n}\right)$. As the difference quotient is bounded by C in $L^p(J_n, X)$ we obtain that $\|f_n'(t)\|_{L^p(J_n,X)} \leq C$ for all $n \in \mathbb{N}$ and almost all $t \in \left(-\frac{1}{n}, \frac{1}{n}\right)$. We fix such a $t = t_n$ and define $g_n(x) := f_n'(t_n)(x - t_n)$. We have that $g_n \in L^p(J_n + t_n, X)$ with $\|g_n\|_{L^p(J_n+t_n,X)} \leq C$ and by the choice of J_n and t_n it is obvious that $\bigcup_{n \in \mathbb{N}} J_n + t_n = I$ and that the sequence $J_n + t_n$ is increasing. Now let $n, m \in \mathbb{N}$. There exists a sequence $h_k \to 0$ such that for almost all $x \in (J_n + t_n) \cap (J_m + t_m)$

$$\begin{aligned}
g_n(x) &= f_n'(t_n)(x - t_n) \\
&= \lim_{k \to \infty} \frac{f_n(t_n + h_k)(x - t_n) - f_n(t_n)(x - t_n)}{h_k} \\
&= \lim_{k \to \infty} \frac{u(x + h_k) - u(x)}{h_k} \\
&= \lim_{k \to \infty} \frac{f_m(t_m + h_k)(x - t_m) - f_m(t_m)(x - t_m)}{h_k} = g_m(x),
\end{aligned}$$

hence the functions g_n satisfy the conditions of Lemma 3.19. We conclude that there exists a function $g \in L^p(I, X)$ with $\|g\|_{L^p(I,X)} \leq C$ which extends g_n. Let $\varphi \in C_c^\infty(I, \mathbb{R})$, then there exists an $n \in \mathbb{N}$ such that $\operatorname{supp} \varphi \subset J_n + t_n$. Choose a sequence $h_k \to 0$ as above, then

$$g(x) = \lim_{k\to\infty} \frac{u(x + h_k) - u(x)}{h_k}$$

pointwise a.e. on $J_n + t_n$ as well as in $L^p(J_n + t_n, X)$. At the same time $\varphi'(x) = \lim_{k\to\infty} \frac{\varphi(x+h_k)-\varphi(x)}{h_k}$ uniformly. Using first Hölder's inequality and then the Dominated Convergence Theorem we compute

$$\begin{aligned}
\int_I g(x)\varphi(x)\,dx &= \lim_{k\to\infty} \int_{J_n+t_n} \frac{u(x + h_k) - u(x)}{h_k} \varphi(x)\,dx \\
&= \lim_{k\to\infty} \frac{1}{h_k} \left(\int_{J_n+t_n} u(x + h_k)\varphi(x)\,dx - \int_{J_n+t_n} u(x)\varphi(x)\,dx \right) \\
&= \lim_{k\to\infty} \frac{1}{h_k} \left(\int_{J_n+t_n} u(x)\varphi(x - h_k)\,dx - \int_{J_n+t_n} u(x)\varphi(x)\,dx \right) \\
&= \lim_{k\to\infty} - \int_I u(x) \frac{\varphi(x - h_k) - \varphi(x)}{-h_k}\,dx \\
&= - \int_I u(x)\varphi'(x)\,dx,
\end{aligned}$$

hence $u \in W^{1,p}(I, X)$ with $u' = g$.

Now let $p = \infty$ and let J_n be as above. As J_n is bounded, the functions $u_n := u_{|J_n}$ are in $L^q(J_n, X)$ for any $1 \leq q < \infty$. For any $J \subset\subset J_n$ and any h as in (3.2) we have

$$\|\tau_h u_n - u_n\|^q_{L^q(J,X)} = \int_J \|u(x + h) - u(x)\|^q\,dx \leq C^q |h|^q \lambda(J_n),$$

hence by the first part of the proof we obtain $u_n \in W^{1,q}(J_n, X)$ with $\|u_n'\|_{L^q(J_n,X)} \leq C\lambda(J_n)^{\frac{1}{q}}$. Letting $q \to \infty$ we obtain $u_n' \in L^\infty(J_n, X)$ with $\|u_n'\|_{L^\infty(J_n,X)} \leq C$. Let u' be defined via $u'_{|J_n} = u_n'$ for any $n \in \mathbb{N}$. By the uniqueness of the weak derivative this is well defined. We have $u' \in L^\infty(I, X)$ with $\|u'\|_{L^\infty(I,X)} \leq C$. Let $\varphi \in C_c^\infty(I, \mathbb{R})$, then $\operatorname{supp} \varphi \subset J_n$ and $u_n \in W^{1,q}(J_n, X)$ for some q imply that

$$\int_I \varphi u' = \int_{J_n} \varphi u_n' = - \int_{J_n} \varphi' u_n = - \int_I \varphi u,$$

hence $u \in W^{1,\infty}(I, X)$ with weak derivative u'. □

From this theorem and Example 3.18 we can deduct a full characterization of the Radon-Nikodym property.

Corollary 3.21. *A Banach space X has the Radon-Nikodym property if and only if criterion (3.2) characterizes the space $W^{1,p}(I, X)$ for one, equivalently all, $1 < p \leq \infty$.*

We want to point out that the case $p = 1$ is not true for any Banach space X as the following counterexample shows.

Example 3.22. Let X be an arbitrary Banach space and define $f : I := \left(-\frac{1}{2}, 1\frac{1}{2}\right) \to X$ via $f = 1_{[0,1]}x$ for some vector $x \in X$. Then for any J and h as in criterion (3.2) we have

$$\int_J \|f(t+h) - f(t)\|\, dt = \int_{([-h,0)\cup(1-h,1])\cap J} \|x\|\, dt = 2h\|x\|$$

and similarly for negative h. Thus f satisfies criterion (3.2) with $C = 2\|x\|$. But f is not continuous and thus not in $W^{1,1}(\mathbb{R}, X)$ by Proposition 3.8.

We can also extend a previous result.

Corollary 3.23. *Let $1 < p \leq \infty$ and let X and Y be Banach spaces. Then Y has the Radon-Nikodym property if and only if for every $u \in W^{1,p}(I, X)$ and every Lipschitz continuous function $F : X \to Y$ it follows that $F \circ u \in W^{1,p}(I, Y)$ for every open interval I.*

Proof. If $u \in W^{1,p}(I, X)$ then u satisfies (3.2). If L is the Lipschitz constant of F then $F \circ u$ satisfies the prerequisites of Theorem 3.20 with the constant $L \cdot C$. Thus $F \circ u \in W^{1,p}(I, Y)$ if Y has the Radon-Nikodym property.

Conversely assume that $F \circ u \in W^{1,p}(I, Y)$ for every $u \in W^{1,p}(I, X)$ and every Lipschitz continuous function $F : X \to Y$. Let $f : \mathbb{R} \to Y$ be Lipschitz continuous. Choose $x_0 \in X$ and $x_0' \in X'$ such that $\langle x_0', x_0 \rangle = 1$ and define

$$\begin{aligned} F :& X \to Y \\ & x \mapsto f(\langle x_0', x \rangle). \end{aligned}$$

Let L be the Lipschitz constant of f, then F is Lipschitz continuous with Lipschitz constant $L \cdot \|x_0'\|$. Let I be a bounded open interval, then the function defined via $u(t) := tx_0$ is in $W^{1,p}(I, X)$. The assumption yields that $f_{|I} = F \circ u \in W^{1,p}(I, Y)$ and thus f is differentiable a.e. on I. As I was chosen arbitrary, the assertion follows. $\square$

3.4 Notes

The first section about vector-valued distributions is oriented on [CH98]. The spaces $W^{1,p}(I, X)$ can be found in many books about non-linear analysis, usually taking $I = [0, T]$ for some $T > 0$. We used the books [CH98], [Bré73] and [GP06]. Especially the extension of the Fundamental Theorem of Calculus 3.7 respectively 3.8 from real-valued to vector-valued functions can be found in all of these books.

With the Examples 3.9 - 3.11 we answer an interesting question: Given a vector-valued function f, does a property of $\langle x', f\rangle$ for all $x' \in X'$ imply the same property for f itself? One example is Pettis' measurability Theorem 2.1: If X is separable, then the property 'measurable' carries over from $\langle x', f\rangle$ to f. Another classical example was found by Grothendieck [Gro53]: If $\langle x', f\rangle$ is holomorphic, then f is holomorphic. Here we wanted to ask the same question for weak differentiability: Given $\langle x', f\rangle \in W^{1,p}(I, \mathbb{R})$ for any $x' \in X'$, does this imply that $f \in W^{1,p}(I, X)$? The examples above tell us that this is false in general. The next question would be whether a geometric property of the space X would yield the implication, but the counterexamples give a broad variety of spaces whose weakly differentiable functions cannot be characterized in this way. While the first example is given on c_0, a space that does not even have the Radon-Nikodym property, the second example uses the spaces $L^p([0, 1])$ which are reflexive if $1 < p < \infty$. For $p = 2$ we even have a counterexample on a separable Hilbert space and the last example is on an inseparable Hilbert space. It would be interesting to know whether there exist infinite-dimensional spaces for which the weak differentiability of $\langle x', f\rangle$ implies that $f \in W^{1,p}(I, X)$. It is not known to the author that this question was ever treated by other authors.

There exists a variety of criteria for weak differentiability in the scalar-valued case and some have been extended to vector-valued functions. The first criterion we gave is taken from [CH98]. While this criterion yielded some nice corollaries, we wanted to give a criterion on more general spaces. The criterion (3.2) has already been extended to vector-valued functions to some degree. A version of this theorem can be found in [GP06], but just as Theorem 3.13 this theorem is restricted to reflexive spaces. We could not find any source for this theorem in the case of a vector space that has the Radon-Nikodym property, thus we assume that the proof of Theorem 3.20 is the first one ever given for this fact. We found it surprising that the criterion is even equivalent to the Radon-Nikodym property.

4 Sobolev Spaces in Higher Dimensions

We now come to weak differentiability of functions whose domain is a subset of $\mathbb{R}^d$. These functions are not as regular as in the case $d = 1$, but we will prove structure and embedding theorems that tell us how these functions behave. Instead of just looking at the space $W^{1,p}(\Omega, X)$, we will also introduce derivatives of higher order.

4.1 The Spaces $W^{m,p}(\Omega, X)$

We first recall the notation for partial differential operators. A vector $\alpha = (\alpha_k)_{k=1}^d \in \mathbb{N}^d$ is called a *multi-index.* Its *length* is defined as $|\alpha| := \sum_{k=1}^d \alpha_k$. For another vector $z \in \mathbb{R}^d$ we define $z^\alpha := z_1^{\alpha_1} \cdot \ldots \cdot z_d^{\alpha_d}$. Multi-indexes can be partially ordered via $\alpha \leq \beta \;\Leftrightarrow\; \alpha_k \leq \beta_k \;\forall k$. Let $D_k := \frac{\partial}{\partial x_k}$, then for a multi-index α we have

$$D^\alpha = D_1^{\alpha_1} \cdot \ldots \cdot D_d^{\alpha_d} = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdot \ldots \cdot \partial z_d^{\alpha_d}}.$$

Let $\Omega \subset \mathbb{R}^d$ be open. As in the one-dimensional case we define the space of *vector-valued distributions* as

$$\mathcal{D}'(\Omega, X) := \mathcal{L}(C_c^\infty(\Omega, \mathbb{R}), X),$$

where the space $C_c^\infty(\Omega, \mathbb{R})$ is topologized in the following way: a sequence (φ_n) converges to φ if and only if all φ_n and φ are supported in the same compact set and on this set $\varphi_n^{(k)} \to \varphi^{(k)}$ uniformly for any $k \in \mathbb{N}_0$. For a precise discussion we again refer to [RS81].

