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De Giorgi-Nash-Moser estimates for linear parabolic partial differential equations

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1 Introduction

Parabolic partial differential equations describe the evolution of a system over time. They are ubiquitous in any field of science as they are used to model a variety of phenomena but are also an object of independent interest in the field of mathematics. As for all kinds of equations, the main question is the existence and uniqueness of solutions to the equation. However, as it turned out, when dealing with partial differential equations, finding the right concept of a solution can be quite delicate. The most natural one would surely be a function that satisfies the equation pointwise, i.e. is sufficiently times continuously differentiable and satisfies the equation at every point of the domain. Unluckily, as history has shown, it is not possible to directly obtain good existence and uniqueness theorems for a broader range of equations in this class of functions.

It turned out that is is more fruitful to reformulate a partial differential equation as an integral equation and drastically alter the most intuitive notion of a solution. This leads to the concept of weak solutions, weak derivatives and Sobolev spaces. Using functional analytic methods, it is often possible to show the existence (and uniqueness) of a weak solution. But one questions naturally emerges: Are weak solutions actually classical solutions?

Long before the concept of weak solutions was formed, Hilbert published his famous and influential list of 23 problems. The 19th one deals with solutions to variational problems. Hilbert asked whether, under some natural and reasonable assumptions, solutions to it will always be smooth.
The above problem leads to an elliptic partial differential equation in divergence form with rough coefficients; i.e. coefficients that were merely assumed to be bounded and measurable. Until the end of the 1950's, two things were known: First, the existence of a weak solution to this equation was settled. Secondly, the answer to Hilbert's question is affirmative, provided that weak solutions to the partial differential equation are locally Hölder continuous. This seemingly tiny step, however, turned out to be the most intricate one. In [28, p. 3], the author testifies *"It was, arguably, the most important problem in the analysis of partial differential equations at that time."*

This gap was eventually closed by De Giorgi in [6], and almost simultaneously but independently by Nash [24]. Later, Moser gave another proof of this result [20]. All three contributions were very original, using completely new methods. De Giorgi derived certain integral inequalities for weak solution and succeeded in showing that all functions which satisfy these are indeed Hölder continuous. Nash studied fundamental solutions of the corresponding parabolic problem and derived the result for weak solutions of this type of equations. From there, it was easy to deduce the assertion for the original elliptic problem. And Moser's proof is based on a Harnack-type inequality.

Especially the ideas of De Giorgi and Moser turned out to be fruitful, as their original techniques could subsequently be applied to a vast number of further problems, for instance equations including lower-order terms and parabolic and quasi-linear equations. Both techniques are nowadays the cornerstone of regularity theory.

This thesis is concerned with proving interior Hölder continuity of weak solutions to parabolic equations. It is our aim to present both ideas of De Giorgi and Moser in detail. We do not intend to treat the most general kind of equations, but rather concentrate on linear equations in divergence form with no lower-order terms. This class of problems still comprises the main difficulties to overcome and is well-suited to present the techniques that can be adapted to more complex parabolic equations. One might wonder why we desist from presenting the Nash's ideas who first solved the problem for parabolic equations. While it turned out that the other two methods can be extended to nonlinear problems by imposing certain structural conditions, Nash's arguments heavily rely on the linear structure of the equation. For this reason, his approach is not in fashion anymore these days.

This thesis is organized as follows: We first introduce some basic notation and recall various basic facts about partial differential equations. Espacially, we spend a great time discussing several function spaces that show up in the theory.
Next, we state the main theorem, the De Giorgi-Nash-Moser estimates, followed by a detailed proof using the methods of De Giorgi and Moser in the subsequent two chapters.
We assume the reader to be already familiar with weak derivatives and their basic properties, Sobolev spaces and the concept of weak formulations of (elliptic) partial differential equations. Since we will make use of Bochner spaces and Sobolev spaces of vector-valued functions, which might not be standard knowledge, we give a very short presentation in the appendix.

2 Notations and preliminaries

This chapter is devoted to fixing our notation and to state several fundamental results that will be used in this thesis. Most of the following is standard and can be found in any book treating the modern theory of partial differential equations, e.g. [11, 13]. Nonetheless, we list them here for the sake of easier reference and we give proofs if it is possible with reasonable effort.

In the first two sections, basic tools for the treatment of partial differential equations are collected, but our treatment will be very brief as we already expect the reader to be familiar with those concepts. Also, we recall the definitions and elementary properties of various function spaces appearing in the theory.

Finally, we prove several (technical) lemmas that are used elsewhere in this thesis, but do not fit thematically to the previous sections.

Before moving on, we fix some very basic notations. The number $N \in \mathbb{N}$ will always denote the dimension of the Euclidean space. For technical reasons, we assume that $N \geq 2$. Permanently, Ω will stand for a non-trivial bounded domain in $\mathbb{R}^N$. For $x \in \mathbb{R}^N$, its Euclidean norm is denoted by $|x|$. We write $(\cdot|\cdot)$ for the Euclidian scalar product in $\mathbb{R}^N$. The symbol $B(x_0, r)$ stands for the open ball of radius $r > 0$, centered at $x_0 \in \mathbb{R}^N$.

For sets $U, V \subset \mathbb{R}^N$, where U is bounded, we write $U \Subset V$ if $\overline{U} \subset V$.

For an arbitrary measure space (U, μ) and $0 < p \leq \infty$, the Lebesgue spaces $L_p(U)$ are endowed with their usual (quasi-)norm. The Lebesgue measure of a measurable set $O \subset \mathbb{R}^N$ will also be denoted by $|O|$. In case $U = O$ or $U = I \times O$ for an interval I, these spaces are always understood with respect to the Borel σ-algebra and the Lebesgue measure dx, the product measure $dx\, dt$, respectively.
All functions are assumed to be real-valued. For a measurable function $u : U \to \mathbb{R}$ and some $k \in \mathbb{R}$, we write $[u > k]$ for $\{x \in U : \ u(x) > k\}$. The sets $[u < k]$, $[u \geq k]$, etc. are defined analogously.

2.1 Tools from real analysis

The following inequalities are well-known ([11, B.2]):

Proposition 2.1.1 (Young's inequality). *Let $a,b \geq 0,\ p,p' > 1$ with $\frac{1}{p}+\frac{1}{p'}=1$. Then*

$$ab \leq \frac{a^p}{p}+\frac{b^{p'}}{p'}.$$

For the special case $p=q=2$, this is called *Cauchy's inequality*. For given $p>1$, the number p' is called the *conjugated index* of p.

More generally:

Proposition 2.1.2 (Young's inequality with epsilon). *Let $a,b \geq 0,\ \varepsilon>0,\ p,p' > 1$ with $\frac{1}{p}+\frac{1}{p'}=1$. Then*

$$ab \leq \frac{\varepsilon a^p}{p}+\frac{b^{p'}}{\varepsilon^{p'/p}p'}.$$

The particular case $p=p'=2$ is called *Cauchy's inequality with epsilon* .

Proposition 2.1.3 (Hölder's inequality, general version). *Let (U,μ) be a measure space and $p_1,\dots,p_n \in [1,\infty]$, and $\sum_{i=1}^n \frac{1}{p_i}=\frac{1}{r}$ with $r\geq 1$. If $f_i \in L_{p_i}(O),\ i=1,\dots,n,$ then $\prod_{i=1}^n f_i \in L_r(U)$ and $\|\prod_{i=1}^n f_i\|_{L_r(U)} \leq \prod_{i=1}^n \|f_i\|_{L_{p_i}(I)}$.*

For vector-valued functions $f=(f_1,\dots,f_N)\in L_p(U;\mathbb{R}^N),\ p\in[1,\infty)$ the L_p-norms are defined via

$$\|f\|_{L_p(U)} = \left(\sum_{i=1}^N \|f_i\|_{L_p(U)}^p\right)^{1/p}.$$

We will often use the following interpolation inequality for L_p-norms:

Proposition 2.1.4 ([13, p. 149]). *Let (U,μ) be a finite measure space and $1\leq p\leq q\leq r<\infty$ and $\varepsilon>0$. Then for $u\in L_r(U)$*

$$\|u\|_{L_q(U)} \leq \varepsilon\|u\|_{L_r(U)}+\varepsilon^{-\mu}\|u\|_{L_p(U)},$$

where $\mu=\left(\frac{1}{p}-\frac{1}{q}\right)/\left(\frac{1}{q}-\frac{1}{r}\right)$.

We further need the following version of Lebesgue's famous differentiation theorem:

Theorem 2.1.5 ([27, Corollary 1.7]). *Let u be a locally integrable function on $\mathbb{R}\times\mathbb{R}^N$ and let $\tau>0$ be fixed. For arbitrary $(t_0,x_0)\in\mathbb{R}\times\mathbb{R}^N$ denote by $Q(r)=[t_0-\tau r^2,t_0]\times B(x_0,r),\ r>0$. Then*

$$\lim_{r\searrow 0}\frac{1}{|Q(r)|}\int_{Q(r)} u(t,x)\,dx\,dt = u(t_0,x_0)$$

for almost all (t_0,x_0). Further, for fixed $r>0$, the mapping $(t_0,x_0)\mapsto \frac{1}{|Q(r)|}\int_{Q(r)}u(t,x)\,dx\,dt$ is continuous.

2.2 Notes on function spaces

2.2.1 Spaces of continuous functions

For $k \in \mathbb{N}_0$, the set $C^k(\Omega)$ consists of all real-valued functions u such that u and all its partial derivatives with maximal order k are continuous. We will always write $C(\Omega)$ instead of $C^0(\Omega)$; the same holds for all further spaces. For bounded Ω, $C^k(\overline{\Omega})$ stands for the set of functions $u \in C^k(\Omega)$ such that u and all its partial derivatives of order up to k have a continuous extension to the boundary of Ω. We note that we will not (symbolically) distinguish between f and its extension to the boundary. These spaces are always endowed with their canonical norms. We set $C^\infty(\Omega) = \bigcap_{k=0}^\infty C^k(\Omega)$ and define $C^\infty(\overline{\Omega})$ analogously. We call elements of these spaces smooth.

We also need Hölder spaces. For a function $u \in C(\overline{\Omega})$ and $\alpha \in (0,1)$, we define the quantity

$$[u]_\alpha = \sup_{\substack{x_1,x_2\in\overline{\Omega} \\ x_1\neq x_2}} \frac{|u(x_1)-u(x_2)|}{|x_1-x_2|^\alpha}.$$

A function $u \in C(\overline{\Omega})$ is said to be *Hölder continuous*, and we write $u \in C^{0,\alpha}(\overline{\Omega}) = C^\alpha(\overline{\Omega})$, if there exists $\alpha \in (0,1)$ such that $[u]_\alpha < \infty$. This space is endowed with the norm

$$\|u\|_{C^\alpha(\overline{\Omega})} = \|u\|_{C(\overline{\Omega})} + [u]_\alpha.$$

At one point in this thesis, we also need the space $C^{1,\alpha}(\overline{\Omega})$ which we define as

$$C^{1,\alpha}(\overline{\Omega}) = \left\{ u \in C^1(\overline{\Omega}) : \frac{\partial}{\partial x_i} u \in C^\alpha(\overline{\Omega}),\ 1 = 1,\dots,N \right\},$$

where we set

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} = \|u\|_{C^1(\overline{\Omega})} + \max_{i=1,\dots,N} \left[\frac{\partial}{\partial x_i} u \right]_\alpha.$$

We fix the following convention which simplifies our notation: If $\beta \in (1,2)$, we denote by $C^\beta(\overline{\Omega})$ the space $C^{\lfloor\beta\rfloor,\beta-\lfloor\beta\rfloor}(\overline{\Omega})$. Here, $\lfloor\cdot\rfloor$ is the standard floor function.
It is well-known that these spaces are nested, i.e. $C^\beta(\overline{\Omega}) \hookrightarrow C^\alpha(\overline{\Omega})$ whenever $\alpha < \beta, \alpha, \beta \in [0,2]$.

A function $u \in C(\Omega)$ is said to have *compact support*, if the set $\operatorname{supp} u = \overline{\{x \in \Omega : u(x) \neq 0\}} \subset \Omega$ and $\operatorname{supp} u$ is compact. For $k \in \mathbb{N}_0 \cup \{\infty\}$, the set $C_c^k(\Omega)$ contains all functions $u \in C^k(\Omega)$ with compact support.

The following terms are required for the treatment of parabolic equations: Let $I = (a,b]$ or $[a,b]$ be a bounded interval. We denote by Ω_I the *parabolic cylinder* $I \times \Omega$. For it, we introduce two parts of its boundary: The *lateral boundary* $S_I = I \times \partial\Omega$ and the *parabolic boundary* $\Gamma_I = S_I \cup \left(\{a\} \times \overline{\Omega}\right)$, which can be interpreted as the side surface and the whole surface without the top of a cylinder, respectively.
Of particular importance will be parabolic cylinders of the form $(0,T] \times \Omega$ for some $T > 0$, for which we

will simply write Ω_T and S_T, Γ_T denote its lateral and parabolic boundary, respectively.

We introduce the space $\mathring{C}_c(\Omega_{\overline{I}})$ of all functions $u \in C(\Omega_{\overline{I}})$ with $\operatorname{dist}(S_I, \operatorname{supp} u) > 0$. Vividly speaking, the support of these functions has a positive distance from the lateral boundary, but not (necessarily) from the *bottom* $\{a\} \times \Omega$ and the *top* $\{b\} \times \Omega$, respectively. Further, $\dot{C}_c(\Omega_{\overline{I}})$ is the subset of all $u \in \mathring{C}_c(\Omega_{\overline{I}})$ with the stronger property $\operatorname{dist}(\Gamma_I, \operatorname{supp} u) > 0$, i.e. they also vanish in a neighborhood of the bottom of the cylinder.

We say that a function $u \in C(\Omega)$ is piecewise smooth, if for all $x = (x_1, \dots, x_N) \in \Omega$ and all $i = 1, \dots, N$, the mapping $x_i \mapsto u(x_1, \dots, x_i, \dots, x_N)$ is smooth up to only finitely many points. We extend this definition to elements of the spaces $C(\overline{\Omega})$ and $C(\Omega_I)$ in an obvious manner.

2.2.2 Parabolic Hölder spaces

For technical reasons, we have to define Hölder spaces of functions depending on time slightly differently than in the previous subsection.
Let $O \subset \mathbb{R} \times \mathbb{R}^N$ be an arbitrary set containing an inner point. For $\alpha \in (0,1]$, $u \in C(O)$ and $z_1 = (t_1, x_1)$, $z_2 = (t_2, x_2) \in O$, define the seminorm

$$[u]_{\alpha/2,\alpha} = \sup_{\substack{z_1, z_2 \in O \\ z_1 \neq z_2}} \frac{|u(z_1) - u(z_2)|}{(|t_1 - t_2|^{1/2} + |x_1 - x_2|)^\alpha}. \tag{2.1}$$

The exponent $1/2$ added for scaling reasons - its purpose will become clear in Subsection 4.4.

We say that $u : O \to \mathbb{R}$ is Hölder continuous (of order α) and write $u \in C^{\alpha/2,\alpha}(O)$, if

$$\|u\|_{C^{\alpha/2,\alpha}(O)} = \sup_O |u| + [u]_{\alpha/2,\alpha} < \infty.$$

2.2.3 Sobolev spaces

Definition 2.2.1 (Weakly differentiable functions)**.** *We say that a function $u \in L_1(\Omega)$ is* weakly differentiable *with respect to the variable x_i, $i = 1, \dots N$, if there exist functions $v_i \in L_1(\Omega)$ such that for all $\varphi \in C_c^\infty(\Omega)$ the following holds:*

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega v_i \varphi\, dx. \tag{2.2}$$

In this case, we write $\partial_i u = v_i$. If u is weakly differentiable with respect to all of its variables, we put ∇u for the column vector $(\partial_1 u, \dots, \partial_N u)^\intercal$.

Every with respect to x_i (classically) differentiable function u is also weakly differentiable if $u, \frac{\partial u}{\partial x_i} \in L_1(\Omega)$. To shorten our notation, we shall also write $\partial_i u$ instead of $\frac{\partial u}{\partial x_i}$, $i = 1, \dots, N$ for the classical partial derivative. Likewise, every continuous and piecewise smooth function is weakly differentiable and its two types of derivatives coincide up to the set of measure zero where the function is not differentiable (see [19, Corollary 1.73]). Of course, the same integrability conditions have to be imposed.

The *Sobolev space* $W_p^1(\Omega),\ 1 \le p \le \infty$ consists of all $u \in L_p(\Omega)$ that have weak derivatives with respect to every variable which belong to $L_p(\Omega)$ as well. We endow this space with the canonical norm

$$\|u\|_{W_p^1(\Omega)} = \begin{cases} \left(\|u\|_{L_p(\Omega)}^p + \sum_{i=1}^N \|\partial_i u\|_{L_p(\Omega)}^p\right)^{1/p}, & p \in [1,\infty), \\ \max\{\|u\|_{L_\infty(\Omega)}, \|\partial_1 u\|_{L_\infty(\Omega)}, \ldots, \|\partial_N u\|_{L_\infty(\Omega)}\}, & p = \infty. \end{cases}$$

Higher-order Sobolev spaces are defined inductively; for our purposes, we only need the space $W_p^2(\Omega),\ p \in [1,\infty]$ which we define as

$$W_p^2(\Omega) = \left\{u \in W_p^1(\Omega) : \partial_i u \in W_p^1(\Omega),\ i = 1, \ldots, N\right\}.$$

We equip this space with its obvious norm.

We shall write $\mathring{W}_p^1(\Omega) = \overline{C_c^\infty(\Omega)}$, where the closure is taken with respect to the norm $\|\cdot\|_{W_p^1(\Omega)}$.

Basic Facts

We list some well-known results that are used in the subsequent chapters.

Proposition 2.2.2 (Product rule, [13, p. 150])**.** *If $u \in W_p^1(\Omega),\ v \in W_\infty^1(\Omega),\ 1 \le p \le \infty$, then $uv \in W_p^1(\Omega)$ and*

$$\partial_i(uv) = (\partial_i u)v + u\partial_i v$$

for all $i = 1, \ldots, N$.

Proposition 2.2.3 (Chain rule, [30, Theorem 2.1.11])**.** *Let Ω be bounded. Let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous and $u \in W_p^1(\Omega),\ 1 \le p < \infty$. If $f \circ u \in L_p(\Omega)$, then $f \circ u \in W_p^1(\Omega)$ and almost everywhere*

$$\nabla(f \circ u) = (f' \circ u)\nabla u. \tag{2.3}$$

Proposition 2.2.4 (Change of variables, [30, Theorem 2.2.2])**.** *Let $\Omega,\ \tilde{\Omega} \subset \mathbb{R}^N$ be arbitrary domains and let $f : \Omega \to \tilde{\Omega}$ be a diffeomorphism. If $u \in W_p^1(\tilde{\Omega}),\ 1 \le p < \infty$, then $u \circ f \in W_p^1(\Omega)$ and $\nabla(u \circ f) = (\nabla u \circ f) f'$.*

We will also need to know how elements of a Sobolev space interact with lattice operations. For a measurable function u, we set $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0) = (-u)^+$. For a (measurable) set $A \subset \mathbb{R}^N$, we denote by χ_A the *characteristic function* of A.

Proposition 2.2.5 ([30, Corollary 2.1.8])**.** *Let $1 \le p < \infty$ and $u \in W_p^1(\Omega),\ \mathring{W}_p^1(\Omega)$, respectively . Then $u^+,\ u^- \in W_p^1(\Omega),\ \mathring{W}_p^1(\Omega)$, respectively, and $\nabla u^+ = \nabla u \chi_{[u>0]}$ and $\nabla u^- = -\nabla u \chi_{[u<0]}$.*

This proposition, more precisely the following immediate consequence, is of great importance for us: A considerable amount of Chapter 4 consists of a fine study of the functions $u_k^+ = (u-k)^+$ and

$u_k^- = ((-u)-k)^+$ for $k \in \mathbb{R}$. These can be interpreted as the part of the function u that are above the 'level' k and below $-k$, respectively. By Proposition 2.2.5 and the chain rule, we have:

Proposition 2.2.6. *Let $u \in W_p^1(\Omega)$, $\mathring{W}_p^1(\Omega)$ with $1 \le p < \infty$ and $k \in \mathbb{R}$. Then u_k^+, $u_k^- \in W_p^1(\Omega)$, $\mathring{W}_p^1(\Omega)$, respectively, and $\nabla u_k^+ = \nabla u \chi_{[u>k]}$ and $\nabla u_k^- = -\nabla u \chi_{[u<-k]}$.*

Inequalities

We shall need some altered versions of the well-known Poincaré inequality. The first result is taken from [8, Chapter 1, Lemma 2.2] and [17, p. 91].

Proposition 2.2.7. *Let $u \in W_1^1(B(x_0,r))$ for some open ball $B(x_0,r)$. Let $m > n$ be arbitrary numbers. Then there is a constant $C = C(N)$ such that*

$$(m-n)\,|[u>m]| \le C\frac{r^{N+1}}{|[u<n]|}\int_{[n\le u\le m]} |\nabla u|\,dx.$$

The following weighted Poincaré inequality is taken from [21, Lemma 3]. Compare also [18, Lemma 6.12].

Proposition 2.2.8. *Let $\psi \in C(B(x_0,r))$ for some open ball $B(x_0,r)$. Assume that $0 \le \psi \le 1$ and that ψ has non-empty compact support of diameter d. Further, let the sets $[\psi \ge a]$ be convex for all $a \le 1$. Then for all $u \in W_2^1(B(x_0,r))$ there holds*

$$\int_{B(x_0,r)} \left(u-u_\psi\right)^2 \psi\,dx \le \frac{2d^2|\text{supp}\ \psi|}{\|\psi\|_{L_1(B(x_0,r))}}\int_{B(x_0,r)} |\nabla u|^2 \psi\,dx,$$

where

$$u_\psi = \frac{\int_{B(x_0,r)} u\psi\,dx}{\|\psi\|_{L_1(B(x_0,r))}}.$$

The following result is the famous Gagliardo–Nirenberg inequality that we shall not formulate in full generality but in a version that serves our purposes. See for example [17, Chapter 2, Theorem 2.2] or [8, Chapter 1, Theorem 2.1] for details.

Theorem 2.2.9 (Gagliardo–Nirenberg)**.** *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain with piecewise smooth boundary and $u \in \mathring{W}_2^1(\Omega)$. Then there is a constant $C = C(N)$ such that*

$$\|u\|_{L_q(\Omega)} \le C\|\nabla u\|_2^\alpha \|u\|_2^{1-\alpha}, \tag{2.4}$$

where $\alpha = N\left(\frac{1}{2}-\frac{1}{q}\right)$ and

$$\begin{cases} q \in [2,\infty), & \text{if } N = 2, \\ q \in \left[2, \frac{2N}{N-2}\right], & \text{if } N \ge 3. \end{cases} \tag{2.5}$$

Here, piecewise smooth boundary means that it can locally be expressed as the graph of a piecewise smooth function. In particular, the theorem holds for open balls.

2.2.4 Sobolev–Slobodeckij spaces

At one point in this thesis, we need the theory of fractional Sobolev spaces and we only introduce the portion of this theory that is needed later. See [7] and the references therein for a more thorough presentation.

Definition 2.2.10. *Let $p \in [1,\infty)$ and $s \in (0,1)$. For $u \in L_p(\Omega)$, denote*

$$[u]_{W_p^s(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \right)^{1/p}.$$

We say that u belongs to the Sobolev–Slobodeckij space $W_p^s(\Omega)$, *if*

$$\|u\|_{W_p^s(\Omega)} = \left(\|u\|_{L_p(\Omega)}^p + [u]_{W_p^s(\Omega)}^p \right)^{1/p} < \infty.$$

Let us now define these spaces for $s \in (1,2)$:

Definition 2.2.11. *Let $p \in [1,\infty)$, $s = 1+\sigma \in (1,2)$, $\sigma \in (0,1)$. Set*

$$W_p^s(\Omega) = \left\{ u \in W_p^1(\Omega) : \partial_i u \in W_p^\sigma(\Omega),\ i = 1,\dots,N \right\}$$

and define the norm

$$\|u\|_{W_p^s(\Omega)} = \left(\|u\|_{W_p^1(\Omega)}^p + \sum_{i=1}^N \|\partial_i u\|_{W_p^\sigma(\Omega)}^p \right)^{1/p}.$$

We stress the point that $W_p^k(\Omega),\ k = 1,2$ denote the usual Sobolev spaces introduced previously.

We need the following imbedding theorem:

Theorem 2.2.12. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with piecewise smooth boundary and let $p \in [1,\infty),\ s \in (0,2]$ be such that $sp > N$. Then $W_p^s(\Omega) \hookrightarrow C^\alpha(\overline{\Omega})$, for all $\alpha \in (0, s-N/p]$, if $s - N/p \neq 1$ and for all $\alpha \in (0, s-N/p)$ else.*

Proof. Compare [7, Theorem 8.2]. See also [1, p. xviii] . □

2.2.5 Weakly differentiable functions depending on time

Here we introduce various spaces of weakly differentiable functions that depend on time. It turns out to be more convenient to define them by using the language of vector-valued functions. Especially, we will make use of Bochner spaces and Sobolev spaces of vector-valued functions and we assume the reader to be already familiar with these objects. A very brief introduction is given in the Appendix A. For a more extensive treatment, we refer to the references listed there. The reader is advised to go through the appendix first as most of the notation used in the following is introduced there. We restrict ourselves to the spaces that are actually needed in this thesis, not intending at all to give the most general definition possible.

In the following, let $\Omega_I = I \times \Omega$ be a bounded parabolic cylinder. We introduce the two spaces that are important for our purposes:

$$\begin{aligned} W_2^{1,1}(\Omega_I) &= L_2(I; W_2^1(\Omega)) \cap W_2^1(I; L_2(\Omega)), \\ V_2(\Omega_I) &= C(\bar{I}; L_2(\Omega)) \cap L_2(I; W_2^1(\Omega)). \end{aligned}$$

By Theorem $A.10$, we have $W_2^{1,1}(\Omega_I) \subset V_2(\Omega_I)$. On the other hand, elements of the space $V_2(\Omega_I)$ are in general not weakly differentiable with respect to time t.

Using the identification $L_p(I; L_p(\Omega)) = L_p(I \times \Omega)$ for $1 \le p < \infty$ by Proposition $A.7$, we endow these spaces with the respective norms

$$\begin{aligned} \|u\|_{W_2^{1,1}(\Omega_I)} &= \left(\|u\|^2_{L_2(\Omega_I)} + \|\nabla u\|^2_{L_2(\Omega_I)} + \|\partial_t u\|^2_{L_2(\Omega_I)} \right)^{1/2} \\ \|u\|_{V_2(\Omega_I)} &= \left(\max_{t \in \bar{I}} \|u(t,\cdot)\|^2_{L_2(\Omega)} + \|\nabla u\|^2_{L_2(\Omega_I)} \right)^{1/2}. \end{aligned}$$

We emphasize that for $u \in W_2^{1,1}(\Omega_I)$, ∇u shall always stand for the column vector of the weak derivatives of u with respect to the spatial variables $x_1, \dots, x_N$. The weak derivative with respect to time t will be denoted by $\partial_t u$.

Further, we define the following subspaces of $W_2^{1,1}(\Omega_I), V_2(\Omega_I)$, respectively:

$$\begin{aligned} \overset{\bullet}{W}{}_2^{1,1}(\Omega_I) &= L_2(I; \mathring{W}_2^1(\Omega)) \cap \mathring{W}_2^1(I; L_2(\Omega)), \\ \mathring{V}_2(\Omega_I) &= C(\bar{I}; L_2(\Omega)) \cap L_2(I; \mathring{W}_2^1(\Omega)), \end{aligned}$$

see (A.4) for the definition of the space $\mathring{W}_2^1(I; L_2(\Omega))$. Vividly speaking, elements of these subspaces vanish on the parabolic, respectivly, lateral boundary of Ω_I in a weak sense.

Basic Properties

We list some basic results concerning elements of the above spaces.

For $u \in V_2(\Omega_I)$, also $u_k^+,\ u_k^- \in V_2(\Omega_I)$, where the mentioned function are defined as in the previous section. This follows directly from Proposition 2.2.6. The same holds true for $u \in W_2^{1,1}(\Omega_I)$, which is a consequence of Proposition $A.12$. These two assertions stay true if one considers the smaller subspaces

$\mathring{V}_2(\Omega_I)$ and $\overset{\bullet}{W}{}_2^{1,1}(\Omega_I)$ instead.
Further, if $\varphi \in \mathring{C}_c(\Omega_{\bar{I}})$ is piecewise smooth and $u \in V_2(\Omega_I)$, then $u\varphi \in \mathring{V}_2(\Omega_I)$. On the other hand, if $u \in W_2^{1,1}(\Omega_I)$ and $\varphi \in \overset{\bullet}{C}_c(\Omega_{\bar{I}})$ is piecewise smooth, then $u\varphi$ also belongs to $\overset{\bullet}{W}{}_2^{1,1}(\Omega_I)$. This follows from Proposition $A.13$. In the last two assertions, we assume that φ has bounded partial derivatives.

Proposition 2.2.13 ([2, Lemma 6])**.** *Let $u \in W_2^{1,1}(\Omega_I)$ and $t_1, t_2 \in I$ be arbitrary. Then*

$$\int_{t_1}^{t_2} \int_\Omega \partial_t u \, dx \, dt = \int_\Omega u \, dx \Big|_{t_1}^{t_2} = \int_\Omega u(t_2, x) \, dx - \int_\Omega u(t_1, x) \, dx.$$

The well-known Sobolev embedding theorem states that (weak) differentiability implies higher integrability. The following comprises a result about the embeddability into (higher-order) L_p-spaces for the time-dependent case.

Theorem 2.2.14 ([8, Chapter 2, Proposition 3.3])**.** *Let Ω_I be a bounded parabolic cylinder with piecewise smooth boundary and $u \in \mathring{V}_2(\Omega_I)$. Then there is a constant $C = C(N)$ such that*

$$\|u\|_{L_{2+4/N}(\Omega_I)} \leq C \|u\|_{V_2(\Omega_I)}. \tag{2.6}$$

Proof. Since $u(t, \cdot) \in \mathring{W}_2^1(\Omega)$ for $t \in \bar{I}$, we can apply Theorem 2.2.9 to it. Set $q = 2 + 4/N$ and observe that this number satisfies the requirement (2.5) in any dimension. This gives

$$\begin{aligned}
\|u\|_{L_q(\Omega_I)}^2 &= \left(\int_I \|u(t,\cdot)\|_{L_q(\Omega)}^q \, dt \right)^{2/q} \leq C^2 \left(\int_I \|\nabla u(t,\cdot)\|_{L_2(\Omega)}^{\alpha q} \, dt \right)^{2/q} \max_{t \in \bar{I}} \|u(t,\cdot)\|_{L_2(\Omega)}^{2(1-\alpha)} \\
&= C^2 \left(\int_I \|\nabla u(t,\cdot)\|_{L_2(\Omega)}^2 \, dt \right)^{2/q} \max_{t \in \bar{I}} \|u(t,\cdot)\|_{L_2(\Omega)}^{2(1-2/q)} \\
&\leq C^2 \left(\frac{2}{q} \|\nabla u\|_{L_2(\Omega_I)}^2 + \left(1 - \frac{2}{q}\right) \max_{t \in \bar{I}} \|u(t,\cdot)\|_{L_2(\Omega)}^2 \right) \leq C \|u\|_{V_2(\Omega_I)}^2,
\end{aligned}$$

where we used $\alpha q = 2$ and Young's inequality with $p = q/2$. The constant C is the same as in (2.2.9). ☐

A straightforward consequence of this is the following corollary:

Corollary 2.2.15. *Let Ω_I be a bounded parabolic cylinder with piecewise smooth boundary and $u \in \mathring{V}_2(\Omega_I)$ be non-negative. Then*

$$\|u\|_{L_2(\Omega_I)} \leq C \|u\|_{V_2(\Omega_I)} \left| [u > 0] \right|^{1/(N+2)},$$

where $C = C(N)$ is the same constant as in Theorem 2.2.14.

Proof. The proof is a simple consequence of (the general) Hölder's inequality with $r = 2$ and the previous Theorem 2.2.14:

$$\|u\|_{L_2(\Omega_I)} \leq \|u\|_{L_{2+4/N}(\Omega_I)} \|\chi_{[u>0]}\|_{L_{N+2}(\Omega_I)} \leq C \|u\|_{V_2(\Omega_I)} \left| [u > 0] \right|^{1/(N+2)}.$$

□

In Section 3.2, the following deep result will be needed. For it, see $A.14$ for the definition of vector-valued Sobolev–Slobodeckij spaces.

Theorem 2.2.16 ([15, Proposition 3.9, and 3.11])**.** *Let $p \in (1,\infty)$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Then*

$$W_p^1(I;L_p(\Omega)) \cap L_p(I;W_p^2(\Omega)) \hookrightarrow W_p^{1-\theta}(I;W_p^{2\theta}(\Omega)), \qquad \theta \in [0,1].$$

Steklov Averages

Elements of the space $V_2(\Omega_I)$ possess very little regularity, especially they are not necessarily differentiable with respect to time t. Due to this lack of regularity, it is very hard to work with them. It is desirable to approximate them by functions that have this property.
In this section, let $T > 0$, $I = (0,T]$ and $\Omega_I = I \times \Omega$. Depending on what is more convenient in a particular situation, we may consider elements of the space $L_p(\Omega_I)$ ($p \in [1,\infty)$) as elements of the abstract space $L_p(I;L_p(\Omega)) = L_p([0,T];L_p(\Omega))$ and vice versa.

Definition 2.2.17. *For a function $u \in L_1(I;L_1(\Omega))$ and $h \in (0,T)$, we define the* Steklov average $S_h u \in L_1(I;L_1(\Omega))$ *of u by*

$$(S_h u)(t,\cdot) = \begin{cases} \frac{1}{h}\int_{t-h}^{t} u(s,\cdot)\,ds, & \text{if } t \in [h,T], \\ 0, & \text{if } t \in (0,h). \end{cases} \tag{2.7}$$

Occasionally, we call S_h the Steklov operator.

The following theorem will play a crucial role for us. It makes a statement about the convergence of the Steklov averages to the original function in various norms. The price we have to pay is that the convergence is only guaranteed on a smaller sub-cylinder.

Theorem 2.2.18. *Let Ω_I be as above, $\varepsilon \in (0,T)$, $0 < h < \varepsilon$ and $p \in [1,\infty)$.*

(a) If $u \in L_p(I;L_p(\Omega))$, then $S_h u \in C([\varepsilon,T],L_p(\Omega))$.
(b) If $u \in L_p(I;L_p(\Omega))$, then $S_h u \in L_p(I;L(\Omega))$ and $\|u - S_h u\|_{L_p([\varepsilon,T];L_p(\Omega))} \to 0$ as $h \to 0$.
(c) If $u \in C([0,T],L_p(\Omega))$, then $S_h u \in C([\varepsilon,T],L_p(\Omega))$ and $\|u(t,\cdot) - S_h u(t,\cdot)\|_{L_p(\Omega)} \to 0$ as $h \to 0$ for all $t \in [\varepsilon,T]$.

Proof. See [17, Chapter 2, § 4] and [8, Chapter 1, Lemma 3.2]. Compare also [12, Lemma 1.14]. □

We collect some consequences of this theorem:

Proposition 2.2.19. *Let Ω_I be as above. For $\varepsilon \in (0,T)$, consider the sub-cylinder $\Omega_\varepsilon = [\varepsilon,T] \times \Omega$ and let $0 < h < \varepsilon$. For $u \in V_2(\Omega_I)$ there holds:*

(a) $S_h u \in W_2^{1,1}(\Omega_\varepsilon)$ *and*

$$\partial_t (S_h u)(t,\cdot) = (u(t,\cdot) - u(t-h,\cdot))/h \tag{2.8}$$

for almost all $t \in [\varepsilon, T]$. *Furthermore,* $\partial_i (S_h u)(t,\cdot) = S_h(\partial_i u)(t,\cdot)$ *for all* $i = \{1,\ldots,N\}$ *for all* $t \in [\varepsilon, T]$.

(b) *For all* $k \in \mathbb{R}$, $(S_h u)_k^+ \to u_k^+$ *in* $L_2(\Omega_\varepsilon)$ *as* $h \to 0$. *Also, for all* $t \in [\varepsilon, T]$, $(S_h u)_k^+(t,\cdot) \to u_k^+(t,\cdot)$ *in* $L_2(\Omega)$ *as* $h \to 0$. *All assertions remain valid if one consideres* u_k^- *instead.*

Proof. The first assertion in (a) is a direct consequence of Proposition $A.9$, while the second one follows from Fubini's Theorem. Assertion (b) is implied by Theorem 2.2.19. □

2.3 Auxiliary lemmas

This section is a conglomeration of results that do not fit thematically to any of the previous sections. We collect them here in order to follow a straight line in some of our later reasonings, instead of having to pause for some technical arguments.

