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Holomorphic Semigroups and the Geometry of Banach Spaces

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Introduction

This thesis carries the same title as Pisier's famous paper [Pis82] in which he proves the equivalence of two geometric properties of Banach spaces: *B-convexity* and *K-convexity*. These two concepts - among others of course - are investigated in the so called *local theory* of Banach spaces. This subdiscipline of the theory of the geometry of Banach spaces investigates the relation between the structure of a Banach space and the properties of its finite dimensional subspaces whose study was initiated by A. Grothendick in the 1950s. Among other deep results in this area Pisier's equivalence of B- and K-convexity (Theorem 2.4.11) probably stands out for its beauty and elegance. In his historical overview of the subject [Mau03] B. Maurey writes

> "..., Pisier proved what I consider the most beautiful result in this area ..."

Indeed, the proof uses elegantly the theory of holomorphic operator semigroups. To be more precise, the key tool is a qualitative version of a result by T. Kato and A. Beurling (Theorem 1.2.6) which states that a strongly continuous semigroup of linear operators $(T(t))$ is holomorphic if for some natural number N

$$\limsup_{t\downarrow 0} \left\| (T(t) - \mathrm{Id})^N \right\|^{1/N} < 2.$$

Pisier's equivalence therefore is an astonishing example of the interplay between semigroup theory and the geometry of Banach spaces which is also the leitmotif of my thesis.

In the first chapter we develop the theory of holomorphic semigroup from scratch using functional calculus methods and only assuming some basic knowledge in the theory of strongly continuous semigroups of linear operators. We prove the above mentioned Kato-Beurling theorem by the help of a famous characterization of holomorphic semigroups by Kato. In this context a detailed investigation of the connection between the approximation of the identity and the holomorphy of the semigroup is given. As a further application, we use the Kato-Beurling theorem to prove a weak form of the Stein interpolation theorem for semigroups on interpolation spaces.

The second chapter is devoted to Pisier's proof of the equivalence of B- and K-convexity. We introduce these two notions of convexity together with the important concepts of *type* and *cotype* and prove their elementary properties. Thereafter we give a detailed and complete proof of Pisier's theorem. As applications of the developed concepts and results we present a complete duality theory (Theorem 2.5.7) for type and cotype and H. König's beautiful characterization of B/K-convexity (Theorem 2.5.22) in terms of absolutely summing Fourier coefficients of vector-valued functions. We conclude with a historical overview of the so called *B-convexity vs. reflexivity* problem which could finally be resolved by R.C. James. An even more sophisticated negative solution to this problem was given by G. Pisier and Q. Xu using interpolation theory. In the very last section we present their construction of non-reflexive B-convex spaces.

While we assume some basic knowledge in functional analysis throughout the thesis, more uncommon mathematical tools used are subsumed - mostly without proofs - in the appendix.

1 Holomorphic Semigroups and Uniformly Convex Spaces

In this chapter we investigate the interplay between holomorphic extensions of one-parameter semigroups of bounded linear operators, their approximations of the identity operator as the parameter goes to zero and the geometric structure of the underlying Banach spaces. After providing the main tools from the theory of holomorphic semigroups, we show in Theorem 1.2.1 that a strongly continuous semigroup $(T(t))$ possesses a holomorphic extension if

$$\limsup_{t \downarrow 0} \|T(t) - I\| < 2. \tag{APX}$$

Thereafter generalizations and variants of this result partially due to A. Beurling are proved. Further, we show that (APX) does in general not imply the holomorphy of the semigroup. However, an observation made by A. Pazy shows that this is true if the underlying space is uniformly convex. In the last section we use this observation to prove a weak form of the Stein interpolation theorem for holomorphic semigroups on interpolation spaces.

1.1 Holomorphic Semigroups

In this section we develop the theory of holomorphic semigroups from scratch. We shortly recall the basic notions and facts from the theory of strongly continuous one-parameter semigroups of bounded operators. We then impose an additional restriction on the spectrum of the infinitesimal generator and on the resolvent: the sectoriality of the generator. For such operators we develop an elementary functional calculus which allows us to extend the semigroup to a sector in the complex plane in which the semigroup depends analytically on the parameter. Semigroups for which such an extension is possible are called *analytic* or *holomorphic*. Thereafter we present the basic very well-known criterions for holomorphy with complete proofs.