For a function $f \in L^1_{\mathrm{loc}}(\Omega, X)$ we define the distribution T_f via

$$T_f \varphi := \int_\Omega f \varphi \, d\lambda \quad (\varphi \in C_c^\infty(\Omega, \mathbb{R})).$$

For any distribution $T \in \mathcal{D}'(\Omega, X)$ and any multi-index α we define the *distributional derivative* $D^\alpha T \in \mathcal{D}'(\Omega, X)$ via

$$D^\alpha T \varphi := (-1)^{|\alpha|} T(D^\alpha \varphi) \quad (\varphi \in C_c^\infty(\Omega, \mathbb{R}))$$

Suppose $u, v \in L^1_{\mathrm{loc}}(\Omega, X)$ and that $D^\alpha T_u = T_v$ then we will use the notation $v = D^\alpha u$ and say that v is the *weak derivative* of u of *order* α. This is equivalent to

$$\int_\Omega v \varphi \, d\lambda = (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi \, d\lambda$$

The following proposition implies that the weak derivative is unique in $L^p(\Omega, X)$.

Proposition 4.1. *Let $f \in L^1_{loc}(\Omega, X)$ such that $T_f = 0$, then $f = 0$ a.e.*

Proof. Let $x \in \Omega$, then there exists an $R > 0$ such that $B(x,r) \subset \Omega$ for all $0 < r < R$. Choose a sequence $\varphi_n \in C_c^\infty(\Omega, \mathbb{R})$ such that $\operatorname{supp} \varphi_n \subset K$ for some fixed compact set $K \subset \Omega$ and such that $\varphi_n \to 1_{B(x,r)}$ a.e. Using the Dominated Convergence Theorem, we compute

$$\int_{B(x,r)} f(s)\, ds = \lim_{n\to\infty} \int_\Omega f(s)\varphi_n(s)\, ds = 0.$$

As x was chosen arbitrary, Lebesgue's Differentiation Theorem implies that $f(x) = 0$ a.e. $\square$

Let $1 \leq p \leq \infty$ then the *Sobolev space* $W^{m,p}(\Omega, X)$ is defined as

$$W^{m,p}(\Omega, X) := \{u \in L^p(\Omega, X),\ D^\alpha u \in L^p(\Omega, X)\ \forall \alpha \in \mathbb{N}^d, |\alpha| \leq m\}.$$

We equip $W^{m,p}(\Omega, X)$ with the norm

$$\|u\|_{W^{m,p}(\Omega,X)} := \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^p(\Omega,X)}$$

Analogously to the one-dimensional case one proves

Theorem 4.2. *$W^{m,p}(\Omega, X)$ equipped with the $W^{m,p}(\Omega, X)$-norm is a Banach space. If X is a Hilbert space, then $H^m(\Omega, X) := W^{m,2}(\Omega, X)$, endowed with an equivalent norm, is a Hilbert space. If X is reflexive and $1 < p < \infty$, then $W^{m,p}(\Omega, X)$ is reflexive.*

Again the term 'weak derivative' is justified as this notion extends the usual derivative. We have

Proposition 4.3. *Let $f \in C^m(\Omega, X)$ such that $D^\alpha f \in L^p(\Omega, X)$ for any multi-index α with $|\alpha| \leq m$, then $f \in W^{m,p}(\Omega, X)$ and the derivatives coincide with the weak derivatives.*

Proof. Let $x' \in X'$, then $\langle x', f\rangle \in C^m(\Omega, \mathbb{R})$ with $D^\alpha \langle x', f\rangle = \langle x', D^\alpha f\rangle$. For a function $\varphi \in C_c^\infty(\Omega, \mathbb{R})$ we apply integration by parts and gain

$$\int_\Omega \langle x', D^\alpha f\rangle \varphi\, d\lambda = (-1)^{|\alpha|} \int_\Omega \langle x', f\rangle D^\alpha \varphi\, d\lambda$$

as φ is compactly supported. We have that both $(D^\alpha f)\varphi$ and $f(D^\alpha \varphi)$ are in $L^1(\Omega, X)$ and thus the above equation and the Hahn-Banach Theorem yield

$$\int_\Omega D^\alpha f \varphi\, d\lambda = (-1)^{|\alpha|} \int_\Omega f D^\alpha \varphi\, d\lambda,$$

from which the assertions follow. $\square$

4.2 Mollification and the Meyers-Serrin Theorem

We now extend the notion of convolution and mollification to vector-valued functions.

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that

(i) $\varphi \geq 0$

(ii) supp $\varphi \subset B(0,1)$

(iii) $\int_{\mathbb{R}^d} \varphi = 1$

We say that φ is a *mollifier*. For $r > 0$ let $\varphi_r := \frac{1}{r^d}\varphi(\frac{\cdot}{r})$, then φ_r has the same properties as φ apart from being supported on the ball $B\,(0,r)$. Note that any function $u \in L^1_{\text{loc}}(\Omega, X)$ can be extended to $L^1_{\text{loc}}(\mathbb{R}^d, X)$ by setting it zero outside of Ω. The same is true for L^p- and C_c^∞-functions and we will implicitely use this throughout in this context. For $u \in L^1_{\text{loc}}(\Omega, X)$ we define the *convolution* with φ_r as in the scalar-valued case via

$$\begin{aligned} u * \varphi_r(z) &:= \int_{\mathbb{R}^d} u(z-y)\varphi_r(y)\,dy \\ &= \int_{\mathbb{R}^d} u(y)\varphi_r(z-y)\,dy = \varphi_r(z) * u. \end{aligned}$$

Note that in the second integral representation of $u * \varphi_r$ the variable z is only plugged into φ_r and not into u. It follows that $u * \varphi_r \in C^\infty(\mathbb{R}^d, X)$. As in the scalar case we define the *support* of an L^p-fucntion via

$$\text{supp } f := \Omega \backslash \bigcup_{\substack{U \subset \mathbb{R}^d \text{ open} \\ f_{|U} = 0}} U.$$

For a continuous function, this and the ususal definition of the support coincide. The following lemma describes the support of $u * \varphi_r \in C^\infty$.

Lemma 4.4. *supp $u * \varphi_r \subset$ supp $u +$ supp $\varphi_r =$ supp $u + B\,(0,r)$*

Proof. We have that supp u is closed and supp φ_r is compact, thus supp $u +$ supp φ_r is closed. Let $x \in (\text{supp } u + \text{supp } \varphi_r)^C$, then we have that $(x - \text{supp } \varphi_r) \cap \text{supp } u = \emptyset$. Indeed assume that $y \in (x - \text{supp } \varphi_r) \cap \text{supp } u$ then there exists a $z \in \text{supp } \varphi_r$ such that $y = x - z$. From this we infer that $x \in \text{supp } u + \text{supp } \varphi_r$ contradictory to our assumption. With this we obtain

$$u * \varphi_r(x) = \int_{\mathbb{R}^d} u(y)\varphi_r(x-y)\,dy = \int_{(x - \text{supp } \varphi_r) \cap \text{supp } u} u(y)\varphi_r(x-y)\,dy = 0.$$

As supp $u +$ supp φ_r is closed, we have that

$$(\text{supp } u + \text{supp } \varphi_r)^C \subset \bigcup_{\substack{U \subset \mathbb{R}^d \text{ open} \\ u * \varphi_{r|U} = 0}} U = (\text{supp } u * \varphi_r)^C,$$

from which the assertion follows. ☐

Proposition 4.5. *Let $1 \leq p \leq \infty$, then the mapping defined via $u \mapsto u * \varphi_r$ is a linear contraction $L^p(\mathbb{R}^d, X) \to L^p(\mathbb{R}^d, X)$.*

Proof. The linearity is evident. First let $p < \infty$. For $u \in L^p(\mathbb{R}^d, X)$ and $x \in \mathbb{R}^d$ we have

$$\|u * \varphi_r(x)\| \leq \int_{\mathbb{R}^d} \|\varphi_r(x-y)u(y)\| \, dy = \frac{1}{r^d} \int_{\mathbb{R}^d} \left\| \varphi_r \left(\frac{x-y}{r} \right) u(y) \right\| \, dy$$

by the fundamental estimate. Now let $p \neq 1$. Using the transformation formula for the Lebesgue integral and Hölder's inequality we obtain

$$\begin{aligned} \|u * \varphi_r(x)\| &\leq \int_{B(0,1)} \|\varphi(z) u\left(x - rz\right)\| \, dz \\ &= \int_{B(0,1)} |\varphi(z)|^{\frac{1}{p}} \|u\left(x - rz\right)\| |\varphi(z)|^{\frac{1}{q}} \, dz \\ &\leq \left(\int_{B(0,1)} |\varphi(z)| \|u\left(x - rz\right)\|^p \, dz \right)^{\frac{1}{p}} \left(\int_{B(0,1)} |\varphi(z)| \, dz \right)^{\frac{1}{q}}, \end{aligned}$$

where the last integral is equal to 1. We conclude that

$$\|u * \varphi_r(x)\|^p \leq \int_{B(0,1)} |\varphi(z)| \|u\left(x - rz\right)\|^p \, dz$$

for all $1 \leq p < \infty$. Using this and Fubini's theorem, we can estimate the L^p-norm via

$$\begin{aligned} \int_{\mathbb{R}^d} \|u * \varphi_r(x)\|^p \, dx &\leq \int_{\mathbb{R}^d} \int_{B(0,1)} |\varphi(z)| \|u\left(x - rz\right)\|^p \, dz \, dx \\ &= \int_{B(0,1)} |\varphi(z)| \int_{\mathbb{R}^d} \|u\left(x - rz\right)\|^p \, dx \, dz = \|u\|^p_{L^p(\mathbb{R}^d, X)}. \end{aligned}$$

For $p = \infty$ we compute

$$\|u * \varphi_r(x)\| \leq \int_{\mathbb{R}^d} |\varphi_r(y)| \|u(x-y)\| \, dy \leq \|u\|_{L^\infty(\mathbb{R}^d, X)} \int_{\mathbb{R}^d} |\varphi_r(y)| \, dy,$$

from which the assertion follows. ☐

As in the scalar-valued case the C^∞-functions $u * \varphi_r$ approximate the function $u \in L^p(\Omega, X)$. This justifies the term 'mollifier'.