2.3.1 Fast geometric convergence

The following will be needed for De Giorgi's iterative argument in Chapter 4.

Lemma 2.3.1 ([17, Chapter 2, Lemma 5.7])**.** *Let* Y_ℓ, $\ell \in \mathbb{N}_0$ *be a sequence of non-negative numbers, satisfying*

$$Y_{\ell+1} \le C b^\ell Y_\ell^{1+\gamma}, \tag{2.9}$$

where $b > 1$ *and* $\gamma, C > 0$ *are given numbers. If* $Y_0 \le C^{-1/\gamma} b^{-1/\gamma^2}$, *then* $Y_\ell \to 0$ *as* $\ell \to \infty$.

Proof. We shall show that the elements of the sequence obey $Y_\ell \le C^{-1/\gamma} b^{-1/\gamma^2} b^{-\ell/\gamma}$, from which the assertion follows, since $b > 1$. The proof of this claim is by induction.
For $\ell = 0$, there is nothing to do.
Next,

$$\begin{aligned} Y_{\ell+1} &\le C b^\ell Y_\ell^{1+\gamma} \le C b^\ell \left[C^{-1/\gamma} b^{-1/\gamma^2} b^{-\ell/\gamma} \right]^{1+\gamma} = C b^\ell \left[C^{-1/\gamma} C^{-1} b^{-1/\gamma} b^{-1/\gamma} b^{-1/\gamma - \ell} \right] \\ &= C^{-1/\gamma} b^{-1/\gamma^2} b^{-(\ell+1)/\gamma}. \end{aligned}$$

□

Lemma 2.3.2 ([8, Chapter 1, Lemma 4.2])**.** *Let* Y_ℓ *and* Z_ℓ, $\ell \in \mathbb{N}_0$ *be two sequences of non-negative numbers, satisfying*

$$Y_{\ell+1} \le C b^\ell \left(Y_\ell^{1+\beta} + Y_\ell^\beta Z_\ell^{1+\gamma} \right), \tag{2.10}$$

$$Z_{\ell+1} \leq Cb^{\ell}\left(Y_\ell + Z_\ell^{1+\gamma}\right) \tag{2.11}$$

for all $\ell \in \mathbb{N}_0$, where $b, C > 1$ and $\beta,\ \gamma > 0$ are given numbers. If

$$Y_0 + Z_0^{1+\gamma} \leq (2C)^{-(1+\gamma)/\delta} b^{-(1+\gamma)/\delta^2}, \tag{2.12}$$

where $\delta = \min\{\beta, \gamma\}$, then $Y_\ell,\ Z_\ell \to 0$ as $\ell \to \infty$.

Proof. Define the auxillary sequence $X_\ell = Y_\ell + Z_\ell^{1+\gamma}$, $\ell \in \mathbb{N}_0$. Fix an arbitrary index $\ell \in \mathbb{N}_0$. Note that assumption (2.11) implies

$$Z_{\ell+1}^{1+\gamma} \leq C^{1+\gamma} b^{\ell(1+\gamma)} X_\ell^{1+\gamma}. \tag{2.13}$$

There are two cases: Either $Y_{\ell+1} \leq Z_{\ell+1}^{1+\gamma}$ or $Y_{\ell+1} > Z_{\ell+1}^{1+\gamma}$.
If the first case is true, then by (2.13)

$$X_{\ell+1} \leq 2Z_{\ell+1}^{1+\gamma} \leq 2C^{1+\gamma} b^{\ell(1+\gamma)} X_\ell^{1+\gamma} \leq (2C)^{(1+\gamma)} b^{\ell(1+\gamma)} X_\ell^{1+\gamma}.$$

On the other hand, if the second case holds, then by assumption (2.10)

$$\begin{aligned} X_{\ell+1} < 2Y_{\ell+1} &\leq 2Cb^{\ell}\left(Y_\ell + Z_\ell^{1+\gamma}\right) Y_\ell^{\beta} \leq 2Cb^{\ell} X_\ell Y_\ell^{\beta} \\ &\leq 2Cb^{\ell} X_\ell^{1+\beta} \leq (2C)^{1+\gamma} b^{\ell(1+\gamma)} X_\ell^{1+\beta}. \end{aligned}$$

Hence, in either case,

$$X_{\ell+1} \leq (2C)^{1+\gamma} \left(b^{1+\gamma}\right)^{\ell} X_\ell^{1+\delta}, \qquad \ell \in \mathbb{N}_0.$$

Assumption (2.12) and Lemma 2.3.1 imply that $X_\ell \to 0$ as $\ell \to \infty$, from which the assertion follows. □

2.3.2 Functional inequalities

The following two results are taken from [5, Section 2.1].

Lemma 2.3.3. *Let $0 \leq s_0 < s_1$ and suppose $g : [s_0, s_1] \to [0, \infty)$ is bounded and satisfies*

$$g(\sigma') \leq \theta g(\sigma) + \frac{A}{(\sigma - \sigma')^{\alpha}} + B \tag{2.14}$$

for all $s_0 \leq \sigma' < \sigma \leq s_1$ and for certain constants $A, B \geq 0$, $\theta \in [0,1)$ and $\alpha > 0$. Then there is a constant $C = C(\alpha, \theta)$ such that for all $s_0 \leq \sigma' < \sigma \leq s_1$ there holds

$$g(\sigma') \leq C\left(\frac{A}{(\sigma - \sigma')^{\alpha}} + B\right).$$

Proof. The case $\theta = 0$ is trivial, so we assume $\theta > 0$ in the following. Let $s_0 \leq \sigma' < \sigma \leq s_1$ be fixed and pick an arbitrary $s \in (0,1)$ such that $\theta s^{-\alpha} < 1$. Define the sequence of numbers

$$\sigma_0 = \sigma', \qquad \sigma_{i+1} = \sigma' + (1-s)(\sigma - \sigma') \sum_{j=0}^{i} s^j, \ i \geq 0.$$

Note that $\sigma_{i+1} - \sigma_i = (1-s)s^i(\sigma - \sigma')$ and $\sigma_i < \sigma, \ i \geq 0$. So we have by the assumption of the mapping g:

$$\begin{aligned} g(\sigma') = g(\sigma_0) &\leq \theta g(\sigma_1) + \frac{A}{(\sigma_1 - \sigma_0)^\alpha} + B \\ &\leq \theta \left(\theta g(\sigma_2) + \frac{A}{(\sigma_2 - \sigma_1)^\alpha} + B \right) + \frac{A}{(\sigma_1 - \sigma_0)^\alpha} + B \leq \dots \\ &\leq \theta^k g(\sigma_k) + \left(\frac{A}{(1-s)^\alpha (\sigma - \sigma')^\alpha} + B \right) \sum_{j=0}^{k-1} \left(\theta s^{-\alpha}\right)^j, \end{aligned} \tag{2.15}$$

for all $k \geq 1$. By our particular choice of the number s, the assertion follows for $k \to \infty$. □

Lemma 2.3.4. *Let $g, h : (0,R] \to \mathbb{R}$ be non-decreasing, $R > 0$. Suppose that for $\eta, \theta \in (0,1)$ there holds*

$$g(\eta r) \leq \theta g(r) + h(r), \qquad 0 < r \leq R.$$

Then for any $\mu \in (0,1)$

$$g(r) \leq \frac{1}{\theta} \left(\frac{r}{R} \right)^{(1-\mu)\frac{\log\theta}{\log\eta}} g(R) + \frac{h\left(R^{1-\mu} r^\mu\right)}{1-\theta}, \qquad 0 < r \leq R.$$

Remark 2.3.5. Note that the exponent $(1-\mu)\frac{\log\theta}{\log\eta}$ is strictly positive.

Proof. Pick an arbitrary $r_1 \in (0,R]$. By monotonicity,

$$g(\eta r) \leq \theta g(r) + h(r_1), \qquad 0 < r \leq r_1.$$

Take $n \in \mathbb{N}$. Then inductively

$$\begin{aligned} g(\eta^n r_1) &\leq \theta g(\eta^{n-1} r_1) + h(r_1) \leq \theta \left(\theta g(\eta^{n-2} r_1) + h(r_1) \right) + h(r_1) \\ &\leq \dots \leq \theta^n g(r_1) + h(r_1) \sum_{i=0}^{n-1} \theta^i \leq \theta^n g(R) + \frac{h(r_1)}{1-\theta}, \end{aligned}$$

since $\theta \in (0,1)$.
Now, for any $0 < r \leq r_1$ there is a $n \in \mathbb{N}$ with $\eta^n r_1 < r \leq \eta^{n-1} r_1$. We deduce

$$\eta^n r_1 < r \Leftrightarrow n \log \eta < \log\left(\frac{r}{r_1}\right) \Leftrightarrow n \log\eta \log\theta > \log\theta \log\left(\frac{r}{r_1}\right)$$

$$\Leftrightarrow \log\left(\theta^{n\log\eta}\right) > \log\left(\left(\frac{r}{r_1}\right)^{\log\theta}\right) \Leftrightarrow \theta^{n\log\eta} > \left(\frac{r}{r_1}\right)^{\log\theta} \Leftrightarrow \theta^n < \left(\frac{r}{r_1}\right)^{\frac{\log\theta}{\log\eta}}.$$

Hence

$$g(r) \le g(\eta^{n-1}r_1) \le \frac{\theta^n}{\theta}g(R) + \frac{h(r_1)}{1-\theta} \le \frac{1}{\theta}\left(\frac{r}{r_1}\right)^{\frac{\log\theta}{\log\eta}} + \frac{h(r_1)}{1-\theta}.$$

Selecting $r_1 = R^{1-\mu}r^{\mu}$ proves the assertion. □

2.3.3 Moser iteration and an abstract lemma

Here, we develop the machinery needed in Chapter 5. Since the following tools are used in different situations, it is more convenient to formulate them in a more abstract setting. We follow [5, Section 2.3] and [29, Section 2.5] here.
Throughout this section, $U_\sigma,\ 0<\sigma\le 1$ is a nested family of measurable subsets with non-zero measure of a fixed finite measure space (U,μ), i.e. $U_{\sigma'} \subset U_\sigma$ whenever $0<\sigma'\le\sigma\le 1$.

The first three results are about Moser's iteration technique. They all assume essentially the same: A higher L_p-(quasi-)norm on a smaller set of a function u can be estimated in terms of u in a lower (quasi-)norm on a larger set. It then turns out that these gains in integrability can be accumulated.

It is convenient to work with the function

$$\Phi(p,\sigma) = \|u\|_{L_p(U_\sigma)} \tag{2.16}$$

in the following. The function u will be clear from the context.

Lemma 2.3.6. *Let $\kappa>1$, $\overline{p}\ge 1$, $C\ge 1$ and $\nu>0$. Let $u\in L_{\overline{p}}(U_1)$ and suppose that for all $\gamma>0$ there holds*

$$\|u\|_{L_{\gamma\kappa}(U_{\sigma'})} \le \left(\frac{C(1+\gamma)^\nu}{(\sigma-\sigma')^\nu}\right)^{1/\gamma} \|u\|_{L_\gamma(U_\sigma)}, \qquad 0<\sigma'<\sigma\le 1. \tag{2.17}$$

Then there are numbers $M = M(\kappa,\nu,\overline{p},C)$ and $\nu_0 = \nu_0(\kappa,\nu)$ such that

$$\operatorname*{ess\,sup}_{U_\theta}|u| \le \left(\frac{M}{(1-\theta)^{\nu_0}}\right)^{1/p}\|u\|_{L_p(U_1)} \text{ for all } \theta\in(0,1) \text{ and } p\in(0,\overline{p}]. \tag{2.18}$$

Proof. Let p and θ be fixed and define the increasing sequence of exponents $p_i = p\kappa^i,\ i\ge 0$ and a sequence of indices σ_i via

$$\sigma_0 = 1, \qquad \sigma_i = 1-\sum_{j=1}^{i}2^{-j}(1-\theta), \qquad i\ge 1. \tag{2.19}$$

Now observe that $\sigma_i > \theta,\ i \geq 0$ and $\sigma_{i-1} - \sigma_i = 2^{-i}(1-\theta) > 0,\ i \geq 1$. So

$$1 = \sigma_0 > \sigma_1 > \cdots > \sigma_i > \cdots > \theta.$$

We make use of the function (2.16) in the following. Pick an arbitrary $n \in \mathbb{N}$. Since (2.17) holds for arbitrary $\gamma > 0$, we have for $\gamma = p_i,\ i = 0,\ldots,n-1$

$$\begin{aligned}
\Phi(p_n,\theta) &\leq \Phi(p_{n-1}\kappa,\sigma_n) \leq \left(\frac{C\left(1+p\kappa^{n-1}\right)^{\nu}}{\left(2^{-n}(1-\theta)\right)^{\nu}}\right)^{1/p_{n-1}} \Phi(p_{n-1},\sigma_{n-1}) && (2.20)\\
&\leq \left(\frac{C^n\left(2\overline{p}\kappa^{n-1}\right))^{\nu}}{\left(2^{-n}(1-\theta)\right)^{\nu}}\right)^{1/p_{n-1}} \Phi(p_{n-1},\sigma_{n-1}) \leq \left(\frac{C_1^n\kappa^{\nu(n-1)}}{(1-\theta)^{\nu}}\right)^{1/p_{n-1}} \Phi(p_{n-1},\sigma_{n-1}) \leq \ldots\\
&\leq \left(C_1^{\sum_{j=0}^{n-1}(j+1)\kappa^{-j}}\kappa^{\nu\sum_{j=0}^{n-1}j\kappa^{-j}}(1-\theta)^{-\nu\sum_{j=0}^{n-1}\kappa^{j}}\right)^{1/p}\Phi(p_0,\sigma_0)\\
&\leq \left(\frac{M}{(1-\theta)^{\nu_0}}\right)^{1/p}\Phi(p,1). && (2.21)
\end{aligned}$$

Here, $C_1 = C_1(\nu,\overline{p},C) = \left(\frac{2\overline{p}}{2^{-1}}\right)^{\nu} C,\ M = M(\kappa,\nu,\overline{p},C) = C_1^{\kappa^2/(\kappa-1)^2}\kappa^{\gamma\kappa/(\kappa-1)^2}$ and $\nu_0 = \nu_0(\kappa,\nu) = \kappa\nu/(\kappa-1)$.
As (2.21) is independent of n and finite, we let $n \to \infty$. Since in this case $\lim_{n\to\infty}\Phi(p_n,\theta) = \|u\|_{L_\infty(U_\theta)}$, the proof is complete.

□

If (2.17) only holds for sufficiently large γ, this does not change much:

Lemma 2.3.7. *Let $\kappa > 1,\ C \geq 1$ and $\nu,\ p > 0$. Let $u \in L_p(U_1)$ and suppose that for all $\gamma \geq p$*

$$\|u\|_{L_{\gamma\kappa}(U_{\sigma'})} \leq \left(\frac{C(1+\gamma)^{\nu}}{(\sigma-\sigma')^{\nu}}\right)^{1/\gamma}\|u\|_{L_\gamma(U_\sigma)}, \qquad 0 < \sigma' < \sigma \leq 1.$$

Then there are numbers $M = M(\kappa,\nu,p,C)$ and $\nu_0 = \nu_0(\kappa,\nu)$ such that for all $\theta \in (0,1)$

$$\operatorname*{ess\,sup}_{U_\theta}|u| \leq \left(\frac{M}{(1-\theta)^{\nu_0}}\right)^{1/p}\|u\|_{L_p(U_1)}.$$

The proof of this assertion is the same, except that one has to replace $\overline{p}$ by p in the lines following (2.20).

We now discuss the situation that an estimate like (2.17) does not hold for arbitrary large values of γ:

Lemma 2.3.8. *Let $\mu(U_1) \leq 1,\ \kappa > 1,\ 0 < p_0 < \kappa,\ \nu > 0$ and $C \geq 1$. Assume that $u : U_1 \to \mathbb{R}$ is measurable and satisfies*

$$\|u\|_{L_{\gamma\kappa}(U_{\sigma'})} \leq \left(\frac{C}{(\sigma-\sigma')^{\nu}}\right)^{1/\gamma}\|u\|_{L_\gamma(U_\sigma)}, \qquad 0 < \sigma' < \sigma \leq 1,\ 0 < \gamma \leq \frac{p_0}{\kappa} < 1. \tag{2.22}$$

Then there are numbers $M = M(\kappa,\nu,C)$ and $\nu_0 = \nu_0(\kappa,\nu)$ such that

$$\|u\|_{L_{p_0}(U_\theta)} \le \left(\frac{M}{(1-\theta)^{\nu_0}}\right)^{1/p-1/p_0} \|u\|_{L_p(U_1)} \text{ for all } \theta\in(0,1),\ p\in\left(0,\frac{p_0}{\kappa}\right].$$

Proof. Again, we use the sequence σ_i defined by (2.19) but this time define the sequence of exponents differently, namely, put $p_i = p_0\kappa^{-i},\ i\ge 1$. Clearly, these numbers fulfill the requirements of (2.22). Pick an arbitrary $n\in\mathbb{N}$. For the function (2.16) there holds for $i = 1,\dots,n$

$$\Phi(p_0,\theta) \le \Phi(p_1\kappa,\sigma_n) \le \frac{C^{\kappa/p_0}}{(2^{-n}(1-\delta))^{\kappa\nu/p_0}}\Phi(p_1,\sigma_{n-1}) \le \dots$$
$$\le \frac{C^{\frac{1}{p_0}\sum_{j=1}^n \kappa^j}}{2^{-\frac{\nu}{p_0}\sum_{j=1}^n j\kappa^{n+1-j}}(1-\theta)^{\frac{\nu}{p_0}\sum_{j=1}^n \kappa^j}}\Phi(p_n,1).$$

Using $\kappa^n = p_0/p_n$ gives

$$\frac{1}{p_0}\sum_{j=1}^n \kappa^j = \frac{\kappa(\kappa^n-1)}{p_0(\kappa-1)} = \frac{\kappa}{\kappa-1}\frac{1}{p_0}\left(\frac{p_0}{p_n}-1\right) = \frac{\kappa}{\kappa-1}\left(\frac{1}{p_n}-\frac{1}{p_0}\right).$$

Since

$$\sum_{j=1}^n j\kappa^{j-1} = \frac{1-(n+1)\kappa^n + n\kappa^{n+1}}{(\kappa-1)^2},$$

we infer

$$\begin{aligned}\sum_{j=1}^n (n+1-j)\kappa^j &= (n+1)\sum_{j=1}^n \kappa^j - \sum_{j=1}^n j\kappa^j\\ &= (n+1)\kappa\frac{\kappa^n-1}{\kappa-1} - \kappa\frac{1-(n+1)\kappa^n+n\kappa^{n+1}}{(\kappa-1)^2}\\ &= \kappa\frac{\kappa^{n+1}-(n+1)\kappa+n}{(\kappa-1)^2} \le \frac{\kappa}{(\kappa-1)^2}\kappa^{n+1}\\ &\le \frac{\kappa^3}{(\kappa-1)^3}(\kappa^n-1) \le \frac{\kappa^3}{(\kappa-1)^3}\left(\frac{p_0}{p_n}-1\right).\end{aligned}$$

So

$$\frac{1}{p_0}\sum_{j=1}^n (n+1-j)\kappa^j \le \frac{\kappa^3}{(\kappa-1)^3}\left(\frac{1}{p_n}-\frac{1}{p_0}\right),$$

which in turn implies

$$\Phi(p_0,\theta) \le \left(\frac{2^{\frac{\nu\kappa^3}{(\kappa-1)^3}}C^{\frac{\kappa}{\kappa-1}}}{(1-\theta)^{\frac{\nu\kappa}{(\kappa-1)}}}\right)^{\frac{1}{p_n}-\frac{1}{p_0}}$$

Since the sequence p_i is decreasing, there is a $n \geq 2$ such that $p_n < p \leq p_{n-1}$ for $p \in (0, p_0/\kappa]$. Therefore

$$\frac{1}{p_n} - \frac{1}{p_0} = \frac{\kappa^n - 1}{p_0} \leq \frac{\kappa^n + \kappa^{n-1} - \kappa - 1}{p_0} = \frac{(1+\kappa)(\kappa^{n-1} - 1)}{p_0}$$
$$= (1+\kappa)\left(\frac{1}{p_{n-1}} - \frac{1}{p_0}\right) \leq (1+\kappa)\left(\frac{1}{p} - \frac{1}{p_0}\right).$$

Also, $1 < p/p_n$. Since we assume $\mu(U_1) = 1$, Hölder's inequality yields

$$\Phi(p_n, 1) \leq \Phi(p, 1)\mu(U_1)^{1-p_n/p} = \Phi(p, 1).$$

Combining all above results reveals

$$\Phi(p_0, \theta) \leq \left(\frac{M}{(1-\theta)^{\nu_0}}\right)^{1/p-1/p_0} \Phi(p, 1),$$

where $M = M(\kappa, \nu, C) = \left(2^{\frac{\nu\kappa^3}{(\kappa-1)^3}} C^{\frac{\kappa}{\kappa-1}}\right)^{1+\kappa}$ and $\nu_0 = \nu_0(\kappa, \nu) = \left(\frac{\kappa\nu}{\kappa-1}\right)^{1+\kappa}$. □

The following abstract lemma, due to Bombieri and Giusti [3], is the main device in the proof of the weak Harnack inequality in Section 5.2. The following formulation is taken from [29, Lemma 2.5.3], whereas we follow [12, Lemma 5.13] in the proof. See also [25, Lemma 2.2.6].

Lemma 2.3.9. *Let $\eta,\ \theta \in (0,1)$, $\nu > 0$, $C \geq 1$ and $0 < \gamma_0 \leq \infty$. Let $u \in L_{\gamma_0}(U_1)$ be strictly positive. Further, suppose u satisfies the following two conditions:*

(a)

$$\|u\|_{L_{\gamma_0}(U_{\sigma'})} \leq \left(\frac{C}{\mu(U_1)(\sigma - \sigma')^\nu}\right)^{1/\gamma - 1/\gamma_0} \|u\|_{L_\gamma(U_\sigma)}, \tag{2.23}$$

for all σ', σ and γ with $0 < \theta \leq \sigma' < \sigma \leq 1$ and $0 < \gamma \leq \min(1, \eta\gamma_0)$.

(b)

$$\mu(U_1 \cap [\log u > \alpha]) \leq C\mu(U_1)\alpha^{-1} \tag{2.24}$$

for all $\alpha > 0$.

Then there is a constant $M = M(\gamma_0, \eta, \nu, \theta, C)$ such that

$$\|u\|_{L_{\gamma_0(U_\theta)}} \leq M\mu(U_1)^{1/\gamma_0}. \tag{2.25}$$

Proof. By dividing both sides of (2.23) by $\mu(U_1)^{1/\gamma_0}$, we see that we may assume $\mu(U_1) = 1$.
For $\sigma \in [\theta, 1]$, define the monotone function $g(\sigma) = \log \|u\|_{L_{\gamma_0}(U_\sigma)}$. Note that g is bounded by our positivity and integrability assumptions. We may further assume that g is a strictly positive function. If this were not the case, then $g(\theta) \leq g(\sigma) \leq 0$ for a σ in the above interval. Hence, $\|u\|_{L_{\gamma_0}(U_\theta)} \leq \|u\|_{L_{\gamma_0}(U_\sigma)} \leq 1$ and the proof of the lemma would already be completed.

Now fix $\sigma \in (\sigma', 1]$ and decompose the set U_σ into

$$U_\sigma^1 = U_\sigma \cap [\log u > g(\sigma)/2] \text{ and } U_\sigma^2 = U_\sigma \cap [\log u \le g(\sigma)/2].$$

Since $\gamma_0/\gamma > 1$, Hölder's inequality and (2.24) yield

$$\begin{aligned}\|u\|_{L_\gamma(U_\sigma^1)} &\le \|u\|_{L_{\gamma_0}(U_\sigma^1)} \mu(U_\sigma^1)^{1/\gamma - 1/\gamma_0} = \exp\left(\log \|u\|_{L_{\gamma_0}(U_\sigma^1)}\right) \mu(U_\sigma^1)^{1/\gamma-1/\gamma_0} \\ &\le e^{g(\sigma)} \left(\frac{2C}{g(\sigma)}\right)^{1/\gamma-1/\gamma_0}.\end{aligned}$$

Further, since $\mu(U_\sigma^2) \le 1$,

$$\|u\|_{L_\gamma(U_\sigma^2)} = \|\exp(\log u)\|_{L_\gamma(U_\sigma^2)} \le e^{g(\sigma)/2)}.$$

So

$$\|u\|_{L_\gamma(U_\sigma)} \le e^{g(\sigma)} \left(\frac{2C}{g(\sigma)}\right)^{1/\gamma-1/\gamma_0} + e^{g(\sigma)/2}. \tag{2.26}$$

Step 2 We examine whether it is possible to find a $\gamma^* \in (0, \min(1, \eta\gamma_0)]$ such that both terms in the right-hand side of (2.26) are equal. Clearly, this can only be achieved if

$$2C < g(\sigma). \tag{2.27}$$

Furthermore, the second factor in the first term has to be equal to $e^{-g(\sigma)/2}$. Solving this for γ^* and recalling the admissible range gives

$$0 < \gamma^* = \left(\frac{g(\sigma)}{2\log\left(\frac{g(\sigma)}{2C}\right)} + \frac{1}{\gamma_0}\right)^{-1} \text{ and } \gamma^* \le \min(1, \eta\gamma_0). \tag{2.28}$$

Clearly, the two conditions (2.27) and (2.29) could be both fulfilled if $g(\sigma)$ is sufficiently large. Let us assume that

$$g(\sigma) > M_1, \tag{2.29}$$

for a suitable number $M_1 = M_1(\gamma_0, \eta, C)$ such that both (2.27) and (2.29) hold. Then, such a γ^* exists. So

$$\|u\|_{L_{\gamma^*}(U_\sigma)} \le 2e^{g(\sigma)/2}.$$

This and (2.23) yield an estimate for $\theta \le \sigma' < \sigma \le 1$, namely

$$\begin{aligned} g(\sigma') &\le \log\left(\left(\frac{C}{(\sigma-\sigma')^\nu}\right)^{1/\gamma^*-1/\gamma_0} \|u\|_{L_{\gamma^*}(U_\sigma)}\right) = \left(\frac{1}{\gamma^*} - \frac{1}{\gamma_0}\right)\log\left(\frac{C}{(\sigma-\sigma')^\nu}\right) + \log\left(\|u\|_{L_{\gamma^*}(U_\sigma)}\right) \\ &\le \left(\frac{1}{\gamma^*} - \frac{1}{\gamma_0}\right)\log\left(\frac{C}{(\sigma-\sigma')^\nu}\right) + \frac{g(\sigma)}{2} + \log(2)\end{aligned}$$

$$= \frac{g(\sigma)}{2}\left(\frac{\log\left(\frac{g(\sigma)}{(\sigma-\sigma')^{\nu}}\right)}{\log\left(\frac{g(\sigma)}{2C}\right)}+1\right)+\log(2),$$

where we used the definition of γ^* in (2.28) in the last step. Now, suppose further that

$$\frac{4C^3}{(\sigma-\sigma')^{2\nu}} \le g(\sigma), \tag{2.30}$$

ich which case we can estimate

$$g(\sigma') \le \frac{3}{4}g(\sigma)+\log(2).$$

Step 3 Now, if (2.29) is violated, we conclude $g(\sigma') \le g(\sigma) \le M_1 \le \frac{M_1}{(\sigma-\sigma')^{2\nu}}$. Also, if (2.30) fails to hold, we also have the similar upper bound $g(\sigma') \le \frac{4C^3}{(\sigma-\sigma')^{2\nu}}$. So in any case, there is a constant $M_2 = M_2(\gamma_0,\eta,C)$ such that

$$g(\sigma') \le \frac{3}{4}g(\sigma)+\frac{M_2}{(\sigma-\sigma')^{2\nu}}, \text{ for all } \theta \le \sigma' < \sigma \le 1.$$

Applying Lemma 2.3.3 with $\sigma' = \theta$ and $\sigma = 1$ finishes the proof. □

3 De Giorgi-Nash-Moser estimates

In this chapter we state the famous theorem of De Giorgi, Nash and Moser that will be tackled by various strategies in the subsequent chapters.

3.1 Statement of the theorem

For the rest of this thesis, let $T > 0$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain . We consider the equation

$$\partial_t u - \operatorname{div}(A\nabla u) = f \qquad \text{on } \Omega_T, \tag{3.1}$$

where $\Omega_T = (0,T] \times \Omega$.

We make the following assumptions:

(A1) $A \in L_\infty\left(\Omega_T; \mathbb{R}^{N\times N}\right)$, $A = (a_{k\ell})$. In particular, there is a number $\Lambda > 0$ such that

$$\sum_{k,\ell=1}^{N} |a_{k\ell}(t,x)|^2 \leq \Lambda^2, \qquad \text{for almost all } (t,x) \in \Omega_T.$$

(A2) There is a number $\lambda > 0$ such that

$$(A(t,x)\xi \,|\, \xi) \geq \lambda |\xi|^2, \qquad \text{for almost all } (t,x) \in \Omega_T \text{ and all } \xi \in \mathbb{R}^N.$$

(A3) The inhomogenity f satisfies $f \in L_q(\Omega_T)$ for a $q \in (1+N/2, \infty)$.

Remark 3.1.1. Note that the numerical criterion in **(A3)** is equivalent to the existence of a fixed number $\kappa_1 \in (0,1)$ such that

$$\frac{1}{q} + \frac{N}{2q} = 1 - \kappa_1. \tag{3.2}$$

We will use this alternative frequently in the subsequent chapters.

Clearly, we first have to define what is meant by a weak solution to this equation.

Definition 3.1.2. *A function $u \in V_2(\Omega_T)$ is said to be a weak solution (sub-/super-)solution to* (3.1) *on Ω_T, if*

$$\int_\Omega u(t_1,x)\varphi(t_1,x)\,dx + \int_0^{t_1}\int_\Omega -u\partial_t\varphi + (A\nabla u|\nabla\varphi)\,dx\,dt = (\leq,\geq)\int_0^{t_1}\int_\Omega f\varphi\,dx\,dt \tag{3.3}$$

for all $t_1 \in (0,T]$ and all $\varphi \in \overset{\bullet}{W}{}_2^{1,1}(\Omega_T)$ ($0 \leq \varphi \in \overset{\bullet}{W}{}_2^{1,1}(\Omega_T)$ in case of sub-/supersolutions).

Clearly, if u is a solution, then it is also a sub- and super-solution. Also, if u is a sub-solution to equation (3.1), then $-u$ is a also super-solution if one replaces the right-hand side by $-f$.

We fix the convention that throughout this thesis, whenever we refer to (sub-, super-)solutions to equation (3.1), we shall always assume that the assumptions **(A1)-(A3)** are in force, without mentioning that explicitly.

We can now state the main theorem of this thesis:

Theorem 3.1.3 (De Giorgi-Nash-Moser)**.** *Let $u \in V_2(\Omega_T)$ be a weak solution to* (3.1) *on Ω_T and let $\Omega'_T = [a,b]\times\Omega' \Subset \Omega_T,\ 0 < a < b \leq T,\ \Omega' \Subset \Omega$ be an arbitrary parabolic cylinder that is compactly contained in Ω_T.*
Then there are numbers $\alpha = \alpha(\lambda,\Lambda,q,N) \in (0,1)$ and $C = C(\lambda,\Lambda,\Omega'_T,d,q,N)$, where $d = \operatorname{dist}(\Omega'_T,\Gamma_T)$ is the distance between Ω'_T and the parabolic boundary of Ω_T, such that

$$u \in C^{\alpha/2,\alpha}(\overline{\Omega'_T}) \text{ and } \|u\|_{C^{\alpha/2,\alpha}(\overline{\Omega'_T})} \leq C\left(\|u\|_{L_2(\Omega_T)} + \|f\|_{L_q(\Omega_T)}\right). \tag{3.4}$$

In other words, every weak solution to (3.1) has a representative that is locally Hölder continuous.
Note carefully that both α and C are independent of u and f. Also, this is a result about interior regularity only: The cylinder Ω'_T is assumed to have a positive distance from the parabolic boundary Γ_T of Ω_T (it is however admissible that the intersection with the top $\{T\}\times\Omega$ is non-empty).

3.2 Remark on assumption (A3)

The integrability assumption **(A3)** requires some explanation. First, it can be shown that this assumption is optimal, i.e. Theorem 3.1.3 fails to hold if the right-hand side f only belongs to $L_q(\Omega_T)$ with $q \leq 1+N/2$, see [17, Chapter 1, §3, Example 3]. As this result alone provides no intuition for why the theorem might be true under assumption **(A3)**, we elaborate on this assumption further.

We consider the simplest example of equation (3.1), namely we assume the matrix A to be the identity matrix. Then (3.1) becomes the well-known heat equation

$$u_t - \Delta u = f \qquad \text{on } \Omega_T, \tag{3.5}$$

where we impose the condition $f \in L_q(\Omega_T),\ q \in \left(\frac{3}{2}, \infty\right)$. This restriction has the following motivation, namely the solution u will possesses even higher regularity in the interior of the cylinder Ω_T:

Theorem 3.2.1 ([17, Chapter 3, Theorem 12.1]). *Let $u \in V_2(\Omega_T)$ be a weak solution to* (3.5) *on Ω_T with $f \in L_q(\Omega_T)$ for $q \in \left(\frac{3}{2}, \infty\right)$. Then for any $[a,b] \times \Omega' \Subset \Omega_T$, $0 < a < b \leq T$, $\Omega' \Subset \Omega$ there holds*

$$u \in L_q([a,b]; W_q^2(\Omega')) \cap W_q^1([a,b]; L_q(\Omega'))$$

For the sake of simplicity, we assume the set Ω' to have smooth boundary. If the conditions of this result are satisfied, we can apply Theorem 2.2.16 and see that

$$u \in W_q^{1-\theta}([a,b]; W_q^{2\theta}(\Omega')), \qquad \theta \in [0,1],\ q \in [2, \infty). \tag{3.6}$$

For the moment being, we fix an arbitrary $\theta \in (0,1)$. The space $W_q^{1-\theta}([a,b]; W_q^{2\theta}(\Omega'))$ will be embedded in $C^{1-\theta-\frac{1}{q}}([a,b]; W_q^{2\theta}(\Omega'))$, while $W_q^{2\theta}(\Omega') \hookrightarrow C^{\alpha}(\overline{\Omega'})$ for any $\alpha \in (0, 2\theta - N/q)$ provided that

$$\begin{aligned} (1-\theta)q > 1 &\Leftrightarrow (1-\theta) > \frac{1}{q}, \\ 2\theta q > N &\Leftrightarrow \theta > \frac{N}{2q}, \end{aligned} \tag{3.7}$$

by Theorem $A.15$ and Theorem 2.2.12.
Adding the two latter inequalities in (3.7) gives us, keeping (3.6) in mind,

$$1 > \frac{1}{q} + \frac{N}{2q} \text{ and } q < \infty. \tag{3.8}$$

This inequality is independent of the number θ and exactly condition **(A3)**. It is easily verified that for q in that range and for $\theta = N/(N+2)$, the conditions in (3.7) will both be satisfied.
Therefore, by virtue of (3.6), the two embedding theorems mentioned above and Remark $A.3$, we see that $u \in C^{\delta/2,\delta}([a,b] \times \overline{\Omega'})$ for a certain $\delta \in (0,1)$.

So if assumption **(A3)** is in force, we see that (neglecting the additional assumption added for the sake of simplicity) Theorem 3.1.3 is true for the simplest example of equation (3.1). Hence, we can *hope for* the validity of the theorem for more general equations under the assumptions **(A1)**-**(A3)**. That this is indeed the case is of course the aim of this thesis.

3.3 A weak formulation based on Steklov averages

It is remarkable that elements of the space V_2 are not necessarily weakly differentiable with respect to time t, and yet it is possible to state a reasonable weak formulation to (3.1) in this class of functions. However, as already pointed out, this minimality assumption on the regularity makes it hard to work with these functions. We thus deliver a short paragraph in which we derive a weak formulation based on Steklov averages which turns out to be more practical. Its main benefit is that the time derivative appearing in the second term of (3.3) is shifted from the test function φ to the Steklov average of u (which has the required regularity for this operation). In the subsequent chapters, we will then be able to derive workable identities for the limit $h \to 0$, using Theorem 2.2.18.

The following two lemmas are based on [12, Section 5.2] and [8, p. 17-18].