1.1.1 From Strongly Continuous to Holomorphic Semigroups

Before introducing holomorphic semigroups, we shortly recall some elementary facts from the strongly continuous case. Proofs of the stated facts can be found in [Paz83] or [EN00]. Given a linear evolution equation, it can often be rewritten in the form of an *abstract Cauchy problem*, that is an equation of the form

$$\begin{cases} \dot{u}(t) &= Au(t) \qquad t > 0 \\ u(0) &= x_0, \end{cases} \tag{ACP}$$

where A is usually assumed to be a closed operator - because boundedness of A is too strict for most applications - on a Banach space X and $x_0 \in X$.

Example 1.1.1 (Heat Equation)**.** The *heat equation* on the real line is given by

$$u_t = u_{xx} = \Delta u.$$

We now rewrite the equation as an abstract Cauchy problem on the Hilbert space $X = L^2(\mathbb{R};\mathbb{C})$. Notice that it is natural to define A as the classical second derivative for a certain space of functions like the *Schwartz space* $\mathcal{S}$. Simply defining A with $D(A) = \mathcal{S}$ in this way would not be the right choice as A would not be closed. However, one can define A as the closure of the Laplace operator on $\mathcal{S}$. Then A is given by

$$\begin{aligned} A : D(A) = H^2(\mathbb{R};\mathbb{C}) &\to L^2(\mathbb{R};\mathbb{C}) \\ f &\mapsto f'', \end{aligned}$$

where $H^2(\mathbb{R};\mathbb{C}) = W^{2,2}(\mathbb{R};\mathbb{C})$ is the *Sobolev space* of all twice weakly differentiable functions on $\mathbb{R}$ and the derivative in the definition of A is understood in the weak sense. This can be seen as follows: recall that the Fourier transform is an automorphism on $\mathcal{S}$ which can be extended to a unitary operator $\mathcal{F}_2$ - the so called *Fourier-Plancherel transformation* on $L^2(\mathbb{R};\mathbb{C})$ (see [Wer09, Satz V.2.8f.]). Under the ordinary Fourier transformation $\mathcal{F}$ the (one-dimensional) Laplace operator Δ on $\mathcal{S}$ transforms into the multiplication operator

$$\mathcal{F} \circ \Delta \circ \mathcal{F}^{-1} : \mathcal{S} \ni f \mapsto -x^2 f \in \mathcal{S}.$$

The closure of this operator is obviously given by

$$D \coloneqq \left\{ f \in L^2(\mathbb{R};\mathbb{C}) : x^2 f \in L^2(\mathbb{R};\mathbb{C}) \right\} \ni f \mapsto -x^2 f.$$

It is known that $\mathcal{F}_2(H^2(\mathbb{R};\mathbb{C})) = D$ and that for $f \in H^2(\mathbb{R};\mathbb{C})$ one has $\mathcal{F}_2(f')(x) = ix\mathcal{F}_2(f)(x)$ almost everywhere (see [AU10, Satz 6.45]). Applying the inverse Fourier transform, we see that the closure of the Laplace operator on $\mathcal{S}$ is indeed given by A.

Observe that the above argument shows that A is unitary equivalent to the multiplication operator $f \mapsto -x^2 f$ on $L^2(\mathbb{R};\mathbb{C})$. From this one can infer that A is self-adjoint (see [Wer09, p. 342 & 345]).

Definition 1.1.2 (Classical Solution)**.** We say that a continuously differentiable function $u : [0,\infty) \to X$ is a *classical solution* of (ACP) if $u(t) \in D(A)$ for $t \geq 0$ and if it satisfies the initial value problem (ACP).