Theorem 4.6. *Let $1 \leq p < \infty$. For all $u \in L^p(\mathbb{R}^d, X)$ we have that*

$$\|u - u * \varphi_r\|_{L^p(\mathbb{R}^d,X)} \to 0 \quad (r \to 0).$$

Proof. First let $u \in C_c^\infty(\mathbb{R}^d, X)$ and let $\varepsilon > 0$. As u is uniformly continuous we find a $\delta > 0$ such that for all $|x - y| \leq \delta$ we have $\|u(x) - u(y)\| \leq \varepsilon$. Thus for all $x \in \Omega$ and all $r < \delta$ we have

$$\begin{aligned}
\|u * \varphi_r(x) - u(x)\| &= \left\| \int_{\mathbb{R}^d} \varphi_r(x - y)u(y)\, dy - u(x) \int_{\mathbb{R}^d} \varphi_r(y)\, dy \right\| \\
&\leq \int_{\mathbb{R}^d} \left\| \frac{1}{r^n} \varphi\left(\frac{x-y}{r}\right)(u(y) - u(x)) \right\| dz \\
&= \int_{\mathbb{R}^d} \|\varphi(z)\,(u\,(x - rz) - u(x))\|\; dz \leq \varepsilon
\end{aligned}$$

and thus $\|u * \varphi_r - u\|_{L^\infty(\mathbb{R}^d,X)} \leq \varepsilon$. Letting $\varepsilon \to 0$ we obtain that $u * \varphi_r$ converges to u uniformly. As u is compactly supported, the convergence is also valid in $L^p(\mathbb{R}^d, X)$ for any $1 \leq p < \infty$.

Now let $u \in L^p(\Omega, X)$ and let $v \in C_c^\infty(\Omega, X)$ such that $\|u - v\|_{L^p(\mathbb{R}^d,X)} \leq \varepsilon$. In addition choose an r such that $\|v - v * \varphi_r\|_{L^p(\Omega,X)} \leq \varepsilon$. Using the contraction property of the convolution we obtain

$$\begin{aligned}
&\|u - u * \varphi_k\|_{L^p(\mathbb{R}^d,X)} \\
&\leq \|u - v\|_{L^p(\mathbb{R}^d,X)} + \|v - v * \varphi_k\|_{L^p(\mathbb{R}^d,X)} + \|v * \varphi_k - u * \varphi_k\|_{L^p(\mathbb{R}^d,X)} \leq 3\,\varepsilon,
\end{aligned}$$

which yields the result letting $\varepsilon \to 0$. ☐

We do not only want to apply the above mollification process to L^p-functions, but to $W^{m,p}$-functions as well. For this we need to analyze how convolution interacts with weak derivatives.

Proposition 4.7. *Let $u \in W^{m,p}(\Omega, X)$ and α be a multi-index with $|\alpha| \leq m$, then*

$$D^\alpha(u * \varphi_r)(x) = (D^\alpha u) * \varphi_r(x)$$

for all $x \in \Omega$ such that dist$(x, \partial\Omega) > r$.

Proof. Using the Dominated Convergence Theorem, one can easily see that differentiation with respect to x and integration with respect to y of the function $x \mapsto \varphi_r(x-y)u(y)$ can be interchanged. Note that $\operatorname{dist}(x, \partial\Omega) > r$ implies that $\varphi_r(x-\cdot) \in C_c^\infty(\Omega, \mathbb{R})$. Using these two properties we compute

$$\begin{aligned}D^\alpha(u*\varphi_r)(x) &= D^\alpha\int_{\mathbb{R}^d}\varphi_r(x-y)u(y)\,dy\\&=\int_{\mathbb{R}^d}D_x^\alpha\varphi_r(x-y)u(y)\,dy\\&=(-1)^{|\alpha|}\int_{\mathbb{R}^d}\left(D_y^\alpha\varphi_r(x-y)\right)u(y)\,dy\\&=\int_{\mathbb{R}^d}\varphi_r(x-y)D^\alpha u(y)\,dy\\&=(D^\alpha u)*\varphi_r(x),\end{aligned}$$

hence the result. □

We will use the following simple version of the product rule for weakly differentiable functions.

Lemma 4.8. *Let $u\in W^{m,p}(\Omega,X)$ and $\varphi\in C_c^\infty(\Omega,\mathbb{R})$, then $\varphi u\in W^{m,p}(\Omega,X)$ and the weak derivatives are given by the usual Leibniz formula*

$$D^\alpha(\varphi u)=\sum_{\sigma\le\alpha}\binom{\alpha}{\sigma}D^\sigma\varphi D^{\alpha-\sigma}u,$$

where $|\alpha|\le m$ and $\binom{\alpha}{\sigma}=\prod_{i=1}^d\binom{\alpha_i}{\sigma_i}$.

Proof. First assume that $|\alpha|=1$, then for any $\psi\in C_c^\infty(\Omega,\mathbb{R})$ the usual product rule yields that $D^\alpha(\varphi\psi)=(D^\alpha\varphi)\,\psi+\varphi D^\alpha\psi$ and thus using $\varphi\psi\in C_c^\infty(\Omega,\mathbb{R})$ we obtain

$$\int_\Omega\varphi uD^\alpha\psi=\int_\Omega uD^\alpha(\varphi\psi)-u\,(D^\alpha\varphi)\,\psi=-\int_\Omega\left((D^\alpha u)\,\varphi+uD^\alpha\varphi\right)\psi,$$

hence $\varphi u\in W^{1,p}(\Omega,X)$ and the rule holds in this case. For $|\alpha|>1$ we prove the result via induction. Let β and γ be multi-indexes with $|\gamma|=1$ and $\alpha=\beta+\gamma$. Using the induction hypothesis first for D^β and then for D^γ we compute

$$\begin{aligned}\int_\Omega\varphi uD^\alpha\psi &= \int_\Omega(\varphi u)D^\beta D^\gamma\psi\\&=(-1)^{|\beta|}\int_\Omega\sum_{\sigma\le\beta}\binom{\beta}{\sigma}D^\sigma\varphi D^{\beta-\sigma}u(D^\gamma\psi)\\&=(-1)^{|\beta|+|\gamma|}\int_\Omega\sum_{\sigma\le\beta}\binom{\beta}{\sigma}D^\gamma(D^\sigma\varphi D^{\beta-\sigma}u)\psi\\&=(-1)^{|\beta|+|\gamma|}\int_\Omega\sum_{\sigma\le\beta}\binom{\beta}{\sigma}(D^{\sigma+\gamma}\varphi D^{\beta-\sigma}u+D^\sigma\varphi D^{\beta+\gamma-\sigma}u)\psi.\end{aligned}$$

Now splitting the sum and shifting the indexes we are left with

$$(-1)^{|\alpha|}\int_\Omega \left(D^\alpha \varphi u + \varphi D^\alpha u + \sum_{\gamma\le\sigma\le\beta} \left[\binom{\beta}{\sigma-\gamma} + \binom{\beta}{\sigma} \right] D^\sigma \varphi D^{\alpha-\sigma} u \right) \psi.$$

As $|\gamma| = 1$ we have that $\gamma_i = \delta_{i,j}$ for some j. Using this we can easily compute that $\binom{\beta}{\sigma-\gamma} + \binom{\beta}{\sigma} = \binom{\beta+\gamma}{\sigma} = \binom{\alpha}{\sigma}$. Hence the above simplifies to

$$\int_\Omega \varphi u D^\alpha \psi = (-1)^{|\alpha|} \int_\Omega \sum_{\sigma\le\alpha} \binom{\alpha}{\sigma} D^\sigma \varphi D^{\alpha-\sigma} u \psi,$$

which is equivalent to the Leibniz formula. □

Before we come to the main theorem of this section we want to recall the following fact.

Theorem 4.9. *Let $(U_i)_{i\in I}$ be an open cover of Ω, then there exists a sequence $(\psi_n)_{n\in\mathbb{N}} \subset C_c^\infty(\Omega, \mathbb{R}^+)$ such that*

(i) supp $\psi_n \subset U_i$ for some $i \in I$

(ii) $\sum_{n=1}^\infty \psi_n(x) = 1$ for all $x \in \Omega$

(iii) for every compact set $K \subset \Omega$ there exists an $m \in \mathbb{N}$ and an open set $W \supset K$ such that $\psi_n(x) = 0$ for all $n \ge m$ and all $x \in W$ (i.e. the series in (ii) is locally a finite sum)

Proof. We refer to [Rud91, Theorem 6.20] for a proof of this theorem. □

A sequence of functions as in Theorem 4.9 is called a *partition of unity* . It can be used to extend local results globally. As in the one-dimensional case we denote $\omega \subset\subset \Omega$ for a subset ω of Ω such that $\overline{\omega} \subset \Omega$ is compact.

Corollary 4.10. *Suppose that in Theorem 4.9 we have that $I = \mathbb{N}$ and $U_i \subset\subset \Omega$, then the partition of unity can be chosen such that supp $\psi_n \subset U_n$. We say that the partition of unity is* subordinate to $(U_n)_{n\in\mathbb{N}}$.

Proof. For each $n \in \mathbb{N}$ sum up all functions of the partition of unity whose support is a subset of U_n. As $\overline{U_n}$ is compact in Ω we have that this summation is finite by (*iii*). Hence the resulting function is in $C_c^\infty(\Omega, \mathbb{R}^+)$ with support in U_n. As any function of the partition of unity is supported in some set U_n by (*i*), the resulting functions satisfy the demands. □

In Chapter 3 we have already seen that the C^∞-functions on a bounded interval I are dense in $W^{1,p}(I, X)$, using the Fundamental Theorem for weakly differentiable functions. This result stays true in the more general setting of this chapter (although note that we can in general not go up to the bundary this time). Instead of the Fundamental Theorem, we will apply the results of convolution that we have proven above. For $1 \leq p < \infty$ we use the notation $H^{m,p}(\Omega, X)$ for the closure of the set $C^\infty(\Omega, X) \cap W^{m,p}(\Omega, X)$ in the $W^{m,p}(\Omega, X)$-norm.