Lemma 3.3.1. *Let $u \in V_2(\Omega_T)$ be a weak (sub-/super-)solution to equation* (3.1) *on Ω_T. Then for all $0 < t_0 < t_1 \le T$ and all $\varphi \in \overset{\bullet}{W}{}_2^{1,1}(\Omega_T)$ (for all $0 \le \varphi \in \overset{\bullet}{W}{}_2^{1,1}(\Omega_T)$)*

$$\int_\Omega u(t,x)\varphi(t,x)\,dx\Big|_{t_0}^{t_1} + \int_{t_0}^{t_1}\int_\Omega -u\partial_t\varphi + (A\nabla u|\nabla\varphi)\,dx\,dt = (\le,\ \ge)\int_{t_0}^{t_1}\int_\Omega f\varphi\,dx\,dt. \tag{3.9}$$

Proof. We only prove the assertion for weak sub-solutions as the other cases are treated similarly.
Let $t_0,\ t_1 \in (0,T]$, $t_0 < t_1$ be fixed and take $h \in (0,t_0)$. We define the piecewise smooth function $\chi : (0,T] \to [0,1]$ which is equal to zero for $t \le t_0 - h$, constantly equal to one for $t \ge t_0$ and linear on the intermediate interval (so $\partial_t\chi(t) = 1/h$ on this segment). Pick an arbitrary $0 \le \psi \in \overset{\bullet}{W}{}_2^{1,1}(\Omega_T)$ and insert the function $\varphi = \chi\psi \in \overset{\bullet}{W}{}_2^{1,1}(\Omega_T)$ in (3.3). This gives

$$\int_\Omega u(t_1,x)\psi(t_1,x)\,dx + \int_{t_0-h}^{t_1}\int_\Omega -u\partial_t(\chi\psi)\,dx\,dt + \int_{t_0-h}^{t_1}\chi\int_\Omega (A\nabla u|\nabla\psi)\,dx\,dt \tag{3.10}$$

$$\le \int_{t_0-h}^{t_1}\int_\Omega f\chi\psi\,dx\,dt. \tag{3.11}$$

By virtue of assumption **(A1)** concerning the matrix A, it is clear that for $h \to 0$, the third term in (3.10) converges to its corresponding term in (3.9). The same is true for (3.11), using assumption **(A3)**.
We turn our attention to the second term in (3.10) and deploy the Steklov averages introduced in Section 2.2.5. From the product rule A.13 and Fubini's Theorem it follows

$$\begin{aligned}\int_{t_0-h}^{t_1}\int_\Omega -u\partial_t(\psi\chi)\,dx\,dt &= -\int_{t_0-h}^{t_0}\int_\Omega \frac{1}{h}u\psi\,dx\,dt + \int_{t_0-h}^{t_1}\int_\Omega -u\chi\partial_t\psi\,dx\,dt\\ &= -\int_\Omega S_h(u\psi)(t_0,x)\,dx + \int_{t_0-h}^{t_1}\int_\Omega -u\chi\partial_t\psi\,dx\,dt\\ &\longrightarrow -\int_\Omega u(t_0,x)\psi(t_0,x)\,dx + \int_{t_0}^{t_1}\int_\Omega -u\chi\partial_t\psi\,dx\,dt\end{aligned}$$

as $h \to 0$, by Theorem 2.2.18. Since ψ was arbitrary and $\chi(t) = 1$ for $t \ge t_0$, the proof is complete. □

Unfortunately, we are forced to abuse our notation a bit, since the next proposition involves Steklov averages of vector-valued functions. More concretely, we have to apply it to the vector $A\nabla u$. We fix the convention that in this situation, the Steklov operator is applied to every component of the vector.

Proposition 3.3.2. *Let $u \in V_2(\Omega_T)$ be a weak (sub-/super-)solution to equation* (3.1) *on Ω_T. Then for all*

$\varepsilon \in (0,T)$, $h \in (0,\varepsilon)$, $\psi \in \mathring{W}_2^1(\Omega)$ $(0 \le \psi \in \mathring{W}_2^1(\Omega))$ *and almost all* $t \in [\varepsilon, T]$ *there holds*

$$\int_\Omega \partial_t (S_h u)(t,x)\psi(x)\,dx + \int_\Omega \left(S_h(A\nabla u)(t,x)|\nabla\psi(x)\right)\,dx = (\le,\ge)\int_\Omega (S_h f)(t,x)\psi(x)\,dx, \tag{3.12}$$

where S_h *is the Steklov operator.*

Note that by Proposition 2.2.19(a), all integrands in (3.12) are well-defined.

Proof. Again, we only consider weak sub-solutions u. By Proposition 2.2.19, for all $t \in [\varepsilon,T]\setminus N$, where N is a set of zero measure, the identity (2.8) holds. Take h as above and t such that $t, t-h \in [\varepsilon,T]\setminus N$ and insert t for t_1 and $t-h$ for t_0 in (3.9). Also, we plug in a specialized function $0 \le \varphi \in \dot{W}_2^{1,1}(\Omega_T)$, namely, we assume φ to be independent on time on the interval $[t-h,t]$. More precisely, we take $\varphi(s,x) = \chi(s)\psi(x)$, where $0 \le \psi \in \mathring{W}_2^1(\Omega)$ and $0 \le \chi(s)$ is a piecewise smooth function that is equal to one for $s \ge t-h$ and vanishes for $s < \varepsilon$.
In this particular situation, (3.9) becomes, using Proposition 2.2.13 and dividing both sides by h,

$$\begin{aligned}
\frac{1}{h}\int_\Omega [u(t,x)-u(t-h,x)]\psi(x)\,dx + \frac{1}{h}\int_{t-h}^t\int_\Omega (A\nabla u(s,x)|\nabla\psi(x))\,dx\,ds \\
= \int_\Omega \partial_t(S_h(u\psi))(t,x)\,dx + \int_\Omega S_h\left(A\nabla u|\nabla\psi\right)(t,x)\,dx \\
= \int_\Omega \partial_s(S_h u)(t,x)\psi(x)\,dx + \int_\Omega \left(S_h(A\nabla u)(t,x)|\nabla\psi(x)\right)\,dx \\
\le \frac{1}{h}\int_{t-h}^t\int_\Omega f(s,x)\psi(x)\,dx\,ds = \int_\Omega S_h(f\psi)(t,x)\,dx = \int_\Omega (S_h f)(t,x)\psi(x)\,dx.
\end{aligned}$$

We made use of the linearity of the Steklov operator. Also, since the function ψ depends on x only, we can move it out of the integral in the definition of the Steklov averages. This proves (3.12).

□

4 The De Giorgi-method for parabolic equations

We give the first proof of Theorem 3.1.3 by adapting De Giorigi's original ideas in [6] to the parabolic case. As the proof is quite lengthy and evolved, we first give a brief, informal outline of the proof.

As the Hölder norm consists of two terms, we split the problem in showing that both parts are finite. We first deal with the local boundedness. The whole proof is based on the idea that a function u will be bounded from, say, above, if u_k^+ is zero for sufficiently large k. Of course, the main question is how to find such a number k.

The first step in this direction is the establishment of a certain inequality for the truncated function u_k^+, on sub-cylinders of a special form, where u is a weak sub-solution. This inequality has an interesting structure: It allows us to estimate the V_2-norm of u_k^+ on a shrinked cylinder in terms of u_k^+ on the original, larger cylinder. By using the same inequality again with yet another smaller sub-cylinder, we set up an iterative argument, which ultimately leads to a condition that reveals how to select a number k with $u_k^+ = 0$. This is De Giorgi's famous iteration technique.

Having established the local boundedness, we approach the second part of the Hölder norm. Clearly, the term (2.1) does not make sense for arbitrary functions of some Sobolev space, so we must look for a measure-theoretic counterpart. Recall that for bounded and continuous functions u, its *oscillation* on a set O is definded by $\operatorname{osc}_O u = \sup_O u - \inf_O u$. As the numerator of (2.1) is bounded from above by the oscillation of u on O and since this term can be re-defined for measurable functions, we opt to study the (essential) oscillation of weak solutions. This is a long and difficult step. We first deal with two lemmas about the local behaviour of these functions in a separate section. They form the key device for obtaining good estimates for the oscillation.

In the fourth (and short!) section, all previous results come on stage to answer one final question: Given a weak solution, how can one construct a representative that is Hölder continuous?

4.1 Local boundedness of weak solutions

Roughly speaking, we establish the first half of the estimate (3.4) in this section. Namely, we show that every weak solution to equation (3.1) is locally bounded.

We start with some notation: For arbitrary $t_0 \in (0,T]$, $x_0 \in \Omega$, and $\tau, r > 0$ we consider parabolic cylinders of the form $Q(t_0,x_0,\tau,r) = [t_0 - \tau r^2, t_0] \times B(x_0,r)$. The reason for considering this particular type of cylinders will become clear below. We will often write Q, B instead of $Q(t_0,x_0,\tau,r)$, $B(x_0,r)$, respectively, if no confusion is to be expected. We fix the convention that whenever we write $Q \Subset \Omega_T$, we implicitly assume that t_0, x_0, τ and r are in such a way that this expression is meaningful, without mentioning this directly. Note carefully that Q and $\{T\} \times \Omega$ may have a nonempty intersection.

For functions $u \in V_2(\Omega_T)$ with $Q \Subset \Omega_T$ and $k \in \mathbb{R}$, we adopt the notation u_k^+, u_k^- of Chapter 2, i.e.

$$u_k^+ = \max(u-k,0), \qquad u_k^- = \max(-u-k,0).$$

Furthermore, we introduce the sets

$$\begin{aligned} A_{k,r}^+(t) &= \{x \in B(x_0,r) : \ u(t,x) - k > 0\}, \\ A_{k,r}^-(t) &= \{x \in B(x_0,r) : \ -u(t,x) - k > 0\}, \end{aligned}$$

where $t \in [t_0 - \tau r^2, t_0]$.
For $\delta \in (0,1)$, we consider the shrinked cylinders $\delta Q = [t_0 - \delta \tau r^2, t_0] \times B(x_0, \delta r)$.

We state our first result:

Lemma 4.1.1. *Let $u \in V_2(\Omega_T)$ be a weak sub-solution to equation* (3.1) *on Ω_T and let $Q \Subset \Omega_T$. Then there is a constant $C = C(\lambda, \Lambda, N)$, such that for any $\delta \in (0,1)$ and $k \in \mathbb{R}$ there holds*

$$\begin{aligned} \|u_k^+\|_{V_2(\delta Q)}^2 \le C \Bigg(&\int_Q \left(\frac{1}{(1-\delta)\tau r^2} + \frac{1}{(1-\delta)^2 r^2} \right) \left(u_k^+\right)^2 \, dx\, dt \\ &+ \|f\|_{L_q(Q)}^2 \left(\int_{t_0 - \tau r^2}^{t_0} |A_{k,r}^+(t)| \, dt \right)^{2(1+2\kappa)/\hat{q}} \Bigg), \end{aligned} \tag{4.1}$$

where $\kappa = 2\kappa_1/N$ with κ_1 as in Remark 3.1.1, *$\hat{q} = 2q'(1+\kappa) = 2 + 4/N$ and q' is the conjugated index of q.*
The same inequality holds for super-solutions u if one consideres u_k^- and $A_{k,r}^-(t)$ instead.

Recall that if u is a super-solution to (3.1), then $-u$ is a sub-solution to (3.1) with the right-hand side replaced by $-f$. So it suffices to show the first part of the assertion.

Proof. Let $\psi \in \dot{C}_c(\Omega_T)$ be piecewise smooth, $0 \le \psi \le 1$ and assume its restriction to Q (which we also denote by ψ) vanishes on the lateral boundary of Q. Since $Q \Subset \Omega_T$, there is an $\varepsilon > 0$ with $Q \Subset [\varepsilon, T] \times \Omega$, so Proposition 3.3.2 and Theorem 2.2.18 are applicable on the cylinder Q. As $(S_h u)_k^+ \in W_2^{1,1}(Q)$ it follows $(S_h u)_k^+ \psi^2 \in \mathring{W}_2^{1,1}(Q)$, where $h > 0$ is sufficiently small. Thus, for $t \in [t_0 - \tau r^2, t_0]$, we have $(S_h u)_k^+(t,\cdot)\psi^2(t,\cdot) \in \mathring{W}_2^1(B)$. Pick an arbitrary $t_1 \in [t_0 - \tau r^2, t_0]$, insert this function in (3.12) and integrate both sides from $t_0 - \tau r^2$ to t_1. This gives

$$\begin{aligned} &\int_{t_0-\tau r^2}^{t_1} \int_B \partial_t (S_h u)(S_h u)_k^+ \psi^2 \,dx\,dt + \int_{t_0-\tau r^2}^{t_1} \int_B \left(S_h(A\nabla u) | \nabla (S_h u)_k^+\right) \psi^2 \,dx\,dt \\ &+ \int_{t_0-\tau r^2}^{t_1} \int_B 2\left(S_h(A\nabla u)|\nabla \psi\right) \psi (S_h u)_k^+ \,dx\,dt \le \int_{t_0-\tau r^2}^{t_1} \int_B (S_h f)(S_h u)_k^+ \psi^2 \,dx\,dt. \end{aligned} \tag{4.2}$$

Notice that the first integrand in (4.2) can be replaced by $\partial_t (S_h u)_k^+ (S_h u)_k^+ \psi^2$, as integration is actually only performed on the set where $(S_h u)_k > 0$ holds. By applying the chain and product rule $A.12$ and $A.13$, we observe that this term can be written as

$$\partial_t (S_h u)_k^+ (S_h u)_k^+ \psi^2 = \frac{1}{2} \partial_t \left((S_h u)_k^+ \psi\right)^2 - \left((S_h u)_k^+\right)^2 \psi \partial_t \psi.$$

We return to (4.2) and use this identity and Proposition 2.2.13:

$$\frac{1}{2} \int_B \left((S_h u)_k^+ \psi\right)^2 (t,x)\,dx \Big|_{t_0-\tau r^2}^{t_1} + \int_{t_0-\tau r^2}^{t_1} \int_B \left(S_h(A\nabla u)|\nabla (S_h u)_k^+\right) \psi^2 \,dx\,dt \tag{4.3}$$

$$+ \int_{t_0-\tau r^2}^{t_1} \int_B 2\left(S_h(A\nabla u)|\nabla \psi\right) \psi (S_h u)_k^+ - \left((S_h u)_k^+\right)^2 \psi \partial_t \psi \,dx\,dt \tag{4.4}$$

$$\le \int_{t_0-\tau r^2}^{t_1} \int_B (S_h f)(S_h u)_k^+ \psi^2 \,dx\,dt. \tag{4.5}$$

From this, we pass to the limit $h \to 0$, using Theorem 2.2.18 and Proposition 2.2.19. Here, the assumptions **(A1)** and **(A3)** come into play: The former one implies that the second integrand in (4.3), respectively the first integrand in (4.4) belong to $L_1(Q)$. The same holds for (4.5), using assumption **(A3)**. This gives

$$\begin{aligned} &\frac{1}{2} \int_B \left(u_k^+\right)^2 \psi^2 \,dx \Big|_{t_0-\tau r^2}^{t_1} + \int_{t_0-\tau r^2}^{t_1} \int_B \left(A\nabla \left(u_k^+\right) | \left(u_k^+\right)\right) \psi^2 \,dx\,dt \\ &+ \int_{t_0-\tau r^2}^{t_1} \int_B 2\left(A\nabla \left(u_k^+\right)|\nabla \psi\right) \psi u_k^+ - \left(u_k^+\right)^2 \psi \partial_t \psi \,dx\,dt \\ &\le \int_{t_0-\tau r^2}^{t_1} \int_B f u_k^+ \psi^2 \,dx\,dt. \end{aligned} \tag{4.6}$$

Step 2 From now on, we write u_k instead of $(u_k)^+$.
Note that by assumption **(A2)**,

$$\lambda \int_{t_0-\tau r^2}^{t_1} \int_B |\nabla u_k|^2 \psi^2 \,dx\,dt \le \int_{t_0-\tau r^2}^{t_1} \int_B (A\nabla u_k|\nabla u_k)\, \psi^2 \,dx\,dt.$$

By Cauchy's inequality with $\varepsilon = \frac{\lambda}{2}$, the term $2u_k\psi\,|(A\nabla u_k|\nabla\psi)|$ can be estimated from above by $2\frac{\Lambda^2}{\lambda} u_k^2 |\nabla\psi|^2 + \frac{\lambda}{2}|\nabla u_k|^2\psi^2$, where Λ is the constant appearing in assumption **(A1)**.

Combining these observations with (4.6) yields, after shifting some terms to the right-hand side

$$\frac{1}{2}\|(u_k\psi)(t,\cdot)\|^2_{L_2(B)}\Big|_{t_0-\tau r^2}^{t_1} + \lambda\int_{t_0-\tau r^2}^{t_1}\int_B |\nabla u_k|^2\psi^2\,dx\,dt$$
$$\leq \int_{t_0-\tau r^2}^{t_1}\int_B \frac{\lambda}{2}|\nabla u_k|^2\psi^2 + u_k^2\left(2\frac{\Lambda^2}{\lambda}|\nabla\psi|^2 + \psi|\partial_t\psi|\right)dx\,dt + \int_{t_0-\tau r^2}^{t_1}\int_B |f|u_k\psi^2\,dx\,dt,$$

so,

$$\frac{1}{2}\|(u_k\psi)(t,\cdot)\|^2_{L_2(B)}\Big|_{t_0-\tau r^2}^{t_1} + \frac{\lambda}{2}\int_{t_0-\tau r^2}^{t_1}\int_B |\nabla u_k|^2\psi^2\,dx\,dt$$
$$\leq \int_{t_0-\tau r^2}^{t_1}\int_B u_k^2\left[2\frac{\Lambda^2}{\lambda}|\nabla\psi|^2 + \psi|\partial_t\psi|\right]dx\,dt + \int_{t_0-\tau r^2}^{t_1}\int_B |f|u_k\psi^2\,dx\,dt.$$

Evaluating $\frac{1}{2}\|(u_k\psi)(t,\cdot)\|^2_{L_2(B)}\big|_{t_0-\tau r^2}^{t_1} = \frac{1}{2}\|(u_k\psi)(t_1,\cdot)\|^2_{L_2(B)} - \frac{1}{2}\|(u_k\psi)(t_0-\tau r^2,\cdot)\|^2_{L_2(B)}$ and moving the latter term to the right-hand side gives us, after recalling that t_1 was arbitrary,

$$\max_{t_0-\tau r^2\leq t\leq t_0}\|(u_k\psi)(t,\cdot)\|^2_{L_2(B)} + \lambda\int_Q |\nabla u_k|^2\psi^2\,dx\,dt \leq \|(u_k\psi)(t_0-\tau r^2,\cdot)\|^2_{L_2(B)}$$
$$+C_1\int_Q \left(|\nabla\psi|^2 + \psi|\partial_t\psi|\right)u_k^2\,dx\,dt + 2\int_Q |f|u_k\psi^2\,dx\,dt, \tag{4.7}$$

with a constant $C_1 = C_1(\lambda,\Lambda)$.

Step 3 We now take a closer look at the last term in (4.7). Let us first describe the idea behind the next step: As we know that $u_k\psi$ is an element of the space $\mathring{V}_2(Q)$ (by the assumptions we posed on ψ), it seems wise to use this fact to proceed further. Theorem 2.2.14 gives us the embedding $\mathring{V}_2(Q) \hookrightarrow L_{2+4/N}(Q)$. We opt to apply the general Hölder inequality to the last term in (4.7), choosing the number $2+\frac{4}{N}$ as one of the exponents, whence we shall be able to apply Theorem 2.2.14 to this term in the right-hand side Hölder's inequality.

First, we perform some algebraic manipulations: By Remark 3.1.1, there is a fixed number $\kappa_1 \in (0,1)$ with

$$\frac{1}{q} + \frac{N}{2q} = 1 - \kappa_1.$$

This implies

$$\frac{1}{2q'} + \frac{N}{4q'} = \frac{1}{2}\left(1-\frac{1}{q}\right)\left(1+\frac{N}{2}\right) = \frac{N}{4} + \frac{\kappa_1}{2} = \frac{N}{4}(1+\kappa),$$

in which q' is the conjugated index of q and $\kappa = 2\kappa_1/N$. From this, we immediately obtain the equality $\hat{q} = 2q'(1+\kappa) = 2+4/N$. Setting $q^* = \hat{q}/(1+2\kappa)$, one easily checks the validity of

$$1 = \frac{1}{q} + \frac{1}{\hat{q}} + \frac{1}{q^*}.$$

We denote by χ for the characteristic function of the subset of Q, where $u_k > 0$ and return to (4.7) :

$$\begin{aligned} 2\int_{t_0-\tau r^2}^{t_0}\int_{A_k^+(t)} |f|u_k\psi^2\,dx\,dt &\le 2\|f\|_{L_q(Q)}\|u_k\psi\|_{L_{\hat{q}}(Q)}\|\psi\chi\|_{L_{q^*}(Q)} \\ &\le 2C_2\|f\|_{L_q(Q)}\|u_k\psi\|_{V_2(Q)}\|\psi\chi\|_{L_{q^*}(Q)} \\ &\le \varepsilon C_2^2\|u_k\psi\|^2_{V_2(Q)} + \frac{1}{\varepsilon}\|f\|^2_{L_q(Q)}\|\psi\chi\|^2_{L_{q^*}(Q)}, \end{aligned} \tag{4.8}$$

for an arbitrary $\varepsilon > 0$ and a constant $C_2 = C_2(N)$. Further,

$$\|u_k\psi\|^2_{V_2(Q)} \le 2\left(\max_{t_0-\tau r^2\le t\le t_0}\|(u_k\psi)(t,\cdot)\|^2_{L_2(B)} + \|(\nabla u_k)\psi\|^2_{L_2(Q)} + \|u_k\nabla\psi\|^2_{L_2(Q)}\right). \tag{4.9}$$

Next, we combine all intermediate results (4.7), (4.8) and (4.9): First, we choose $\varepsilon = \varepsilon(\lambda,\Lambda,N)$ so small that $2\varepsilon C_2^2 = \frac{1}{2}\min(1,\lambda)$. Further, we impose the function ψ to vanish on the whole parabolic boundary on Q. In the following order, we shift the first two terms in the parenthesis of (4.9) to the left-hand side of (4.7), combine similar remaining terms and subsequently divide by $\frac{1}{2}\min(1,\lambda)$. This gives

$$\begin{aligned} &\max_{t_0-\tau r^2\le t\le t_0}\|(u_k\psi)(t,\cdot)\|^2_{L_2(B)} + \|(\nabla u_k)\psi\|^2_{L_2(Q)} \\ &\quad\le C_3\left(\|u_k\nabla\psi\|^2_{L_2(Q)} + \int_Q u_k^2\psi|\partial_t\psi|\,dx\,dt + \|f\|^2_{L_q(Q)}\|\psi\chi\|^2_{L_{q^*}(Q)}\right) \end{aligned}$$

for a certain constant $C_3 = C_3(\lambda,\Lambda,N)$.

Step 4 The final step of the proof is very simple: We take ψ as before but further impose that it can be decomposed (on the cylinder Q) as $\psi(t,x) = \varphi(t)\omega(x)$, where $\varphi : [t_0-\tau r^2, t_0] \to [0,1]$ and $\omega : B(x_0,r) \to [0,1]$ are both continuous and piecewise smooth. Additionally, we assume φ to be constantly equal to one on $[t_0-\delta\tau r^2, t_0]$, vanishes for $t = t_0-\tau r^2$ and that φ is linear on the segment in between. Similar restrictions are put on ω. This ensures $|\partial_t\psi| \le \frac{1}{(1-\delta)\tau r^2}$ and $|\nabla\psi|^2 \le \frac{N}{(1-\delta)^2 r^2}$. This concludes the proof.

□

We now make the key step towards the local boundedness of weak solutions.

Theorem 4.1.2. *Let $u \in V_2(\Omega_T)$ be a weak sub-solution to* (3.1) *on Ω_T and let $Q = [t_0-\tau r^2, t_0] \times B(x_0,r) \Subset \Omega_T$. Then there is a constant $C = C(\lambda,\tau,\Lambda,q,N)$ such that*

$$\operatorname*{ess\,sup}_{\frac{1}{2}Q} u \le C\left(\left(\frac{1}{|Q|}\int_Q u^2\,dx\,dt\right)^{1/2} + r^{2-(N+2)/q}\|f\|_{L_q(Q)}\right) < \infty. \tag{4.10}$$

An immediate consequence of this assertion is also the boundedness from below of weak super-solutions and altogether, the boundedness of solutions on the cylinder $\frac{1}{2}Q$.

But before we begin with the actual proof, we spend a few words on a crucial feature of equation (3.1):

scaling invariance. This property will considerably simplify most of our arguments in this thesis.

Take u and Q as above and put $\tilde{x} = \frac{x-x_0}{r}$, $\tilde{t} = \frac{t-t_0}{r^2}$ for $x \in B(x_0,r)$ and $t \in [t_0 - \tau r^2, t_0]$. Obviously, $\tilde{x} \in B(0,1)$ and $\tilde{t} \in [-\tau, 0]$. With these new variables, consider the function $\tilde{u}(\tilde{t},\tilde{x}) = u(r^2\tilde{t} + t_0, r\tilde{x} + x_0)$, which is clearly in $V_2(Q(0,0,\tau,1))$ by Proposition 2.2.4. Likewise, with the aid of the same Proposition, $\tilde{u}$ satisfies the definition of a weak sub-solution (3.3) on $Q(0,0,\tau,1)$ (where we replace $\varphi(t,x)$ by $\tilde{\varphi}(\tilde{t},\tilde{x}) = \varphi(r^2\tilde{t} + t_0, r\tilde{x} + x_0))$ for the new right-hand $\tilde{f}(\tilde{t},\tilde{x}) = r^2 f(r^2\tilde{t} + t_0, r\tilde{x} + x_0) \in L_q(Q(0,0,\tau,1))$ and matrix $\tilde{A}(\tilde{t},\tilde{x}) = A(r^2\tilde{t} + t_0, r\tilde{x} + x_0)$. Notice that the key assumptions **(A1)-(A2)** remain unchanged, in particular, there is no alteration of the numbers λ and Λ.
So if we assume (4.10) to be true, then

$$
\begin{aligned}
\operatorname*{ess\,sup}_{\frac{1}{2}Q(0,0,\tau,1)} \tilde{u} = \operatorname*{ess\,sup}_{\frac{1}{2}Q} u &\le C\left(\left(\frac{1}{|Q|}\int_Q u^2\,dx\,dt\right)^{1/2} + r^{2-(N+2)/q}\|f\|_{L_q(Q)}\right)\\
&= C\left(\left(\frac{r^{-(N+2)}}{|Q|}\int_{Q(0,0,\tau,1)} \tilde{u}^2\,d\tilde{x}\,d\tilde{t}\right)^{1/2} + r^{-(N+2)/q} r^{(N+2)/q}\|\tilde{f}\|_{L_q(Q(0,0,\tau,1))}\right)\\
&= C\left(\left(\frac{1}{|Q(0,0,\tau,1)|}\int_{Q(0,0,\tau,1)} \tilde{u}^2\,d\tilde{x}\,d\tilde{t}\right)^{1/2} + \|\tilde{f}\|_{L_q(Q(0,0,\tau,1))}\right).
\end{aligned}
$$

So to summarize, we may assume without loss of generality that $(t_0,x_0) = (0,0)$ and $r = 1$ in the above theorem.

Since the proof of Theorem 4.1.2 is quite sophisticated, we first spend a few words on the key idea of it, which might otherwise be concealed by technicalities. The estimate above is simply the statement that u is bounded from above on $\frac{1}{2}Q$ in disguise. So far, however, we are only in possession of estimates of u in some versions of the L_2-norm and it is by no means obvious how to turn them into estimates in the L_∞-setting.

The basis of the whole proof is the crucial but strikingly simple observation that u will be bounded from above if and only if $u_{2M}^+ = 0$ for sufficiently a large number M (the factor 2 will be explained soon) - or equivalently that $\|u_{2M}^+\| = 0$ for some norm $\|\cdot\|$. This different way of looking at the problem seems to be quite promising, taking Lemma 4.1.1 into account, which appears to be custum-tailored for our situation: The V_2-norm of u_{2M}^+ on a smaller cylinder can be estimated from above in terms of u_{2M}^+ on a larger cylinder. However, we face a serious problem here: The structure of the above inequality provides no information at all for which number $2M$ its right-hand side vanishes.

So the direct approach to go down from the larger cylinder Q to the smaller one $\frac{1}{2}Q$ is not working - the step in the scale is simply too huge and prevents us from obtaining viable information. The crucial idea is to squeeze in a carefully constructed sequence of nested cylinders $\frac{1}{2}Q \Subset \cdots \Subset Q_{\ell+1} \Subset Q_\ell \Subset \cdots \Subset Q_0 \Subset Q$. Then, instead of making one large step from $\frac{1}{2}Q$ to Q, we opt to make (infinitely) many smaller steps from one of the intermediate cylinders to the next. With the aid of Lemma 4.1.1 and by a careful construction of the cylinders, we will end up with a recursive system of sequences. These will (finally!) tell us how to choose the number M: If chosen correctly, both sequences converge to zero, which will be enough to deduce the assertion of the theorem.

Let us make this more precise and also introduce some notation in order to keep the proof short and clean:

For $\ell = -1, 0, 1, \dots$ the decreasing sequence of radii τ_ℓ, r_ℓ are defined by $\tau_\ell = \tau/2 + \tau/2^{\ell+2}$, $r_\ell = 1/2 + 1/2^{\ell+2}$, with which we associate the nested family of cylinders $Q_\ell = [-\tau_\ell, 0] \times B(0, r_\ell)$. Further, by $\hat{r}_\ell = \frac{1}{2}(r_\ell + r_{\ell+1})$, $\hat{\tau}_\ell = \frac{1}{2}(\tau_\ell + \tau_{\ell+1})$ we denote the median of two subsequent radii.

Also, we introduce an increasing sequence of numbers $k_\ell = 2M - \frac{M}{2^\ell} \nearrow 2M$ for a positive number $M \geq \|f\|_{L_q(Q)}$ to be specified later. It is handy to write F for $\|f\|_{L_q(Q)}$.

We fix the convention that for the rest of this subsection - in contrast to the previous usage - we will write $u_\ell = (u - k_\ell)^+$ instead of $u^+_{k_\ell}$ in order to avoid too many indices.
Again, the sets $A_{k_\ell, r_\ell}(t) = \{x \in B(0, r_\ell) : \ u(t,x) - k_\ell > 0\}$, $t \in [-\tau_\ell, 0]$ will be used. As a shorthand notation, we write $A_\ell(t) = A_{k_\ell, r_\ell}(t)$. Lastly, the set $\{x \in B(0, \hat{r}_\ell) : \ u(t,x) - k_{\ell+1} > 0\}$ will be denoted by $\hat{A}_\ell(t)$ and we write $\hat{Q}_\ell$ for the cylinder $[-\hat{\tau}_\ell, 0] \times B(0, \hat{r}_\ell)$.

We shall frequently make use of special cutoff functions $\psi_\ell, \ell = -1, 0, 1, \dots$ with the following properties: We demand that $0 \leq \psi_\ell \leq 1$ is piecewise smooth function equal to 1 on the cylinder $Q_{\ell+1}$, supp $\psi_\ell = \hat{Q}_\ell$ and that $|\partial_t \psi| \leq 2^{\ell+4}/\tau$ and $|\nabla \psi_\ell| \leq C 2^{\ell+4}$ with a constant $C = C(N)$.

Also, we introduce the following sequences of quantities, where $\ell \in \mathbb{N}_0$:

$$
\begin{aligned}
X_\ell &= \int_{-\hat{\tau}_\ell}^0 |\hat{A}_\ell(t)| \, dt, \\
Y_\ell &= \frac{1}{M^2} \int_{-\tau_\ell}^0 \int_{A_\ell(t)} u_\ell^2 \, dx \, dt, \\
Z_\ell &= \left(\int_{-\tau_\ell}^0 |A_\ell(t)| \, dt \right)^{2/\hat{q}}.
\end{aligned}
$$

The latter two are the major components of the right-hand side of the inequality in Lemma 4.1.1, while the first one is an auxillary quantity to shorten our notation.

So here is our plan: We aim to show that $\|u^+_{2M}\|_{L_2(\frac{1}{2}Q)} = 0$. Since $2M > k_\ell$ and $\frac{1}{2}Q \Subset Q_\ell$ for any $\ell \in \mathbb{N}_0$, we have the trivial estimate $\|u^+_{2M}\|^2_{L_2(\frac{1}{2}Q)} \leq M^2 Y_\ell$. Since this holds for every ℓ, it suffices to show that $Y_\ell \to 0$ for $\ell \to \infty$. It then follows that $u^+_{2M} = 0$ on $\frac{1}{2}Q$ - or in other words, $\operatorname{ess\,sup}_{\frac{1}{2}Q} u \leq 2M$. It turns out that $2M = 2C(\|u\|_{L_2(Q)} + \|f\|_{L_q(Q)})$, where C is a certain constant. This is exactly (4.10) up to the factor $|Q|^{-1/2}$ which can be added by altering the constant C.
So we do not proceed exactly as discussed before at the beginning of this introduction: We do not aim for showing that $\|u^+_{2M}\|_{V_2(\frac{1}{2}Q)}$ vanishes, but rather use Lemma 4.1.1 as a tool to estimate $Y_{\ell+1}, Z_{\ell+1}$ in terms of Y_ℓ, Z_ℓ. As already mentioned above, the values of Y_0, Z_0 will determine whether these two sequences converge to zero. In order to ensure that these quantities are small enough, it will be necessary to use Lemma 4.1.1 once more. That also explains why we do not take the cylinder Q for Q_0 - we simply need some more space.

Having finished this long introduction of our notation, we are now able to start with the actual proof.

Proof. Let $\ell \in \mathbb{N}_0$ be fixed. Applying Corollary 2.2.15 yields

$$\begin{aligned} Y_{\ell+1} &= \frac{1}{M^2}\int_{-\tau_{\ell+1}}^{0}\int_{A_{\ell+1}(t)} u_{\ell+1}^2 \, dx\, dt = \frac{1}{M^2}\int_{-\tau_{\ell+1}}^{0}\int_{A_{\ell+1}(t)} u_{\ell+1}^2 \psi_\ell^2 \, dx\, dt \\ &\le \frac{1}{M^2}\|u_{\ell+1}\psi_\ell\|_{L_2(\hat{Q}_\ell)}^2 \le \frac{C_1}{M^2} X_\ell^{2/(N+2)} \|u_{\ell+1}\psi_\ell\|_{V_2(\hat{Q}_\ell)}^2, \end{aligned} \tag{4.11}$$

with the constant $C_1 = C_1(N)$. We estimate the right-hand side of (4.11) further:

$$\begin{aligned} &\frac{C_1}{M^2} X_\ell^{2/(N+2)} \|u_{\ell+1}\psi_\ell\|_{V_2(\hat{Q}_\ell)}^2 \\ &= \frac{C_1}{M^2} X_\ell^{2/(N+2)} \left(\max_{-\hat{\tau}_\ell \le t \le 0} \|(u_{\ell+1}\psi_\ell)(t,\cdot)\|_{B(0,\hat{r}_\ell)}^2 + \|\nabla(u_{\ell+1}\psi_\ell)\|_{L_2(\hat{Q}_\ell)}^2 \right) \\ &\le 2\frac{C_1}{M^2} X_\ell^{2/(N+2)} \left(\max_{-\hat{\tau}_\ell \le t \le 0} \|(u_{\ell+1}\psi_\ell)(t,\cdot)\|_{B(0,\hat{r}_\ell)}^2 + \|\nabla u_{\ell+1}\psi_\ell\|_{L_2(\hat{Q}_\ell)}^2 + \|u_{\ell+1}\nabla\psi_\ell\|_{L_2(\hat{Q}_\ell)}^2 \right) \\ &\le 2C_2 X_\ell^{2/(N+2)} \left(\frac{1}{M^2} \max_{-\hat{\tau}_\ell \le t \le 0} \|u_{\ell+1}(t,\cdot)\|_{B(0,\hat{r}_\ell)}^2 + \frac{1}{M^2}\|\nabla u_{\ell+1}\|_{L_2(\hat{Q}_\ell)}^2 + 4^{\ell+4} Y_\ell \right), \end{aligned} \tag{4.12}$$

since $0 \le \psi_\ell \le 1$ and by the construction of ψ_ℓ. The constant C_2 depends on N only.
We proceed by examining the terms appearing in the right-hand side of (4.12), applying Lemma 4.1.1 with $\hat{Q}_\ell$ and Q_ℓ. A simple calculation shows that in this setting, $(1-\delta)^{-1} \le 2^{\ell+4}$. So for some constants $C_3 = C_3(\lambda,\Lambda,N)$, $C_4 = C_4(\lambda,\tau,\Lambda,N)$ we have

$$\frac{1}{M^2} \max_{-\hat{\tau}_\ell \le t \le 0} \|u_{\ell+1}(t,\cdot)\|_{B(0,\hat{r}_\ell)}^2 + \frac{1}{M^2}\|\nabla u_{\ell+1}\|_{L_2(\hat{Q}_\ell)}^2 + 4^{\ell+4} Y_\ell \tag{4.13}$$

$$\le \frac{C_3}{M^2} 4^{\ell+4} \left(1 + \frac{1}{\tau}\right) \|u_{\ell+1}\|_{L_2(Q_\ell)}^2 + C_3 \frac{F^2}{M^2} Z_\ell^{1+2\kappa} + 4^{\ell+4} Y_\ell \tag{4.14}$$

$$\le 4^{\ell+4} \left((C_3+1)\left(1+\frac{1}{\tau}\right) Y_\ell + C_3 Z_\ell^{1+2\kappa} \right) \le C_4 4^{\ell+4} \left(Y_\ell + Z_\ell^{1+2\kappa} \right). \tag{4.15}$$

We made use of the assumption $F \le M$, and $0 \le u_{\ell+1} \le u_\ell$.