In the case $X = \mathbb{K}^n$ and $A \in \mathbb{K}^{n \times n}$, (ACP) is a system of first order linear differential equations. It is well known from the theory of ordinary differential equations that the unique classical solution is given by $u(t) = e^{tA}x_0$, where $e^{tA} \coloneqq \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$ is the matrix exponential function. So $T(t) \coloneqq e^{tA} \in \mathcal{L}(X)$ is a family of linear mappings containing the whole solution structure. The matrix exponential can directly be generalized to the infinite dimensional case provided A is a bounded operator. However, in most applications A will be unbounded. The concept of a strongly continuous semigroup is one natural generalization to the infinite dimensional case. We recall its definition.

Definition 1.1.3 (Strongly Continuous Semigroup)**.** Let X be a Banach space. A family $(T(t))_{t\geq 0}$ of bounded linear operators on X is called a *strongly continuous (one-parameter) semigroup (or C_0-semigroup)* if

(i) $T(0) = I$,

(ii) $T(t+s) = T(t)T(s) \quad t, s \geq 0$,

(iii) $\lim_{t\downarrow 0} T(t)x = x \; \forall x \in X$.

Example 1.1.4 (Left Shift)**.** Let $X = (UC_b[0,\infty); \|\cdot\|_\infty)$ be the space of complex-valued uniformly continuous bounded functions endowed with the supremum norm. Let

$$(T(t)f)(s) \coloneqq f(t+s).$$

Then $(T(t))_{t\geq 0}$ defines a family of contractions that obeys the semigroup law. Moreover, we have $T(0) = \mathrm{Id}$ and

$$\lim_{t\downarrow 0} \|T(t)f - f\|_\infty = \lim_{t\downarrow 0} \sup_{s\geq 0} |f(t+s) - f(s)| = 0$$

by the uniform continuity of f. Hence, $(T(t))$ is a strongly continuous semigroup.

Example 1.1.5 (Heat Semigroup)**.** We return to the heat equation on the line (Example 1.1.1). One can show that for a given continuous and exponentially bounded initial data $u_0 : \mathbb{R} \to \mathbb{R}$

$$u(t,x) \coloneqq \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-(x-y)^2/4t} u_0(y)\, dy$$

is the unique classical exponentially bounded solution of the heat equation for $t > 0$ [AU10, Satz 3.40]. Moreover, one has

$$\lim_{t\downarrow 0} u(t,x) = u_0(x)$$

uniformly on bounded subsets. Hence, the solution is given by the convolution of the initial data u_0 with the so called *Gauß kernel*

$$g(t,x) \coloneqq \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \qquad x \in \mathbb{R},\, t > 0.$$

Therefore it is natural to try to define a semigroup on $L^p(\mathbb{R};\mathbb{C})$ by $T(0) \coloneqq \mathrm{Id}$ and

$$(T(t)f)(x) \coloneqq \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-(x-y)^2/4t} f(y)\, dy \qquad \text{for } t > 0.$$

Indeed, one can show that $(T(t))$ is a strongly continuous semigroup of contractions on $L^p(\mathbb{R};\mathbb{C})$ for $1 \leq p < \infty$ using the same methods as in Example 1.1.1 (see [EN00, p. 69f.]).

One important property of a strongly continuous semigroup $(T(t))$ is its exponential boundedness: there exist constants $M \geq 0$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leq Me^{\omega t}.$$

This is a consequence of (iii) and the uniform boundedness principle. The infinitesimal generator A of a C_0-semigroup is directly connected to some abstract Cauchy problem.

Definition 1.1.6 (Infinitesimal Generator)**.** Let $(T(t))$ be a C_0-semigroup. The *infinitesimal generator* of $(T(t))$ is the linear operator defined by

$$\begin{aligned} D(A) &:= \left\{ x \in X : \lim_{h \downarrow 0} \frac{T(h)x - x}{h} \text{ exists} \right\}, \\ Ax &:= \lim_{h \downarrow 0} \frac{T(h)x - x}{h}. \end{aligned}$$

One can show that A is a closed operator. Then one sees that for $x_0 \in D(A)$ the unique solution of (ACP) is given by $u(t) := T(t)x_0$. Since this shows that strongly continuous semigroups give us solutions to the abstract Cauchy problems given by their infinitesimal generators, it is a natural and important question to ask under which conditions a given operator A is the infinitesimal generator of a strongly continuous semigroup. This question is completely answered by the Hille-Yosida Generation theorem.