Theorem 4.11 (Meyers-Serrin). $H^{m,p}(\Omega, X) = W^{m,p}(\Omega, X)$

Proof. Let $\varepsilon \geq 0$ and for $n \in \mathbb{N}$ let $\Omega_n := \left\{x \in \Omega,\ \|x\| < n, \operatorname{dist}(x, \partial\Omega) > \frac{1}{n}\right\}$. Additionally we define $\Omega_0 = \Omega_{-1} = \emptyset$. The sets $U_n := \Omega_{n+1} \backslash \overline{\Omega_{n-1}}$ form an open cover of Ω. Let $(\psi_n)_{n\in\mathbb{N}}$ be a partition of unity subordinate to this cover. Choose a sequence (r_n) such that

$$r_n \leq \frac{1}{(n+1)(n+2)}.$$

Note that this number is a lower bound for both $\operatorname{dist}(\Omega_{n+1}, \partial\Omega_{n+2})$ and $\operatorname{dist}(\Omega_{n-2}, \partial\Omega_{n-1})$, hence $\operatorname{supp} \varphi_{r_n} * \psi_n u \subset \Omega_{n+2} \backslash \Omega_{n-2}$ by Lemma 4.4. Thus any x in the support satisfies the conditions of Propositon 4.7 with respect to the radius r_n. By Lemma 4.8 and Theorem 4.6 we infer that we can choose r_n small enough that

$$\|\varphi_{r_n} * \psi_n u - \psi_n u\|_{W^{m,p}(\Omega,X)} \leq \frac{\varepsilon}{2^n}.$$

Define $v(x) := \sum_{n=1}^\infty \varphi_{r_n} * \psi_n u(x)$. For fixed $k \in \mathbb{N}$ and all $x \in \Omega_k \backslash \overline{\Omega_{k-1}}$ we have that $v(x) = \sum_{n=k-2}^{k+2} \varphi_{r_n} * \psi_n u(x)$ thus the series $v(x)$ is actually a locally finite sum which implies that $v \in C^\infty(\Omega, X)$. We now have that

$$\begin{aligned}
\|u - v\|_{W^{m,p}(\Omega,X)} &= \left\| \sum_{n=1}^\infty \psi_n u - \varphi_{r_n} * \psi_n u(x) \right\|_{W^{m,p}(\Omega,X)} \\
&\leq \sum_{n=1}^\infty \|\psi_n u - \varphi_{r_n} * \psi_n u(x)\|_{W^{m,p}(\Omega,X)} \\
&\leq \sum_{n=1}^\infty \frac{\varepsilon}{2^n} = \varepsilon,
\end{aligned}$$

from which we infer that $v \in W^{m,p}(\Omega, X)$ and thus the asserted denseness property holds. $\square$

4.3 A Criterion for Weak Differentiability and the Sobolev Embedding Theorem

We now want to extend the criterion for weak differentiability and its applications from the one dimensional to the d-dimensional case. The criterion is essentially the same and the proofs work with slight alterations. Note that the calculations in Section 3.3 utilized the Fundamental Theorems 3.7 and 3.8. These theorems are not true in higher dimensions. Instead, we will use the structure of the spaces $W^{1,p}(\Omega, X)$ given by the Meyers-Serrin Theorem. Let $\{e_i\}_{i=1}^d$ be the standard base of $\mathbb{R}^d$. For a shorter notation we define $D^{e_i} := D^{(\delta_{i,j})_{j=1}^d}$.

Proposition 4.12. *If $u \in W^{1,p}(\Omega, X)$ with $1 \leq p \leq \infty$, then there exists a constant C such that for all $\omega \subset\subset \Omega$ and all $h \in \mathbb{R}$ with $|h| < dist(\omega, \partial\Omega)$ we have*

$$\|\tau_{he_i} u - u\|_{L^p(\omega,X)} \leq C|h|,$$

for $i = 1, \dots, d$. Moreover we can choose $C = \max_{i=1,\dots,d} \|D^{e_i} u\|_{L^p(\Omega,X)}$.

Proof. First let $u \in C^\infty(\Omega, X) \cap W^{1,p}(\Omega, X)$ and let i be fixed, then

$$\frac{1}{h} \int_0^h \frac{\partial}{\partial x_i} u(x + te_i)\, dt = \frac{u(x + he_i) - u(x)}{h}.$$

Let $1 < p < \infty$ and $h > 0$. The case $h < 0$ is handled analogously. Using the fundamental estimate first and then Hölder's inequality we obtain

$$\begin{aligned} \left\| \frac{u(x + he_i) - u(x)}{h} \right\|^p &\leq \frac{1}{|h|^p} \left(\int_0^h \left\| \frac{\partial}{\partial x_i} u(x + te_i) \right\| dt \right)^p \\ &\leq \frac{1}{|h|} \int_0^h \left\| \frac{\partial}{\partial x_i} u(x + te_i) \right\|^p dt. \end{aligned}$$

The computation also includes the case $p = 1$. Using this, we compute

$$\frac{1}{|h|^p} \|\tau_{he_i} u - u\|^p_{L^p(\omega,X)} \leq \frac{1}{|h|} \int_{\mathbb{R}^d} \int_0^h \left\| \frac{\partial}{\partial x_i} u(x + te_i) \right\|^p dt\, dx,$$

where we extend the function by zero outside of its support. Now with Fubini's Theorem we can simplify the above to

$$\frac{1}{|h|} \int_0^h \int_{\mathbb{R}^d} \left\| \frac{\partial}{\partial x_i} u(x + te_i) \right\|^p dx\, dt = \|D^{e_i} u\|^p_{L^p(\Omega,X)},$$

from which we infer the estimate as well the value of C in this case. For general $u \in W^{1,p}(\Omega, X)$ we can choose a sequence u_n in $C^\infty(\Omega, X) \cap W^{1,p}(\Omega, X)$ converging to u in $W^{1,p}(\Omega, X)$ via the Meyers-Serrin Theorem. By the above computation, the estimate holds for u_n and hence for u as well. Now let $p = \infty$. As ω is bounded we have that $u \in W^{1,p}(\omega, X)$ for every $p < \infty$, hence by the above computation, the inequality holds for any such p. As $\|f\|_{L^\infty(\omega,X)} = \lim_{p\to\infty} \|f\|_{L^p(\omega,X)}$ for any $f \in L^\infty(\omega, X)$ the estimate also holds in this case. □

As in the one-dimensional case the converse is true if X has the Radon-Nikodym property.

Theorem 4.13. *Let $1 < p \leq \infty$ and let $u \in L^p(\Omega, X)$ where X is a Banach space that has the Radon-Nikodym property. Assume that there exists a C such that for all $\omega \subset\subset \Omega$ and $h \in \mathbb{R}$ with $|h| < dist(\omega, \partial\Omega)$ we have*

$$\|\tau_{he_i} u - u\|_{L^p(\omega,X)} \leq C|h|,$$

for $i = 1, \dots, d$. Then $u \in W^{1,p}(\Omega, X)$.

Proof. Proceed as in the proof of Theorem 3.20. □

Again we infer the same corollaries as in the one-dimensional case. This time, the embedding theorems are much more interesting. These theorems are often referred to as *Sobolev inequalities* or *Sobolev Embedding Theorems.*

Corollary 4.14. *Let $1 < p \leq \infty$, $u \in W^{1,p}(\Omega, X)$ and $F : X \to Y$ be Lipschitz continuous. If Y has the Radon-Nikodym property, then $F \circ u \in W^{1,p}(\Omega, Y)$. In particular $\|u\| \in W^{1,p}(\Omega, \mathbb{R})$.*

Theorem 4.15 (Embedding Theorems)**.** *Let $\Omega = \mathbb{R}^d$ or $\Omega \subset \mathbb{R}^d$ with C^1-boundary, then we have the following embeddings*

(i) if $1 < p < d$, then $W^{1,p}(\Omega, X) \hookrightarrow L^{p^*}(\Omega, X)$ where p^* is given by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$

(ii) if $p = d$, then $W^{1,p}(\Omega, X) \hookrightarrow L^q(\Omega, X)$ where $p \leq q < \infty$

(iii) if $p > d$, then $W^{1,p}(\Omega, X) \hookrightarrow L^\infty(\Omega, X)$

and all these injections are continuous.

Proof. It is known that all assertions above are true if $X = \mathbb{R}$, see [Bré10, Corollary 9.14]. Now let $u \in W^{1,p}(\Omega, X)$, then by Corollary 4.14 we have that $\|u\| \in W^{1,p}(\Omega, \mathbb{R})$ and hence $\|u\| \in L^r(\Omega, X)$ for $r = p^*, q, \infty$ depending on the case. From this we infer that $u \in L^r(\Omega, X)$. Finally we show that the graph of the injection is closed. Suppose that $u_n \in W^{1,p}(\Omega, X)$ such that $u_n \to u$ in $W^{1,p}(\Omega, X)$ and $u_n \to v$ in $L^r(\Omega, X)$. Then there exists a subsequence u_{n_k} such that the above convergences hold pointwise almost everywhere on the same set and thus $u = v$. Now the Closed Graph Theorem implies that the embedding is continuous. □

The remaining section is dedicated to the converses of Theorem 4.13 and Corollary 4.14. As in the one-dimensional case, these fail if X does not have the Radon-Nikodym property. Before we can proof this, we need to give an alternative to the Fundamental Theorem in the d-dimensional case.

Theorem 4.16. *Let $u \in W^{1,p}(\Omega, X)$ for $1 \le p \le \infty$ and let $\omega \subset\subset \Omega$. Then u has a representative such that for every $i = 1, \dots, d$ and almost every $x \in \omega$ the function*

$$\{(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_d), t \in \mathbb{R}\} \cap \omega \to X$$
$$t \mapsto u(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_d)$$

is absolutely continuous and the derivative of this function coincides with the weak derivative $D^i u$ for almost all t. Conversely if $u \in L^p(\Omega, X)$ satisfies the above and the partial derivatives $\frac{\partial}{\partial x_i} u$, extended by 0 where they don't exist, are in $L^p(\omega, X)$ as well, then $u \in W^{1,p}(\omega, X)$ for all ω as above.

Proof. Choose compact sets K_1 and K_2 such that $\omega \subset K_1 \subset K_2 \subset \Omega$ and $\partial K_1 \cap \partial K_2 = \emptyset$. If $\varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ such that $\varphi_{|K_1} = 1$ and $\varphi_{|K_2^c} = 0$ then by Lemma 4.8 we have that $\varphi u \in W^{1,p}(\mathbb{R}^d, X)$ and this function coincides with u on ω. Hence we may w.l.o.g. assume that u is compactly supported and defined on $\mathbb{R}^d$. In this setting we have that $u, D^i u \in L^1(\mathbb{R}^d, X)$. Using the mollification process described in Section 4.2 we find a sequence $(\varphi_n) \subset C_c^\infty(\mathbb{R}^d, X)$ supported by a fixed compact set such that

$$\|\varphi_n - u\|_{W^{1,1}(\mathbb{R}^d, X)} \le \frac{1}{2^{n+1}}$$

and such that the convergence also holds pointwise a.e. Let G be the set of all points on which φ_n converges pointwise and denote the pointwise limit by u^*. Letting $u^* = 0$ outside of G we obtain that $u^* = u$ a.e. We fix a direction e_i and assume w.l.o.g. that $i = d$. Let

$$f_n(x_1, \dots, x_{d-1}) := \int_{-\infty}^{\infty} \|\varphi_{n+1}(x) - \varphi_n(x)\| + \sum_{j=1}^{d} \|D^j \varphi_{n+1}(x) - D^j \varphi_n(x)\| \, dx_d,$$

where $x = (x_1, \dots, x_d)$ and let

$$f(x_1, \dots, x_{d-1}) = \sum_{n=1}^{\infty} f_n(x_1, \dots, x_{d-1}).$$

All functions f_n are real-valued and positive, hence the Monotone Convergence Theorem implies that

$$\begin{aligned} \int_{\mathbb{R}^{d-1}} f \, dx_1 \dots dx_{d-1} &= \sum_{n=1}^{\infty} \int_{\mathbb{R}^{d-1}} f_k \, dx_1 \dots dx_{d-1} \\ &= \sum_{n=1}^{\infty} \|\varphi_{n+1} - \varphi_n\|_{W^{1,1}(\mathbb{R}^d, X)} < \infty, \end{aligned}$$

where we used Fubini's Theorem in the last equality. The estimate implies that $f \in L^1(\mathbb{R}^{d-1}, \mathbb{R})$ and in particular that f is finite a.e. on $\mathbb{R}^{d-1}$. Let $\hat{x} = (x_1, \dots, x_{d-1})$

such that $f(\hat{x}) < \infty$ and denote $g_n(t) := \varphi_n(\hat{x}, t)$ and $g(t) = u^*(\hat{x}, t)$. Applying the Fundamental Theorem to the differentiable function g_n we compute

$$\begin{aligned}\|g_{n+1}(t) - g_n(t)\| &\leq \int_{-\infty}^{t} \|g'_{n+1}(s) - g'_n(s)\| \, ds \\ &\leq \int_{-\infty}^{\infty} \|D^d\varphi_{n+1}(\hat{x}, s) - D^d\varphi_n(\hat{x}, s)\| \, ds \\ &\leq f_n(\hat{x}) < \infty,\end{aligned}$$

which does not depend on the value of t. We have that the series $\sum f_n(\hat{x})$ converges to $f(\hat{x})$, hence the series

$$g_1 + \sum_{n=1}^{\infty} g_{n+1} - g_n$$

converges uniformly. This implies that $\{\hat{x}\} \times \mathbb{R} \subset G$ and that g is the limit of the series. Furthermore g is the uniform limit of continuous functions and thus continuous as well. From the uniform convergence and the common compact support of g_n we also obtain that the the limit

$$Dg := \lim_{n\to\infty} g'_n = g'_1 + \sum_{n=1}^{\infty} g'_{n+1} - g'_n$$

exists in $L^1(\mathbb{R}, X)$. For all $n \in \mathbb{N}$ we have that

$$g_n(t) = \int_{-\infty}^{t} g'_n(s) \, ds,$$

and using the Dominated Convergence Theorem this carries over to g and Dg. Hence g is the primitive of Dg and thus absolutely continuous. Now Corollary 3.3 implies that Dg is both the weak and the pointwise a.e. derivative of g, i.e.