As $k_{\ell+1} > k_\ell$, we infer $\left|\hat{A}_\ell(t)\right| \le \left|\{x \in B(0,\hat{r}_\ell) : u(t,x) > k_\ell\}\right|$. This implies

$$X_\ell \le \int_{-\tau_{\ell+1}}^{0} \left| \left\{ x \in B(0,\hat{r}_\ell) : \frac{u(t,x)-k_\ell}{k_{\ell+1}-k_\ell} > 0 \right\} \right| dt < \int_{-\tau_\ell}^{0}\int_{A_\ell(t)} \frac{u_\ell^2}{(k_{\ell+1}-k_\ell)^2} \, dx\, dt \tag{4.16}$$

$$= \frac{M^2}{(k_{\ell+1}-k_\ell)^2} Y_\ell \le 4^{\ell+1} Y_\ell \tag{4.17}$$

by the mere definiton of k_ℓ.
We put $\beta = 2/(N+2)$ and finally, by combining the above estimates, arrive at

$$Y_{\ell+1} \le C_5 16^\ell \left(Y_\ell^{1+\beta} + Y_\ell^\beta Z_\ell^{1+2\kappa} \right),$$

where $C_5 = C_5(\tau, \lambda, \Lambda, N) = 4^6 C_2 C_4$.
Step 2 We now derive some estimates for Z_ℓ by proceeding similarly as in (4.16):

$$Z_{\ell+1} = \left(\int_{-\tau_{\ell+1}}^{0} \int_{A_{\ell+1}(t)} 1 \, dx \, dt \right)^{2/\hat{q}} \leq \left(\int_{\hat{Q}_\ell} \left(\frac{u_\ell \psi_\ell}{k_{\ell+1} - k_\ell} \right)^{2+4/N} dx \, dt \right)^{2/(2+4/N)}$$
$$\leq C_1 \left(\frac{1}{k_{\ell+1} - k_\ell} \right)^2 \|u_\ell \psi_\ell\|^2_{V_2(\hat{Q}_\ell)} = C_1 \frac{4^{\ell+1}}{M^2} \|u_\ell \psi_\ell\|^2_{V_2(\hat{Q}_\ell)}$$

Here, we used $\hat{q} = 2 + 4/N$ and Theorem 2.2.14. We remark that the constant in the last term coincides with the one in (4.11). From this, we move on as in the last step:

$$C_1 \frac{4^{\ell+1}}{M^2} \|u_\ell \psi_\ell\|^2_{V_2(\hat{Q}_\ell)} \leq 2 \frac{C_2}{M^2} 4^{\ell+1} \left(C_3 4^{\ell+4} \left(1 + \frac{1}{\tau} \right) M^2 Y_\ell + C_3 F^2 Z_\ell^{1+2\kappa} + M^2 Y_\ell \right)$$
$$\leq C_5 16^\ell \left(Y_\ell + Z_\ell^{1+2\kappa} \right).$$

Step 3 We summarize the results of Step 1 and 2: The sequences Y_ℓ, Z_ℓ satisfy for $\ell \geq 0$

$$Y_{\ell+1} \leq C_5 16^\ell \left(Y_\ell^{1+\beta} + Y_\ell^\beta Z_\ell^{1+2\kappa} \right), \tag{4.18}$$
$$Z_{\ell+1} \leq C_5 16^\ell \left(Y_\ell + Z_\ell^{1+2\kappa} \right), \tag{4.19}$$

i.e. they fulfill the same recurrence relation as in Lemma 2.3.2. Hence, the above sequences will converge to zero, provided that (2.12) holds - this is what we aim for in the following. We set $C_6 = (2C_5)^{1/\delta} 16^{1/\delta^2}$ for $\delta = \min(\beta, 2\kappa)$. So the goal is to choose M in such a way that both

$$Y_0 \leq \frac{1}{2} C_6^{-(1+2\kappa)} \text{ and} \tag{4.20}$$
$$Z_0 \leq 2^{-1/(1+2\kappa)} C_6^{-1} \tag{4.21}$$

hold, i.e. both Y_0 and Z_0 do not exceed one half of the right-hand side of (2.12).
Step 4 First, we have

$$Y_0 = \frac{1}{M^2} \int_{Q_0} \left((u - M)^+ \right)^2 \, dx \, dt \leq \frac{1}{M^2} \|u\|^2_{L_2(Q_0)} \leq \frac{1}{M^2} \|u\|^2_{L_2(Q)} \leq \frac{1}{M^2} \left(\|u\|_{L_2(Q)} + F \right)^2$$

So (4.20) holds, provided that

$$\sqrt{2} C_6^{(1+2\kappa)/2} \left[\|u\|_{L_2(Q)} + F \right] \leq M. \tag{4.22}$$

Step 5 In order to obtain an estimate for Z_0, we proceed in the very same way as in Step 2:

$$Z_0 \leq \left(\int_{\hat{Q}_{-1}} \left(\frac{u \psi_{-1}}{M} \right)^{2+4/N} dx \, dt \right)^{2/(2+4/N)} \leq \frac{C_1^2}{M^2} \|u \psi\|^2_{V_2(\hat{Q}_{-1})}$$
$$\leq 2 \frac{C_2}{M^2} \left(\max_{-\frac{7}{8}\tau \leq t \leq 0} \|u(t, \cdot)\|^2_{L_2\left(B\left(0, \frac{7}{8}\right)\right)} + \|\nabla u\|^2_{L_2(\hat{Q}_{-1})} + 4^3 \|u\|^2_{L_2(\hat{Q}_{-1})} \right)$$

$$
\begin{aligned}
&\le 2\frac{C_2}{M^2}\left(4^3C_3\left(1+\frac{1}{\tau}\right)\|u\|^2_{L_2(Q)}+C_3F^2|Q|^{2(1+2\kappa)/\hat{q}}+4^3\|u\|^2_{L_2(Q)}\right)\\
&\le 4^4\frac{C_2}{M^2}C_4\left(\|u\|^2_{L_2(Q)}+F^2|Q|^{2(1+2\kappa)/\hat{q}}\right)=4^{-4}\frac{C_5}{M^2}16\left(\|u\|^2_{L_2(Q)}+F|Q|^{2(1+2\kappa)/\hat{q}}\right)\\
&\le 4^{-2}\frac{C_5}{M^2}\max\left(1,(\tau|B|)^{2(1+2\kappa)/\hat{q}}\right)\left(\|u\|_{L_2(Q)}+F\right)^2.
\end{aligned}
$$

Hence, if

$$
\frac{1}{4}2^{1/2(1+2\kappa)}(C_5C_6)^{1/2}\max\left(1,(\tau|B|)^{(1+2\kappa)/\hat{q}}\right)\left(\|u\|_{L_2(Q)}+F\right)\le M, \tag{4.23}
$$

(4.21) will be satisfied.
We now choose $M=C_7\left(\|u\|_{L_2(Q)}+F\right)$, where $C_7=C_7(\lambda,\tau,\Lambda,q,N)=\max\left(\sqrt{2}C_6^{(1+2\kappa)/2},\frac{1}{4}2^{1/2(1+2\kappa)}\right.$ $\left.(C_5C_6)^{1/2}\max\left(1,(\tau|B|)^{(1+2\kappa)/\hat{q}}\right)\right)$ and (4.22) and (4.23) are both fulfilled.
The proof is complete. □

We wish to generalize this result to scaling factors different from $1/2$. This will make the proof of the next Theorem 4.1.4 easier. The following corollary shows that this is indeed possible and one can almost exhaust the full cylinder.

Corollary 4.1.3. *Let $u\in V_2(\Omega_T)$ be a weak sub-solution to* (3.1) *on Ω_T and let $Q=[t_0-\tau r^2,t_0]\times B(x_0,r)\Subset\Omega_T$. Fix $\delta\in(0,1)$. Then there is a constant $C=C(\delta,\lambda,\tau,\Lambda,q,N)$ such that*

$$
\operatorname*{ess\,sup}_{\delta Q}u\le C\left((1-\delta)^{-(N+2)/2}\left(\frac{1}{|Q|}\int_Q u^2\,dx\,dt\right)^{1/2}+r^{2-(N+2)/q}\|f\|_{L_p(Q)}\right). \tag{4.24}
$$

Proof. For arbitrary points $(t',x')\in\delta Q$, consider the cylinders $Q\supset Q'=[t'-(1-\delta)\tau r^2,t']\times B(x',(1-\delta)r)=[t'-\tau'(r')^2,t']\times B(x',r')$, where $r'=(1-\delta)r$ and $\tau'=\tau'(\delta,\tau)=\tau/(1-\delta)$. Then by Theorem 4.1.2:

$$
\begin{aligned}
\operatorname*{ess\,sup}_{\delta Q}u&\le\sup_{(t',x')\in\delta Q}\ \operatorname*{ess\,sup}_{\frac{1}{2}Q'}u\\
&\le\sup_{(t',x')\in\delta Q}C\left(\left(\frac{1}{|Q'|}\int_{Q'}u^2\,dx\,dt\right)^{1/2}+(r')^{2-(N+2)/q}\|f\|_{L_q(Q')}\right)\\
&\le C\left(((1-\delta))^{-(N+2)/2}\left(\frac{1}{|Q|}\int_Q u^2\,dx\,dt\right)^{1/2}+r^{2-(N+2)/q}\|f\|_{L_q(Q)}\right).
\end{aligned}
$$

Here, $C=C(\lambda,\tau',\Lambda,q,N)=C(\delta,\lambda,\tau,\Lambda,N)$. We used $2-(N+2)/q>0$ in the last step. □

We are now in possession of all tools needed to prove the main theorem of this section: The local boundedness of weak solutions.

Theorem 4.1.4. *Let $u \in V_2(\Omega_T)$ be a weak sub-solution to equation* (3.1) *and let $\Omega'_T = [a,b] \times \Omega' \Subset \Omega_T$, $0 < a < b \le T$, $\Omega' \Subset \Omega$ be an arbitrary parabolic cylinder that is compactly contained in Ω_T. Then u is bounded from above on $\overline{\Omega'_T}$ and*

$$\operatorname*{ess\,sup}_{\overline{\Omega'_T}} u \le C\left(\|u\|_{L_2(\Omega_T)} + \|f\|_{L_q(\Omega_T)}\right), \tag{4.25}$$

with a constant $C = C(\lambda, \Lambda, \Omega'_T, d, q, N)$, where $d = \operatorname{dist}(\Omega'_T, \Gamma_T) > 0$ is the distance between Ω'_T and the parabolic boundary of Ω_T.
Similarly, weak (super-)solutions to (3.1) *are locally bounded (from below).*

Proof. The proof is a simple consequence of Corollary 4.1.3 and a covering-argument. Put $r = d/2$ and take $\tau = \tau(\Omega'_T, d)$ such that $0 < b - \tau r^2 < a$ - this is possible since $\Omega'_T \Subset \Omega_T$. Also, there are finitely many points $x_1, \ldots, x_n \in \overline{\Omega'}$, $n = n(\Omega'_T)$ such that $\overline{\Omega'_T} \Subset \bigcup_{k=1}^n Q_k \Subset \Omega_T$, where $Q_k = [b - \tau r^2, b] \times B(x_k, r)$. Now choose $\delta = \delta(\Omega'_T) \in (0,1)$ so large that $\overline{\Omega'_T} \Subset \bigcup_{k=1}^n \delta Q_k$. As we are only dealing with a finite number of sets, this is possible by the compactness of the inclusions.
Now let u be a weak sub-solution to equation (3.1) on Ω_T and pick an arbitrary index k. Then by Corollary 4.1.3, there is a constant $C_1 = C_1(\delta, \lambda, \tau, \Lambda, q, N)$ such that

$$\operatorname*{ess\,sup}_{\delta Q_k} u \le C_1\left((1-\delta)^{-(N+2)/2} \frac{1}{|Q_k|^{1/2}} \|u\|_{L_2(Q_k)} + r^{2-(N+2)/q} \|f\|_{L_q(Q_k)}\right).$$

Taking the definition of the numbers r and δ into account yields

$$\operatorname*{ess\,sup}_{\delta Q_k} u \le C_2\left(\|u\|_{L_2(Q_k)} + \|f\|_{L_q(Q_k)}\right)$$

with a constant $C_2 = C_2(\lambda, \Lambda, \Omega'_T, d, q, N)$ independent of k. Eventually summing over all k proves (4.25). □

We point out that the proof reveals that the constant C becomes exceedingly worse the smaller the distance d is.

4.2 Density estimates

Having established local boundedness of weak solutions, we gradually approach the second component of the Hölder norm. This section is devoted to the acquisition of suitable tools to accomplish this feat. We first derive two further integral inequalities for weak solutions, now deploying our hard-earned boundedness-result. From them, two further results will arise and these are the central instruments to establish the local Hölder continuity.
Basically, the two results are statements about the local behaviour of weak solutions. Note carefully that boundedness alone does not rule out the possibility of locally highly oscillatory solutions. Gaining control

of the oscillation is the central topic of the next section. The two mentioned lemmas already point in that direction as they, in a broader sense, make statements about the local growth and decay behaviour of weak solutions.

We use the same notation as in the previous section. As the following result holds both for u_k^+ and u_k^- (with the same constants for either of them) we simply use the superscript $'\pm'$ to treat both functions simultaneously.

Lemma 4.2.1. *Let $u \in V_2(\Omega_T)$ be a weak solution to* (3.1) *on Ω_T and let $Q \Subset \Omega_T$. Fix $\delta \in (0,1)$. Then there are constants C_1, C_2 dependent only on λ, Λ, N such that for all $k \in \mathbb{R}$ the following two estimates hold:*

$$\begin{aligned}\|u_k^\pm\|^2_{V_2(\delta Q)} \le C_1 \Bigg(\int_Q \left(\frac{1}{(1-\delta)\tau r^2} + \frac{1}{(1-\delta)^2 r^2} \right) \left(u_k^\pm\right)^2 \, dx\, dt \\ + \|u_k^\pm\|_{L_\infty(Q)} \|f\|_{L_q(Q)} \left(\int_{t_0-\tau r^2}^{t_0} |A^\pm_{k,r}(t)| \, dt \right)^{2(1+\kappa)/\hat{q}} \Bigg),\end{aligned} \tag{4.26}$$

and

$$\begin{aligned}&\max_{t_0-\tau r^2 \le t \le t_0} \|u_k^\pm(t,\cdot)\|^2_{L_2(B(x_0,\delta r))} \le \|u_k^\pm(t_0-\tau r^2,\cdot)\|^2_{L_2(B(x_0,r))} \\ &+ C_2 \Bigg(\int_Q \frac{1}{(1-\delta)^2 r^2} \left(u_k^\pm\right)^2 \, dx\, dt + \|u_k^\pm\|_{L_\infty(Q)} \|f\|_{L_q(Q)} \left(\int_{t_0-\tau r^2}^{t_0} |A^\pm_{k,r}(t)| \, dt \right)^{2(1+\kappa)/\hat{q}} \Bigg),\end{aligned} \tag{4.27}$$

where κ_1 is the number in 3.1.1, *$\kappa = 2\kappa_1/N$ and $\hat{q} = 2q'(1+\kappa)$ is the same as in Lemma* 4.1.1.

The proof is very similar to the one of Lemma 4.1.1 and we use the very same notation, up to an obvious adjustment of the domains of definition of the appearing functions. As before, it suffices to prove the assertion for u_k^+ and we simply write u_k and $A_{k,r}(t)$ for u_k^+ and $A^+_{k,r}(t)$, respectively.

Proof. We argue exactly as in the first two steps of the proof of Lemma 4.1.1 and obtain the estimate

$$\begin{aligned}&\max_{t_0-\tau r^2 \le t \le t_0} \|(u_k\psi)(t,\cdot)\|^2_{L_2(B)} + \lambda \int_Q |\nabla u_k|^2 \psi^2 \, dx\, dt \\ &\le \|(u_k\psi)(t_0-\tau r^2,\cdot)\|^2_{L_2(B)} + C_3 \int_Q u_k^2 \left(|\nabla\psi|^2 + \psi|\partial_t\psi|\right) + 2\int_Q |f| u_k \psi^2 \, dx\, dt.\end{aligned} \tag{4.28}$$

Here, $0 \le \psi \le 1$ is a piecewise smooth function that vanishes on the lateral boundary of Q and C_3 depends on λ and Λ only.

Step 2 We estimate the last term in (4.28) further. For this, we set $M = \|u_k\|_{L_\infty(Q)}$ and χ for the characteristic function of the subset of Q on which $u > k$ holds. Then

$$2\int_Q |f| u_k \psi^2 \, dx\, dt \le 2M \int_Q |f| \chi \, dx\, dt \le 2\|f\|_{L_q(Q)} \|\chi\|^2_{L_{2q'}(Q)},$$

since $\psi \le 1$. We rewrite the exponent in the last term as

$$\frac{2}{2q'} = \frac{2(1+\kappa)}{\hat{q}}.$$

Thus

$$2\int_Q |f| u_k \psi^2 \, dx\, dt \le 2M\|f\|_{L_q(Q)} \left(\int_{t_0-\tau r^2}^{t_0} |A_{k,r}(t)| \, dt \right)^{2(1+\kappa)/\hat{q}}.$$

So (4.28) becomes

$$\begin{aligned} \max_{t_0-\tau r^2 \le t \le t_0} \|(u_k\psi)(t,\cdot)\|^2_{L_2(B)} + \lambda \|\nabla u_k \psi\|^2_{L_2(Q)} \le \|(u_k\psi)(t_0-\tau r^2,\cdot)\|^2_{L_2(B)} \\ + C_3 \left(\int_Q \left(|\nabla\psi|^2 + \psi|\partial_t\psi| \right) u_k^2 \, dx\, dt + 2M\|f\|_{L_q(Q)} \left(\int_{t_0-\tau r^2}^{t_0} |A_{k,r}(t)| \, dt \right)^{2(1+\kappa)/\hat{q}} \right). \end{aligned} \tag{4.29}$$

Step 3 The proof of inequality (4.26) is the same as the final step in the proof of Lemma 4.1.1: We take ψ as before but impose further that ψ is constantly equal to one on δQ, vanishes on the parabolic boundary of Q and $|\partial_t\psi| \le \frac{1}{(1-\delta)\tau r^2}$, $|\nabla\psi| \le \frac{C_4}{(1-\delta)r}$, where $C_4 = C_4(N)$. Recalling $\psi \le 1$ and estimating the left-hand side of (4.29) from below by $\min(1,\lambda)\|u\|^2_{V_2(\delta Q)}$ concludes the proof of (4.26).
Step 4 To prove (4.27), we proceed as in the previous step. We take ψ as above except that we demand ψ to be independent of time t. This concludes the proof. □

We now formulate the first lemma that is fundamental for the subsequent section. It is concerned with the construction of special parabolic cylinders on which we have a certain amount of control of the growth behaviour of weak solutions.

Lemma 4.2.2. *Let $u \in V_2(\Omega_T)$ be a weak solution to* (3.1) *on Ω_T and let $k \in \mathbb{R}$. Then there is a strictly positive number $\tau_1 = \tau_1(\lambda,\Lambda,q,N) \le 1$ and a cylinder $Q = Q(t_0,x_0,\tau_1,r) \Subset \Omega_T$ such that, if the two conditions*

$$\left| A^{\pm}_{k,r}(t_0-\tau_1 r^2) \right| \le \frac{1}{2}|B(x_0,r)| \tag{4.30}$$

and

$$M^{\pm} = \operatorname*{ess\,sup}_Q u_k^{\pm} > r^{2-(N+2)/q}\|f\|_{L_q(Q)}, \tag{4.31}$$

are fulfilled, then for all $t \in [t_0-\tau_1 r^2, t_0]$ the following holds:

$$\left| B(x_0,r) \setminus A^{\pm}_{k+\frac{3}{4}M^{\pm},r}(t) \right| \ge \frac{1}{36}|B(x_0,r)|. \tag{4.32}$$

Note carefully that the number τ_1 is independent of k.

Let us informally discuss the assertion of this result. For the sake of simplicity, we only consider u_k^+ and assume that the conditions are fulfilled for $k = 0$.

In (4.31), we assume that the essential supremum of u on the whole parabolic cylinder is sufficiently large and in particular, strictly positive. However, at 'starting-time' $t = t_0 - \tau_1 r^2$, the function u is mostly non-positive. This is what we assume in (4.30): The fraction of the ball, where the function u exceeds zero at time $t_0 - \tau_1 r^2$ is at most $1/2$. In this situation, we can guarantee that at least for some short time, there is a fixed fraction of the measure of the ball, where the function u stays away from the supremum . So it is impossible that u gets close to the supremum everywhere on the ball instantaneously.

Proof. We consider the cylinder $Q = [t_0 - \tau_1 r^2, t_0] \times B(x_0, r) \Subset \Omega_T$, where τ_1 will be chosen later. We only prove the assertion for u_k^+ as the other case is treated analogously. As before, we leave out the superscript '+'. Using the very same rescaling-argument as in the proof of Theorem 4.1.2, we see from the structure of the inequalities (4.30)-(4.32) that we may assume $(t_0, x_0) = (0,0)$ and $r = 1$. In order to shorten our notation, we write $B = B(0,1)$ and $\delta B = B(0,\delta)$.

Pick an arbitrary $\delta \in (0,1)$. Using (4.27) and assumptions (4.30) and (4.31) yields

$$\begin{aligned}
&\max_{-\tau_1 \le t \le 0} \|u_k(t,\cdot)\|^2_{L_2(\delta B)} \le \|u_k(-\tau_1,\cdot)\|^2_{L_2(B)} \\
&+ C_1 \left((1-\delta)^{-2} \|u_k\|^2_{L_2(Q)} + \|u_k\|_{L_\infty(Q)} \|f\|_{L_q(Q)} \left(\int_{-\tau}^0 |A_{k,1}(t)|\, dt \right)^{2(1+\kappa)/\hat{q}} \right) \\
&\le \frac{1}{2} M^2 |B| + C_1 \left(M^2 (1-\delta)^{-2} \tau_1 |B| + M \|f\|_{L_q(Q)} (\tau_1 |B|)^{2(1+\kappa)/\hat{q}} \right), \qquad (4.33)
\end{aligned}$$

where $C_1 = C_1(\lambda, \Lambda, N)$.

Also, we have the following for arbitrary $t \in [-\tau_1, 0]$:

$$\begin{aligned}
\left| A_{k+\frac{3}{4}M,\delta}(t) \right| &= \int_{A_{k+\frac{3}{4}M,\delta}(t)} 1\, dx \le \int_{A_{k+\frac{3}{4}M,\delta}(t)} \left(\frac{u-k}{\frac{3}{4}M} \right)^2 dx \\
&\le \left(\frac{1}{\frac{3}{4}M} \right)^2 \int_{A_{k,\delta}(t)} (u-k)^2\, dx. \qquad (4.34)
\end{aligned}$$

We combine these intermediate results (4.33) and (4.34), use that $|B| = C_2$ with $C_2 = C_2(N)$, and the identity $2(1+\kappa)/\hat{q} = 1/q'$. We end up with

$$\begin{aligned}
\left| A_{k+\frac{3}{4}M,\delta}(t) \right| &\le \frac{8}{9}|B| + \frac{16}{9} C_1 \left((1-\delta)^{-2} \tau_1 |B| + \frac{1}{M} (C_2 \tau_1)^{2(1+\kappa)/\hat{q}} \|f\|_{L_q(Q)} \right) \\
&\le \frac{8}{9}|B| + \frac{16}{9} C_1 |B| \left((1-\delta)^{-2} \tau_1 + C_2^{1/q'-1} \tau_1^{1/q'} \right), \qquad (4.35)
\end{aligned}$$

where we used (4.31) in the last estimate.

To complete the proof, we notice that the measure of the set $A_{k+3/4,1}(t)$ can be estimated from above by

$\left|A_{k+3/4,\delta}(t)\right| + |B\backslash\delta B|$ for any $\delta \in (0,1)$. So this observation and (4.35) give us

$$\left|A_{k+\frac{3}{4}M,1}(t)\right| \leq \frac{8}{9}|B| + \frac{16}{9}C_1|B|\left((1-\delta)^{-2}\tau_1 + C_2^{1/q'-1}\tau_1^{1/q'}\right) + |B| - |\delta B|. \tag{4.36}$$

Since we are free in the choice of δ, we take $\delta = \delta(N)$ sufficiently large such that $|B| - |\delta B| \leq \frac{1}{36}|B|$. Having fixed δ, we take $\tau_1 = \tau_1(\lambda,\Lambda,q,N) \leq 1$ sufficiently small such that

$$\frac{16}{9}C_1\left((1-\delta)^{-2}\tau_1 + C_2^{1/q'-1}\tau_1^{1/q'}\right)|B| \leq \frac{1}{18}|B|.$$

Putting everything together, we conclude $|A_{k+\frac{3}{4}M,1}(t)| \leq \frac{35}{36}|B|$ for all $t \in [-\tau_1, 0]$, or said differently, $\left|A_{k+\frac{3}{4}M,1}(t)\right| \leq (1-\frac{1}{36})|B|$. Thus,

$$\left|B \setminus A_{k+\frac{3}{4}M,1}(t)\right| = |B| - \left|A_{k+\frac{3}{4}M,1}(t)\right| \geq |B| - \left(1 - \frac{1}{36}\right)|B| = \frac{1}{36}|B|.$$

□

Remark 4.2.3. From (4.36) we infer that we could have taken any number in $(0,1/9)$ instead of $1/36$. Of course, this would alter the number τ_1. However, this the exact value of the prefactor is not important for us, the strict positivity is what matters.
Also, the proof does not reveal why we take $\tau_1 \leq 1$. This will be crucial in the very last step of the proof of Theorem 3.1.3 in Section 4.4.

We now come to the second important result of this section.

Lemma 4.2.4. *Let $u \in V_2(\Omega_T)$ be a weak solution to* (3.1) *on Ω_T and let $Q \Subset \Omega_T$. Then there exists a number $\theta = \theta(\lambda,\tau,\Lambda,q,N) \in (0,1)$ such that, whenever $k^\pm \in \mathbb{R}$ satisfies the two conditions*

$$M^\pm = \operatorname*{ess\,sup}_Q u^\pm_{k^\pm} \geq r^{2-(N+2)/q}\|f\|_{L_q(Q)} \tag{4.37}$$

and

$$\left|\left\{(t,x) \in Q : u^\pm_{k^\pm} > 0\right\}\right| \leq \theta|Q|, \tag{4.38}$$

then

$$\operatorname*{ess\,sup}_{\frac{1}{2}Q} \pm u \leq \frac{M^\pm}{2} + k^\pm. \tag{4.39}$$

Again, we take u_k^+ and assume the two conditions to be correct for $k^+ = 0$ to illustrate the quintessence of this assertion. For technical reasons, we assume the essential supremum of u on the whole cylinder to be sufficiently large. On the other hand, however, if the fraction of the cylinder, where u is positive is sufficiently small, then u cannot exceed half of that supremum on the smaller cylinder $\frac{1}{2}Q$.

Let us discuss our setup for the proof first. We shall only prove the assertion for $u^+_{k^+}$, as the proof in the other case is the same. Clearly, we may assume $(t_0, x_0) = (0,0)$ and $r = 1$, by deploying the familiar rescaling-argument and the structure of (4.37)-(4.39). Also, we divide both sides of (4.26) by $(M^+)^2$ and use (4.37). This gives

$$\|v^+_{\tilde{k}}\|^2_{V_2(\delta Q)} \leq C_1 \left(\left((1-\delta)^{-1} + (1-\delta)^{-2}\right) \|v^+_{\tilde{k}}\|^2_{L_2(Q)} + \left(\int_{-\tau}^{0} |\tilde{A}^+_{\tilde{k},1}(t)| \, dt \right)^{2(1+\kappa)/\hat{q}} \right). \tag{4.40}$$

Here, $v = \left(\frac{u}{M}\right)$, $\tilde{k} = \frac{k^+}{M}$ and $\tilde{A}^+_{\tilde{k},1}(t) = \{x \in B(0,1) : v^+_{\tilde{k}} > 0\}$. Hence, we may assume that (4.26) takes the form (4.40). Also, it suffices to prove the assertion for $M^+ = 1$. From now on, we leave out the superscript '+' in our argument in order to shorten the notation.

The proof is basically a replication of the proof of Theorem 4.1.2, utilizing essentially the same iterative argument. Exactly the same notation as in the above mentioned proof will be used in the following, up to two minor changes: We set $k_0 = k$ and consider the sequences

$$k_\ell = k + \frac{1}{2} - \frac{1}{2^{\ell+1}}, \qquad \ell \geq 1, \text{ and}$$
$$Y_\ell = 4 \int_{Q_\ell} u_\ell^2 \, dx \, dt, \qquad \ell \geq 0$$

instead. Again, we aim to show that the sequence Y_ℓ converges to zero as $\ell \to \infty$. This suffices to prove the assertion, since for all $\ell \geq 0$

$$\left\| (u - (k + 1/2))^+ \right\|^2_{L_2(\frac{1}{2}Q)} \leq \frac{1}{4} Y_\ell.$$

However, the situation is slightly different this time: Before, we utilized this argument to show that u is bounded from above at all. The convergence to zero of the above sequences was established by an appropriatly large guess for the upper bound. But now, we already know about the boundedness and are interested in a finer erstimate for the essential supremum of u on the sub-cylinder $\frac{1}{2}Q$. In particular, we are not permitted, as in the argument before, to vary in the proposed, fixed upper bound. This time, however, we achieve the convergence of the two sequences by an assumption on the smallness of the set, where $u > k$ holds.

Proof. Again, we establish the recurrence relations of Lemma 2.3.2 for the sequences Y_ℓ, Z_ℓ and enforce condition (2.12) to hold. Here, we put $\beta = 2/(N+2)$, $\gamma = \kappa$ and the constant C will be chosen later. Since the derivation of these relations follows exactly the same steps and ideas as in the first two steps in the proof before, we refrain from literally reinserting the same lines here and simply point out where minor adjustments in the argument have to be made.

By replacing the number M by $1/2$, all considerations from the first line to (4.13) remain valid. In order to move from there to (4.14), we deploy (4.40) instead of (4.1). Note that in (4.40), there is no prefactor F^2, so that number can be thought of to be equal to one in (4.14). Also, the exponent of Z_ℓ has to be changed accordingly. As the number $M = 1/2$ is already determined, we simply include this number in the constant C_3. This only alters the size of the subsequent constants but clearly that is not important as we are only interested in the parameters on which the constants depend. The same changes are made in Step 2.

Eventually, we end up with the desired inequalities

$$Y_{\ell+1} \leq C_1 16^\ell \left(Y_\ell^{1+\beta} + Y_\ell^\beta Z_\ell^{1+\kappa} \right), \tag{4.41}$$

$$Z_{\ell+1} \leq C_1 16^\ell \left(Y_\ell + Z_\ell^{1+\kappa} \right), \tag{4.42}$$

with a constant $C_1 = C_1(\lambda, \tau, \Lambda, N)$.
Again, we set $C_2 = (2C_1)^{1/\delta} 16^{1/\delta^2}$, where $\delta = \min(\beta, \kappa)$. We have to enforce the validity of

$$\begin{aligned} Y_0 &\leq \frac{1}{2} C_2^{-(1+\kappa)}, \\ Z_0 &\leq 2^{-1/(1+\kappa)} C_2^{-1}. \end{aligned} \tag{4.43}$$

In light of (4.37) and (4.38), we see

$$\begin{aligned} Y_0 &= 4 \int_{Q_0} \left((u-k)^+ \right)^2 \, dx \, dt \leq 4 \int_Q \left((u-k)^+ \right)^2 \, dx \, dt \leq 4 |\{(t,x) \in Q : u(t,x) - k > 0\}| \leq 4\theta |Q|, \\ Z_0 &\leq |\{(t,x) \in Q : u(t,x) - k > 0\}|^{2/\hat{q}} \leq (\theta |Q|)^{2/\hat{q}}. \end{aligned}$$

So for sufficiently small $\theta = \theta(\lambda, \tau, \Lambda, q, N) \in (0,1)$, (4.43) holds. □

4.3 Oscillation estimates

This section is the heart of the proof of Theorem 3.1.3. We use our density results of the previous section to derive a chain of statements - becoming stronger from step to step - about the essential oscillation of weak solutions u to equation (3.1). The main result is Theorem 4.3.5, which roughly asserts the following: Consider a nested family of cylinders $Q(r) = [t_0 - \tau r^2, t_0] \times B(x_0, r)$ for $r \in (0, R]$ with $Q(R) \Subset \Omega_T$. Then $\operatorname{ess\,osc}_{Q(r)} u$ behaves like $r^\alpha \operatorname{ess\,osc}_{Q(R)} u$. From this there will be only a tiny step left to establish the long-promised Hölder continuity. This will be settled in the next section.

We will work mostly with cylinders of the form $\delta Q = [t_0 - \tau \delta r^2, t_0] \times B(x_0, \delta r)$ with $\delta \in \left\{ \frac{1}{2}, 1, 2 \right\}$. It turns out to be more convenient to work with these scaling factors as there would be too many fractions otherwise. Obviously, we will write Q instead of $1Q$.

We fix the following important convention for the rest of this chapter: Whenever we write $2Q \Subset \Omega_T$, we will always assume that $\tau = \tau(\lambda, \Lambda, q, N)$ is the number τ_1 of Lemma 4.2.2. This ensures that if the conditions of Lemma 4.2.2 are fulfilled, (4.32) holds for the whole cylinder Q.

Lemma 4.3.1. *Let $u \in V_2(\Omega_T)$ be a weak solution to* (3.1) *on Ω_T and let $2Q \Subset \Omega_T$. Set $\omega = \operatorname{ess\,osc}_{2Q} u$.*

Then for any $\theta \in (0,1)$ *there is a number* $s = s(\lambda,\tau,\theta,\Lambda,q,N) = s(\lambda,\theta,\Lambda,q,N) \in \mathbb{N}$, $s \geq 3$ *such that at least one of the following inequalities holds:*

$$\omega \leq 2^s r^{2-(N+2)/q} \|f\|_{L_q(2Q)}, \text{ or} \tag{4.44}$$

$$\left| \left\{ (t,x) \in Q : u(t,x) > \operatorname*{ess\,sup}_{2Q} u - \frac{\omega}{2^s} \right\} \right| \leq \theta |Q|, \text{ or} \tag{4.45}$$

$$\left| \left\{ (t,x) \in Q : -u(t,x) > \operatorname*{ess\,sup}_{2Q} -u - \frac{\omega}{2^s} \right\} \right| \leq \theta |Q|. \tag{4.46}$$

Remark 4.3.2. The proof reveals that the number s depends on τ, however, as remarked before $\tau = \tau(\lambda,\Lambda,q,N)$.

In the following proof, we use the shorthand notation

$$M_1 = \operatorname*{ess\,sup}_{Q} u, \qquad M_2 = \operatorname*{ess\,sup}_{2Q} u, \qquad m_2 = \operatorname*{ess\,inf}_{2Q} u.$$

Remark 4.3.3. Clearly, we could have written $u(t,x) < \operatorname{ess\,inf}_{2Q} u + \omega/2^s$ in (4.46) instead. However, it turns out to be more handy this way.

The interpretation of this lemma is as follows: Suppose that the number θ is small. Then the number s will be large (in fact, the proof shows that s behaves like θ^{-2}). Suppose that the first alternative (4.44) is violated, i.e. the oscillation of u on the larger cylinder $2Q$ is huge. In that case, however, one of the other two assertions must be true - let us assume that the second one holds. Then, the violation of (4.44) implies

$$\left| \left\{ (t,x) \in Q : u(t,x) > M_2 - r^{2-(N+2)/q} \|f\|_{L_q(2Q)} \right\} \right| \leq \left| \left\{ (t,x) \in Q : u(t,x) > M_2 - \frac{\omega}{2^s} \right\} \right| \leq \theta |Q|.$$

But θ is assumed to be small, so the fraction of the cylinder, where $u > M_2 - r^{2-(N+2)/q} \|f\|_{L_q(2Q)}$ holds is small. Thus, on most part of the cylinder Q, $u \leq u + r^{2-(N+2)/q} \|f\|_{L_q(2Q)} \leq M_2 = \operatorname{ess\,sup}_{2Q} u$ holds. So although the oscillation of u on the cylinder $2Q$ is large, we still have some control of u on the smaller cylinder Q: The solution u, restricted to the smaller cylinder Q, *tends* to stay away from its essential supremum on the larger cylinder $2Q$. A similar conclusion follows if (4.46) holds instead.