Since from now on we are interested in properties of strongly continuous semigroups described in the complex plane, we make the following convention.

Convention 1.1.7. Until the end of this chapter X always denotes a *complex* Banach space.

Theorem 1.1.8 (Hille-Yosida Generation Theorem)**.** *Let A be a linear operator on X and let $\omega \in \mathbb{R}, M \geq 1$. The following conditions are equivalent.*

(a) A generates a C_0-semigroup satisfying

$$\|T(t)\| \leq Me^{\omega t} \qquad \text{for } t \geq 0.$$

(b) A is closed, densely defined and the resolvent set $\rho(A)$ contains the half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \omega\}$ and

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\operatorname{Re} \lambda - \omega)^n}.$$

Proof. See [Paz83, Theorem I.5.3 & Remark I.5.4] or [EN00, Theorem II.3.8]. □

Example 1.1.9 (Left Shift)**.** We continue Example 1.1.4. Now, we want to determine its infinitesimal generator A. Let $f \in D(A)$. Then $u(t) := T(t)f$ is the unique solution of the associated Cauchy problem. A fortiori, $u(t)$ is differentiable and

$$u'(t) = \lim_{h \to 0} \frac{T(t+h)f - Tf}{h} = \lim_{h \to 0} \frac{f(t+h+\cdot) - f(t+\cdot)}{h}.$$

Hence,

$$\left|u'(t)(0) - \frac{f(t+h) - f(t)}{h}\right| \leq \left\|u'(t) - \frac{f(t+h+\cdot) - f(t+\cdot)}{h}\right\|_\infty \xrightarrow[h\to 0]{} 0.$$

This shows that f is differentiable. Moreover, there exists a $g \in UC_b[0,\infty)$ such that

$$g = \lim_{h\downarrow 0} \frac{T(h)f - f}{h}.$$

A fortiori, the pointwise limit exists for all $t \in [0,\infty)$ and we have

$$g(t) = \lim_{h\downarrow 0} \frac{(T(h)f)(t) - f(t)}{h} = \lim_{h\downarrow 0} \frac{f(t+h) - f(t)}{h} = f'(t).$$

Therefore $f' \in UC_b[0,\infty)$. Conversely, let $f \in UC_b[0,\infty)$ be differentiable such that $f' \in UC_b[0,\infty)$. Then the mean value theorem shows

$$\left|f'(t) - \frac{(T(h)f)(t) - f(t)}{h}\right| = \left|f'(t) - \frac{f(t+h) - f(t)}{h}\right| \overset{\xi(t)\in(t,t+h)}{=} \left|f'(t) - f'(\xi(t))\right|.$$

Since f' is uniformly continuous, we conclude

$$\lim_{h\downarrow 0} \left\|f' - \frac{T(t)f - f}{h}\right\|_\infty = \lim_{h\downarrow 0} \sup_{t\geq 0} \left|f'(t) - f'(\xi(t))\right| = 0.$$

Therefore the infinitesimal generator A is given by

$$\begin{aligned} D(A) &= \{f \in UC_b[0,\infty) : f \text{ is differentiable and } f' \in UC_b[0,\infty)\}, \\ A : D(A) &\ni f \mapsto f'. \end{aligned}$$

Let us determine the resolvent set of A. By Theorem 1.1.8, the open right half-plane is contained in the resolvent set of A. Moreover, for $\operatorname{Re}\lambda \leq 0$ the non-trivial solution $t \mapsto e^{\lambda t}$ of

$$\lambda f - Af = \lambda f - f' = 0$$

lies in $D(A)$ (the function and its derivative are even Lipschitz continuous). Hence, $\lambda - A$ is not injective and is therefore not invertible. Thus the closed left half-plane lies in the spectrum of A. This shows

$$\rho(A) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}.$$

Example 1.1.10 (Heat Semigroup)**.** One can show that the infinitesimal generator of the heat semigroup defined in Example 1.1.5 is the closed operator A used in the formulation of the abstract Cauchy problem in Example 1.1.1 (see [Wer09, p. 359f.]), that is

$$\begin{aligned} A : D(A) = H^2(\mathbb{R};\mathbb{C}) &\to L^2(\mathbb{R};\mathbb{C}) \\ f &\mapsto f''. \end{aligned}$$