$$\begin{aligned}\int_{\mathbb{R}} u^*(\hat{x}, x_d)\psi'(x_d) \, dx_d &= \int_{\mathbb{R}} g(t)\psi'(t) \, dt \\ &= -\int_{\mathbb{R}} Dg(t)\psi(t) \, dt = -\int_{\mathbb{R}} \frac{\partial}{\partial x_d} u^*(\hat{x}, x_d)\psi(x_d) \, dx_d,\end{aligned}$$

for all $\psi \in C_c^\infty(\mathbb{R}, \mathbb{R})$. As this holds for almost all $\hat{x}$ we conclude

$$\int_{\mathbb{R}^d} u^* D^d\psi = -\int_{\mathbb{R}^d} \frac{\partial}{\partial x_d} u^*\psi,$$

for all $\psi \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$. Hence $\frac{\partial}{\partial x_d} u^*$ is equal to the weak derivative $D^d u$ a.e. as claimed.

Now suppose that u has a representative u^* as stated. Again we assume w.l.o.g. that $i = d$. If $\varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$, then φu^* has the same properties a u on the line segments, hence we compute

$$\int_{\mathbb{R}} \frac{\partial}{\partial x_d} \left(\varphi(\hat{x}, t) u(\hat{x}, t) \right) \, dt = 0$$

for almost all $\hat{x} \in \mathbb{R}^{d-1}$. Using this and applying the product rule a.e. on $\mathbb{R}$ we see that

$$\int_{\mathbb{R}} u(\hat{x}, t) \frac{\partial}{\partial x_d} \varphi(\hat{x}, t) \, dt = - \int_{\mathbb{R}} \frac{\partial}{\partial x_d} u(\hat{x}, t) \varphi(\hat{x}, t) \, dt.$$

Again we obtain that $\frac{\partial}{\partial x_d} u \in L^p(\omega, X)$ is the weak derivative of u using Fubini's theorem. □

Corollary 4.17. *Let X be a Banach-space. X has the Radon-Nikodym property if and only if the criterion given in Theorem 4.13 characterizes the spaces $W^{1,p}(\Omega, X)$.*

Proof. It remains to show that if X does not have the Radon-Nikodym property, then there exists a function $u \notin W^{1,p}(\Omega, X)$ which satisfies the conditions of Theorem 4.13. We may assume that $\Omega = (0,1)^d$ as every open set Ω contains some cube and we may cut off the function outside of this cube via multiplication with a C_c^∞-function as in the one-dimensional case. Let $f : (0,1) \to X$ be a Lipschitz continuous function that is not differentiable a.e. and define

$$u(x_1, \ldots, x_d) = f(x_1).$$

As f is not differentiable a.e. but continuous, it follows that u cannot have a representative as in Theorem 4.16 and hence $u \notin W^{1,p}(\Omega, X)$. But we have that $u \in L^p(\Omega, X)$ and for all $\omega \subset\subset \Omega$ and h small enough

$$\begin{aligned} \int_\omega \|u(x + he_i) - u(x)\|^p \, dx &= \int_\omega \|f(x_1 + h\delta_{1,i}) - f(x_1)\|^p \, dx \\ &\leq \int_\Omega (L|h|)^p \, dx = (L|h|)^p, \end{aligned}$$

hence u satisfies the conditions of Theorem 4.13. □

Corollary 4.18. *Let $1 \leq p \leq \infty$ and X and Y be Banach spaces. If $F \circ W^{1,p}(\Omega, X) \subset W^{1,p}(\Omega, Y)$ for every Lipschitz continuous mapping $F : X \to Y$, then Y has the Radon-Nikodym property.*

Proof. Let $f : \mathbb{R} \to Y$ be Lipschitz continuous and choose an arbitrary interval I. Further choose a vector $x_0 \in X$ and a functional $x_0' \in X'$ such that $\langle x_0', x_0 \rangle = 1$. Let

$$\begin{aligned} F :& X \to Y \\ & x \mapsto f(\langle x_0', x \rangle), \end{aligned}$$

then F is Lipschitz continuous as in the one-dimensional case. We may w.l.o.g. assume that $I \times (0,1)^{d-1} \subset \Omega$. Define $\tilde{u} : \Omega \to X$ via

$$\tilde{u}(t) := t_1 \cdot x_0.$$

Choose a function $\varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ such that $\varphi_{|I\times(0,1)^{d-1}} \equiv 1$ and let $u := \varphi\tilde{u} \in W^{1,p}(\Omega, X)$. By assumption we have that $f(\langle x_0', t_1 x_0\rangle) = F(u(t)) \in W^{1,p}(\Omega, Y)$ hence by Theorem 4.16 we have that its partial derivative with repect to t_1 exists for almost all $t \in I \times (0,1)^{d-1}$. But for any such t this derivative is equal to $\frac{d}{dt_1} f(t_1)$, hence f is differentiable a.e. on I. As I was chosen arbitrary, we infer that f is differentiable a.e. and hence Y has the Radon-Nikodym property. □

4.4 Notes

We have found several books which contain treatises about the spaces $W^{1,p}(I, X)$ but we could not find any book containing an extension from $W^{m,p}(\Omega, \mathbb{R})$ to $W^{m,p}(\Omega, X)$. There exist a few articles which deal with Sobolev spaces of vector-valued functions in higher dimensions but it seems that no author has undertaken the work to give a detailed discussion of their basics. This is the main purpose of this chapter which is based on the books [Eva98], [GP06], [Bré10] and [Sau12] containing treatises of the spaces $W^{m,p}(\Omega, \mathbb{R})$. The first section covers the general definition of the spaces $W^{m,p}(\Omega, X)$ analogously to these sources. The results about vector-valued distributions are not contained in these books. Here we followed the one-dimensional case and the sources given in Chapter 3
Mollification has been proven to be a useful tool in the field of Sobolev spaces and beyond. Our approach extends this to vector-valued functions where we have to mention that all proofs are basically the same as in the scalar-valued case. The Meyers-Serrin Theorem is only one possible application of mollifiers. This theorem helps us to understand the structure of the spaces $W^{m,p}(\Omega, X)$ as the Fundamental Theorem of Chapter 3 is not true in the general setting. Meyers and Serrin proved the result in 1964. Up to this point, it was not clear that the spaces $H^{m,p}(\Omega, X)$ of so called *strongly differentiable* functions and the spaces $W^{m,p}(\Omega, X)$ of weakly differentiable functions always coincide, though it was known in special cases. Hence the notation $H^{m,p}(\Omega, X)$ is outdated and we use it solely to pay tribute to the name of the original paper [MS64], "H=W". Meyers and Serrin suggested that mathematicians should use the term *strong derivative* in the future, but the term weak derivative has prevailed. Again the proof for vector-valued functions is established analogously to the original proof. We used the original article as well as the presentations in the books.

The last section is a generalization of the corresponding section in Chapter 3. The first proof is taken from [Sau12] and shows how one can use the Meyers-Serrin Theorem as a substitute for the Fundamental Theorem of Calculus. The rest is based on what we have already proven in Chapter 3. The embedding theorems for the scalar-valued case are a useful tool in the field of partial differential equations. With the criterion for weak differentiability they can easily be extended to vector-valued functions while the original proofs take up much more work. The proof of Theorem 4.16 is taken from [MZ97] which again covers the result only for the case $X = \mathbb{R}$. Using this theorem the corollaries are simply a reformulation of the results we have proven in Section 3.3.

5 Functions with Values in Banach Lattices

In the last two chapters we have seen that the composition of a Lipschitz continuous function and a weakly differentiable function results in a weakly differentiable function if the space has the Radon-Nikodym property. However we did not investigate what the weak derivative of this function looks like and in many cases this question might be hard to answer. In this chapter we want to consider special examples of Lipschitz maps in Banach lattices, Banach spaces which are equipped with a well behaving partial ordering. We will compute the weak derivatives in these cases and also show that the Radon-Nikodym property may be dropped if we instead require the functions to be more regular. Before we come to this we will give an introduction to the basic results of Banach lattice theory, which we will need later on.

5.1 Banach Lattices and Projection Bands

Let $(X, \leq)$ be a partially ordered set. X is called a *lattice* if for all $x, y \in X$ the least upper bound and the greatest lower bound of x and y, denoted by

$$x \vee y \quad \text{and} \quad x \wedge y$$

exist. If in addition X is a vector space such that

$$x \leq y \quad \Rightarrow \quad x + z \leq y + z \text{ and } ax \leq ay$$

holds for all $x, y, z \in X$ and all $a \in \mathbb{R}^+$, then X is called a *Riesz space* or a *vector lattice.* In a Riesz space we can define the *positive and negative part* of any element $x \in X$ as well as its *absolute value* via

$$x^+ := x \vee 0, \quad x^- := (-x) \vee 0, \quad |x| := x \vee -x.$$

Now assume that in addition X is a Banach space such that

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\|,$$

then we say that X is a *Banach lattice.* In the following, X will always denote a Banach lattice.