Concerning the proof, we could start by stating the exact formula of the number s and show that this number fulfills the assertions of this lemma. But due to its rather complicated form, it appears to be more convenient to work with an arbitrary natural number $s \geq 3$ first. At the end of the proof it will become clear how to choose the number s.

Proof. Again, by rescaling, we may assume $(t_0,x_0) = (0,0)$ and $r = 1$. Set $B = B(0,1)$.
As already said, let $s \geq 3$ to be specified later. If (4.44) is true, then the proof of the lemma is already completed. We aim to show in the following that if (4.44) is violated, then either (4.45) or (4.46) must

hold, hence, the assertion of the lemma is correct. So we will assume

$$\omega > 2^s \|f\|_{L_q(2Q)} \geq 2^s \|f\|_{L_q(Q)} \tag{4.47}$$

hereafter.

If $M_1 \leq M_2 - \frac{\omega}{2^s}$, then obviously $\left|\left\{(t,x) \in Q : u(t,x) > M_2 - \frac{\omega}{2^s}\right\}\right| = 0$. So (4.45) is fulfilled for any positive θ. We can thus assume that

$$M_1 > M_2 - \frac{\omega}{2^s} \tag{4.48}$$

holds in the following.

Step 2 Clearly, at least one of the following two claims must be true:

$$\left|\left\{x \in B : u(-\tau,x) > M_2 - \frac{\omega}{2}\right\}\right| \leq \frac{1}{2}|B|, \text{ or} \tag{4.49}$$

$$\left|\left\{x \in B : u(-\tau,x) < M_2 - \frac{\omega}{2}\right\}\right| \leq \frac{1}{2}|B|. \tag{4.50}$$

Suppose (4.49) holds. Clearly, as the numbers $M_2 - \frac{\omega}{2^\ell}$, $\ell = 1,\dots,s-2$ are increasing

$$\left|\left\{x \in B : u(-\tau,x) > M_2 - \frac{\omega}{2^\ell}\right\}\right| \leq \frac{1}{2}|B|, \qquad \ell = 1,\dots,s-2. \tag{4.51}$$

This implies that the quantities M^ℓ, $\ell = 1,\dots,s-2$ defined by $M^\ell = M_1 - M_2 + \frac{\omega}{2^\ell}$, satisfy

$$M^\ell > \frac{\omega}{2^l} - \frac{\omega}{2^s} = \frac{\omega}{2^s}\left(\frac{1}{2^{\ell-s}} - 1\right) > \frac{\omega}{2^s} > \|f\|_{L_q(Q)},$$

where we used the assumptions (4.47) and (4.48).

The purpose of these calculations is to verify that for every ℓ, the assumptions of Lemma 4.2.2 are fulfilled, where we choose Q to be the underlying cylinder, $k = M_2 - \frac{\omega}{2^\ell}$ and $M^+ = M^\ell$. So

$$\left|B \setminus \left\{x \in B :\ u(t,x) > M_2 - \frac{\omega}{2^\ell} + \frac{3}{4}M^\ell\right\}\right| \geq \frac{1}{36}|B| \tag{4.52}$$

for every $t \in [-\tau, 0]$ and $\ell = 1,\dots,s-2$.

Since $M_1 - M_2 \leq 0$, the mere definition of M^ℓ implies $M^\ell \leq \frac{\omega}{2^\ell}$, and this amounts to $M_2 - \frac{\omega}{2^\ell} + \frac{3}{4}M^\ell \leq M_2 - \frac{\omega}{2^\ell} + \frac{3\omega}{2^{\ell+2}} = M_2 - \frac{\omega}{2^{\ell+2}}$. This consideration, together with (4.52) leads us to the conclusion that for t in the above interval

$$\left|B \setminus \left\{x \in B :\ u(t,x) \geq M_2 - \frac{\omega}{2^{\ell+2}}\right\}\right| \geq \frac{1}{36}|B|, \qquad \ell = 1,\dots,s-2 \tag{4.53}$$

holds.

Step 3 In the following, we fix an arbitrary index $\ell \in \{1,\dots,s-2\}$ and put $m(\ell) = M_2 - \frac{\omega}{2^{\ell+2}}$ and

$n(\ell) = M_2 - \frac{\omega}{2^{\ell+1}}$. Also, we introduce the difference sets

$$\Delta_\ell(t) = \{x \in B : n(\ell) \leq u(t,x) \leq m(\ell)\}$$

to shorten our notation. So Proposition 2.2.7 and (4.53) imply that for $t \in [-\tau, 0]$

$$\frac{\omega}{2^{\ell+2}} |\{x \in B : u(t,x) > m(\ell)\}| \leq \frac{36C_1}{|B|} \int_{\Delta_\ell(t)} |\nabla u(t,x)| \, dx = 36C_2 \int_{\Delta_\ell(t)} |\nabla u(t,x)| \, dx, \tag{4.54}$$

for certain constants C_1, C_2 depending only on N.
From this, it follows

$$\begin{aligned}
&\left(\frac{\omega}{2^{\ell+2}}\right)^2 |\{(t,x) \in Q : \ u(t,x) > m(\ell)\}|^2 \leq (36C_2)^2 \left(\int_{-\tau}^0 \int_{\Delta_\ell(t)} |\nabla u| \, dx \, dt\right)^2 \\
&\leq (36C_2)^2 \left(\int_{-\tau}^0 \int_{\Delta_\ell(t)} |\nabla u|^2 \, dx \, dt\right) \left(\int_{-\tau}^0 \int_{\Delta_\ell(t)} 1 \, dx \, dt\right) \\
&\leq (36C_2)^2 \left\| \nabla \left(u^+_{n(\ell)}\right) \right\|^2_{L_2(Q)} \left(\int_{-\tau}^0 |\Delta_\ell(t)| \, dt\right).
\end{aligned} \tag{4.55}$$

The second term in (4.55) can be estimated further, using (4.27) with the cylinders Q and $2Q$ (i.e. with $\delta = 1/2$). Also, observe $u^+_{n(\ell)} \leq \omega/2^{\ell+1}$. So

$$\begin{aligned}
\left\|\nabla(u_n^+)\right\|^2_{L_2(Q)} &\leq C_3 \left(4\left(\frac{1}{\tau}+1\right)\left(\frac{\omega}{2^{\ell+1}}\right)^2 |2Q| + \|u^+_{n(\ell)}\|_{L_\infty(2Q)} \|f\|_{L_q(2Q)} (|2Q|)^{2(1+\kappa)/\hat{q}}\right) \\
&\leq C_3 \left(16\left(\frac{1}{\tau}+1\right)\left(\frac{\omega}{2^{\ell+2}}\right)^2 + 2\left(\frac{\omega}{2^{\ell+2}}\right)^2 (|2Q|)^{2(1+\kappa)/\hat{q}-1}\right) |2Q| \\
&= C_4 \left(\frac{\omega}{2^{\ell+2}}\right)^2 |Q|.
\end{aligned} \tag{4.56}$$

Here, $C_3 = C_3(\lambda, \Lambda, N)$ and $C_4 = C_4(\lambda, \tau, \Lambda, q, N) = C_4(\lambda, \Lambda, q, N)$. In fact, (4.56) suggests that C_4 is dependent on τ, but by construction, τ can be expressed in terms of λ, Λ, q and N. We made use of assumption (4.47) in the second step.
By plugging this back into (4.55), we deduce that for $\ell \in \{1, \ldots, s-2\}$ and for a constant $C_5 = C_5(\lambda, \tau, \Lambda, q, N)$

$$|\{(t,x) \in Q : \ u(t,x) > m(\ell)\}|^2 \leq C_5 |Q| \left(\int_{-\tau}^0 |\Delta_\ell(t)| \, dt\right). \tag{4.57}$$

Step 4 The proof is almost complete. Since for $\ell \in \{1, \ldots, s-2\}$ there holds

$$|\{(t,x) \in Q : \ u(t,x) > M_2 - \omega/2^s\}|^2 \leq |\{(t,x) \in Q : \ u(t,x) > m(\ell)\}|^2,$$

estimate (4.57) implies

$$\begin{aligned}(s-2)\,|\{(t,x)\in Q:\ u(t,x)>M_2-\omega/2^s\}|^2 &\le \sum_{\ell=1}^{s-2}|\{(t,x)\in Q:\ u(t,x)>m(\ell)\}|^2\\ &\le C_5|Q|\sum_{\ell=1}^{s-2}\left(\int_{-\tau}^{0}|\Delta_\ell(t)|\,dt\right)\\ &\le C_5|Q|^2. \end{aligned}\tag{4.58}$$

Finally, the quantity θ comes into play: In order to prove (4.45), it suffices to choose s in such a way that $C_5/(s-2)\le\theta^2$. We achieve this by taking $s=s(\theta,\lambda,\Lambda,p,N)\in\mathbb{N}$ with

$$3\le\left\lceil\frac{C_5}{\theta^2}\right\rceil+2\le s.$$

Here, $\lceil\cdot\rceil$ denotes the standard ceiling function.

Step 4 The whole proof relied on the assumption (4.49). So let us suppose that this assumption is violated, i.e. (4.50) holds instead. The following little consideration

$$u(-\tau,x)<M_2-\frac{\omega}{2}\iff -u(-\tau,x)>-\operatorname*{ess\,sup}_{2Q}u+\frac{\omega}{2}=-\operatorname*{ess\,inf}_{2Q}u-\frac{\omega}{2}=\operatorname*{ess\,sup}_{2Q}-u-\frac{\omega}{2}$$

reveals that an assumption similar to (4.49) holds for $-u$. So in that case, we can repeat the whole proof with the function $-u$ and eventually obtain (4.46). □

Lemma 4.3.4. *Let $u\in V_2(\Omega_T)$ be a weak solution to* (3.1) *on Ω_T and let $2Q\Subset\Omega_T$. Then there is a number $s=(\lambda,\Lambda,q,N)\ge 3$ such that at least one of the following holds:*

$$\operatorname*{ess\,osc}_{\frac{1}{2}Q}u\le 2^{s+1}r^{2-(N+2)/q}\|f\|_{L_q(2Q)},\ \text{or}\tag{4.59}$$

$$\operatorname*{ess\,osc}_{\frac{1}{2}Q}u\le\left(1-\frac{1}{2^{s+1}}\right)\operatorname*{ess\,osc}_{2Q}u.\tag{4.60}$$

The estimate (4.60) is very significant: Of course, one always has that $\operatorname{ess\,osc}_{\frac{1}{2}Q}u$ is bounded from above by $\operatorname{ess\,osc}_{2Q}u$. However, (4.60) is a much stronger result, namely it asserts that the essential oscillation of u actually decreases when one passes from $2Q$ to $\frac{1}{2}Q$. Additionally, we can quantify this oscillation decay.

The abbreviations $M_2=\operatorname*{ess\,sup}_{2Q}u$ and $\omega_2=\operatorname*{ess\,osc}_{2Q}u$ will be used in the following.

Proof. Again, by rescaling, we may assume $r=1$.

Let $\theta=\theta(\lambda,\Lambda,q,N)\in(0,1)$ be the number proclaimed by Lemma 4.2.4 (we omit the dependency on τ, see Remark 4.3.2), i.e. for all $k^\pm\in\mathbb{R}$ with $M^\pm=\operatorname{ess\,sup}_Q u^\pm_{k^\pm}\ge\|f\|_{L_q(Q)}$ and $\left|\{(t,x)\in Q: u^\pm_{k^\pm}>0\}\right|\le\theta|Q|$, there holds $\operatorname{ess\,sup}_{\frac{1}{2}Q}u^\pm\le\frac{M^\pm}{2}+k^\pm$.

For this θ, according to the previous Lemma 4.3.1, there is a number $s=s(\lambda,\Lambda,q,N)\ge 3$ such that at least one of the three alternatives (4.44)-(4.46) is fulfilled. If assertion (4.59) holds for the number $s+1$, we are already done. So let us assume $\operatorname{ess\,osc}_{\frac{1}{2}Q} u > 2^{s+1}\|f\|_{L_q(2Q)}$ in the following. Clearly, this assumption implies

$$\omega_2 > 2^{s+1}\|f\|_{L_q(2Q)} > 2^s\|f\|_{L_q(Q)} > \|f\|_{L_q(Q)}.$$

Step 2 We first assume (4.45) to be correct and take a look at the function $u^+_{k^+}$, where we set $k^+ = M_2 - \omega_2/2^s$. So by Lemma 4.2.4 either

$$M^+ = \operatorname*{ess\,sup}_{Q} u^+_{k^+} < \|f\|_{L_q(Q)}, \text{ or} \tag{4.61}$$

$$\operatorname*{ess\,sup}_{\frac{1}{2}Q} u \leq \frac{M^+}{2} + k^+. \tag{4.62}$$

If (4.61) is true, we then have the estimate

$$\operatorname*{ess\,sup}_{\frac{1}{2}Q} u - k^+ \leq \operatorname*{ess\,sup}_{Q} u - k^+ = \operatorname*{ess\,sup}_{Q} u^+_{k^+} < \|f\|_{L_q(Q)},$$

so

$$\operatorname*{ess\,sup}_{\frac{1}{2}Q} u < M_2 - \frac{\omega_2}{2^s} + \|f\|_{L_q(Q)} < M_2 - \frac{\omega_2}{2^s} + \frac{\omega}{2^{s+1}} = M_2 - \frac{\omega_2}{2^{s+1}}. \tag{4.63}$$

Now consider the case that (4.62) holds instead. Then

$$\operatorname*{ess\,sup}_{\frac{1}{2}Q} u \leq \frac{1}{2}\operatorname*{ess\,sup}_{Q}\left(\max\left(u - M_2 + \frac{\omega_2}{2^s}, 0\right)\right) + k^+ \leq \frac{\omega_2}{2^{s+1}} + M_2 - \frac{\omega_2}{2^s} = M_2 - \frac{\omega_2}{2^{s+1}}.$$

So in either case we have the estimate $\operatorname{ess\,sup}_{\frac{1}{2}Q} u \leq M_2 - \frac{\omega_2}{2^{s+1}}$. However, if the alternative (4.46) holds instead of (4.45), then we argue exactly the same way and end up with the estimate

$$-\operatorname*{ess\,inf}_{\frac{1}{2}Q} u = \operatorname*{ess\,sup}_{\frac{1}{2}Q} -u < \operatorname*{ess\,sup}_{2Q} -u - \frac{\omega_2}{2^{s+1}} = -\operatorname*{ess\,inf}_{2Q} u - \frac{\omega_2}{2^{s+1}}. \tag{4.64}$$

Step 3 By considering $u^-_{k^-}$ instead of $u^+_{k^+}$ in the previous step, where $k^- = \operatorname{ess\,sup}_{2Q} -u - \frac{\omega}{2^s}$, we see that if (4.45) is true (for the function $-u$), we obtain the estimate (4.64) instead of (4.63) and vice versa. Hence, - and this is the main point - no matter which of the two alternatives (4.45)-(4.46) is correct, we end up with the same two estimates:

$$\operatorname*{ess\,sup}_{\frac{1}{2}Q} u \leq M_2 - \frac{\omega_2}{2^{s+1}}, \qquad \text{and} \qquad -\operatorname*{ess\,inf}_{\frac{1}{2}Q} u \leq -\operatorname*{ess\,inf}_{2Q} u - \frac{\omega_2}{2^{s+1}}.$$

Adding these two estimates concludes the proof:

$$\operatorname*{ess\,osc}_{\frac{1}{2}Q} u \leq M_2 - \operatorname*{ess\,inf}_{2Q} - \frac{\omega_2}{2^s} = \left(1 - \frac{1}{2^s}\right)\omega_2 < \left(1 - \frac{1}{2^{s+1}}\right)\omega_2.$$

□

We now come to the main theorem of this section:

Theorem 4.3.5. *Let* $u \in V_2(\Omega_T)$*, let* $\tau = \tau(\lambda,\Lambda,q,N)$ *be the number of Lemma* 4.2.2 *and let* $Q = Q(t_0,x_0,\tau,R) \Subset \Omega_T$*. For* $r \in (0,R]$*, set* $Q(r) = Q(t_0,x_0,\tau,r) \subset Q$*. Then there are numbers* $\alpha = \alpha(\lambda,\Lambda,q,N) \in (0,1)$ *and* $C = C(\lambda,\Lambda,q,N)$ *such that*

$$\operatorname*{ess\,osc}_{Q(r)} u \le C\left(\frac{r}{R}\right)^{\alpha}\left(\operatorname*{ess\,osc}_{Q(R)} u + R^{2-(N+2)/q}\|f\|_{L_q(Q(R))}\right). \tag{4.65}$$

Hence, keeping the cylinder Q fixed, we can control the oscillation of u on a subcylinder $Q(r)$ in terms of its radius. More importantly, it behaves like r^{α}. One can see a certain analogy between (4.65) and (2.1) So it is not surprising that we will be very close to finally finishing the proof of Theorem 3.1.3, once we have the above result at our disposal.

Proof. Without loss of gerneralilty, we assume $(t_0,x_0) = (0,0)$. Pick an arbitrary $r \in (0,R]$. By Lemma 4.3.4, there is a number $s = s(\lambda,\Lambda,q,N) \ge 4$ such that

$$\operatorname*{ess\,osc}_{\frac{1}{4}Q(r)} u \le \left(1-\frac{1}{2^s}\right)\operatorname*{ess\,osc}_{Q(r)} + 2^s r^{2-(N+2)/q}\|f\|_{L_q(Q(r))}. \tag{4.66}$$

Observe

$$Q\left(\frac{r}{4}\right) = \left[-\frac{1}{16}\tau r^2, 0\right] \times B\left(0,\frac{r}{4}\right) \subset \left[-\frac{1}{4}\tau r^2, 0\right] \times B\left(0,\frac{r}{4}\right) = \frac{1}{4}Q(r).$$

So by (4.66)

$$\operatorname*{ess\,osc}_{Q(\frac{r}{4})} u \le \left(1-\frac{1}{2^s}\right)\operatorname*{ess\,osc}_{Q(r)} + 2^s r^{2-(N+2)/q}\|f\|_{L_q(Q(r))}. \tag{4.67}$$

Set $\theta = \theta(\lambda,\Lambda,q,N) = 1-\frac{1}{2^s} \in [15/16,1)$ and for $r \in (0,R]$, define the non-decreasing functions

$$g(r) = \operatorname*{ess\,osc}_{Q(r)} u, \qquad h(r) = 2^s r^{2-(N+2)/q}\|f\|_{L_q(Q(r))}.$$

Hence, (4.67) reads

$$g\left(\frac{r}{4}\right) \le \theta g(r) + h(r). \tag{4.68}$$

From Lemma 2.3.4 we infer

$$g(r) \le \frac{1}{\theta}\left(\frac{r}{R}\right)^{(1-\mu)\frac{\log\theta}{\log 1/4}} g(R) + \frac{h(R^{1-\mu}r^{\mu})}{1-\theta}, \qquad r \in (0,R], \tag{4.69}$$

where $\mu \in (0,1)$ is arbitrary. Recall that $2-(N+2)/q > 0$ and that $(1-\mu)\frac{\log\theta}{\log 1/4} \in (0,1)$ is fixed. Hence, it is possible to take $\mu \in (0,1)$ so large that

$$\alpha = \alpha(\lambda,\Lambda,q,N) = (1-\mu)\frac{\log\theta}{\log 1/4} \leq \left(2-\frac{N+2}{q}\right)\mu.$$

Note $\alpha \in (0,1)$. Plugging this into (4.69) concludes the proof:

$$\begin{aligned}\operatorname*{ess\,osc}_{Q(r)} u &\leq \frac{1}{\theta}\left(\frac{r}{R}\right)^{\alpha}\operatorname*{ess\,osc}_{Q(R)} u + \frac{2^s}{1-\theta}\left(R\left(\frac{r}{R}\right)^{\mu}\right)^{2-(N+2)/q}\|f\|_{L_q(Q(R^{1-\mu}r^{\mu}))} \\ &\leq C\left(\frac{r}{R}\right)^{\alpha}\left(\operatorname*{ess\,osc}_{Q(R)} u + \left(\frac{r}{R}\right)^{(2-(N+2)/q)\mu-\alpha} R^{2-(N+2)/q}\|f\|_{L_q(Q(R))}\right) \\ &\leq C\left(\frac{r}{R}\right)^{\alpha}\left(\operatorname*{ess\,osc}_{Q(R)} u + R^{2-(N+2)/q}\|f\|_{L_q(Q(R))}\right),\end{aligned}$$

where $C = C(\lambda,\Lambda,q,N)$. □

4.4 Hölder continuity

This is the final step of the proof of Theorem 3.1.3, which we restate here for the sake of convenience:

Theorem 4.4.1 (De Giorgi-Nash-Moser). *Let $u \in V_2(\Omega_T)$ be a weak solution to* (3.1) *on Ω_T and let $\Omega'_T = [a,b]\times\Omega' \Subset \Omega_T$, $0 < a < b \leq T$, $\Omega' \Subset \Omega$ be an arbitrary parabolic cylinder that is compactly contained in Ω_T.*
Then there are numbers $\alpha = \alpha(\lambda,\Lambda,q,N) \in (0,1)$ and $C = C(\lambda,\Lambda,\Omega'_T,d,q,N)$, where $d = \operatorname{dist}(\Omega'_T,\Gamma_T)$, such that:

$$u \in C^{\alpha/2,\alpha}(\overline{\Omega'_T}) \text{ and } \|u\|_{C^{\alpha/2,\alpha}(\overline{\Omega'_T})} \leq C\left(\|u\|_{L_2(\Omega_T)} + \|f\|_{L_q(\Omega_T)}\right).$$

Let us first recall what it means for a function $u \in V_2(\Omega_T)$ to be locally Hölder continuous: It means that there is a function $\hat{u}$ which has this property and coincides with u almost everywhere. We proceed in three steps: Given an arbitrary weak solution u to (3.1), we first construct a particular representative $\hat{u}$ in the equivalence class of u. Next, es an intermediate step, we show the continuity of it and then, finally, we establish the whole assertion of the theorem.

Proof. Let $\tau = \tau(\lambda,\Lambda,q,N) \in (0,1]$ be the number of Lemma 4.2.2 and let $R = R(\tau,d) = R(\lambda,\Lambda,d,q,N) > 0$ be such that

$$R \leq d/2, \text{ and } Q(t_0,x_0,\tau,R) \Subset \Omega_T \text{ for all } (t_0,x_0) \in \overline{\Omega'_T}. \tag{4.70}$$

This is possible, since $\Omega'_T \Subset \Omega_T$.
Now fix an arbitrary point $(t_0,x_0) \in \overline{\Omega'_T}$ with its associated cylinder $Q = Q(t_0,x_0,\tau,R)$ and family of subcylinders $Q(r) = Q(t_0,x_0,\tau,r)$, where $r \in (0,R]$. Further, set

$$u(t_0,x_0,r) = \frac{1}{|Q(r)|}\int_{Q(r)} u\,dx\,dt, \qquad r \in (0,R].$$

Theorem 4.3.5 implies that for certain numbers $\alpha = \alpha(\lambda,\Lambda,q,N) \in (0,1)$ and $C = C(\lambda,\Lambda,q,N)$ there holds

$$\operatorname*{ess\,inf}_{Q(r)} u \leq u(t_0,x_0,r) \leq \operatorname*{ess\,sup}_{Q(r)} u \leq \operatorname*{ess\,inf}_{Q(r)} u + C\left(\frac{r}{R}\right)^{\alpha}\left(\operatorname*{ess\,osc}_{Q(R)} u + R^{2-(N+2)/q}\|f\|_{L_q(Q(R))}\right). \quad (4.71)$$

Note that $\operatorname{ess\,inf}_{Q(r)} u$ is bounded from above for all $r \in (0,R]$ and monotonically increasing for $r \searrow 0$. So the limit $\lim_{r\searrow 0} \operatorname{ess\,inf}_{Q(r)} u$ exists. Also, the second term in the right-hand side of (4.71) converges to zero for $r \searrow 0$. Hence, the limit $\lim_{r\searrow 0} u(t_0,x_0,r)$ exists. Denote this limit by $\hat{u}(t_0,x_0)$. But Lebesgue's differentiation theorem 2.1.5 implies that $u(t_0,x_0)$ and $\hat{u}(t_0,x_0)$ are equal for almost all $(t_0,x_0) \in \overline{\Omega'_T}$.

Step 2 Having constructed our candidate for the Hölder continuous representative of u, we first establish the continuity of $\hat{u}$ as an intermediate step.

Observe $\operatorname{ess\,inf}_{Q(r)} u \leq \hat{u}(t_0,x_0) \leq \operatorname{ess\,sup}_{Q(r)} u$ for all $r \in (0,R]$. So this and the first two inequalities in (4.71) imply

$$|u(t_0,x_0,r) - \hat{u}(t_0,x_0)| \leq \operatorname*{ess\,osc}_{Q(r)} u \leq C\left(\frac{r}{R}\right)^{\alpha}\left(\operatorname*{ess\,osc}_{Q(R)} u + R^{2-(N+2)/q}\|f\|_{L_q(Q(R))}\right) \to 0 \quad (4.72)$$

as $r \to 0$. Recall that Lebesgue's Differentiation Theorem also proclaims that, keeping r fixed, the mapping $(t_0,x_0) \mapsto u(t_0,x_0,r)$ is continuous. Thus, for a sequence $r_k \subset (0,R]$ with $r_k \to 0$ for $k \to \infty$, $u(t_0,x_0,r_k)$ is a sequence of continuous functions on $\overline{\Omega'_T}$ that converge uniformly to $\hat{u}$ by (4.72). This proves the continuity of $\hat{u}$ on $\overline{\Omega'_T}$.

Step 3 We now discuss the Hölder continuity of $\hat{u}$. Let $(t,x),(t',x') \in \overline{\Omega'_T}$. Without loss of generality, we may assume $t \leq t'$. First, consider the case that

$$r = \max\left(\frac{\sqrt{t'-t}}{\sqrt{\tau}}, |x'-x|\right) < \frac{R}{2}.$$

Clearly, by (4.70), $Q(r) = Q(t',x',\tau,r) \Subset \Omega_T$ and $(t,x) \in Q(r)$. From (4.65), applied to the cylinder $Q(R/2)$ we infer

$$\begin{aligned}
|\hat{u}(t',x') - \hat{u}(t,x)| &\leq \operatorname*{osc}_{Q(r)} \hat{u} \leq 2^{\alpha}C\left(\frac{r}{R}\right)^{\alpha}\left(\operatorname*{osc}_{Q(R/2)} \hat{u} + \left(\frac{R}{2}\right)^{2-(N+2)/q}\|f\|_{L_q(Q(R/2)))}\right) \\
&\leq \frac{2^{\alpha}C}{(\sqrt{\tau}R)^{\alpha}}\left(\sqrt{t'-t} + |x'-x|\right)^{\alpha}\left(\operatorname*{osc}_{Q(R/2)} \hat{u} + \left(\frac{R}{2}\right)^{2-(N+2)/q}\|f\|_{L_q(Q(R/2))}\right) \quad (4.73) \\
&\leq C_1\left(\sqrt{t'-t} + |x'-x|\right)^{\alpha}\left(\|\hat{u}\|_{L_\infty(Q(R/2))} + \|f\|_{L_q(Q(R/2))}\right) \\
&\leq C_2\left(\sqrt{t'-t} + |x'-x|\right)^{\alpha}\left(\|\hat{u}\|_{L_2(\Omega_T)} + \|f\|_{L_q(\Omega_T)}\right). \quad (4.74)
\end{aligned}$$

We made use of $\tau \leq 1$ in order to derive (4.73). Further, $Q(R/2) \subset \frac{1}{2}Q$, which enabled us to use Corollary 4.1.3 to obtain (4.74). Here, $C_1, C_2 = C_1, C_2(\lambda,\tau,\Lambda,d,q,N) = C_1, C_2(\lambda,\Lambda,d,q,N)$.

Now suppose $\max\left(\frac{\sqrt{t'-t}}{\sqrt{\tau}}, |x'-x|\right) \geq \frac{R}{2}$. Then there are finitely many points $(t_0,x_0) = (t,x)$, (t_1,x_1),

$\dots,(t_{n-1},x_{n-1}),(t_n,x_n)=(t',x')\in\overline{\Omega'_T}$ with $t_0\le t_1\le\dots\le t_n$ and

$$r_i=\max\left(\frac{\sqrt{t_i-t_{i-1}}}{\sqrt{\tau}},|x_i-x_{i-1}|\right)<\frac{R}{2}$$
$$t_i-t_{i-1}\le t'-t$$
$$|x_i-x_{i-1}|\le|x'-x|$$

for $i=1,\dots,n$. Here, $n=n(\tau,\Omega'_T,R)=n(\lambda,\Lambda,\Omega'_T,d,q,N)$.

Define the cylinders $Q_i=Q(t_i,x_i,\tau,r_i)$. As before, $(t_{i-1},x_{i-1})\in Q_i$ for $i=1,\dots,n$. Then, by (4.74),

$$\begin{aligned}|\hat{u}(t',x')-\hat{u}(t,x)|&\le\sum_{i=1}^{n}|\hat{u}(t_i,x_i)-\hat{u}(t_{i-1},x_{i-1})|\le\sum_{i=1}^{n}\operatorname*{osc}_{Q_i}\hat{u}\\&\le\sum_{i=1}^{n}C_2\left(\sqrt{t_i-t_{i-1}}+|x_i-x_{i-1}|\right)^{\alpha}\left(\|\hat{u}\|_{L_2(\Omega_T)}+\|f\|_{L_q(\Omega_T)}\right)\\&\le C_3\left(\sqrt{t'-t}+|x'-x|\right)^{\alpha}\left(\|\hat{u}\|_{L_2(\Omega_T)}+\|f\|_{L_q(\Omega_T)}\right)\end{aligned}\tag{4.75}$$

for a constant $C_3=C_3(\lambda,\Lambda,\Omega'_T,d,q,N)$.
A combination of the two estimates (4.1.4) and (4.75) then (finally!) concludes the proof of the theorem.

□

4.5 Concluding remarks and references

It is noteworthy that we actually showed more than Theorem 3.1.3: We showed that if $u\in V_2(\Omega_T)$ is a weak solution, then both $\pm u$ satisfy inequality (4.1). And this inequality sufficed to established the local boundedness, i.e. we did not draw on the fact that u is a weak solution to (3.1) thereafter. Hence, any function $u\in V_2(\Omega_T)$ such that both $\pm u$ satisfy an inequality of the type (4.1) will be locally bounded (of course, the numbers $\|f\|_{L_q(Q)}$ and q have to be replaced suitably as they have nothing to do with a general function $u\in V_2(\Omega_T)$).
Likewise, any *locally bounded* function $u\in V_2(\Omega_T)$ that satisfies inequalities of type (4.26) and (4.27) will be locally Hölder continuous.
This motivates the introduction of the so-called parabolic De Giorgi classes. See for instance [18, Chapter VI, Section 13] for a broader discussion of them.

The proof we gave in this chapter was based on [17]. We followed [17, Chapter 2, §6 and Chapter 3, §8] in the proof of the local boundedness. Section 4.2 and 4.3 are based on [17, Chapter 2, §7 and Chapter 3, §10]. The establishment of the Hölder regularity is based on [14, Theorem 11.2.1] and [25, Section 5.4.4]. Compare also [18, Chapter VI, Section 13] for a related variant of the proof.

5 Moser's iteration technique and the weak Harnack inequality

This chapter is the second main part of this thesis. In it, we give a different proof of Theorem 3.1.3, based on Moser's ideas developed in [21]. Comparing it with the proof given in the previous chapter, one can say that the intermediate objectives are the same: The establishment of local boundedness and suitable oscillation estimates. In fact, we 'only' give two new proofs of Theorem 4.1.2 and 4.3.5. However, that's it already with the similarity, as the strategies to obtain these results differ in every respect. We begin with a brief overview, first discussing how to establish the local boundedness this time.

The proof of that heavily relies on the embedding Theorem 2.2.14, which gives elements of the space $\mathring{V}_2$ additional integrability. This leads to the idea of studying higher powers of weak sub-solutions. We consider a weak sub-solution u on a cylinder $Q \Subset \Omega_T$ and insert a suitable test function φ in the definition of a weak sub-solution. Very roughly speaking, φ is the product a certain power of u and a cutoff-function of a smaller subcylinder. Eventually, by using the embedding theorem, we end up with an estimate of the following form: A higher L_p-norm of u on a smaller cylinder is dominated by a lower L_p-norm on a larger cylinder. It turns out that the particular structure of this estimate allows us to iterate this procedure, leading ultimately to the desired boundedness result.

As before, the more difficult part of the proof of Theorem 3.1.3 is the derivation of good estimates for the oscillation of u. While the previous approach was more direct and involved hands-on estimates of the oscillation, we argue in a roundabout way this time. To grasp the idea, recall the following fundamental result in the field of partial differential equations:

Theorem 5.0.1 (Harnack inequality for harmonic functions)**.** *Given domains $\Omega' \Subset \Omega \subset \mathbb{R}^N$, there is a constant $C = C(\Omega', \Omega, \operatorname{dist}(\Omega', \Omega))$ such that for any non-negative $u \in C^2(\Omega)$ with $\Delta u = 0$*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u. \tag{5.1}$$

Let us demonstrate how this theorem easily yields oscillation estimates for u on different subdomains: Let

u be as above but not necessarily non-negative and consider another domain $\Omega'' \Subset \Omega'$. Clearly, $\sup_{\Omega'} u - u$ and $u - \inf_{\Omega'} u$ are harmonic and non-negative on Ω'. Applying (5.1) to these two functions on Ω'' and adding the resulting inequalities then quickly shows

$$\operatorname*{osc}_{\Omega''} u \leq \frac{C-1}{C} \operatorname*{osc}_{\Omega'} u.$$

Hence, the oscillation of u decays when one passes to the smaller domain Ω'' - the principal assertion of Theorem 4.3.5 for which we had to invest enormous efforts.

The major part of this chapter is concerned with the derivation an inequality that resembles (5.1). Once we have this tool in our hands, the proof of Theorem 4.3.5 is fairly simple and is based on the just mentioned strategy. This way, we can avoid the tedious oscillation estimates of the previous chapter and obtain a result of independent interest as a byproduct. Establishing this inequality is very far from trivial, though.

5.1 Yet another proof of local boundedness

In this first section, we give another proof of the fact that weak sub-solutions are locally bounded from above. As discussed in the introduction, the main idea is to study various powers of these functions. However, this should alert us, since we assume these weak sub-solutions to be (roughly speaking) merely square ingegrable. The trick to circumvent the obstacle that arbitrary high powers of these functions might not be integrable, is to cut them off for large values. In some sense, we proceed conversely to De Giorgi's proof where we studied the parts of a weak sub-solution above a certain level.

Here is our setup: Let $m \geq k > 0$ and $x \in \mathbb{R}$. Define $\bar{x} = x^+ + k$ and

$$\bar{x}_m = \begin{cases} \bar{x}, & x < m, \\ k+m, & x \geq m. \end{cases}$$

Recall $x^+ = \max(x,0)$. Considering $\bar{x}_m$ as a function of x, it is clearly piecewise smooth (hence, classically differentiable almost everywhere) and its derivative is the indicator function of the interval $(0,m)$.
For $\beta \geq 0$, define the function $G : \mathbb{R} \to \mathbb{R}$ via

$$G(x) = \bar{x}_m^\beta \bar{x} - k^{\beta+1} = \begin{cases} \bar{x}^{\beta+1} - k^{\beta+1}, & x < m, \\ (k+m)^\beta \bar{x} - k^{\beta+1}, & x \geq m. \end{cases}$$

Let us discuss the motivation behind G: Since we are interested in studying powers of a weak sub-solution u, we will insert it in the function G. As we do not know yet that arbitrary powers of u are integrable, we cut it off at level m. Of course, we could simply study $\bar{u}_m^\beta$ but by doing so, we would lose all information about u above the level m. For this reason, the factor $\bar{x}$ is included in the definition of G. Note carefully that this function is designed in such a way that $G(u)$ is square-integrable. The scalar k serves as a placeholder for the norm of the inhomogenity f in (3.1). Lastly, it is desirable to have $G(u) = 0$ whenever $u \leq 0$. This explains the term $-k^{\beta+1}$.

Moreover, we will need the derivative and a primitive function of G (note that G is piecewise smooth with corners at the points $x = 0$ and $x = m$). We have

$$G'(x) = \begin{cases} 0, & x \leq 0, \\ (\beta+1)\bar{x}^{\beta}, & 0 < x < m, \\ (k+m)^{\beta}, & x \geq m, \end{cases}$$

and we define

$$H(x) = \begin{cases} \frac{1}{\beta+2}\left(\bar{x}^{\beta+2} - (\beta+2)k^{\beta+1}\bar{x} + (\beta+1)k^{\beta+2}\right), & x < m, \\ \frac{1}{2}(k+m)^{\beta}\bar{x}^2 - k^{\beta+1}\bar{x} + \frac{1}{\beta+2}\left((\beta+1)k^{\beta+2} - \frac{\beta}{2}(k+m)^{\beta+2}\right), & x \geq m. \end{cases}$$

It is easy to see that $H'(x) = G(x)$ for $x < m$ and $x \geq m$. The rather complicated form of the constant terms in the definition of H arises from our desire to avoid a jump discontinuity at the point $x = m$.