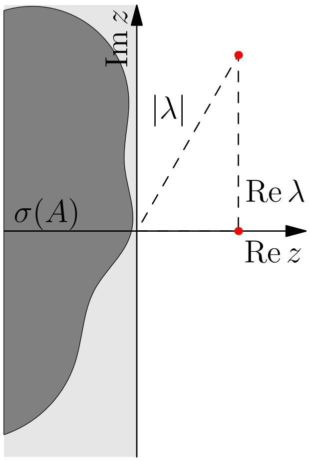

Figure 1.1: Sectorial domain

Remark 1.1.11. Let A be the infinitesimal generator of a bounded C_0-semigroup $(T(t))$. By Theorem 1.1.8 for $\omega = 0$ and $n = 1$,

$$\|R(\lambda, A)\| \leq \frac{M}{\operatorname{Re}\lambda} \qquad \text{for } \operatorname{Re}\lambda > 0. \tag{1.1.1}$$

So the spectrum $\sigma(A)$ of A is contained in the closed left half-plane or in other words in the closed sector with opening angle π and central angle π (see fig. 1.1). Now choose $\alpha > \pi$ and let λ be in the sector given by the same central angle and a bigger opening angle of α. Using elementary trigonometry, we see that $\operatorname{Re}\lambda = \cos(\arg\lambda)\,|\lambda|$. Thus $\cos(\frac{\pi-\alpha}{2}) = \sin(\frac{\alpha}{2})$ is a lower bound for $\cos(\arg\lambda)$. Together with estimate (1.1.1) we get

$$\|\lambda R(\lambda, A)\| \leq \frac{|\lambda|\, M}{\operatorname{Re}\lambda} = \frac{M}{\cos(\arg\lambda)} \leq \frac{M}{\sin\left(\frac{\alpha}{2}\right)}.$$

The above argument shows that $\|\lambda R(\lambda, A)\|$ is uniformly bounded in strictly smaller subsectors of the right half-plane.

Operators with the properties described in Remark 1.1.11 form the important class of *sectorial operators*. In the following we give an exact definition.

Definition 1.1.12 (Sectorial Operator)**.** Let $\alpha \in [0, 2\pi), d \in \mathbb{R}$ and $z_0 \in \mathbb{C}$. We call

$$S(z_0, d, \alpha) := \left\{ z \in \mathbb{C} \setminus \{z_0\} : |\arg(z - z_0) - d| < \frac{\alpha}{2} \right\},$$
$$\overline{S(z_0, d, \alpha)} := \left\{ z \in \mathbb{C} : |\arg(z - z_0) - d| \leq \frac{\alpha}{2} \right\}$$

the open / closed sector with center z_0, central angle d and opening angle α. We sometimes write $S(d, \alpha)$ instead of $S(0, d, \alpha)$. An operator A on X is called z_0*-sectorial* of angle $\omega \in (0, 2\pi)$ (we write in symbols $A \in \operatorname{Sect}(z_0, \omega)$) if

(i) the resolvent set of A is contained in the sector $S(z_0, 0, 2\pi - \omega)$ or equivalently if the spectrum of A is contained in the closed sector $\overline{S(z_0, \pi, \omega)}$, that is

$$\sigma(A) \subset \overline{S(z_0, \pi, \omega)},$$

(ii) $\lambda \mapsto (\lambda - z_0)R(\lambda, A)$ is bounded in strictly smaller subsectors of $S(z_0, 0, 2\pi - \omega)$, that is

$$\sup\{\|(\lambda - z_0)R(\lambda, A)\| : \lambda \in \mathbb{C} \setminus \overline{S(z_0, \pi, \omega')}\} < \infty \; \forall \omega' \in (\omega, 2\pi).$$

We will write *sectorial* and $\mathrm{Sect}(\omega)$ as a shorthand for 0-sectorial and $\mathrm{Sect}(0, \omega)$.