Example 5.1. Let X be an arbitrary Banach lattice (e.g. the obvious example $X = \mathbb{R}$) and $1 \leq p \leq \infty$, then the space $L^p(\Omega, X)$ equipped with the partial ordering

$$f \leq g :\Leftrightarrow f(x) \leq g(x) \ a.e.$$

is a Banach lattice. In particular we have that the Lebesgue spaces $L^p(\Omega, \mathbb{R})$ are Banach lattices. It is clear that this defines a partial ordering and one easily sees that

$$(f \vee g)(x) = f(x) \vee g(x) \quad \text{and} \quad (f \wedge g)(x) = f(x) \wedge g(x),$$

for $f, g \in L^p(\Omega, X)$. If $f(x) \le g(x)$ holds for almost all $x \in \Omega$, then obviously $f(x) + h(x) \le g(x) + h(x)$ and $af(x) \le ag(x)$ holds on the same set for all $f, g, h \in L^p(\Omega, X)$ and all $a \ge 0$. Hence $L^p(\Omega, X)$ endowed with the above ordering becomes a Riesz space. The pointwise comparison leads to

$$f^+(x) = (f \vee 0)(x) = f(x) \vee 0 = f(x)^+, \quad f^-(x) = f(x)^- \quad \text{and} \quad |f|(x) = |f(x)|,$$

for almost all $x \in \Omega$. Now suppose that $|f| \le |g|$, i.e. $|f(x)| \le |g(x)|$ for almost all $x \in \Omega$. The definition of the norm on X implies that $\|f(x)\| \le \|g(x)\|$ holds for all these x, hence we compute

$$\|f\|^p_{L^p(\Omega,X)} = \int_\Omega \|f(x)\|^p \, d\mu(x) \le \int_\Omega \|g(x)\|^p \, d\mu(x) = \|g\|^p_{L^p(\Omega,X)},$$

for $p < \infty$ and obviously $\|f\|_{L^\infty(\Omega,X)} \le \|g\|_{L^\infty(\Omega,X)}$. The spaces $L^p(\Omega, X)$ are complete, hence the above computation shows that $L^p(\Omega, X)$ with the pointwise comparison is indeed a Banach lattice.

The positive and negative parts and the absolute value of an element of X behave as one would assume, namely:

Proposition 5.2. *Let X be a Banach lattice and let $x, y \in X$. Then we have*

(i) $x = x^+ - x^-$
(ii) $|x| = x^+ + x^-$
(iii) $|x + y| \le |x| + |y|$
(iv) $\big| |x| - |y| \big| \le |x - y|$

Proof. From the definition of $\vee$ it follows that for any $x, y, z \in X$ and $a \ge 0$ we have

$$(x \vee y) + z = (x + z) \vee (y + z) \quad \text{and} \quad a(x \vee y) = (ax) \vee (ay).$$

Hence we compute

$$x^+ - x = (x \vee 0) - x = 0 \vee -x = x^-,$$

from which (i) follows. With this we prove (ii) via

$$|x| = x \vee -x = (2x \vee 0) - x = 2x^+ - (x^+ - x^-) = x^+ + x^-.$$

For (iii) note that it suffices to show that $|x| + |y|$ is an upper bound of $\{x + y, -x - y\}$. But this is obvious as $|x| \ge \pm x$ and $|y| \ge \pm y$. Now the reverse triangular inequality (iv) follows from the triangular inequality (iii) as usual. $\square$

For a Banach lattice X we define the *positive cone* to be $X^+ := \{x \in X, x \geq 0\}$. Note that we refer to its elements as *positive* rather than non-negative.

Proposition 5.3. *The lattice operations $\vee$ and $\wedge$ are jointly continuous and the positive cone is closed.*

Proof. We first show that the mapping $x \mapsto |x|$ is continuous. Let $(x_n), (y_n) \subset X$ with $x_n \to x$ and $y_n \to y$. Using the definition of the norm on a Banach lattice we deduce that $\| \, |x| \, \| = \|x\|$. Hence from the reverse triangular inequality it follows that

$$\| \, |x_n| - |x| \, \| \leq \|x_n - x\| \to 0.$$

Now it follows that $x_n \vee y_n \to x \vee y$ and $x_n \wedge y_n \to x \wedge y$ using the formulas

$$x \vee y = \frac{x + y + |x - y|}{2} \quad \text{and} \quad x \wedge y = \frac{x + y - |x - y|}{2}.$$

Hence $\vee$ and $\wedge$ are jointly continuous. Now let $X^+ \ni x_n \to x$, then by the continuity of the lattice operations we have that $x_n = x_n^+ \to x^+$, hence $x = x^+ \in X^+$, i.e. X^+ is closed. □

Proposition 5.4 (Archimedes)**.** *Let $x \in X$ such that the set $\{nx, n \in \mathbb{N}\}$ has an upper bound. Then $x \leq 0$.*

Proof. Let $y \in X$ be an upper bound. We have that y^+ is an upper bound for $y \geq nx$ and for 0, hence $y^+ \geq nx \vee 0 = nx^+$. By the definition of the lattice norm, we obtain $\frac{1}{n}\|y^+\| \geq \|x^+\|$. Letting $n \to \infty$ it follows that $x^+ = 0$ and thus $x = -x^- \leq 0$. □

The space X is said to be *σ-Dedekind complete* if every sequence that is bounded with respect to the order has a supremum and an infimum in X.

Example 5.5. (i) The space $C([0,1], \mathbb{R})$, ordered by the pointwise comparison of the function values, is a Banach lattice which is not σ-Dedekind complete. Consider a monotonically increasing sequence of continuous functions $0 \leq f_n(x) \leq 1$ converging pointwise to $1_{[0,\frac{1}{2}]}$. This sequence is bounded by the functions $f \equiv 1$ and $g \equiv 0$ but it does not have a supremum in $C([0,1])$.

(ii) For $1 \leq p \leq \infty$ the space $L^p(\Omega, \mathbb{R})$ is σ-Dedekind complete. Let f_n be a sequence of functions in $L^p(\Omega, \mathbb{R})$ such that $f_n \leq g \in L^p(\Omega, \mathbb{R})$. Let $f(x) := \sup_{n \in \mathbb{N}} f_n(x)$, then it is well known that f is measurable. If $p = \infty$ it follows immediately that $f \in L^\infty(\Omega, \mathbb{R})$. If $p < \infty$ then for all $n \in \mathbb{N}$ and almost all $x \in \Omega$ we have that $|f_n(x)|^p \leq |g(x)|^p$, hence $f \in L^p(\Omega, \mathbb{R})$. One can easily see that $f = \sup_{n \in \mathbb{N}} f_n$. The existence of the infimum is proved analogously.

A subspace $Y \subset X$ is called a *sublattice* if for all $x, y \in Y$ it follows that $x \vee y, x \wedge y \in Y$.

Proposition 5.6. *A subspace $Y \subset X$ is a sublattice of X if and only if for each $x \in Y$ it follows that $x^+ \in Y$.*

Proof. If Y is a sublattice then it is obvious that $x^+ \in Y$ for all $x \in Y$. Conversely we have that

$$x \vee y = (x - y \vee 0) + y = (x - y)^+ + y,$$

hence Y is closed under $\vee$. Now note that $x \leq y$ is equivalent to $-x \geq -y$. Hence if $x, y \in X$ and z is a lower bound for both, then $-z$ is an upper bound for $-x$ and $-y$. We conclude that $x \wedge y = -(-x \vee -y)$. As Y is closed under $\vee$ it now follows that Y is also closed under $\wedge$, thus Y is a sublattice. □

A subspace $Y \subset X$ is called an *ideal* of X if for all $y \in Y$ and $x \in X$ we have that $|x| \leq |y|$ implies $x \in Y$. The ideal Y is called a *band* if $\sup A \in Y$ for any subset $A \subset Y$ that has a supremum in X.

Example 5.7. Let $A \subset \Omega$ be a measurable set of positive measure and define $Y = \{f \in L^p(\Omega, \mathbb{R}), f_{|A} = 0\}$. Then if $|g| \leq |f|$ it follows immediately that $g(x) = 0$ for almost all $x \in A$. Thus Y is an ideal of $L^p(\Omega, \mathbb{R})$. If $Z \subset Y$ then it is immediately clear that the least upper bound of Z has to be equal to zero a.e. on A, hence Y is also a band.

Given $x, y \in X$ we say that x and y are *disjoint* if $|x| \wedge |y| = 0$ and denote this by $x \perp y$. For a set $A \subset X$ and an element $x \in X$ we denote $A^\perp := \{z \in X, z \perp y \ \forall y \in A\}$ and $x^\perp := \{x\}^\perp$.

Proposition 5.8. *For any $x \in X$ we have that $x^+ \perp x^-$ and the decomposition of x into the difference of two disjoint positive elements is unique.*

Proof. For $x \in X$ we compute

$$x^+ \wedge x^- = x^- + (x \wedge 0) = x^- - (-x) \vee 0 = 0.$$

Now assume that $u, v \in X^+$ are disjoint and that $x = u - v$. We have that $u \geq x$ hence $u \geq x^+$. Now from $u - v = x = x^+ - x^-$ we infer that $u - x^+ = v - x^-$ and thus

$$0 \leq u - x^+ = (u - x^+) \wedge (v - x^-) \leq u \wedge v = 0,$$

from which the assertion follows. □

Proposition 5.9. *For every subset A of X the set $A^\perp$ is a band.*

Proof. Let $x \in A^\perp$ and $y \in X$ such that $|y| \leq |x|$. For all $z \in A$ it follows that

$$0 \leq |y| \wedge |z| \leq |x| \wedge |z| = 0,$$

hence $y \in A^\perp$. For any $x, y, z \in X^+$ we have

$$\begin{aligned}(x+y) \wedge z &= ((x+y) \wedge (z+y)) \wedge z \\ &= ((x \wedge z) + y) \wedge z \\ &\leq ((x \wedge z) + y) \wedge ((x \wedge z) + z) = (x \wedge z) + (y \wedge z).\end{aligned}$$

Let $x, y \in A^\perp$. For any $z \in A$ we use the triangular inequality and the above estimate to compute

$$0 \leq |z| \wedge |x+y| \leq |z| \wedge (|x| + |y|) \leq |z| \wedge |x| + |z| \wedge |y| = 0,$$

i.e. $x + y \in A^\perp$ and thus $A^\perp$ is an ideal. For the final property of a band we need to prove a distributive law first. If $x, y \in X$, then $x \vee y = ((x-y) \vee 0) + y = (x-y)^+ + y$ and $x \wedge y = (0 \wedge (y-x)) + x = -(0 \vee (x-y)) + x = -(x-y)^+ + x$. Adding the above equations we obtain

$$x + y = x \vee y + x \wedge y.$$

Now let $D \subset X$ such that $x_0 = \sup D$ exists. For any $y \in X$ we prove that $y \wedge x_0 = \sup_{x \in d} y \wedge x$. We have that $x_0 \geq x$ for all $x \in D$, hence $x_0 \wedge y \geq x \wedge y$, i.e. $x_0 \wedge y$ is an upper bound for the set $\{x \wedge y,\ x \in D\}$. Suppose that z is another upper bound, then

$$z \geq x \wedge y = x + y - x \vee y \geq x + y - x_0 \vee y,$$

hence

$$z - y + x_0 \vee y \geq x \quad (x \in D),$$

from which we infer that

$$z - y + x_0 \vee y \geq x_0.$$

It follows that $z \geq x_0 + y - x_0 \vee y = x_0 \wedge y$, i.e. $x_0 \wedge y$ is the least upper bound. Now let $D \subset A^\perp$ such that $x_0 = \sup D$. For any $y \in A$ the distributive law above yields that

$$|x_0| \wedge |y| = \sup_{x \in D} |x| \wedge |y| = 0,$$

thus $A^\perp$ is a band as claimed. □

Let $A \subset X$ be a band such that $A + A^\perp = X$. For any $x \in A \cap A^\perp$ we have that $0 \leq |x| \wedge |x| = 0$, hence $X = A \oplus A^\perp$ and the decomposition of $x \in X$ into the sum of $x_1 \in A$ and $x_2 \in A^\perp$ is unique. We say that A is a *projection band*. The *band projection* onto A is given by

$$\begin{aligned}P_A : &X \to X \\ &x \mapsto x_1,\end{aligned}$$

where $x = x_1 + x_2$ as above. It is clear that P_A is linear and that $P_A^2 = P_A$. The projection is also continuous as we will show in the next two propositions.