Next, we list some further properties of the above functions. Although most of them follow directly from the definitions, we state them as a lemma in order not to pause incessantly for some algebraic manipulations in our later argument.

Lemma 5.1.1. *Let the functions $\bar{x}_m, G', G, H : \mathbb{R} \to \mathbb{R}$ be defined as above.*

(a) $G'(x), G(x), H(x)$ are non-negative and vanish whenever $x \leq 0$.
(b) The following inequalities hold:

$$G(x) \leq \bar{x}_m^{\beta/2}\bar{x}\sqrt{G'(x)}, \tag{5.2}$$

$$G(x) \leq \bar{x}_m^{\beta}\bar{x} \tag{5.3}$$

$$\frac{1}{2(\beta+2)}\bar{x}_m^{\beta}\bar{x}^2 - \frac{\beta+1}{\beta+2}k^{\beta+2} \leq H(x) \leq \bar{x}_m^{\beta}\bar{x}^2 \tag{5.4}$$

$$\bar{x}_m \nearrow \bar{x} \text{ for } m \to \infty. \tag{5.5}$$

Proof. We only prove (a) for the function H and (5.4) as the other assertions are trivial.
It is a simple calculation to show that $H(x) = 0$ for $x \leq 0$. The rest of the claim follows from monotonicity, since $G(x) \geq 0$ and H is a primitive function of G.
To show the upper bound in (5.4), we first consider the case $x < m$. Clearly, $k \leq \bar{x}$, so $-(\beta+2)k^{\beta+1}\bar{x} + (\beta+1)k^{\beta+2}$ will be negative and can be dropped. If $x \geq m$, we use $k^{\beta+2} < k^{\beta+1}\bar{x}$. So the sum of the three latter summands in the definition of H will be negative, which proves the assertion.
Concerning the lower bound in (5.4), we first consider the case $x < m$. It suffices to show

$$\frac{1}{2(\beta+2)}\bar{x}^{\beta+2} + 2\frac{\beta+1}{\beta+2}k^{\beta+2} \geq \bar{x}k^{\beta+1}.$$

We use Young's inequality with $\varepsilon = \frac{1}{2}$, $p = \beta+2$ and $p' = \frac{\beta+2}{\beta+1}$. This implies

$$\bar{x}k^{\beta+1} \leq \frac{1}{2(\beta+2)}\bar{x}^{\beta+2} + 2^{1/(\beta+1)}\frac{\beta+1}{\beta+2}k^{\beta+2} \leq \frac{1}{2(\beta+2)}\bar{x}^{\beta+2} + \frac{\beta+1}{\beta+2}k^{\beta+2}.$$

Now assume $x \geq m$ and recall $-(k+m) \geq -\bar{x}$. First observe that, since $\frac{1}{2} - \frac{\beta}{2(\beta+2)} = \frac{1}{\beta+2}$,

$$\frac{1}{2}(k+m)^{\beta}\bar{x}^2 - k^{\beta+1}\bar{x} + \frac{1}{\beta+2}\left((\beta+1)k^{\beta+2} - \frac{\beta}{2}(k+m)^{\beta+2}\right) \geq \frac{1}{\beta+2}(k+m)^{\beta}\bar{x}^2 - k^{\beta+1}\bar{x} + \frac{\beta+1}{\beta+2}k^{\beta+2}.$$

So it suffices to show

$$\frac{1}{2(\beta+2)}(k+m)^{\beta}\bar{x}^2 + 2\frac{\beta+1}{\beta+2}k^{\beta+2} \geq k^{\beta+1}\bar{x}.$$

Since we assume $m \geq k$, and $\frac{2^{\beta}}{2(\beta+2)} \geq 1/4$ there holds

$$\frac{1}{2(\beta+2)}(k+m)^{\beta}\bar{x}^2 + 2\frac{\beta+1}{\beta+2}k^{\beta+2} \geq \frac{1}{4}k^{\beta}\bar{x}^2 + k^{\beta+2} \geq k^{\beta+1}\bar{x},$$

where the last inequality can be shown by a derivative test. We omit this elementary argument. □

As in the previous chapter, we denote for $(t_0,x_0) \in \Omega_T$ and $\tau, r > 0$ the set $[t_0 - \tau r^2, t_0] \times B(x_0,r)$ by Q. Also, for $\delta \in (0,1)$ we write $\delta Q = [t_0 - \delta\tau r^2] \times B(x_0, \delta r)$. We will always assume that $Q \Subset \Omega_T$ is meaningful in the following.

Theorem 5.1.2. *Let $u \in V_2(\Omega_T)$ be a weak sub-solution to* (3.1) *on Ω_T and let $Q = [t_0 - \tau r^2, t_0] \times B(x_0,r) \Subset \Omega_T$. Fix $\delta \in (0,1)$ and $p > 0$. Then there is a constant $C = C(\delta, \lambda, \tau, \Lambda, p, q, N)$ such that*

$$\operatorname*{ess\,sup}_{\delta Q} u^+ \leq C\left((1-\delta)^{-(N+2)/p}\left(\frac{1}{|Q|}\int_Q (u^+)^p \, dx\, dt\right)^{1/p} + r^{2-(N+2)/q}\|f\|_{L_q(Q)}\right). \tag{5.6}$$

Note that this time, we allow the exponent p to be any positive number, which will be needed in a later argument. The proof heavily relies on Moser's iteration scheme, in particular Lemma 2.3.7.

Proof. Just as in the proof of Theorem 4.1.2, we may assume $(t_0,x_0) = (0,0)$ and $r = 1$. Set $B = B(0,1)$. Also, we first prove the assertion in case of $\delta = 1/2$ and $p = 2$.

In the definition of the function G in (5.1), let $\beta \geq 0$ be arbitrary and put $k = \|f\|_{L_q(Q)}$. In the following, we assume this number to be strictly positive. If not, pick any $k > 0$ and let $k \searrow 0$ in the very end - this suffices to deduce (5.6) as the constant C is independent of f.

Again, Steklov averages of u and Proposition 3.3.2 will be used in the following. For it, let $0 \leq \psi \leq 1$ be a piecewise smooth function that vanishes on the parabolic boundary of Q and outside it. Consider the non-negative function $\varphi = \psi^2 G(S_h u)$, where $h > 0$ is sufficiently small in the sense of the above proposition. Since $S_h u \in W_2^{1,1}(Q)$, the same holds true for $G(S_h u)$ by the chain rule $A.12$; in total, $\varphi \in \overset{\bullet}{W}{}_2^{1,1}(Q) \subset \mathring{V}_2(Q)$. So for $t \in [-\tau, 0]$, $\varphi(t,\cdot) \in \mathring{W}{}_2^1(B)$. Pick an arbitrary t_1 in this interval and integrate

(3.12) from $-\tau$ to t_1. This gives, using the chain and product rule,

$$\begin{aligned}
&\int_{-\tau}^{t_1}\int_B \partial_t(S_h u)G(S_h u)\psi^2\,dx\,dt + \int_{-\tau}^{t_1}\int_B \left(S_h(A\nabla u)|\nabla(S_h u)\right)\psi^2 G'(S_h u)\,dx\,dt \\
&+\int_{-\tau}^{t_1}\int_B 2\psi G(S_h u)\left(S_h(A\nabla u)|\nabla\psi\right)\,dx\,dt \le \int_{-\tau}^{t_1}\int_B (S_h f)G(S_h u)\psi^2\,dx\,dt.
\end{aligned} \tag{5.7}$$

Again, from the product and chain rule and Lemma 5.1.1, we deduce

$$\partial_t(S_h u)G(S_h u)\psi^2 = \partial_t\left(H(S_h u)\psi^2\right) - 2\psi\partial_t\psi H(S_h u). \tag{5.8}$$

In this order, we plug this identity in (5.7), apply Proposition 2.2.13 and move the latter term in (5.8) to the right-hand side of (5.7). This gives, since ψ vanishes on the parabolic boundary of Q

$$\begin{aligned}
&\int_B \left(H(S_h u)\psi^2\right)(t_1,x)\,dx + \int_{-\tau}^{t_1}\int_B \left(S_h(A\nabla u)|\nabla(S_h u)\right)\psi^2 G'(S_h u)\,dx\,dt \\
&+\int_{-\tau}^{t_1}\int_B 2\psi G(S_h u)\left(S_h(A\nabla u)|\nabla\psi\right)\,dx\,dt \le \int_{-\tau}^{t_1}\int_B (S_h f)G(S_h u)\psi^2 + 2\psi\partial_t\psi H(S_h u)\,dx\,dt.
\end{aligned}$$

We now let $h \to 0$, arguing exactly the same as in the derivation of (4.6). We end up with

$$\begin{aligned}
&\int_B \left(H(u)\psi^2\right)(t_1,x)\,dx + \int_{-\tau}^{t_1}\int_B (A\nabla u|\nabla u)\psi^2 G'(u)\,dx\,dt \\
&+\int_{-\tau}^{t_1}\int_B 2\psi G(u)(A\nabla u|\nabla\psi)\,dx\,dt \le \int_{-\tau}^{t_1}\int_B fG(u)\psi^2 + 2\psi\partial_t\psi H(u)\,dx\,dt.
\end{aligned} \tag{5.9}$$

Step 2 This step is devoted to the derivation of finer estimates from (5.9). We first restrict ourselves to the part of the domain of integration, where $u > 0$ holds. Note carefully that we have $\nabla u = \nabla\overline{u}$ in this situation. In the forthcoming estimates are valid almost everywhere.
By assumption **(A2)** on the matrix A, there holds

$$\lambda|\nabla\overline{u}|^2\psi^2 G'(u) \le (A\nabla\overline{u}|\nabla\overline{u})\psi^2 G'(u).$$

Furthermore, (5.2), assumption **(A1)** and Young's inequality with $\varepsilon = \lambda/2$ imply

$$2\psi G(u)(A\nabla\overline{u}|\nabla\psi) \ge -2\psi\sqrt{\overline{u}_m^{\beta}\overline{u}^2 G'(u)}\,|(A\nabla\overline{u}|\nabla\psi)| \ge -\frac{\lambda}{2}\psi^2 G'(u)|\nabla\overline{u}|^2 - \frac{2\Lambda^2}{\lambda}|\nabla\psi|^2\overline{u}_m^{\beta}\overline{u}^2.$$

We insert these two results in (5.9) and apply in the following chain of inequalities (5.3), $k \le \overline{u}$ and (5.4):

$$\begin{aligned}
&\int_B \left(H(u)\psi^2\right)(t_1,x)\,dx + \frac{\lambda}{2}\int_{-\tau}^{t_1}\int_B |\nabla\overline{u}|^2 G'(u)\psi^2\,dx\,dt \\
&\le \int_{-\tau}^{t_1}\int_B fG(u)\psi^2\,dx\,dt + 2\int_{-\tau}^{t_1}\int_B \psi\partial_t\psi H(u)\,dx\,dt + \frac{2\Lambda}{\lambda}\int_{-\tau}^{t_1}\int_B |\nabla\psi|^2\overline{u}_m^{\beta}\overline{u}^2\,dx\,dt \\
&\le \int_{-\tau}^{t_1}\int_B \left(\frac{f}{k}\psi^2 + 2\psi\partial_t\psi + \frac{2\Lambda}{\lambda}|\nabla\psi|^2\right)\overline{u}_m^{\beta}\overline{u}^2\,dx\,dt
\end{aligned} \tag{5.10}$$

$$\leq \int_Q \left(\frac{|f|}{k} \psi^2 + 2\psi|\partial_t \psi| + \frac{2\Lambda}{\lambda} |\nabla \psi|^2 \right) \overline{u}_m^{\beta} \overline{u}^2. \tag{5.11}$$

The derivation relied on the positivity of u. However, on the set where $u \leq 0$ is true, the functions $G'(u)$ and $H(u)$ vanish, see Lemma 5.1.1(a). So our intermediate result also remains valid in this case.
Put $v = \overline{u}_m^{\beta/2} \overline{u}$. By (5.4),

$$\frac{1}{(\beta+2)} \int_B \left(\frac{1}{2} k^{\beta+2} \psi^2 - (\beta+1) v^2 \psi^2 \right)(t_1, x)\, dx \leq \int_B \left(H(u) \psi^2 \right)(t_1, x)\, dx. \tag{5.12}$$

Pick an arbitrary $j \in \{1, \ldots, N\}$ and observe

$$\partial_j v = \begin{cases} \frac{\beta}{2} \overline{u}_m^{\beta/2-1} \partial_j \overline{u}_m \overline{u} + \overline{u}_m^{\beta/2} \partial_j \overline{u}, & 0 < u < m, \\ 0 & \text{else}, \end{cases} = \left(1 + \frac{\beta}{2}\right) \overline{u}_m^{\beta/2} \partial_j \overline{u}.$$

This implies

$$(1+\beta/2)^{-2} \int_{-\tau}^{t_1} \int_B |\nabla v|^2 \psi^2 \, dx\, dt \leq \int_{-\tau}^{t_1} \int_B |\nabla \overline{u}|^2 G'(u) \psi^2 \, dx\, dt. \tag{5.13}$$

Note carefully that both integrands in (5.10) are non-negative. So by dropping either of them, using (5.12), (5.13), respectively, and recalling that $t_1 \in [-\tau, 0]$ is arbitrary, we see (since $\beta + 2 \leq 2(1+\beta/2)^2$)

$$\max_{-\tau \leq t \leq 0} \int_B \left(\frac{1}{2} v^2 \psi^2 - (\beta+1) k^{\beta+2} \psi^2 \right)(t,x)\, dx \leq C_1 (1+\beta/2)^2 \int_Q \left(\frac{|f|}{k} \psi^2 + \psi |\partial_t \psi| + |\nabla \psi|^2 \right) v^2 \, dx\, dt, \tag{5.14}$$

$$\frac{\lambda}{2} \int_Q |\nabla v|^2 \psi^2 \, dx\, dt \leq C_1 (1+\beta/2)^2 \int_Q \left(\frac{|f|}{k} \psi^2 + \psi |\partial_t \psi| + |\nabla \psi|^2 \right) v^2 \, dx\, dt, \tag{5.15}$$

where $C_1 = C_1(\lambda, \Lambda)$.
Step 3 Quite clearly, we are now interested in an estimate for the right-hand sides of the above inequalities. To this end, observe that $\int_Q \frac{f}{k} v^2 \psi^2 \, dx\, dt \leq \|v\psi\|^2_{L_{2q'}(Q)}$, where we used Hölder's inequality and the definition of the number k. The interpolation iequality 2.1.4 provides us with

$$\|v\psi\|^2_{L_{2q'}(Q)} \leq \left(\varepsilon \|v\psi\|_{L_{2+4/N}(Q)} + \varepsilon^{-\mu} \|v\psi\|_{L_2(Q)} \right)^2 \leq 2 \left(\varepsilon^2 \|v\psi\|^2_{L_{2+4/N}(Q)} + \varepsilon^{-2\mu} \|v\psi\|^2_{L_2(Q)} \right).$$

It is easily verified that our chosen constellation of exponents fulfill the requirements of Proposition 2.1.4, using assumption **(A3)**. Here, $\varepsilon > 0$ will be fixed later and $\mu = \mu(q, N) > 0$. Since $v\psi \in \mathring{V}_2(Q)$, there is a constant $C_2 = C_2(N)$ by Theorem 2.2.14 such that

$$\begin{aligned} \|v\psi\|^2_{L_{2q'}(Q)} &\leq 2 \left(\varepsilon^2 C_2 \|v\psi\|^2_{L_{V_2(Q)}} + \varepsilon^{-2\mu} \|v\psi\|^2_{L_2(Q)} \right) \\ &\leq 4 \left(\varepsilon^2 C_2 \left(\max_{-\tau \leq t \leq 0} \|(v\psi)(t, \cdot)\|^2_{L_2(B)} + \|\nabla v \psi\|^2_{L_2(Q)} + \|v \nabla \psi\|^2_{L_2(Q)} \right) + \varepsilon^{-2\mu} \|v\psi\|^2_{L_2(Q)} \right). \end{aligned} \tag{5.16}$$

Now choose ε in such a way that $2C_1(1+\beta/2)^2 4\varepsilon^2 C_2 = \frac{1}{4}\min(1,\lambda)$. We add the two inequalities (5.14) and (5.15), move the first two summands in (5.16) the left-hand side and subjoin another term $\frac{1}{4}\min(1,\lambda)\|v\nabla\psi\|^2_{L_2(Q)}$. This gives

$$\begin{aligned}
&\frac{1}{4}\min(1,\lambda)\left(\max_{-\tau\le t\le 0}\|(v\psi)(t,\cdot)\|^2_{L_2(B)}+\|\nabla v\psi\|^2_{L_2(Q)}+\|v\nabla\psi\|^2_{L_2(Q)}\right)\\
\le&\frac{1}{2}\min(1,\lambda)\|v\nabla\psi\|^2_{L_2(Q)}+(1+\beta/2)^2k^{\beta+2}\max_{-\tau\le t\le 0}\|\psi(t,\cdot)\|^2_{L_2(B)}\\
&+2C_1(1+\beta/2)^2\left(\varepsilon^{-2\mu}\|v\psi\|^2_{L_2(Q)}+\int_Q\left(\psi|\partial_t\psi|+|\nabla\psi|^2\right)v^2\,dx\,dt\right)\\
\le&\frac{1}{2}\min(1,\lambda)\|v\nabla\psi\|^2_{L_2(Q)}+(1+\beta/2)^2k^{\beta+2}\max_{-\tau\le t\le 0}\|\psi(t,\cdot)\|^2_{L_2(B)}\\
&+C_3(1+\beta/2)^2\left(\|v\psi\|^2_{L_2(Q)}+\int_Q\left(\psi|\partial_t\psi|+|\nabla\psi|^2\right)v^2\,dx\,dt\right),
\end{aligned}$$

for a certain constant $C_3 = C_3(\lambda,\Lambda,q,N)$, where we used the definition of ε and $-2\mu+2<2$. From this, we infer

$$\begin{aligned}
&\max_{-\tau\le t\le 0}\|(v\psi)(t,\cdot)\|^2_{L_2(B)}+\|\nabla v\psi\|^2_{L_2(Q)}+\|v\nabla\psi\|^2_{L_2(Q)}\\
\le&\, C_4(1+\beta/2)^2\left(\max_{-\tau\le t\le 0}k^{\beta+2}\|\psi(t,\cdot)\|^2_{L_2(B)}+\|v\nabla\psi\|^2_{L_2(Q)}+\|v\psi\|^2_{L_2(Q)}+\int_Q\psi|\partial_t\psi|v^2\,dx\,dt\right)
\end{aligned}\tag{5.17}$$

with $C_4 = C_4(\lambda,\Lambda,q,N)$.

Step 4 For $0<\sigma\le 1$, define the family of cylinders U_σ via $U_\sigma = [-\sigma\tau,0]\times\sigma B,\ \sigma B = B(0,\sigma)$. Pick an arbitrary $\sigma'\in(0,\sigma)$ and impose the following further restrictions on the function ψ: We require ψ to be equal to one on $U_{\sigma'}$, supp $\psi\subset U_\sigma$ and linear in between. This way, we can ensure that $|\nabla\psi|\le\frac{2}{\sigma-\sigma'}$ and $|\partial_t\psi|\le\frac{4}{(\sigma-\sigma')\tau}\le\frac{4}{(\sigma-\sigma')^2\tau}$. Note carefully that under these assumptions there holds

$$\max_{-\tau\le t\le 0}k^{\beta+2}\|\psi(t,\cdot)\|^2_{L_2(B)}\le k^{\beta+2}|\sigma B| = \frac{1}{\sigma\tau}\|k^{1+\beta/2}\|^2_{L_2(U_\sigma)}\le\frac{1}{\sigma\tau}\|v\|^2_{L_2(U_\sigma)}\le\frac{1}{(\sigma-\sigma')^2\tau}\|v\|^2_{L_2(U_\sigma)},$$

since $k^{\beta+2}\le v^2$.

Set $\kappa = 1+2/N$. Again, by Theorem 2.2.14 with the constant C_2 from above and by (5.17),

$$\begin{aligned}
&\|v\|^2_{L_{2\kappa}(U_{\sigma'})}\le\|v\psi\|^2_{L_{2\kappa}(U_\sigma)}\le C_2\|v\psi\|^2_{V_2(U_\sigma)} &(5.18)\\
&\le 2C_2\left(\max_{-\tau\le t\le 0}\|(v\psi)(t,\cdot)\|^2_{L_2(\sigma B)}+\|\nabla v\psi\|^2_{L_2(U_\sigma)}+\|v\nabla\psi\|^2_{L_2(U_\sigma)}\right)\\
&\le C_5(1+\beta/2)^2\left(\frac{1}{(\sigma-\sigma')^2\tau}\|v\|^2_{L_2(U_\sigma)}+\frac{4}{(\sigma-\sigma')^2}\|v\|^2_{L_2(U_\sigma)}+\|v\|^2_{L_2(U_\sigma)}+\frac{4}{(\sigma-\sigma')^2\tau}\|v\|^2_{L_2(U_\sigma)}\right)\\
&\le C_6\frac{(1+\beta/2)^2}{(\sigma-\sigma')^2}\|v\|^2_{L_2(U_\sigma)}\le C_6\frac{(1+(\beta+2))^2}{(\sigma-\sigma')^2}\|\overline{u}^{1+\beta/2}\|^2_{L_2(U_\sigma)}, &(5.19)
\end{aligned}$$

where we used $1 \leq (\sigma-\sigma')^{-2}$, the definition of v and $\overline{u}_m \leq \overline{u}$. Here, $C_5 = C_5(\lambda,\Lambda,q,N)$, $C_6 = C_6(\lambda,\tau,\Lambda,q,N)$ are suitable constants.
Notice that (5.19) is is independent of the number m in the definition of v. So with the aid of (5.5), we let $m \to \infty$ and see $\overline{u}_m^{\beta/2}\overline{u} \to \overline{u}^{1+\beta/2}$ in $\|\cdot\|_{L_{2\kappa}(U_{\sigma'})}$ by the monotone convergence theorem.
Set $\gamma = \beta+2 \geq 2$. Since $\|\overline{u}^{1+\beta/2}\|^2_{L_{2\kappa}(U_{\sigma'})} = \|\overline{u}\|^{\gamma}_{L_{\gamma\kappa}(U_{\sigma'})}$ and $\|\overline{u}^{1+\beta/2}\|^2_{L_2(U_\sigma)} = \|\overline{u}\|^{\gamma}_{L_\gamma(U_\sigma)}$, we have

$$\|\overline{u}\|_{L_{\gamma\kappa}(U_{\sigma'})} \leq \left(\frac{C_6(1+\gamma)^2}{(\sigma-\sigma')^2}\right)^{1/\gamma} \|\overline{u}\|_{L_\gamma(U_\sigma)}, \qquad \gamma \geq 2,\ 0<\sigma'<\sigma\leq 1,$$

if $\overline{u} \in \|\overline{u}\|_{L_\gamma(U_{\sigma'})}$. The structure of this estimate enables us to apply Lemma 2.3.7 with $p=2$ and $\theta = 1/2$. So there is a constant $M_1 = M_1(C_6,\kappa) = M_1(\lambda,\tau,\Lambda,q,N)$ such that

$$\operatorname*{ess\,sup}_{U_{1/2}} |\overline{u}| \leq M_1 |\overline{u}|_{L_2(U_1)}.$$

Now write $|U_1| = C_7\tau$, where $C_7 = C_7(N)$, is the volume of the unit ball and denote by $M_2 = M_1(C_7\tau)^{1/2}$. Recalling the definition of the number k concludes the proof:

$$\begin{aligned}\operatorname*{ess\,sup}_{U_{1/2}} u^+ &\leq \operatorname*{ess\,sup}_{U_{1/2}} \overline{u} \leq M_1\left(\|u\|_{L_2(U_1)} + \|f\|_{L_q(U_1)}|U_1|^{1/2}\right) \leq M_2\left(\frac{1}{|U_1|^{1/2}}\|u\|_{L_2(U_1)} + \|f\|_{L_q(U_1)}\right)\\ &\leq M_2\left((1-1/2)^{-(N+2)/2}|U_1|^{-1/2}\|u\|_{L_2(U_1)} + \|f\|_{L_q(U_1)}\right). \end{aligned} \tag{5.20}$$

Step 5 The case $\delta \in (0,1)$ and $p=2$ is treated exactly the same way as in Corollary 4.1.3.
Step 6 We now discuss the case $\delta \in (0,1)$ and $p>2$. By Hölder's inequality

$$|U_1|^{-1/2}\|u\|_{L_2(U_1)} \leq |U_1|^{-1/2}|U_1|^{1/2-1/p}\|u\|_{L_p(U_1)} = |U_1|^{-1/p}\|u\|_{L_p(U_1)}.$$

Further,

$$(1-\delta)^{-(N+2)/2} = (1-\delta)^{-(N+2)/p}(1-\delta)^{(N+2)/p-(N+2)/2}.$$

Inserting both in (4.24) yields the assertion with a constant $M_3 = M_3(\delta,\lambda,\tau,\Lambda,p,q,N)$ $= (1-\delta)^{N/p-N/2}M_2$.
Step 7 The final case $\delta \in (0,1)$ and $p \in (0,2)$ requires some more work. The idea is to use the already established inequality (5.6) for $\tilde{p} = 2$. But this time, we pick an arbitrary $\sigma \in (0,1]$ and $\eta \in (0,1)$ and define the even smaller cylinder $U_{\eta\sigma}$ and apply the above result to it. So for a constant $M_4 = M_4(\eta,\lambda,\tau,\Lambda,\tilde{p},q,N)$

$$\begin{aligned}\|u^+\|_{L_\infty(U_{\eta\sigma})} &\leq M_4\left((1-\eta)^{-(N+2)/2}\left(\frac{1}{|U_\sigma|}\int_{U_\sigma}(u^+)^2\,dx\,dt\right)^{1/2} + \sigma^{2-(N+2)/q}\|f\|_{L_q(U_1)}\right)\\ &\leq M_4\left((1-\eta)^{-(N+2)/2}\left(\frac{1}{|U_\sigma|}\int_{\sigma}(u^+)^2\,dx\,dt\right)^{1/2} + \|f\|_{L_q(U_1)}\right)\end{aligned}$$

$$\leq M_4 \left((1-\eta)^{-(N+2)/2} \|u^+\|_{L_\infty(U_\sigma)}^{1-p/2} \left(\frac{1}{|U_\sigma|} \int_{U_\sigma} (u^+)^p \, dx \, dt \right)^{1/2} + \|f\|_{L_q(U_1)} \right)$$

$$\leq M_4 \left(((1-\eta)\sigma)^{-(N+2)/2} \|u^+\|_{L_\infty(U_\sigma)}^{1-p/2} \left(\frac{1}{|U_1|} \int_{U_1} (u^+)^p \, dx \, dt \right)^{1/2} + \|f\|_{L_q(U_1)} \right), \quad (5.21)$$

where we used assumption **(A3)**. Clearly, we could further estimate $\|u^+\|_{L_\infty(U_\eta)}^{1-p/2} \leq \|u^+\|_{L_\infty(U_1)}^{1-p/2}$, but this would spoil our forthcoming argument.
All that is left is processing the first term in the parenthesis. We split it via

$$a = \frac{1}{2} \|u^+\|_{L_\infty(U_\sigma)}^{1-p/2}, \qquad b = \frac{2M_4}{((1-\eta)\sigma)^{-(N+2)/2}} \left(\frac{1}{|U_1|} \int_{U_1} (u^+)^p \, dx \, dt \right)^{1/2}.$$

and proceed with Young's inequality with exponents $s = 2/(2-p) = 1/(1-p/2)$ and $s' = 2/p$. Note that $(1/2)^s/s \leq 1/2$. This gives

$$ab \leq \frac{1}{2} \|u^+\|_{L_\infty(U_\sigma)} + \frac{p}{2} \frac{(2M)^{p/2}}{((1-\eta)\sigma)^{(N+2)/p}} \left(\frac{1}{|U_1|} \int_{U_1} (u^+)^p \, dx \, dt \right)^{1/p}. \quad (5.22)$$

Define the function $g : (0,1] \to \mathbb{R},\ \sigma \mapsto \|u^+\|_{L_\infty(U_\sigma)}$. Then (5.21) and (5.22) imply that for all $0 < \sigma' < \sigma \leq 1$ and for a certain constant $M_5 = M_5(\sigma'/\sigma, \lambda, \tau, \Lambda, p, q, N)$

$$g(\sigma') \leq \frac{1}{2} g(\sigma) + \frac{M_5}{(\sigma - \sigma')^{(N+2)/p}} \left(\frac{1}{|U_1|} \int_{U_1} (u^+)^p \, dx \, dt \right)^{1/p} + M_4 \|f\|_{L_q(U_1)}. \quad (5.23)$$

We are thus able to apply Lemma 2.3.3, in which we choose for the endpoints $s_0 = \delta,\ s_1 = 1$. So by this lemma, for $\sigma' = \delta$ and $\sigma = 1$, there holds

$$\|u^+\|_{L_\infty(U_\delta)} = g(\delta) \leq M_6 \left(\frac{1}{(1-\delta)^{(N+2)/p}} \left(\frac{1}{|U_1|} \int_{U_1} (u^+)^p \, dx \, dt \right)^{1/p} + \|f\|_{L_q(U_1)} \right)$$

for a constant $M_5 = M_5(\delta, \lambda, \tau, \Lambda, p, q, N)$.
The proof of Theorem 5.1.2 is complete. □

Since Theorem 5.1.2 with $p = 2$ was all we needed to deduce Theorem 4.1.4, we will assume that weak sub-solutions to (3.1) on Ω_T are locally bounded from above from now on.

5.2 The weak Harnack inequality

Let us fix some notation for the rest of this chapter. Since we will work a lot with lower and upper halves of a given parabolic cylinder, it is more convenient to alter the definition of the cylinders Q used

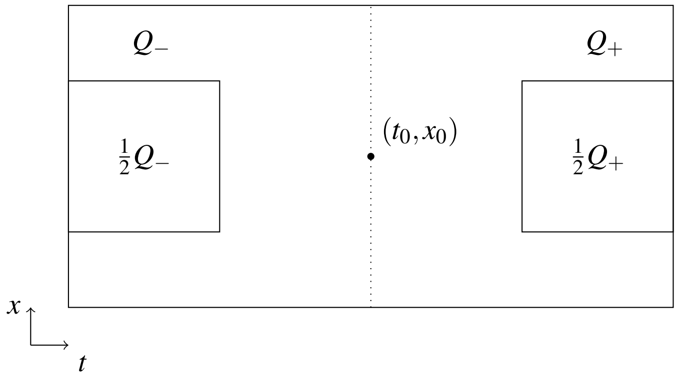

Figure 5.1: Sketch of the cylinders σQ_-, σQ_+ for $\tau = 1$, $\sigma = \frac{1}{2}$ and $r = 2$.

beforehand. So for $\tau, r > 0$ and $(t_0, x_0) \in \Omega_T$, we define

$$Q = [t_0 - \tau r^2, t_0 + \tau r^2] \times B(x_0, r).$$

Figuratively speaking, the point (t_0, x_0) is now in the center of the cylinder Q instead of in the middle of the top and the cylinders now have twice the length of the cylinders considered earlier. Again, we always assume that τ, r, t_0 and x_0 are in such a way that $Q \Subset \Omega_T$ is meaningful.
Next, we define the lower and upper halves of a cylinder Q and for $\sigma \in (0, 1]$ their shrinked versions. Put

$$Q_- = [t_0 - \tau r^2, t_0] \times B(x_0, r), \qquad \sigma Q_- = [t_0 - \tau r^2, t_0 - (1-\sigma)\tau r^2] \times B(x_0, \sigma r),$$
$$Q_+ = [t_0, t_0 + \tau r^2] \times B(x_0, r), \qquad \sigma Q_+ = [t_0 + (1-\sigma)\tau r^2, t_0 + \tau r^2] \times B(x_0, \sigma r).$$

Vividly speaking, the shrinked cylinders move away from the point (t_0, x_0) as σ decreases, which is in sharp contrast to previous shrinked versions of a cylinder. Observe that $\sigma' Q_- \subset \sigma Q_-$ whenever $0 < \sigma' < \sigma \leq 1$. The same properties are shared by σQ_+.

With these definitions understood, we now formulate the main result of this section:

Theorem 5.2.1 (Weak Harnack inequality)**.** *Let $u \in V_2(\Omega_T)$ be a non-negative weak super-solution to* (3.1) *on Ω_T and let $Q \Subset \Omega_T$. Let $\delta \in (0, 1)$ be fixed. Then for any $0 < p_0 < 1 + 2/N$ there is a constant $C = C(\delta, \lambda, \tau, \Lambda, p_0, q, N)$ such that*

$$\left(\frac{1}{|\delta Q_-|} \int_{\delta Q_-} u^{p_0} \, dx \, dt \right)^{1/p_0} \leq C \left(\operatorname*{ess\,inf}_{\delta Q_+} u + r^{2-(N+2)/q} \|f\|_{L_q(Q)} \right). \tag{5.24}$$

In contrast to Theorem 5.0.1, only the the averaged L_{p_0}-norm of u can be bounded by the essential infimum of u. This stems from the fact that weak super-solutions need not to be bounded from above (compare [21]). So the left-hand side cannot be replaced by $\operatorname{ess\,sup}_{\delta Q_-} u$. This explains the notion of a weak Harnack inequality. However, from the local boundedness of weak *solutions*, in particular from Theorem 5.1.2, one immediately deduces the following stronger version:

Corollary 5.2.2 (Strong Harnack inequality)**.** *Let $u \in V_2(\Omega_T)$ be a non-negative weak solution to* (3.1) *on Ω_T and let $Q \Subset \Omega_T$. Fix $\delta \in (0,1)$. Then there is a constant $C = C(\delta, \lambda, \tau, \Lambda, q, N)$ such that*

$$\operatorname*{ess\,sup}_{\delta Q_-} u \leq C \left(\operatorname*{ess\,inf}_{\delta Q_+} u + r^{2-(N+2)/q} \|f\|_{L_q(Q)} \right).$$

The weaker version will be enough for our purposes, though. Lastly, we remark that the global non-negativity assumption can be relaxed. The proof shows that it suffices to assume that $u \geq 0$ on Q.

5.2.1 Proof strategy

Due to the sophistication and length of the proof, we first give a brief overview, sketching the main steps. Unfortunately, it would take us too far afield to elaborate on the motivation and the idea of our plan at the moment. So we decided to postpone this to the end of the chapter and rather concentrate on the main points for the time being.
We set $\overline{u} = u + r^{2-(N+2)/q}\|f\|_{L_q(Q)}$ and for the sake of simplicity, we assume $f \neq 0$ momentarily. We aim to show an even stronger inequality, namely that (5.24) holds when u is replaced by $\overline{u}$ on the left-hand side.

The pivotal element of the proof is the abstract Lemma 2.3.9. We first show that condition (2.23) holds for the function $\overline{u}^{-1}$ on the family of cylinders $\sigma Q_+,\ \delta \leq \sigma \leq 1$ and for $\gamma_0 = \infty$. Subsequently, we check whether (2.23) holds for the function $\overline{u}$ with $\gamma_0 = p_0$, but this time on the family of cylinders $\sigma Q_-,\ \delta \leq \sigma \leq 1$. Note carefully that if (2.23) holds for these two functions, then also for any constant multiples of them.
Afterwards, we turn our attention to logarithmic estimates of the form (2.24) for the functions $\overline{u}^{-1}$ and $\overline{u}$ on $Q_+,\ Q_-$ respectively. However, we are unable to derive them for these two functions, but only for certain scalar multiples of them. More precisely, we show that (2.24) holds for $u_1 = e^{-c}\overline{u}^{-1}$ and $u_2 = e^{c}\overline{u}$ on $Q_+,\ Q_-$, respectively. Here, the number c is a certain constant.
Having established the conditions of the above Lemma for $u_1,\ u_2$, the last step of the proof is fairly simple: The conclusion (2.25) provides us with

$$e^{-c} \leq M_1 \operatorname*{ess\,inf}_{\delta Q_+} \overline{u} \qquad \text{and} \qquad |Q_-|^{-1/p_0} \|\overline{u}\|_{L_{p_0}(\delta Q_-)} \leq M_2 e^{-c},$$

which is, up to another constant dependent on δ and N, exactly (5.24).

5.2.2 On the conditions of Lemma 2.3.9

This section is the heart of the proof of the weak Harnack inequality. Here, we deal with the assumptions of Lemma 2.3.9. These are mainly established by modifications of the iterative argument in Section 5.1. Similarly, we will work with various powers of a super-solution u. In contrast to the preceding section, however, we are mainly interested in negative powers.