Proposition 5.10. *The operator P_A is positive, that is $P_A x \geq 0$ for all $x \geq 0$.*

Proof. Let $x \geq 0$ and $x = x_1 + x_2$ be de decomposition as above. We have that

$$0 \leq x = x_1^+ + x_2^+ - (x_1^- + x_2^-),$$

from which we infer that $x_1^- + x_2^- \leq x_1^+ + x_2^+$. As A and $A^\perp$ are both ideals we have that $x_1^\pm \in A$ and $x_2^\pm \in A^\perp$. Hence we compute

$$0 \leq x_1^- = x_1^- \wedge (x_1^- + x_2^-) \leq x_1^- \wedge (x_1^+ + x_2^+) = 0$$

using Proposition 5.8. Hence $P_A x = x_1 \geq 0$ as claimed. □

Proposition 5.11. *Any positive operator - in particular P_A - is continuous.*

Proof. Let T be a positive operator that is not continuous, i.e. unbounded. There exists a sequence $(x_n) \subset X$ such that $\| |x_n| \| = \|x_n\| \leq 2^{-n}$ and $\|T|x_n| \| \geq \|Tx_n\| \geq n$. By assumption on x_n the element

$$x := \sum_{n=1}^{\infty} |x_n|$$

is well defined. Hence $\|Tx\| \geq \|Tx_n\| \geq n$ for all $n \in \mathbb{N}$, a contradiction. □

Theorem 5.12. *Let $A \subset X$ be a band, then A is a projection band if and only if for any $y \in X^+$ the element*

$$y_1 = \sup\{x \in A, 0 \leq x \leq y\}$$

exists. In this case $y_1 = P_A y$. Analogously the projection onto the band $A^\perp$ is given by

$$P_{A^\perp} y = \sup\{x \in A^\perp, 0 \leq x \leq y\}.$$

Proof. Suppose that A is a projection band and let $y \in X^+$ with decomposition $y = y_1 + y_2$. Let $V := \{x \in A, 0 \leq x \leq y\}$. For all $x \in V$ we have $0 \leq y - x = (y_1 - x) + y_2$, hence $y_1 - x = P_A(y - x)$. As P_A is positive, we obtain that $y_1 \geq x$ for all $x \in V$. But as $y_1 \in V$ it follows that $y_1 = \sup V$. Conversely suppose that $y_1 = \sup V$. As A is a band it follows that $y_1 \in A$. We have to show that $y_2 := y - y_1 \in A^\perp$. Suppose that this is not the case, then as $y_2 = y - y_1 > 0$ there exists a $0 \leq z \in A$ such that $p := y_2 \wedge z > 0$. As A is a band and $0 \leq p \leq z \in A$ it follows that $p \in A$ and hence $y_1 + p \in A$. We also have that $y_1 + p \leq y$ thus $y_1 + p \in V$. It now follows that $y_1 + p \leq \sup V = y_1$, a contradiction. If $y \in X$ is arbitrary, we decompose it into the positive difference $y = y^+ - y^-$. The claim now follows from the first part of the proof. □

It is evident that the intersection of a family of bands is a band itself. Hence for any $x \in X^+$ there exists a smallest band, denoted by B_x such that $x \in B_x$. We say that B_x is generated by x. More general the same holds for ideals. A straightforward computation shows that for any $z \in B_x$ there exists a set A in the ideal generated by x such that $z = \sup A$. The next step will be to characterize the projection $P_x := P_{B_x}$.

Theorem 5.13. *The band B_x is a projection band if and only if for all $y \in X^+$ the element*

$$y_1 = \sup_{n\in\mathbb{N}} y \wedge nx$$

exists. In this case $y_1 = P_x y$.

Proof. Let $V := \{z \in B_x, 0 \leq z \leq y\}$ and $V' := \{y \wedge nx, n \in \mathbb{N}\}$. By Theorem 5.12 it is sufficient to show that $\sup V = \sup V'$. For any $n \in \mathbb{N}$ we have that $0 \leq y \wedge nx \leq y$ and also $y \wedge nx \leq nx \in A$. Hence $y \wedge nx \in B_x$ as B_x is a band. We conclude that $V' \subset V$ and thus $\sup V \geq \sup V'$. Conversely let $z \in V$ and let A be a subset of the ideal generated by x such that $z = \sup A$. For any $a \in A$ there exists a k_a such that $a \leq k_a x$ as a lies in the ideal generated by x. It follows that $a \leq y \wedge k_a x \leq \sup V'$. As z is the least upper bound for A we obtain $z \leq \sup V'$. This implies that $\sup V \leq \sup V'$ as z was chosen arbitrarily. □

Corollary 5.14. *Let X be a σ-Dedekind complete Banach lattice. For any element $x \in X^+$ the band B_x is a projection band and the projection onto this band is given via*

$$P_x y = \sup_{n\in\mathbb{N}} y^+ \wedge nx - \sup_{n\in\mathbb{N}} y^- \wedge nx$$

Proof. In view of the last theorem it suffices to show that for any $y \in X^+$ the element $\sup_{n\in\mathbb{N}} y \wedge nx$ exists. The formula for the projection then follows by linearity. Let $y_n := y \wedge nx$, then $0 \leq y_n \leq y$, hence $\sup_{n\in\mathbb{N}} y_n$ exists as X is σ-Dedekind complete. □

For our work in the next section we will need one more definition. A set $A \subset X$ is called *downwards directed* if for any $x, y \in A$ there exists a $z \in A$ such that $z \leq x, y$. The norm of X is called *order continuous* if for any downwards directed set A such that $\inf A = 0$ it follows that $\inf_{x\in A} \|x\| = 0$. A common example of an order continuous norm is the norm in $L^p(\Omega, \mathbb{R})$ for $1 \leq p < \infty$.

5.2 The Lattice Property of $W^{1,p}(\Omega, X)$

For real-valued functions it is well known that the function u^+ is weakly differentiable if u is. In the last section we have seen that this implies that $W^{1,p}(\Omega, \mathbb{R})$ is a sublattice of $L^p(\Omega, \mathbb{R})$. We have also seen how for $v, w \in W^{1,p}(\Omega, \mathbb{R})$ we can retrieve the functions $v \vee w$ and $v \wedge w$ from the knowledge of u^+ for any $u \in W^{1,p}(\Omega, \mathbb{R})$. Further, by the linearity of the weak differential operator, it is sufficient to know what the weak derivative of u^+ is, in order to be able to compute all derivatives of functions that are the results of lattice operations. In the real-valued case the weak derivatives are given by $D^j u^+ = D^j u \, 1_{\{x\in\Omega, u(x)>0\}}$, see [Sau12, X Theorem 8]. We now want to investigate under which circumstances this is true for vector-valued functions as well. The following example shows that there exist spaces for which u^+ may not be weakly differentiable.

Example 5.15. Let $u : (0,1) \to X = C([0,1],\mathbb{R})$ be given by

$$u(t)(r) = r - t.$$

Computing the difference quotient pointwise shows that the candidate for the weak derivative is $u'(t) = -1_{(0,1)} \in L^p((0,1),X)$. Further fix a $t_0 \in (0,1)$ and let $u_0(r) := r$. It holds that

$$u(t) = u_0 - t \cdot 1_{(0,1)} = u_0 - t_0 \cdot 1_{(0,1)} + \int_{t_0}^{t} u'(s)\, ds,$$

thus $u \in W^{1,p}((0,1),X)$. The function u^+ is given by

$$u^+(t)(r) = \begin{cases} 0 & \text{if } r < t \\ r - t & \text{if } r \geq t \end{cases}$$

Rather than computing the difference quotient in $C([0,1],\mathbb{R})$, we will do this pointwise. This is justified by Proposition 3.8 which states that we may compute the difference quotient of $\langle x', u\rangle$ instead where we choose x' to be the evaluation in a point of $[0,1]$. Computing the difference quotient pointwise leads to

$$\frac{u^+(t+h)(r) - u(t)(r)}{h} = \begin{cases} 0 & \text{if } r < t \\ \frac{t-r}{h} & \text{if } t \leq r < t+h \\ -1 & \text{if } r \geq t+h \end{cases}$$

which converges to $-1_{(t,1)}(r)$. Hence the only possible candidate for the weak derivative is a function which does not have values in X. Thus u^+ cannot be weakly differentiable.

Using the results we have proven in the last chapters we obtain a positive result.

Theorem 5.16. *Let X be a Banach lattice that has the Radon-Nikodym property and let $1 < p \leq \infty$. Then for any $u \in W^{1,p}(\Omega, X)$ the function u^+ is in $W^{1,p}(\Omega,X)$ as well. If $\Omega = I$ is an interval and X is reflexive, then the result is true for $p = 1$ as well.*

Proof. Let $x, y \in X$, then

$$|x^+ - y^+| = |x + x^- - (y + y^-)| \leq |x - y| + |x^- - y^-| \leq |x - y|$$

and thus $\|x^+ - y^+\| \leq \|x - y\|$. This means that the function

$$\begin{aligned} {}^+ : & X \to X \\ & x \mapsto x^+ \end{aligned}$$

is Lipschitz continuous. The Corollaries 3.14, and 4.14 imply that $u^+ \in W^{1,p}(\Omega, X)$. $\square$

We want to note that one has to be a bit careful when using the above theorem. It does not work if X is not a Banach lattice but only a lattice. A common example of a lattice which is not a Banach lattice is the space $X = W^{1,p}(\Omega, \mathbb{R})$. Theorem 5.16 shows that X is a lattice. However it is not closed when endowed with the L^p-norm. If we endow it with the $W^{1,p}$-norm, then it is not a Banach lattice as well. That is because we can easily find functions $u, v \in W^{1,p}(\Omega, \mathbb{R})$ such that $|u| \leq |v|$ holds a.e. but $\|u\|_{W^{1,p}(\Omega,\mathbb{R})} > \|v\|_{W^{1,p}(\Omega,\mathbb{R})}$. It is also easy to see that $^+ : X \to X$ is not Lipschitz continuous in this case.

As stated in the introdcution, one can compute the weak derivative of u^+ in the real-valued case. We will now extend this to Banach lattices.

Theorem 5.17. *Let X be a Banach lattice that is σ-Dedekind complete and let $u \in W^{1,p}(\Omega, X)$. Suppose that $u^+ \in W^{1,p}(\Omega, X)$ as well, then the weak derivative of u^+ is given by*

$$(u^+)'(t) = P_{u^+(t)} u'(t)$$

for almost all $t \in \Omega$.