We first state a lemma that will serve us well. Similar to the previous section, we define $\overline{x} = x + k$, where $x \geq 0$ and $k > 0$. Further, for $\beta < 0$, define the function

$$H(x) = \begin{cases} \frac{1}{\beta+1}\overline{x}^{\beta+1}, & \beta < 0,\ \beta \neq -1, \\ \log \overline{x}, & \beta = -1. \end{cases} \tag{5.25}$$

Lemma 5.2.3. *Let $u \in V_2(\Omega_T)$ be a non-negative weak super-solution to* (3.1) *and let $Q = [t_0 - \tau r^2, t_0 + \tau r^2] \times B(x_0, r) \Subset \Omega_T$. For $\beta < 0$ and $t_1 < t_2$, $t_1, t_2 \in [t_0 - \tau r^2, t_0 + \tau r^2]$ and any non-negative piecewise smooth function $\psi \in \mathring{C}_c(Q)$ there holds*

$$\begin{aligned} &-\int_{B(x_0,r)} \left(H(u)\psi^2\right)(t,x)\, dx\Big|_{t_1}^{t_2} + \frac{|\beta|\lambda}{2}\int_{t_1}^{t_2}\int_{B(x_0,r)} |\nabla \overline{u}|^2 \overline{u}^{\beta-1}\psi^2\, dx\, dt \\ &\leq \int_{t_1}^{t_2}\int_{B(x_0,r)} \left(\frac{|f|}{k}\psi^2 + \frac{2\Lambda^2}{|\beta|\lambda}|\nabla\psi|^2\right)\overline{u}^{\beta+1}\, dx\, dt + 2\int_{t_1}^{t_2}\int_{B(x_0,r)} \psi|\partial_t\psi||H(u)|\, dx\, dt. \end{aligned} \tag{5.26}$$

Remark 5.2.4. Note that all terms in (5.26) are actually finite: This follows directly from the fact that $\overline{u}$ is bounded from below. Hence, negative powers of them will be bounded from above. Also, $|\log \overline{u}| \leq C(1 + \overline{u})$, by the boundedness from below of $\overline{u}$, which is integrable.

For the following three theorems, the full range of negative exponents β is used. First, we consider $\beta < -1$, then $\beta \in (0,1)$ and eventually $\beta = -1$.

Proof. Put $G(u) = \overline{u}^{\beta}$. By the above remark, $\overline{u}$ is bounded on Q. This enables us to argue exactly the same as in the first part of the proof of Theorem 5.1.2 (up the obvious replacements of " $\leq$ " by " $\geq$ " and the limit of integration) and obtain

$$\begin{aligned} &\int_{B(x_0,r)} \left(H(u)\psi^2\right)(t,x)\, dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{B(x_0,r)} (A\nabla\overline{u}|A\nabla\overline{u})\psi^2 G'(u)\, dx\, dt \\ &+ \int_{t_1}^{t_2}\int_{B(x_0,r)} 2\psi G(u)(A\nabla\overline{u}|\nabla\psi)\, dx\, dt \geq \int_{t_1}^{t_2}\int_{B(x_0,r)} fG(u)\psi^2 + 2\psi\partial_t\psi H(u)\, dx\, dt. \end{aligned} \tag{5.27}$$

The rest of the proof is basically a recap of the second part of the above-mentioned proof. The only difference here is that $G'(u) < 0$.
Again, by assumption **(A1)** on the matrix A:

$$\int_{t_1}^{t_2}\int_{B(x_0,r)} (A\nabla\overline{u}|\nabla\overline{u})\psi^2 G'(u)\, dx\, dt \leq \int_{t_1}^{t_2}\int_{B(x_0,r)} \lambda|\nabla\overline{u}|^2\psi^2 G'(u)\, dx\, dt$$

and similarly, by **(A2)**, $G(u) = \sqrt{u^{\beta+1}G'(u)/\beta}$ and Young's inequality with $\varepsilon = \frac{\lambda|\beta|}{2}$

$$\int_{t_1}^{t_2}\int_{B(x_0,r)} 2\psi G(u)(A\nabla\overline{u}|\nabla\psi)\, dx\, dt \leq \int_{t_1}^{t_2}\int_{B(x_0,r)} \frac{\lambda|\beta|}{2\beta}G'(u)|\nabla\overline{u}|^2 + \frac{2\Lambda^2}{|\beta|\lambda}|\nabla\psi|^2\overline{u}^{\beta+1}\, dx\, dt$$

$$\leq \int_{t_1}^{t_2} \int_{B(x_0,r)} -\frac{\lambda}{2} G'(u)|\nabla \overline{u}|^2 + \frac{2\Lambda^2}{|\beta|\lambda} |\nabla \psi|^2 \overline{u}^{\beta+1} \, dx \, dt,$$

since $\beta < 0$. Intserting these two estimates in (5.27) and multiplying both sides by -1 afterwards shows, since $G'(u) = \beta \overline{u}^{\beta-1}$

$$\begin{aligned} &-\int_{B(x_0,r)} \left(H(u)\psi^2\right)(t,x) \, dx \Big|_{t_1}^{t_2} - \frac{\beta\lambda}{2} \int_{t_1}^{t_2} \int_{B(x_0,r)} |\nabla \overline{u}|^2 \overline{u}^{\beta-1} \psi^2 \, dx \, dt \\ &\leq -\int_{t_1}^{t_2} \int_{B(x_0,r)} f G(u)\psi^2 + 2\psi \partial_t \psi |H(u)| \, dx \, dt + \int_{t_1}^{t_2} \int_{B(x_0,r)} \frac{2\Lambda^2}{|\beta|\lambda} |\nabla \psi|^2 \overline{u}^{\beta+1} \, dx \, dt \\ &\leq \int_{t_1}^{t_2} \int_{B(x_0,r)} \left(\frac{|f|}{k} \psi^2 + \frac{2\Lambda^2}{|\beta|\lambda} |\nabla \psi|^2 \right) \overline{u}^{\beta+1} \, dx \, dt + \int_{t_1}^{t_2} \int_{B(x_0,r)} 2\psi |\partial_t \psi| |H(u)| \, dx \, dt. \end{aligned}$$

□

We now make the first step towards the weak Harnack inequality.

Theorem 5.2.5. *Let $u \in V_2(\Omega_T)$ be a non-negative weak super-solution to* (3.1) *on Ω_T. Let $Q \Subset \Omega_T$ and $\delta \in (0,1)$ be fixed. Set $\overline{u} = u + k$, where $k = r^{2-(N+2)/q} \|f\|_{L_q(Q)}$ if $f \neq 0$ on Q and any positive number else. Then there are constants $C = C(\delta, \lambda, \tau, \Lambda, q, N)$ and $\nu_0 = \nu_0(q, N)$ such that*

$$\operatorname*{ess\,sup}_{\sigma' Q_+} \overline{u}^{-1} \leq \left(\frac{C}{|Q_+|(\sigma - \sigma')^{\nu_0}} \right)^{1/p} \|\overline{u}^{-1}\|_{L_p(\sigma Q_+)}, \qquad \delta \leq \sigma' < \sigma \leq 1, \; p \in (0,1]. \tag{5.28}$$

Observe that (5.28) is just (2.23) with $\gamma_0 = \infty$.
Clearly, by rescaling, we may take $Q_+ = [0, \tau] \times B$, $B = B(0,1)$. In the proof, we apply Lemma 2.3.6 to the function $\overline{u}^{-1}$. Note that (2.18) and (5.28) already look similar, but the former is not exactly the result we want. The idea is to keep $\sigma \in (0,1]$ fixed and to work with $U_\eta = \eta \sigma Q_+$, where $\eta \in (0,1]$. We aim to show that (2.17) holds. Selecting θ in (2.18) appropriately concludes the proof.

Proof. The proof is similar to the proof of Theorem 5.1.2 and deploys Lemma 5.2.3 with $\beta < -1$. Set $v = \overline{u}^{(\beta+1)/2} \in V_2(Q)$ and observe $|\nabla v|^2 = \left(\frac{\beta+1}{2}\right)^2 \overline{u}^{\beta-1} |\nabla \overline{u}|^2 = \left(\frac{|\beta+1|}{2}\right)^2 \overline{u}^{\beta-1} |\nabla \overline{u}|^2$. Let $\psi \in \dot{C}_c(Q_+)$ be non-negative, not exceeding one and piecewise smooth. By (5.26), using $\beta + 1 < 0$,

$$\begin{aligned} &\frac{1}{|\beta+1|} \int_B (v\psi)^2 (t_2, x) \, dx + \frac{2|\beta|\lambda}{|\beta+1|^2} \int_0^{t_2} \int_B |\nabla v|^2 \psi^2 \, dx \, dt \qquad (5.29) \\ &\leq \int_0^{t_2} \int_B \left(\frac{|f|}{k} \psi^2 + \frac{2\Lambda^2}{|\beta|\lambda} |\nabla \psi|^2 + \frac{2}{|\beta+1|} \psi |\partial_t \psi| \right) v^2 \, dx \, dt \\ &\leq \int_{Q_+} \left(\frac{|f|}{k} \psi^2 + \frac{2\Lambda^2}{|\beta|\lambda} |\nabla \psi|^2 + \frac{2}{|\beta+1|} \psi |\partial_t \psi| \right) v^2 \, dx \, dt, \qquad (5.30) \end{aligned}$$

for arbitrary $t_2 \in [0, \tau]$.
Since $v\psi \in \mathring{V}_2(Q_+)$, we can argue exactly as in the derivation of (5.16) and see, by the definition of the

number k:

$$\int_{Q_+} \frac{|f|}{k} v^2 \psi^2 \, dx \, dt \le \|v\psi\|^2_{L_{2q'}(Q_+)} \le 2\big(\varepsilon^2 C_2 \|v\psi\|^2_{L_{V_2(Q)_+}} + \varepsilon^{-2\mu} \|v\psi\|^2_{L_2(Q_+)}\big)$$

$$\le 4\left(\varepsilon^2 C_1 \left(\max_{-\tau \le t \le 0} \|(v\psi)(t,\cdot)\|^2_{L_2(B)} + \|\nabla v\psi\|^2_{L_2(Q_+)} + \|\nabla \psi v\|^2_{L_2(Q_+)} \right) + \varepsilon^{-2\mu} \|v\psi\|^2_{L_2(Q_+)} \right), \quad (5.31)$$

where $\varepsilon > 0$ is arbitrary, $C_1 = C_1(N)$ and $\mu = \mu(q,N) > 0$.
As both terms in (5.29) are non-negative, we drop the first one and take in (5.31) ε as

$$\varepsilon^2 = \frac{|\beta|\lambda}{8C_1|\beta+1|^2}.$$

This, (5.30) and (5.31) give, since t_2 is arbitrary and $|\beta| > 1$

$$\begin{aligned}
\|\nabla v\psi\|^2_{L_2(Q_+)} &\le \frac{|\beta+1|^2}{2|\beta|\lambda} \int_{Q_+} \left(\frac{|f|}{k}\psi^2 + \frac{2\Lambda^2}{|\beta|\lambda}|\nabla\psi|^2 + \frac{2}{|\beta+1|}\psi|\partial_t\psi| \right) v^2 \, dx \, dt \\
&\le \frac{1}{4}\left(\max_{-\tau\le t\le 0} \|(v\psi)(t,\cdot)\|^2_{L_2(B)} + \|\nabla v\psi\|^2_{L_2(Q_+)} + \|\nabla\psi v\|^2_{L_2(Q_+)} \right) \\
&\quad + C_2 \frac{|\beta+1|^{2\mu+2}}{|\beta|^{\mu+1}} \|v\psi\|^2_{L_2(Q_+)} + \frac{|\beta+1|^2\Lambda^2}{|\beta|^2\lambda^2} \|\nabla\psi v\|^2_{L_2(Q)} + \frac{|\beta+1|}{|\beta|\lambda} \int_{Q_+} \psi|\partial_t\psi| v^2 \, dx \, dt
\end{aligned} \quad (5.32)$$

$$\begin{aligned}
&\le \frac{1}{4}\left(\max_{-\tau\le t\le 0} \|(v\psi)(t,\cdot)\|^2_{L_2(B)} + \|\nabla v\psi\|^2_{L_2(Q_+)} \right) + C_2 \left(1+|\beta+1|\right)^{2\mu+2} \|v\psi\|^2_{L_2(Q_+)} \\
&\quad + \frac{\Lambda^2}{\lambda^2}\left(1+|\beta+1|\right)^{2\mu+2} \|\nabla\psi v\|^2_{L_2(Q_+)} + \frac{(1+|\beta+1|)^{2\mu+2}}{\lambda} \int_{Q_+} \psi|\partial_t\psi| v^2 \, dx \, dt,
\end{aligned} \quad (5.33)$$

where $C_2 = C_2(\lambda, q, N)$.
Likewise, dropping the second term in (5.29), recalling that t_2 is arbitrary and using (5.31) with

$$\varepsilon^2 = \frac{1}{16C_1|\beta+1|}$$

yields

$$\begin{aligned}
\max_{0\le t\le\tau} \|(v\psi)(t,\cdot)\|^2_{L_2(B)} &\le |\beta+1| \int_{Q_+} \left(\frac{|f|}{k}\psi^2 + \frac{2\Lambda^2}{|\beta|\lambda}|\nabla\psi|^2 \right) v^2 \, dx \, dt + 2\int_{Q_+} \psi|\partial_t\psi| v^2 \, dx \, dt \\
&\le \frac{1}{4}\left(\max_{0\le t\le\tau} \|(v\psi)(t,\cdot)\|^2_{L_2(B)} + \|\nabla v\psi\|^2_{L_2(Q_+)} + \|\nabla\psi v\|^2_{L_2(Q_+)} \right) \\
&\quad + C_3|\beta+1|^{\mu+1} \|v\psi\|^2_{L_2(Q_+)} + \frac{2|\beta+1|\Lambda^2}{|\beta|\lambda} \|\nabla\psi v\|^2_{L_2(Q_+)} + 2\int_{Q_+} \psi|\partial_t\psi| v^2 \, dx \, dt
\end{aligned} \quad (5.34)$$

$$\leq \frac{1}{4}\left(\max_{-\tau\leq t\leq 0}\|(v\psi)(t,\cdot)\|^2_{L_2(B)}+\|\nabla v\psi\|^2_{L_2(Q)}\right)+C_3\,(1+|\beta+1|)^{2\mu+2}\,\|v\psi\|^2_{L_2(Q)}$$
$$+\frac{2\Lambda^2}{\lambda}(1+|\beta+1|)^{2\mu+2}\|\nabla\psi v\|^2_{L_2(Q)}+2(1+|\beta+1|)^{2\mu+2}\int_Q\psi|\partial_t\psi|v^2\,dx\,dt \tag{5.35}$$

with a constant $C_3=C_3(\lambda,q,N)$. We used $\lambda\leq\Lambda^2$ in the last step.

Step 2 Adding (5.33) and (5.35) and an additional term $\|\nabla\psi v\|^2_{L_2(Q)}$ implies that for a certain constant $C_4=C_4(\lambda,\Lambda,q,N)$ there holds

$$\max_{0\leq t\leq\tau}\|(v\psi)(t,\cdot)\|^2_{L_2(B)}+\|\nabla v\psi\|^2_{L_2(Q_+)}+\|\nabla\psi v\|^2_{L_2(Q_+)}$$
$$\leq C_4\,(1+|\beta+1|)^{2\mu+2}\left(\|v\psi\|^2_{L_2(Q_+)}+\|\nabla\psi v\|^2_{L_2(Q_+)}+\int_{Q_+}\psi|\partial_t\psi|v^2\,dx\,dt\right). \tag{5.36}$$

In the following, let σ',σ with $\delta\leq\sigma'<\sigma\leq 1$ be fixed. Set $B_1=B(0,\sigma)$. For $\eta\in(0,1]$, define the increasing family of cylinders $U_\eta=\eta\sigma Q_+$. Further, set $\eta B_1=B(0,\eta\sigma)$.

Like before, we put some further restrictions on the function ψ: For $0<\eta'<\eta\leq 1$ fixed, take ψ such that ψ is constantly equal to one on $[(1-\eta'\sigma)\tau,\tau]\times\eta' B_1$, supp $\psi\subset[(1-\eta\sigma)\tau,\tau]\times\eta B_1$, and $|\nabla\psi|\leq 2/(\sigma(\eta-\eta'))\leq 2/(\delta(\eta-\eta')^{\mu+1})$ and $|\partial_t u|\leq 4/(\sigma\tau(\eta-\eta'))\leq 4/(\delta\tau(\eta-\eta')^{2\mu+2})$. Also, assume $0\leq\psi\leq 1$.

With this particular choice of ψ, the result (5.36) takes the form

$$\max_{0\leq t\leq\eta\sigma\tau}\|(v\psi)(t,\cdot)\|^2_{L_2(\eta B_1)}+\|\nabla v\psi\|^2_{L_2(U_\eta)}+\|\nabla\psi v\|^2_{L_2(U_\eta)}\leq C_5\frac{(1+|\gamma|)^\nu}{(\eta-\eta')^\nu}\|v\|_{L_2(U_\eta)},$$

where $\gamma=\beta+1<0$, $\nu=\nu(q,N)=2\mu+2$ and $C_5=C_5(\delta,\lambda,\tau,\Lambda,q,N)$. Put $\kappa=1+2/N$. Then, again by Theorem 2.2.14, with a constant $C_6=2C_1C_5$,

$$\|v\|^2_{L_{2\kappa}(U_{\eta'})}=\|v\psi\|^2_{L_{2\kappa}(U_{\eta'})}\leq\|v\psi\|^2_{L_{2\kappa}(U_\eta)}$$
$$\leq 2C_1\left(\max_{0\leq t\leq\eta\sigma\tau}\|(v\psi)(t,\cdot)\|^2_{L_2(\eta B_1)}+\|\nabla v\psi\|^2_{L_2(U_\eta)}+\|\nabla\psi v\|^2_{L_2(U_\eta)}\right)\leq\frac{C_6(1+|\gamma|)^\nu}{(\eta-\eta')^\nu}\|v\|^2_{L_2(U_\eta)}. \tag{5.37}$$

So returning to our original function $\overline{u}$, i.e. plugging in $v=\overline{u}^{(\beta+1)/2}$, the above estimates becomes

$$\left(\int_{U_{\eta'}}\overline{u}^{-|\gamma|\kappa}\,dx\,dt\right)^{1/\kappa}\leq\frac{C_6(1+|\gamma|)^\nu}{(\eta-\eta')^\nu}\int_{U_\eta}\overline{u}^{-|\gamma|}\,dx\,dt,$$

or put differently,

$$\|\overline{u}^{-1}\|_{L_{\gamma\kappa}(U_{\eta'})}\leq\left(\frac{C_6(1+|\gamma|)^\nu}{(\eta-\eta')^\nu}\right)^{1/|\gamma|}\|\overline{u}^{-1}\|_{L_\gamma(U_\eta)},\qquad 0<\eta'<\eta\leq 1,\ |\gamma|>0.$$

We are thus able to apply Lemma 2.3.6 with $\overline{p} = 1$. So there is are constants $M = M(\kappa, \nu, C_6) = M(\delta, \lambda, \tau, \Lambda, q, N)$ and $\nu_0 = \nu_0(q,N)$ with

$$\operatorname*{ess\,sup}_{U_\theta} \overline{u}^{-1} \leq \left(\frac{M}{(1-\theta)^{\nu_0}}\right)^{1/p} \|\overline{u}^{-1}\|_{L_p(U_1)}, \qquad \theta \in (0,1),\ p \in (0,1].$$

Set $M_1 = M|U_1|$. So for the particular choice $\theta = \sigma'/\sigma < 1$ this gives, since $(1-\theta)^{-1} \leq (\sigma - \sigma')^{-1}$ and $U_\theta = \sigma' Q$

$$\operatorname*{ess\,sup}_{\sigma' Q} \overline{u}^{-1} \leq \left(\frac{M_1}{|U_1|(\sigma-\sigma')^{\nu_0}}\right)^{1/p} \|\overline{u}^{-1}\|_{L_p(\sigma Q)}, \quad \delta \leq \sigma' < \sigma \leq 1,\ p \in (0,1].$$

□

Remark 5.2.6. The proof shows (by picking $\overline{p} = \infty$ in the application of Lemma 2.3.6) that the estimate (5.28) holds for all $p \in (0,\infty]$. However, we do not need that stronger form.

We now examine whether an estimate of the form (2.23) holds for the function $\overline{u}$:

Theorem 5.2.7. *Let $u \in V_2(\Omega_T)$ be a non-negative weak super-solution to* (3.1) *on Ω_T. Let $Q \Subset \Omega_T$ and $\delta \in (0,1)$ be fixed. Set $\kappa = 1 + 2/N$ and take $p_0 \in (0,\kappa)$. Define $\overline{u} = u + k$, where $k = r^{2-(N+2)/q}\|f\|_{L_q(Q)}$ if $f \neq 0$ on Q and any positive number else. Then there are constants $C = C(\delta, \lambda, \tau, \Lambda, p_0, q, N)$ and $\nu_0 = \nu_0(q,N)$ such that*

$$\|\overline{u}\|_{L_{p_0}(\sigma' Q_-)} \leq \left(\frac{C}{|Q_-|(\sigma-\sigma')^{\nu_0}}\right)^{1/p - 1/p_0} \|\overline{u}\|_{L_p(\sigma Q_-)}, \qquad \delta \leq \sigma' < \sigma \leq 1,\ p \in \left(0, \frac{p_0}{\kappa}\right]. \tag{5.38}$$

The proof is quite similar to the proof of the previous theorem, but let us spend a few words on some crucial differences first. Informally speaking, we picked a function ψ that vanishes near the bottom of the cylinder Q_+ in the above proof. The main point of this was to ensure that all terms appearing in the left-hand side of (5.26) have the same sign. This enabled us to estimate each term separately by dropping the other one.
Again, our argument relies on that estimate but this time, we take $\beta \in (-1,0)$. If we argue as before, i.e. if we assume $\psi(t_1,x) = 0$, then the remaining terms would have different signs, spoiling our entire argument. Hence, in order to make our proof work, we suppose the function ψ to vanish near the top of the cylinder Q_- instead - and this change salvages the idea of the proof. This explains to necessity to work with the two different halves Q_+ and Q_- of the cylinder Q and moreover, why their scaled versions δQ_+ and Q_- have a positive distance from each other.
Let us also briefly discuss why we have to impose a condition on the range of the exponent p this time: Previously, we assumed $|\beta| > -1$ and hence, it was possible to drop it from various denominators. This will no longer work without more ado. So the condition on p ensures that $|\beta|$ will be bounded away from zero.

Proof. Again, by rescaling, we may assume $Q_- = [-\tau, 0] \times B$, $B = B(0,1)$. From (5.38) we see that it suffices to prove the assertion only in case of $|Q_-| = 1$. We use the same machinery as in the proof of Theorem 5.2.5: Let σ' and σ with $\delta \le \sigma' < \sigma \le 1$ be fixed in the following and set $U_\eta = \eta\sigma Q_-$ for $\eta \in (0,1]$, and $B_1 = \sigma B = B(0,\sigma)$.

As before, we impose some further restrictions on the function ψ in Lemma (5.2.3): Fix $0 < \eta' < \eta \le 1$ and take ψ such that $\psi(t,\cdot)$ is constantly equal to one on $[-\tau, -(1-\eta'\sigma)\tau] \times \eta' B_1$, supp $\psi \subset [-\tau, -(1-\eta\sigma)\tau] \times \eta B_1$, and $|\partial_t \psi| \le 4/(\sigma\tau(\eta-\eta'))$ and $|\nabla\psi| \le 2/(\sigma(\eta-\eta'))$.

Set $v = \overline{u}^{(\beta+1)/2}$, where $\beta \in (-1,0)$, let $t_1 \in [-\tau, -(1-\eta\sigma)\tau]$ be arbitrary and $t_2 = -(1-\eta\sigma)\tau$. Then (5.26) becomes, by our choice of ψ,

$$\begin{aligned}
&\frac{1}{\beta+1}\int_{\eta B_1} (v\psi)^2(t_1,x)\,dx + \frac{2|\beta|\lambda}{(\beta+1)^2}\int_{t_1}^{t_2}\int_{\eta B_1} |\nabla v|^2\psi^2\,dx\,dt \\
&\le \int_{t_1}^{t_2}\int_{\eta B_1}\left(\frac{|f|}{k}\psi^2 + \frac{2\Lambda^2}{|\beta|\lambda}|\nabla\psi|^2\right)v^2\,dx\,dt + \frac{2}{\beta+1}\int_{t_1}^{t_2}\int_{\eta B_1}\psi|\partial_t\psi|v^2\,dx\,dt \\
&\le \int_{U_\eta}\left(\frac{|f|}{k}\psi^2 + \frac{2\Lambda^2}{|\beta|\lambda}|\nabla\psi|^2 + \frac{2}{\beta+1}\psi|\partial_t\psi|\right)v^2\,dx\,dt.
\end{aligned} \tag{5.39}$$

Note carefully that both terms in (5.39) have the same sign. From here, we argue exactly the same way as in the lines following (5.30). The only difference now is that $|\beta|$ cannot be dropped in the steps (5.32) to (5.33) and (5.34) to (5.35), as $|\beta|$ is not bounded away from zero. The estimate we obtain now is

$$\begin{aligned}
&\max_{-\tau\le t\le t_2}\|(v\psi)(t,\cdot)\|^2_{L_2(\eta' B_1)} + \|\nabla v\psi\|^2_{L_2(U_\eta)} + \|\nabla\psi v\|^2_{L_2(U_\eta)} \\
&\le C_1\left(\frac{1+|\beta+1|}{|\beta|}\right)^{2\mu+2}\left(\|v\psi\|^2_{L_2(U_\eta)} + \|\nabla\psi v\|^2_{L_2(U_\eta)} + \int_{U_\eta}\psi|\partial_t\psi|v^2\,dx\,dt\right)
\end{aligned} \tag{5.40}$$

for a constant $C_1 = C_1(\lambda,\Lambda,q,N)$ and $\mu = \mu(q,N) > 0$.

Recall that $|\nabla\psi| \le 2/(\sigma(\eta-\eta')) \le 2/(\delta(\eta-\eta')^{\mu+1})$ and $|\partial_t\psi| \le 4/(\sigma\tau(\eta-\eta')) \le 4/(\delta\tau(\eta-\eta')^{2\mu+2}$. Set $\gamma = \beta+1 \in (0,1)$ and $\nu = \nu(q,N) = 2\mu+2$. Then, arguing as in (5.37), there holds for a constant $C_2 = C_2(\delta,\lambda,\tau,\Lambda,q,N)$

$$\|v\|^2_{L_{2\kappa}(U_{\eta'})} \le C_2\left(\frac{1+\gamma}{|\gamma-1|(\eta-\eta')}\right)^{\nu}\|v\|^2_{L_2(U_\eta)}. \tag{5.41}$$

We now wish to apply Lemma 2.3.8, in which we have to restrict ourselves to a $p_0 \in (0,\kappa)$, which will be kept fixed in the following. Then for $0 < \gamma \le p_0/\kappa < 1$, the term $(1+\gamma)/|\gamma-1|$ will be bounded. Hence, for a constant $C_3 = C_3(\delta,\lambda,\tau,\Lambda,p_0,q,N)$ we infer from (5.41) and the definition of the function v

$$\|\overline{u}\|_{L_{\gamma\kappa}(U_{\eta'})} \le \left(\frac{C_3}{(\eta-\eta')^{\nu}}\right)^{1/\gamma}\|\overline{u}\|_{L_\gamma(U_\eta)}, \qquad 0 < \eta' < \eta \le 1,\ \gamma \in \left(0, \frac{p_0}{\kappa}\right].$$

Lemma 2.3.8, now yields the existence of numbers $M = M(\delta,\lambda,\tau,\Lambda,p_0,q,N)$ and $\nu_0 = \nu_0(q,N)$ such that

$$\|\overline{u}\|_{L_{p_0}(U_\theta)} \leq \left(\frac{M}{(1-\theta)^{\nu_0}}\right)^{1/p-1/p_0} \|\overline{u}\|_{L_p(U_1)}, \qquad \theta \in (0,1),\ p \in \left(0, \frac{p_0}{\kappa}\right].$$

Take $\theta = \sigma'/\sigma < 1$, use $(1-\theta)^{-1} \leq (\sigma-\sigma')^{-1}$ and recall $U_\theta = \sigma' Q$. So

$$\|\overline{u}\|_{L_{p_0}(\sigma' Q)} \leq \left(\frac{M}{(\sigma-\sigma')^{\nu_0}}\right)^{1/p-1/p_0} \|\overline{u}\|_{L_p(\sigma Q)}, \qquad \delta \leq \sigma' < \sigma \leq 1,\ p \in \left(0, \frac{p_0}{\kappa}\right],$$

which proves the assertion.

□

We now come to the trickier part, namely deriving suitable estimates for $\log \overline{u}$:

Theorem 5.2.8. *Let $u \in V_2(\Omega_T)$ be a non-negative weak super-solution to* (3.1) *on Ω_T and let $Q \Subset \Omega_T$. Define $\overline{u} = u + k$, where $k = r^{2-(N+2)/q}\|f\|_{L_q(Q)}$ if $f \neq 0$ on Q and any positive number else. Then there are constants $c = c(\lambda, \Lambda, f, q, u, N)$ and $C = C(\lambda, \tau, \Lambda, N)$ such that for all $\alpha > 0$*

$$|\{(t,x) \in Q_+ : \log \overline{u}(t,x) < -\alpha - c\}| \leq C|Q_+|\alpha^{-1} \tag{5.42}$$

and

$$|\{(t,x) \in Q_- : \log \overline{u}(t,x) > \alpha - c\}| \leq C|Q_-|\alpha^{-1}. \tag{5.43}$$

Note carefully that the constant c in the above two estimates is the same. This will be vital in the next section. Also, observe that this theorem asserts that condition (2.24) is satisfied for the functions $e^{-c}\overline{u}^{-1}$ and $e^{c}\overline{u}$.

One may feel uneasy that the constant c depends on the function u (compared to every other constant in this thesis!). However, this constant will eventually be eliminated in the final step of the proof of the weak Harnack inequality.

The proof of the Theorem also very different from the ones encountered so far in this chapter. There will be no iterative argument but some rather quirky and artificial estimates, giving the argument very little appeal. The main tool is the weighted Poincaré inequality 2.2.8.

Proof. Clearly, we may assume $t_0 = 0, x_0 = 0$ and $r = 1$. Set $B = B(0,1)$.
One last time, we make use of Lemma 5.2.3 and take $\beta = -1$. We assume the function $\psi \in \mathring{C}_c(Q)$ to be independent of time, i.e. $\psi(t,x) = \omega(x)$ for all $t \in [-\tau, \tau]$ and for some piecewise smooth function $\omega \in C_c(B)$, $0 \leq \omega \leq 1$. Furthermore, assume that ω is constantly equal to one on $B(0,1/2)$, vanishes for $|x| = 1$ and is linear in between, i.e. $|\nabla \omega| \leq 4$. Note carefully that the sets $\{x \in B : \omega^2(x) \geq a\}$ are convex in this setting, where $a \in [0,1]$.

Put $v = -\log \overline{u} \in V_2(Q)$. The assertion of the above mentioned lemma then reads

$$\int_B (v(t_2,x) - v(t_1,x))\,\omega^2(x)\,dx + \frac{\lambda}{2}\int_{t_1}^{t_2}\int_B |\nabla v|^2\omega^2\,dx\,dt \le \int_{t_1}^{t_2}\int_B \frac{|f|}{k}\omega^2 + \frac{2\Lambda^2}{\lambda}|\nabla\omega|^2\,dx\,dt, \quad (5.44)$$

where $t_1,\ t_2 \in [-\tau,\tau]$ are - for the moment - arbitrary.
Define the function

$$V(t) = \frac{\int_B v(t,x)\omega^2(x)\,dx}{\int_B \omega^2(x)\,dx}, \qquad t \in [-\tau,\tau].$$

Since $v \in V_2(Q)$, V is continuous and hece, uniformly continuous on $[-\tau,\tau]$.
Our main instrument to move on is Proposition 2.2.8 which yields, since $v(t,\cdot) \in W_2^1(Q)$ for $t \in [-\tau,\tau]$,

$$\int_{t_1}^{t_2}\int_B |v - V(t)|^2\omega^2\,dx\,dt \le \frac{8C_1|B|}{\int_B \omega^2(x)\,dx}\int_{t_1}^{t_2}\int_B |\nabla v|^2\omega^2\,dx\,dt, \quad (5.45)$$

where $C_1 = C_1(N)$. From inserting this in (5.44) we infer

$$\int_B (v(t_2,x) - v(t_1,x))\,\omega^2(x)\,dx + \frac{\lambda\int_B \omega^2(x)\,dx}{16C_1|B|}\int_{t_1}^{t_2}\int_B |v - V(t)|^2\omega^2\,dx\,dt$$
$$\le \int_{t_1}^{t_2}\int_B \frac{|f|}{k}\omega^2 + \frac{2\Lambda^2}{\lambda}|\nabla\omega|^2\,dx\,dt$$

We divide both sides by $\int_B \omega^2(x)\,dx \ge |B_1|$. So by the definition of the number k and the properties of ω, there holds

$$V(t_2) - V(t_1) + \frac{\lambda}{16C_1|B|}\int_{t_1}^{t_2}\int_B |v - V(t)|^2\,dx\,dt \le C_2 + C_2(t_2 - t_1) \quad (5.46)$$

for a certain constant $C_2 = C_2(\lambda,\Lambda,N)$.
Step 2 By the uniform continuity of V, for $\varepsilon = 1$ there is a $\eta > 0$ such that for all $t_1 < t_2$ with $t_2 - t_1 < \eta$ and $t \in [t_1,t_2]$ there holds $|V(t) - V(t_2)| < 1$. So

$$\frac{\lambda}{16C_1|B|}\int_{t_1}^{t_2}\int_B |v - V(t_2)|^2\,dx\,dt \le \frac{\lambda}{8C_1|B|}\int_{t_1}^{t_2}\int_B \left(|v - V(t)|^2 + |V(t) - V(t_2)|^2\right)\,dx\,dt$$
$$\le \frac{\lambda}{8C_1|B|}\int_{t_1}^{t_2}\int_B |v - V(t)|^2\,dx\,dt + \frac{\lambda}{8C_1}(t_2 - t_1)$$

Adding $V(t_2) - V(t_1)$ to both sides and using (5.46) gives

$$V(t_2) - V(t_1) + C_3\int_{t_1}^{t_2}\int_B |v - V(t_2)|^2\omega^2\,dx\,dt \le C_4 + C_4(t_2 - t_1),$$

where $C_3 = C_3(\lambda,N)$and $C_4 = C_4(\lambda,\Lambda,N)$.
We introduce the auxillary functions

$$w(t,x) = v(t,x) - C_4t - C_4, \qquad W(t) = V(t) - C_4t - C_4,$$

which fulfill

$$W(t_2)-W(t_1)+C_3\int_{t_1}^{t_2}\int_B |w-W(t_2)+C_4(t_2-t)|^2\,dx\,dt\le 0. \tag{5.47}$$

This already implies $W(t_2)\le W(t_1)$ for $t_2<t_1+\eta,\ t_1,t_2\in[-\tau,\tau]$ and by the uniform continuity, it is easy to see from this that the function W is non-increasing on the whole interval $[-\tau,\tau]$.
We now define the constant c in the assertion of the theorem by

$$c=c(\overline{u},\lambda,\Lambda,N)=c(\lambda,\Lambda,f,q,u,N)=W(0).$$

Step 3 We show (5.42). To this end, let $\alpha>0$ be arbitrary and for $t\in[0,\tau]$, define the sets $A_\alpha^+(t)=\{x\in B: w(t,x)>\alpha+c\}$.
We return to the setting in (5.47) but only consider non-negative $t_1<t_2$ in the following. By monotonicity of the function W, we have for $t\in[t_1,t_2]$ and almost all $x\in A_\alpha^+(t)$

$$w(t,x)-W(t_2)>\alpha+c-W(t_2)\ge\alpha>0.$$

So we may drop the term $C_4(t_2-t)$ in the integrand in (5.47) and see

$$\begin{aligned}\int_{t_1}^{t_2}\int_B |w-W(t_2)+C_4(t_2-t)|^2\,dx\,dt &\ge \int_{t_1}^{t_2}\int_{A_\alpha^+(t)} |w-W(t_2)+C_4(t_2-t)|^2\,dx\,dt\\ \ge\int_{t_1}^{t_2}\int_{A_\alpha^+(t)}(\alpha+c-W(t_2))^2\,dx\,dt &= (\alpha+c-W(t_2))^2\int_{t_1}^{t_2}|A_\alpha^+(t)|\,dt.\end{aligned} \tag{5.48}$$

Hence, by (5.47), for all $0\le t_1<t_2\le\tau,\ t_2-t_1<\eta$,

$$\begin{aligned}\int_{t_1}^{t_2}|A_\alpha^+(t)|\,dt &\le \frac{1}{C_3}\frac{W(t_1)-W(t_2)}{(\alpha+c-W(t_2))^2}\le\frac{1}{C_3}\frac{W(t_1)-W(t_2)}{(\alpha+c-W(t_1))(\alpha+c-W(t_2))}\\ &=\frac{1}{C_3}\left(\frac{1}{\alpha+c-W(t_1)}-\frac{1}{\alpha+c-W(t_2)}\right),\end{aligned} \tag{5.49}$$

where we deployed the monotonicity again. Pick $n\in\mathbb{N}$ with $\frac{\tau}{n}<\eta$. Then, by (5.49)

$$\int_0^\tau |A_\alpha^+(t)|\,dt=\sum_{i=0}^{n-1}\int_{\frac{i}{n}\tau}^{\frac{i+1}{n}\tau}|A_\alpha^+(t)|\,dt\le\frac{1}{C_3}\left(\frac{1}{\alpha}-\frac{1}{\alpha+c-W(\tau)}\right)\le\frac{1}{C_3\alpha}. \tag{5.50}$$

The proof is almost complete: Since $\log\overline{u}=-w-C_4t-C_4$, the result (5.50) yields

$$\begin{aligned}&|\{(t,x)\in Q_+\log\overline{u}<-\alpha-c\}|=|\{(t,x)\in Q_+:\log\overline{u}+(C_4t+C_4)-(C_4t+C_4)<-\alpha-c\}|\\ \le&|\{(t,x)\in Q_+:w>\alpha/2+c\}|+|\{(t,x)\in Q_+:C_4t+C_4>\alpha/2\}|\\ \le&\frac{2}{C_3\alpha}+\left(\tau-\frac{\alpha}{2C_4}\right)|B|\le\frac{C_5}{\alpha},\end{aligned}$$

where $C_5=C_5(\lambda,\tau,\Lambda,N)$.
Step 4 Up to minor changes, to proof of (5.43) is the same. For $t\in[-\tau,0]$ and $\alpha>0$ define the sets

$A_\alpha^-(t) = \{x \in B : w(t,x) < c - \alpha\}$. Like above, for non-positive $t_1 < t_2$ and almost all $t \in [t_1, t_2]$, $x \in A_\alpha^-(t)$ there holds $w(t,x) - W(t_2) < c - \alpha - W(t_2) < 0$. Hence, the terms $w - W(t_2)$ and $C_4(t_2 - t)$ have same sign on $A_\alpha^-(t)$. The proof then follows exactly the same lines after (5.48), except for obvious replacements. □

5.2.3 Concluding the proof

We now have all tools required to finish the proof of the weak Harnack inequality at our disposal. For the convenience of the reader, we state the theorem again:

Theorem 5.2.9 (Weak Harnack inequality)**.** *Let* $u \in V_2(\Omega_T)$ *be a non-negative weak super-solution to* (3.1) *on* Ω_T *and let* $Q \Subset \Omega_T$. *Let* $\delta \in (0,1)$ *be fixed. Then for any* $0 < p_0 < 1 + 2/N$ *there is a constant* $C = C(\delta, \lambda, \tau, \Lambda, p_0, q, N)$ *such that*

$$\left(\frac{1}{|\delta Q_-|} \int_{\delta Q_-} u^{p_0} \, dx \, dt \right)^{1/p_0} \leq C \left(\operatorname*{ess\,inf}_{\delta Q_+} u + r^{2-(N+2)/q} \|f\|_{L_q(Q)} \right).$$

Proof. Set $\overline{u} = u + r^{2-(N+2)/q}\|f\|_{L_q(Q)}$. If $f = 0$ on Q, replace $r^{2-(N+2)/q}\|f\|_{L_q(Q)}$ by any $k > 0$ and let $k \searrow 0$ in the end.
By Theorem 5.2.5 there holds for certain constants $C_1 = C_1(\delta, \lambda, \tau, \Lambda, q, N)$ and $\nu_0 = \nu_0(q, N)$

$$\operatorname*{ess\,sup}_{\sigma' Q_+} \overline{u}^{-1} \leq \left(\frac{C_1}{|Q_+|(\sigma - \sigma')^{\nu_0}} \right)^{1/p} \|\overline{u}^{-1}\|_{L_p(\sigma Q_+)}, \qquad \delta \leq \sigma' < \sigma \leq 1, \; p \in (0,1],$$

which implies that assumption (a) of Lemma 2.3.9 is fulfilled for $\overline{u}^{-1}$ (and hence, also for any positive scalar multiple of it). In it, we choose σQ_+ for the family U_σ and $\gamma_0 = \infty$.
Define the auxillary function $u_1 = e^{-c}\overline{u}^{-1}$, where $c = c(\lambda, \Lambda, f, q, u, N)$ is the constant postulated by Theorem 5.2.8. Since $\log u_1 = -c - \log \overline{u}$, estimate (5.42) implies that for all $\alpha > 0$ and a constant $C_2 = C_2(\lambda, \tau, \Lambda, N)$

$$|\{(t,x) \in Q_+ : \log u_1 > \alpha\}| \leq C_2 |Q_+| \alpha^{-1},$$

holds, which is just condition (b) of Lemma 2.3.9. Hence, we can apply it to u_1 and see that there is a constant $M_1 = M_1(\delta, \lambda, \tau, \Lambda, q, N)$ such that $\operatorname{ess\,sup}_{\delta Q_+} u_1 \leq M_1$. Put differently, this means

$$e^{-c} \leq M_1 \operatorname*{ess\,inf}_{\delta Q_+} u. \tag{5.51}$$

Next, Theorem 5.2.7 comes on stage and yields

$$\|\overline{u}\|_{L_{p_0}(\sigma' Q_-)} \leq \left(\frac{C_3}{(\sigma - \sigma')^{\nu_1}} \right)^{1/p - 1/p_0} \|\overline{u}\|_{L_p(\sigma Q_-)}, \qquad \delta \leq \sigma' < \sigma \leq 1, \; p \in \left(0, \frac{p_0}{\kappa}\right],$$

where $C_3 = C_3(\delta, \lambda, \tau, \Lambda, p_0, q, N)$, $\nu_1 = \nu_1(q, N)$ and $\kappa = 1 + 2/N$. Again, this is just condition (a) of Lemma 2.3.9 with $\eta = 1/\kappa$ and $\gamma_0 = p_0$. This time, define $u_2 = e^c \overline{u}$ with c as above. Arguing the same way, (5.43) yields

$$|\{(t,x) \in Q_- : \log u_2 > \alpha\}| \le C_2|Q_-|\alpha^{-1}$$

with the same constant C_2 as above and for arbitrary $\alpha > 0$. Thus, condition (b) of Lemma 2.3.9 also holds. So for constants $M_2 = M_2(\delta, \lambda, \tau, \Lambda, p_0, q, N)$ and $M_3 = \delta^{(N+2)/p_0} M_2$ we obtain

$$\|u_2\|_{L_{p_0}(\delta Q_-)} \le M_2|Q_-|^{1/p_0} = M_3|\delta Q_-|^{1/p_0},$$

which is nothing else but

$$|\delta Q_-|^{-1/p_0}\|\overline{u}\|_{L_{p_0}(\delta Q_-)} \le M_3 e^{-c}. \tag{5.52}$$

A combination of (5.51) and (5.52) then implies

$$|\delta Q_-|^{-1/p_0}\|u\|_{L_{p_0}(\delta Q_-)} \le |\delta Q_-|^{-1/p_0}\|\overline{u}\|_{L_{p_0}(\delta Q_-)} \le M_3 e^{-c} \le M_1 M_3 \operatorname*{ess\,inf}_{\delta Q_+} \overline{u}. \tag{5.53}$$

The proof of the weak Harnack inequality is (finally!) complete. □

5.3 Oscillation estimates

We now give a second proof of the oscillation estimates as in Theorem 4.3.5. This time, however, the proof is comparatively simple since we have the weak Harnack inequality at our disposal.

We recall the notation used in the above mentioned theorem: For $\tau, R > 0$, $r \in (0,R]$ and (t_0, x_0) define $Q(r) = [t_0 - \tau r^2] \times B(x_0, r)$.

Theorem 5.3.1. *Let $u \in V_2(\Omega_T)$ be a weak solution to* (3.1) *on Ω_T and let $R > 0$ such that $Q(R) \Subset \Omega_T$. Then there are numbers $C = C(\lambda, \tau, \Lambda, q, N)$ and $\alpha = \alpha(\lambda, \tau, \Lambda, q, N) \in (0,1)$ such that for any $r \in (0,R]$ there holds*

$$\operatorname*{ess\,osc}_{Q(r)} u \le C\left(\frac{r}{R}\right)^{\alpha}\left(\operatorname*{ess\,osc}_{Q(R)} u + R^{2-(N+2)/q}\|f\|_{L_q(Q(R))}\right).$$

Proof. We may assume $t_0 = 0$ and $x_0 = 0$. Fix an arbitrary $r \in (0,R]$ and decompose the cylinder $Q(r)$ in $Q_-(r) = [-\tau r^2, -\frac{1}{2}\tau r^2] \times B(0,r)$ and $Q_+(r) = [-\frac{1}{2}\tau r^2, 0] \times B(0,r)$. In conjunction with Theorem 5.2.1, introduce $\frac{1}{2}Q_-(r) = [-\tau r^2, -\frac{3}{4}\tau r^2] \times B(0,\frac{1}{2}r)$ and $\frac{1}{2}Q_+(r) = [-\frac{1}{4}\tau r^2, 0] \times B(0,\frac{1}{2}r)$. Note carefully that $\frac{1}{2}Q_+(r) = Q(\frac{1}{2}r)$.

Define the (finite) numbers

$$M = \operatorname*{ess\,sup}_{Q(r)} u, \qquad m = \operatorname*{ess\,inf}_{Q(r)}, \qquad M_1 = \operatorname*{ess\,sup}_{Q(\frac{r}{2})}, \qquad m_1 = \operatorname*{ess\,inf}_{Q(\frac{r}{2})}.$$

Notice that $u - m$ is non-negative on $Q(r)$ and a weak solution to (3.1). The same holds true for $M - u$ if one replaces the right-hand side of (3.1) by $-f$.

We now apply Theorem 5.2.1 with $p_0 = 1$ to $M-u$ and $u-m$, respectively. This yields

$$\frac{1}{|\frac{1}{2}Q_-(r)|}\int_{\frac{1}{2}Q_-(r)} u-m\,dx\,dt \le C_1\left(m_1 - m + r^{2-(N+2)/q}\|f\|_{L_q(Q(r))}\right)$$
$$\frac{1}{|\frac{1}{2}Q_-(r)|}\int_{\frac{1}{2}Q_-(r)} M-u\,dx\,dt \le C_1\left(M - M_1 + r^{2-(N+2)/q}\|f\|_{L_q(Q(r))}\right)$$

with the same constant $C_1 = C_1(\lambda,\tau,\Lambda,q,N)$. Without loss of generality, we assume $C_1 > 1$. Adding these two inequalities gives us

$$M - m \le C_1\left(m_1 - m + M - M_1 + 2r^{2-(N+2)/q}\|f\|_{L_q(Q(R))}\right),$$

i.e.

$$\operatorname*{ess\,osc}_{Q(\frac{r}{2})} u \le \frac{C_1 - 1}{C_1}\operatorname*{ess\,osc}_{Q(r)} + 2r^{2-(N+2)/q}\|f\|_{L_q(Q(R))}. \tag{5.54}$$

Set $\theta = \theta(\lambda,\tau,\Lambda,q,N) = (C-1)/C \in (0,1)$ and define the non-decreasing functions

$$g(r) = \operatorname*{ess\,osc}_{Q(r)} u, \qquad h(r) = 2r^{2-(N+2)/q}\|f\|_{L_q(Q(R))}, \qquad r \in (0,R].$$

From (5.54) we deduce that these functions satisfy the functional inequality

$$g(r/2) \le \theta g(r) + h(r), \qquad r \in (0,R].$$

Lemma 2.3.4 gives us the estimate

$$g(r) \le \frac{1}{\theta}\left(\frac{r}{R}\right)^{(1-\mu)\frac{\log\theta}{\log 1/2}} g(R) + \frac{h(R^{1-\mu}r^{\mu})}{1-\theta}, \qquad r \in (0,R], \tag{5.55}$$

where $\mu \in (0,1)$ is arbitrary. Recall that $2-(N+2)/q > 0$. Hence, it is possible to find a μ with

$$\alpha = \alpha(\lambda,\tau,\Lambda,q,N) = (1-\mu)\frac{\log\theta}{\log 1/2} \le (2-(N+2)/q)\,\mu$$

and furthermore, $\alpha \in (0,1)$ can be achieved. Substituting this back in (5.55) gives the desired result:

$$\begin{aligned}\operatorname*{ess\,osc}_{Q(r)} u &\le \frac{1}{\theta}\left(\frac{r}{R}\right)^{\alpha}\operatorname*{ess\,osc}_{Q(R)} u + \frac{2}{1-\theta}\left(R\left(\frac{r}{R}\right)^{\mu}\right)^{2-(N+2)/q}\|f\|_{L_q(Q(R))}\\ &\le C_2\left(\frac{r}{R}\right)^{\alpha}\left(\operatorname*{ess\,osc}_{Q(R)} u + \left(\frac{r}{R}\right)^{(2-(N+2)/q)\mu-\alpha} R^{2-(N+2)/q}\|f\|_{L_q(Q(R))}\right)\\ &\le C_2\left(\frac{r}{R}\right)^{\alpha}\left(\operatorname*{ess\,osc}_{Q(R)} u + R^{2-(N+2)/q}\|f\|_{L_q(Q(R))}\right),\end{aligned}$$

where $C_2 = C_2(\lambda,\tau,\Lambda,q,N)$. $\square$

The rest of the proof of Theorem 3.1.3 is exactly the same as in Section 4.4 (this time however, we are free in the choice of the number τ, which can be taken for example as $\tau = 1$).

5.4 Notes and concluding remarks

5.4.1 Further comments on the proof of the weak Harnack inequality

Although the weak Harnack inequality is not in the center of this thesis, we give some additional information. Especially, we catch up on what was left out in Section 5.2, namely depicting the idea behind the proof: It is peculiar that the abstract Lemma 2.3.9 seems to come out of the blue. We provide some more background information about how the proof has evolved over time in order to make the idea behind the proof discernible. However, we focus on conveying the big picture and leave the technicalities aside.

For a measurable set U of finite measure with positive measure and a strictly positive measurable function u, we set

$$\Phi(u,p,U) = \left(\frac{1}{|U|}\int_U u^p \, d\mu\right)^{1/p},$$

where $p \in \mathbb{R} \setminus \{0\}$. This is simply the averaged L_p-(quasi-)norm of u for $p > 0$. An elementary argument shows that this function is monotonically increasing in p and

$$\lim_{p\to-\infty} \Phi(u,p,U) = \operatorname*{ess\,inf}_U u, \qquad \lim_{p\to\infty} \Phi(u,p,U) = \operatorname*{ess\,sup}_U u,$$
$$\lim_{p\to 0} \Phi(u,p,U) = \exp\left(\frac{1}{|U|}\int_U \log u \, d\mu\right), \tag{5.56}$$

whenever these expressions make sense. Hence, Φ extends to a function on $[-\infty,\infty]$.
We showed that (5.24) even holds when u is replaced by $\overline{u}$ on the left-hand side. In the above language, this inequality can be written as

$$\Phi(\overline{u}, p_0, \delta Q_-) \le C\Phi(\overline{u}, -\infty, \delta Q_+) \tag{5.57}$$

Moreover, Theorem 5.2.5, Remark 5.2.6 and 5.2.7 imply, by taking the particular values $\sigma' = \delta$ and $\sigma = 1$

$$\begin{aligned} &\Phi(\overline{u}, p, Q_+) \le C_1\Phi(\overline{u}, -\infty, \delta Q_+), && p \in [-\infty, 0),\\ &\Phi(\overline{u}, p_0, \delta Q_-) \le C_2\Phi(\overline{u}, p, Q_-), && 0 < p \text{ small enough.} \end{aligned} \tag{5.58}$$

We stress three points: In both inequalities, we can take p arbitrarily close to zero. However, we approach zero either from the left or the right. Furthermore, the inequalities involve either the upper or lower halves of the cylinder Q.
Evidently, in order to finish the proof of the weak Harnack inequality, we need to build a bridge between

these two results. More precisely, we are left to show the following: There is an $\varepsilon > 0$ such that:

$$\Phi(\overline{u}, \varepsilon, Q_-) \leq C_3 \Phi(\overline{u}, -\varepsilon, Q_+). \tag{5.59}$$

Note that this inequality is substantially different from the two above: Firstly, it involves a 'jump over zero' in the exponent and secondly, both halves of the cylinder Q.

At this point, it seems appropriate to insert some words on the historical development. Moser first applied his ideas in [20] to elliptic equations of the form

$$-\text{div}\, A \nabla u = 0$$

to give a different proof of De Giorgi's and Nash's result, before he turned his attention to the corresponding parabolic problem in [21]. In particular, he established analogous estimates of the form (5.58) on balls for weak super-solutions of the elliptic problem using the same iteration technique presented here. The adaption of his ideas to the parabolic setting in order to derive (5.58) was relatively straightforward. However, and this is the main point, there is no need to decompose the balls in any form in the elliptic situation and (5.58) then holds for shrinked versions of *the same* ball.

Deriving (5.59) in the elliptic case already required a deep theorem due to John and Nirenberg, involving the function space BMO. An integrable function v on $\mathbb{R}^N$ belongs to this space if for all balls $B \subset \mathbb{R}^N$ the quantity $\frac{1}{|B|} \int_B |v - v_B| \, dx$ is finite, where $v_B = \Phi(v, 1, B)$ is the average of v over B.
The John-Nirenberg inequality asserts that there is a constant $c > 0$ such that

$$\frac{1}{|B|} \int_B \exp\left(c\, |v - v_B|\right) \, dx < \infty,$$

for all $v \in BMO$. See for example [9, Chapter 6] for a direct introduction of the space BMO and a proof of this inequality.
In light of the limit (5.56), it seems reasonable to study $\log \overline{u}$ and in fact, one has that this function belongs to BMO. With the aid of the John-Nirenberg inequality, Moser then succeeded in proving (5.59) on balls in the elliptic case.

It thus appears only natural to try to follow the same route to derive the inequality (5.59) for the parabolic problem. However, one is faced with the additional challenge to compare the function Φ on different domains and not only for exponents with a different sign. This additional ingredient impedes things considerably and calls for an adaption of the above inequality to our situation. This gerneralization turned out to be exceptionally hard to prove; in fact, it makes up the largest part of Moser's originial contribution [21]. Additionally, his first argument turned out to contain a subtle error, which was corrected in [22] some time later. In a later work [23] he writes

> "The proof of this lemma is quite intricate and it was desirable to avoid it entirely."

The purpose of this work was to simplify his own proof by applying Lemma 2.3.9 discovered in [3], where the authors demonstrated its applicability to the corresponding elliptic problem.

Eventually, we can extract the principal statement of Lemma 2.3.9: There is no need to show (5.59) and to generalize the John-Nirenberg inequality! The mean value inequalities (5.28) and (5.38), and the logarithmic estimates (5.2.8) (which are all essentially the same as the ones in Moser's original contribution) already suffice.

For a more detailed discussion of the application of the John-Nirenberg inequality to the elliptic problem, see for example [19, Section 2.3.5].

The reader might wonder where exactly we compared the terms in (5.58). Indeed, this step is very well hidden and takes place when we eliminate the factor e^{-c} in (5.53); but there is no small exponent needed as in (5.59).

5.4.2 References

Section 5.1 is based on the paper [2]. Especially, the particular form of the auxiliary functions G and H and their application in the proof of Theorem 5.1.2 is taken from there. Also, Lemma 5.2.3 can be found in this contribution. The proof of Theorem 5.1.2, especially the usage of Moser's iteration technique in the abstract setting is inspired by [5, Section 3.1].
The particular form of the weak Harnack inequality is derived from [29, Chapter 5], while the the proofs of Theorem 5.2.5 and 5.2.7, and Section 5.2.3 are based on [29, Section 5.3] and [29, Section 5.5], respectively. The proof of the logarithmic estimates in Theorem 5.2.8 are taken from [29, Section 5.4.1] and [25, Proposition 6.9].

6 Outlook and related results

To conclude this thesis, we briefly present several further and related results.

6.1 More general equations

Our main result also holds for equations with lower order terms, e.g. for equations of the form

$$\partial_t u - \operatorname{div}(A\nabla u + au + g) = (b|\nabla u) + cu + f. \tag{6.1}$$

with additional integrability conditions imposed on the new terms, namely

$$a, b, g \in L_{2q}(\Omega_T; \mathbb{R}^N), \ c \in L_q(\Omega_T).$$

For these equations, the estimate (3.4) has to be modified, namely

$$\|u\|_{C^\alpha(\overline{\Omega'_T})} \leq C\left(\|u\|_{L_2(\Omega_T)} + \|f\|_{L_q(\Omega_T)} + \|g\|_{L_{2q}(\Omega_T)}\right)$$

and the constant C will depend on the respective norms of a, b and c.

Further, the linearity of the equation was not the essential element to make the proofs work. It is possible to extend the result to quasi-linear equations of the form

$$u_t - \operatorname{div} \mathscr{A}(t, x, u, \nabla u) = \mathscr{B}(t, x, u, \nabla u)$$

by imposing certain structure conditions on the nonlinearities $\mathscr{A} : \Omega_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $\mathscr{B} : \Omega_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$:

$$(\mathscr{A}(t, x, u, p)|p) \geq \lambda|p|^2 - \varphi_1^2|u|^2 - \varphi_2^2,$$
$$|\mathscr{A}(t, x, u, p)| \leq \Lambda^2|p| + \varphi_3|u| + \varphi_4,$$

$$|\mathscr{B}(t,x,u,p)| \leq \varphi_5|p| + \varphi_6|u| + \varphi_7,$$

with $\varphi_1, \varphi_2, \varphi_3, \varphi_5, \varphi_7 \in L_{2q(\Omega_T)}$ and $\varphi_4, \varphi_6 \in L_q(\Omega_T)$ and the estimate for the Hölder norm has to be modified accordingly.

It was the aim of this thesis to depict the main ideas and steps of De Giorgi's and Moser's technique for the simpler equation (3.1). The proofs of the theorem for these more general equations above is essentially the same: Now, the reader should be familiar with the prinicipal form of the intermediate results and inequalities; it requires no new ideas to treat the additional terms but only some additional paperwork. In that sense, both De Giorgi's and Moser's methods are also nonlinear techniques. We refer to [17, 3, §10 and Chapter 4] and [2] for the details.

6.2 Regularity up to the parabolic boundary

As we saw, it is vital for the given proofs to work that the set Ω'_T in Theorem 3.1.3 has a positive distance from the parabolic boundary. Hence, this result only answers the question of interior Hölder continuity of weak solutions and leaves open what happens at the parabolic boundary. To answer this question, more ingredients have to be added - in particular, a priori information about the weak solution on the parabolic boundary.

We first elaborate on the global boundedness of a weak solution u. We add to our three assumptions **(A1)-(A3)** another one, namely that u does not exceed a number K on the parabolic boundary Γ_T. This information could be derived for instance from the given data, if u is indeed the weak solution to a initial boundary value problem. Of course, one has to make sense of the condition "$u_{|\Gamma_T} \leq K$" which we understand as

$$(u-K)^+ \in \mathring{V}_2(\Omega_T) \text{ and } (u-K)^+(0,x) = 0. \tag{6.2}$$

The main tools used in Section 4.1 were 2.2.18 and the Gagliardo–Nirenberg inequality. The former result does not work on the whole cylinder Ω_T, while the latter requires some smoothness of the domain Ω. The trick to overcome these issues to extend the weak solution u by zero to a larger parabolic cylinder $\tilde{\Omega}_T = [-\varepsilon, T] \times \tilde{\Omega}$, where $\tilde{\Omega}$ is a bounded domain with smooth boundary that contains the original domain Ω. Denote the extension of u by $\tilde{u}$. It can be shown, by using (6.2), that $(\tilde{u}-K)^+ \in \mathring{V}_2(\tilde{\Omega}_T)$, i.e. one can then obtain (3.12) for almost all $t \in [0,T]$. From this, a modification of our argument in Section 4.1 then shows that the weak solution u will in fact be bounded on the whole domain Ω_T and one has the estimate

$$\|u\|_{L_\infty(\Omega_T)} \leq K + C\|f\|_{L_q(\Omega_T)}$$

for a certain constant C. The proof of this estimate is in the same spirit as the proof of the local boundedness in Section 4.1 and requires essentially no new ideas. This result can also be obtained by adapting Moser's iteration technique. See for example [17, Chapter 3, §7], [18, Chapter 6, Section 5] or [2, Section 3] for more details concerning the global boundedness.

Global Hölder regularity requires two additional supplements: One must already know that u is Hölder continuous on the parabolic boundary and certain smoothness assumptions on the boundary $\partial\Omega$ have to be imposed. Deriving this result, however, is more technical, so we refer to [17, Chapter 3, Theorem 10.1] and [18, Theorem 6.32] for a thorough discussion.

A Spaces of vector-valued functions

For the sake of completeness, we include a short treatise on Bochner spaces and Sobolev spaces of vector-valued functions. However, our presentation is rather short and we solely concentrate on what is needed for our purposes. See for example [11, 5.9.2 and E.5], [10, Chapter 7-8] and [16, Chapter 2-3] and the references therein for more details.

In the following, let $I = (a,b]$ or $[a,b]$ be a bounded non-trivial interval and let X,Y be a Banach space with norms $\|\cdot\|_X, \|\cdot\|_Y$, respectively (we will omit the subscript if no confusion is to be expected). For Lebesgue measurable subsets $A \subset I$, we denote by χ_A its characteristic function. Permanently, $\Omega \subset \mathbb{R}^N$ will be a bounded domain.

Definition A.1. *The space $C(\bar{I};X)$ consists of all functions $u : \bar{I} \to X$ that are continuous with respect to the norm of X with*

$$\|u\|_{C(\bar{I};X)} = \max_{t\in\bar{I}} \|u(t)\| < \infty.$$

Further, for $k \in \mathbb{N}_0 \cup \{\infty\}$, the space $C^k(\bar{I};X)$ consists of all k-times continuously differentiable functions $u : \bar{I} \to X$.

For a function $u(t,x) \in C^k(\bar{I} \times \Omega)$, we define the function $\tilde{u}(t)(x) = u(t,x)$ and it can be shown that $\tilde{u} \in C^k(\bar{I};C^k(\Omega))$; even more, these two spaces coincide given this identification (compare [10, p. 258]). The same holds for the spaces $C^k(\bar{I} \times \overline{\Omega}) = C^k(\bar{I};C^k(\overline{\Omega}))$.

We also need vector-valued Hölder spaces:

Definition A.2. *We say that a function $u \in C(\bar{I};X)$ is Hölder continuous, and we write $u \in C^{\alpha/2}(\bar{I};X)$, if there is $\alpha \in (0,1)$ such that*

$$\sup_{\substack{t_1,t_2\in\bar{I}\\ t_1\neq t_2}} \frac{\|u(t_1) - u(t_2)\|_X}{|t_1 - t_2|^{\alpha/2}} < \infty.$$

We equip this space with the norm

$$\|u\|_{C^{\alpha/2}(\bar{I};X)} = \|u\|_{C(\bar{I};X)} + \sup_{\substack{t_1,t_2\in\bar{I}\\ t_1\neq t_2}} \frac{\|u(t_1)-u(t_2)\|_X}{|t_1-t_2|^{\alpha/2}}.$$

The factor $1/2$ in the exponent is added in conjunction with the spaces introduced in Subsection 2.2.2. It is easy to see that for $\alpha,\beta\in(0,1)$ with $\alpha<\beta$, one has the embedding $C^{\beta/2}(\bar{I};X)\hookrightarrow C^{\alpha/2}(\bar{I};X)$.

Remark A.3. If $\alpha\in(0,1)$ and $X=C^{\beta}(\overline{\Omega})$ for a $\beta\in(0,2)$, it is easy to see that

$$C^{\alpha/2}(\bar{I};C^{\beta}(\overline{\Omega}))\subset C^{\min(\alpha,\beta)/2,\min(\alpha,\beta)}(\bar{I}\times\overline{\Omega}).$$

See Subsection 2.2.1.

We now discuss Bochner spaces:

Definition A.4. *Let $u: I\to X$. We call u*

(a) simple, if $u(t)=\sum_{j=1}^{N}\chi_{A_j}(t)x_j$ for measurable $A_1,\dots,A_N\subset I$ and $x_1,\dots,x_N\in X$ fixed;
(b) measurable, if there is a sequence u_n of simple functions such that for almost all $t\in I$ there holds $\|u_n(t)-u(t)\|\to 0$ for $n\to\infty$;
(c) integrable, if there is a sequence u_n of simple functions such that $\int_I \|u_n(t)-u(t)\|\,dt\to 0$ for $n\to\infty$. In that case, if u is simple, we define $\int_I u(t)\,dt=\sum_{j=1}^{N}|A_j|x_j$ and otherwise

$$\int_I u(t)\,dt = \lim_{n\to\infty}\int_I u_n(t)\,dt \tag{A.1}$$

for a sequence u_n of simple functions.

Evidently, it makes no difference if we take the closed interval $\bar{I}$ in the above definition instead. It is easy to see that the limit (A.1) is independent of the sequence u_n. Also, if $u: I\to X$ is continuous, then u is also measurable [16, Corollary 2.3].

Theorem A.5 (Bochner, [11, E.5 Theorem 8])**.** *A function $u: I\to X$ is integrable if and only if u is measurable and $\int_I\|u(t)\|\,dt<\infty$. In this case, $\|\int_I u(t)\,dt\|\le\int_I\|u(t)\|\,dt$.*

Definition A.6. *For $p\in[1,\infty)$, we set*

$$L_p(I;X)=\left\{u: I\to X \text{ measurable with } \|u\|_{L_p(I;X)}=\int_I\|u(t)\|_X^p\,dt<\infty\right\}.$$

If $p=\infty$, then $L_\infty(I;X)$ denotes the set of all measurable functions $u: I\to X$ with $\|u(t)\|_{L_\infty(I,X)}=$ $\operatorname{ess\,sup}_I\|u(t)\|_X<\infty$.

Note that from Bochner's Theorem, Hölder's inequality and the assumption $|I| < \infty$ it follows that $u \in L_p(I;X)$ is integrable for all $p \in [1,\infty]$.

We now discuss the important special case that X is a Lebesgue space. Similar to spaces of continuous functions, given an element $u(t)(x)$ of the abstract space $L_p(I;L_p(\Omega))$, we define on $I \times \Omega$ the function $\tilde{u}(t,x) = u(t)(x)$. We have the following result:

Proposition A.7 ([10, Satz 7.1.24]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Then $L_p(I;L_p(\Omega)) = L_p(I \times \Omega)$, if $p \in [1,\infty)$ and $L_\infty(I;L_\infty(\Omega)) \subsetneq L_\infty(I \times \Omega)$. These inclusions are understood in the sense of the above identification.*

We now discuss Sobolev spaces of vector-valued functions. Recall that $C_c^\infty(\bar{I}) = \{\varphi \in C^\infty(\bar{I}) :$ dist (supp $\varphi, \partial I) > 0\}$.

Definition A.8. *Let $u, v \in L_1(I;X)$. We say that u is weakly differentiable with weak derivative $\partial_t u = v$ if*

$$\int_I u(t)\varphi'(t)\,dt = -\int_I v(t)\varphi(t)\,dt \text{ for all } \varphi \in C_c^\infty(I).$$

We write $W_p^1(I;X) = \{u \in L_p(I;X) : u$ is weakly differentiable with $\partial_t u \in L_p(I;X)\}$ for $p \in [1,\infty]$. We equip this space with the norm

$$\|u\|_{W_p^1(I;X)} = \begin{cases} \left(\|u\|_{L_p(I;X)} + \|\partial_t u\|_{L_p(I;X)}\right)^{1/p}, & p \in [1,\infty), \\ \max\{\|u\|_{L_\infty(I;X)}, \|\partial_t u\|_{L_\infty(I;X)}\}, & p = \infty. \end{cases}$$

Proposition A.9 ([10, Satz 8.1.5] and [11, Section 5.9.2, Theorem 2]). *Let $u, v \in L_p(I;X)$ for $p \in [1,\infty]$. Then $u \in W_p^1(I;X)$ with $\partial_t u = v$ if and only if there is a $u_0 \in X$ such that for almost all $t \in \bar{I}$ there holds*

$$u(t) = u_0 + \int_a^t v(s)\,ds. \tag{A.2}$$

Theorem A.10 ([11, Section 5.9.2, Theorem 2]). *Let $p \in [1,\infty]$. Then $W_p^1(I,X) \hookrightarrow C(\bar{I};X)$, i.e. for all $u \in W_p^1(I;X)$ there is $\tilde{u} \in C(\bar{I};X)$ with $\tilde{u}(t) = u(t)$ for almost all $t \in \bar{I}$ and $\|u\|_{C(\bar{I};X)} \leq C\|u\|_{W_p^1(I;X)}$ for a constant $C = C(I)$.*

It is customary that whenever one writes $u \in W_p^1(I;X)$, it is always assumed that this function is the continuous representative (with the extension to the closure of the interval I).

Corollary A.11. *Let $p \in (1,\infty)$. Then $W_p^1(I;X) \hookrightarrow C^{\alpha/2}(\bar{I};X)$, where $\alpha = 1 - \frac{1}{p}$.*

Proof. Let $u \in W_p^1(I;X)$ and without loss of generality, assume that u is the continuous representative in the sense of Theorem $A.10$. Let $v \in L_p(I;X)$ be the function in Proposition $A.9$. As u is continuous, (A.2) holds for all $t, s \in \bar{I},\ s < t$. Hence

$$\|u(t) - u(s)\|_X \leq \int_s^t \|v(r)\|_X\,dr \leq \|v\|_{L_p(I;X)}|t-s|^\alpha \leq \|v\|_{L_p(I;X)}|\bar{I}|^{\alpha/2}|t-s|^{\alpha/2}, \tag{A.3}$$

where we made use of Theorem $A.5$ and the scalar Hölder's inequality. □

For $p \in [1,\infty]$ and $I = (a,b]$ or $[a,b]$ as above, we set

$$\mathring{W}_p^1(I;X) = \{u \in W_p^1(I;X) : u(a) = 0 \in X\}, \tag{A.4}$$

which is well defined by the above theorem.

The following is a variant of the chain rule that suffices for our purposes:

Proposition A.12 (Chain rule)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous map such that $f(v) \in L_2(\Omega)$ for all $v \in L_2(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain. Let $u \in W_2^1(I;L_2(\Omega))$. Then $f \circ u \in W_2^1(I;L_2(\Omega))$ and $\partial_t(f \circ u)(t) = f'(u(t))\partial_t u(t)$ for almost all $t \in I$.*

Proof. See [4, Lemma A.3] for the proof that $f \circ u \in W_2^1(I;L_2(\Omega))$. The proof of the formula for the derivative is the same as in the scalar-valued case (cf. [30, Theorem 2.1.11]). □

Proposition A.13. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $u \in W_2^1(I;L_2(\Omega))$ and $v \in C(\bar{I} \times \Omega)$ be bounded and piecewise smooth. Then $uv \in W_2^1(I;L_2(\Omega))$ and $\partial_t(uv) = (\partial_t u)v + u\partial_t v$.*

Again, the proof is the same as in the scalar-valued case.

We also need vector-valued Sobolev–Slobodeckij spaces. Their definition is the same as in Subsection 2.2.4:

Definition A.14. *Let $p \in [1,\infty)$ and $s \in (0,1)$. For $u \in L_p(I;X)$, denote*

$$[u]_{W_p^s(I;X)} = \left(\int_I \int_I \frac{\|u(t_1) - u(t_2)\|_X^p}{|t_1 - t_2|^{1+sp}} \, dt_1 \, dt_2 \right)^{1/p}.$$

We say that u belongs to the Sobolev–Slobodeckij space $W_p^s(I;X)$, *if*

$$\|u\|_{W_p^s(I;X)} = \left(\|u\|_{L_p(I;X)}^p + [u]_{W_p^s(I;X)}^p \right)^{1/p} < \infty.$$

There is also a vactor-valued analogue to Theorem 2.2.12:

Theorem A.15. *Let $p \in [1,\infty)$ and $s \in (0,1)$ be such that $sp > 1$. Then $W_p^s(I;X) \hookrightarrow C^{\alpha/2}(\bar{I};X)$, where $\alpha = s - \frac{1}{p}$.*

Proof. See [26, Corollary 26]. However, the additional factor $1/2$ has to be included as in (A.3). □

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Statutory declaration

I hereby declare that I have authored this thesis independently, that I have not used other than the declared sources, and that I have explicitly marked all material which has been quoted either literally or by content from the used sources. I furthermore declare that this thesis has not been submitted to any other board of examiners yet.

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Ulm, June 20th, 2016 ______________________________

Attila Klimmek