Proof. We start with the one-dimensional case $\Omega = I$. In this case u and u^+ are differentiable a.e. thus for almost all t we have

$$\begin{aligned}(u^+)'(t) &= \lim_{h \to 0} \frac{u^+(t+h) - u^+(t)}{h} \\ &= \lim_{h \to 0} \frac{u^+(t+h) - P_{u^+(t)}u(t+h) + P_{u^+(t)}u(t+h) - P_{u^+(t)}u(t)}{h}.\end{aligned}$$

By the linearity of $P_{u^+(t)}$ we obtain

$$\lim_{h \to 0} \frac{P_{u^+(t)}u(t+h) - P_{u^+(t)}u(t)}{h} = P_{u^+(t)}u'(t)$$

and thus the limit

$$\lim_{h \to 0} \frac{u^+(t+h) - P_{u^+(t)}u(t+h)}{h}$$

exists. We have to show that it is equal to 0. We decompose $u^+(t+h)$ into its parts in the projection bands and obtain

$$\frac{u^+(t+h) - P_{u^+(t)}u(t+h)}{h} = \frac{P_{u^+(t)}u^+(t+h) + P_{u^+(t)^\perp}u^+(t+h) - P_{u^+(t)}u(t+h)}{h}.$$

Here we have that

$$\frac{P_{u^+(t)^\perp}u^+(t+h)}{h} = \frac{P_{u^+(t)^\perp}u^+(t+h) - P_{u^+(t)^\perp}u^+(t)}{h} \to P_{u^+(t)^\perp}(u^+)'(t),$$

where $P_{u^+(t)^\perp} u^+(t+h) \geq 0$. Hence we have that the right limit $h \downarrow 0$ is positive and the left limit $h \uparrow 0$ is negative. As both must be equal we obtain that $P_{u^+(t)^\perp}(u^+)'(t) = 0$. We conclude that

$$\begin{aligned}\lim_{h\to 0} \frac{u^+(t+h) - P_{u^+(t)}u(t+h)}{h} &= \lim_{h\to 0} \frac{P_{u^+(t)}u^+(t+h) - P_{u^+(t)}u(t+h)}{h} \\ &= \lim_{h\to 0} \frac{-P_{u^+(t)}u^-(t+h)}{h}.\end{aligned}$$

Again we have that $P_{u^+(t)}u^-(t+h) \geq 0$ and thus letting $h \downarrow 0$ and $h \uparrow 0$ we obtain that the above limit is equal to 0. Hence $(u^+)'(t) = P_{u^+(t)}u'(t)$ as claimed.

For the general case consider representatives of u and u^+ as in Theorem 4.16. For a direction e_i there exists a common null set such that u and u^+ are partially differentiable in direction e_i outside of this set. Further the partial and the weak derivatives coincide. The claim now follows from the one-dimensional case. ☐

Note that the above theorem includes the case $X = \mathbb{R}$. If $u(t) \leq 0$, then $u^+(t) = 0$ and thus $P_{u^+(t)} = 0$. Moreover we can compute the value of $P_{u^+(t)}u'(t)$ in certain other Banach lattices.

Corollary 5.18 (Arendt, Dier, Kramar Fijavz)**.** *Let (S, Σ, μ) be a measure space and let $u \in W^{1,p}(\Omega, L^r(S, \mathbb{R}))$ where $1 < p \leq \infty$ and $1 < r < \infty$. Then $u^+ \in W^{1,p}(\Omega, L^r(S, \mathbb{R}))$ and the weak derivative is given by*

$$(u^+)'(t) = u'(t) \cdot 1_{\{s\in S, u(t)(s)>0\}}.$$

If $\Omega = I$ is an interval, then the case $p = 1$ is true as well.

Proof. The space $L^r(S, \mathbb{R})$ is reflexive, hence Theorem 5.16 implies that $u^+ \in W^{1,p}(\Omega, L^r(S, \mathbb{R}))$. Analogously to Example 5.5 we have that $L^r(S, \mathbb{R})$ is σ-Dedekind complete, thus the weak derivative of u^+ is given by $(u^+)'(t) = P_{u^+(t)}u'(t)$ using Theorem 5.17. Corollary 5.14 implies that

$$P_{u^+(t)}u'(t) = \sup_{n\in\mathbb{N}} u'(t)^+ \wedge nu^+(t) - \sup_{n\in\mathbb{N}} u'(t)^- \wedge nu^+(t).$$

Now a pointwise comparison shows that

$$P_{u^+(t)}u'(t)(\omega) = \begin{cases} u'(t)(s) & \text{if } u(t)(s) > 0 \\ 0 & \text{if } u(t)(s) \leq 0, \end{cases}$$

which is equivalent to the claimed formula. ☐

In the following we want to detect another situation in which u^+ is differentiable. The result in Theorem 5.16 cannot be generalized to other Banach lattices as the proof relies on the criteria for weak differentiability which are equivalent to the Radon-Nikodym property. But as $^+$ is a specific Lipschitz continuous mapping, we do not need to rely on these criteria. As stated in the introduction we may find other cases where u^+ stays weakly differentiable if we assume more regularity. A function $F : X \to Y$ is called *convex* if $F(\lambda x + (1-\lambda)y) \le \lambda F(x) + (1-\lambda)F(x)$ holds for all $x, y \in X$.

Theorem 5.19. *Let X be a Banach space and let $u \in W^{1,p}(\Omega, X)$, where $1 \le p \le \infty$. Let Y be a Banach lattice with order continuous norm such that Y' has a countable, separating subset. Let $F : X \to Y$ be Lipschitz continuous and convex, then $F \circ u \in W^{1,p}(\Omega, Y)$.*

Proof. Again we start with the one-dimensional case $\Omega = I$. We show that the right Gateaux derivatives of F exists, that is $D_y^+ F(x) := \lim_{t \downarrow 0} \frac{F(x+ty)-F(x)}{t}$ exists for all $x, y \in X$. For $0 < s < t$ we have $x + sy = \left(1 - \frac{s}{t}\right) x + \frac{s}{t}(x + ty)$ with $\frac{s}{t} < 1$. Thus by the convexity of F it follows that

$$\frac{F(x+sy) - F(x)}{s} \le \frac{F(x+ty) - F(x)}{t},$$

i.e. the difference quotients are downwards directed and even totally ordered. Further for all $t > 0$ we have

$$F(x) = F\left(\frac{1}{2}(x+ty) + \frac{1}{2}(x-ty)\right) \le \frac{1}{2}F(x+ty) + \frac{1}{2}F(x-ty),$$

which is equivalent to

$$\frac{F(x-tw) - F(x)}{-t} \le \frac{F(x+tw) - F(x)}{t}.$$

Thus the difference quotients are bounded from below which implies that they converge as Y has an order continuous norm. We show that this implies that $F \circ u$ is right differentiable a.e. Let $t_0 \in I$ such that u is differentiable in t_0. There exists an $R_1 : (0, \delta) \to X$, where $\delta > 0$ is chosen appropriately, with $\frac{R_1(h)}{h} \to 0$ as $h \to 0$ such that $u(t_0 + h) = u(t_0) + hu'(t_0) + R_1(h)$. By the right Gateaux differentiability of F there exists an $R_2 : (0, \delta) \to Y$ with $\frac{R_2(h)}{h} \to 0$ as $h \to 0$ such that $F(u(t_0) + h(u'(t_0))) = F(u(t_0)) + hD^+_{u'(t_0)}F(u(t_0)) + R_2(h)$. We compute

$$\begin{aligned}
&\frac{1}{h}\left[F \circ u(t_0+h) - F \circ u(t_0)\right] - D^+_{u'(t_0)}F(u(t_0)) \\
&= \frac{1}{h}\big[F(u(t_0) + hu'(t_0) + R_1(h)) - F(u(t_0) + hu'(t_0)) + F(u(t_0) + hu'(t_0)) - F(u(t_0))\big] \\
&\quad - D^+_{u'(t_0)}F(u(t_0)) \\
&= \frac{1}{h}\big[F(u(t_0) + hu'(t_0) + R_1(h)) - F(u(t_0) + hu'(t_0))\big] + \frac{R_2(h)}{h},
\end{aligned}$$

where the last summand converges to 0. For the first summand we have

$$\frac{1}{h}\|F(u(t_0) + hu'(t_0) + R_1(h)) - F(u(t_0) + hu'(t_0))\| \le \frac{L}{h}\|R_1(h)\| \to 0,$$

where L is the Lipschitz constant of F. We conclude that $F \circ u$ is right differentiable for almost all $t \in I$ with derivative $D_{u'(t)}F(u(t))$. Analogously to the above reasoning one proves that $F \circ u$ is left differentiable as well. Now for any $x' \in Y'$ we have that $x' \circ F$ is Lipschitz continuous, hence $\langle x', F \circ u \rangle \in W^{1,p}(I, \mathbb{R})$ and in particular differentiable a.e. We obtain

$$\left\langle x', \frac{d}{dt^+} F \circ u \right\rangle = \frac{d}{dt^+} \langle x', F \circ u \rangle = \frac{d}{dt^-} \langle x', F \circ u \rangle = \left\langle x', \frac{d}{dt^-} F \circ u \right\rangle,$$

for almost all $t \in I$. For countably many x' we find this to be true for the same $t \in I$. As Y' has a countable separating subset it follows that the left and right derivative of $F \circ u$ coincide a.e. thus $F \circ u$ is differentiable a.e. The function u is locally absolutely continuous, hence $F \circ u$ is locally absolutely continuous as well. Further we have

$$\left\| \frac{d}{dt} F \circ u(t) \right\| \leq L \|u'(t)\|$$

from which we infer that $\frac{d}{dt} F \circ u \in L^p(I, X)$. We conclude that $F \circ u \in W^{1,p}(I, X)$ as claimed.

For general Ω let u be a representant as in Theorem 4.16. The one-dimensional case shows that on any line parallel to the coordinate axes in any $\omega \subset\subset \Omega$ the function $F \circ u$ is still absolutely continuous, differentiable a.e. and that the partial derivatives are L^p-functions. Using the second part of Theorem 4.16 we conclude that $u \in W^{1,p}(\Omega, X)$. $\square$

Example 5.20. The function $^+ : L^1((0,1), \mathbb{R}) \to L^1((0,1), \mathbb{R})$ is Lipschitz continuous and a simple pointwise comparison shows that it is convex as well. Hence for any $u \in W^{1,p}(\Omega, L^1((0,1), \mathbb{R}))$ it follows that $u^+ \in W^{1,p}(\Omega, L^1((0,1), \mathbb{R}))$. Note that $L^1((0,1), \mathbb{R})$ does not have the Radon-Nikodym property.

5.3 Notes

In the first section we tried to give introduction to Banach lattices and prove all results needed for the second section as briefly as possible. Most of the presented results are not the most general ones. We used the books [LZ71], [Sch74] and [MN91].

Apart from the real-valued case, this chapter was mainly motivated by the idea to generalize Corollary 5.18 which was proven directly in [ADKF14] for the case $p = r = 2$, $\Omega = (0, \tau)$ and $S = \Omega$. Note that the proof in that article also works in the more general setting of this thesis. It is not known to the author that the generalization – Theorem 5.17 – was known before. The last theorem of this chapter is based on personal notes of Wolfgang Arendt and [Are82].

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Ich bin mir bewusst, dass eine unwahre Erklärung rechtliche Folgen haben wird.

Ulm, den 15.04.2015 ______________________________

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