Remark 1.1.13. The sectors should be seen as parts of the Riemann surface of the complex logarithm because of the discontinuity of the argument function in the complex plane. However, since the opening angle α is always chosen smaller than 2π, the sectors can always be seen as a subset of the complex plane with an appropriate chosen argument.

Example 1.1.14. We have seen in Remark 1.1.11 that the infinitesimal generator A of a bounded strongly continuous semigroup is sectorial of angle π.

Let us return to the situation in Remark 1.1.11. For the moment we make the stronger assumption that $A \in \mathrm{Sect}(\omega)$ for some $\omega < \pi$. Notice that Example 1.1.9 showed that this is not always the case. Then the Cauchy integral

$$e^{tA} := \frac{1}{2\pi i} \int_\Gamma e^{tz}(z - A)^{-1} \, dz \tag{1.1.2}$$

converges (as we will show later in a slightly more general setting), where Γ denotes the positively oriented boundary of $S(\pi, \omega') \cup B_\delta(0)$ for some $\delta > 0$ and $\omega' \in (\omega, \pi)$ (see fig. 1.2). Remember that the resolvent map $z \mapsto R(z, A) = (z - A)^{-1}$ is holomorphic on the resolvent set (see [EN00, Proposition IV.1.3]). Since the unique solution of (ACP) is given by $u(t) = T(t)x_0$, which can at least be formally written as $e^{tA}x_0$, one may hope that Cauchy's integral formula will give us a representation of the semigroup generated by A.

1.1.2 Functional Calculus for Sectorial Operators

Therefore we are interested in a way that allows us to naturally associate - given an operator A - an operator $f(A)$ to a certain class of holomorphic functions in a rigorous way using formula (1.1.2). This leads to the notion of *functional calculi*. We will present an elementary functional calculus for sectorial operators. Actually, much more can be done with some additional effort, for a complete reference see [Haa06].

We start by defining the algebra of elementary functions.

Definition 1.1.15 (Algebra of Elementary Functions)**.** Let $\varphi \in (0, 2\pi)$, $d \in \mathbb{R}$, $\delta > 0$ and $A \in \mathrm{Sect}(z_0, \omega)$ for $z_0 \in \mathbb{C}$ and $\omega \in (0, \pi)$. We define the algebra of *elementary functions*

$$\begin{aligned} H_\delta(S(z_0, d, \varphi)) := \{ & f : S(z_0, d, \varphi) \cup B_\delta(z_0) \to \mathbb{R} : f \text{ is holomorphic and} \\ & \exists M, R, s > 0 : |f(z)| \leq M\,|z - z_0|^{-s} \text{ for } |z - z_0| > R\}, \end{aligned}$$

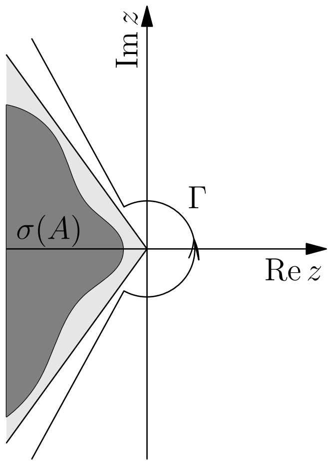

Figure 1.2: Contour integral for a sectorial operator

where $S(z_0, d, \varphi) \cup B_\delta(z_0)$ is called an *extended sector*. Moreover, define the algebra

$$H(A) \coloneqq \bigcup_{\substack{\delta>0 \\ \pi>\varphi>\omega}} H_\delta(S(z_0, \pi, \varphi)).$$

The following lemma is the heart of the functional calculus for sectorial operators.

Lemma 1.1.16. *Let $A \in \mathrm{Sect}(z_0, \omega)$ for $\pi > \varphi > \omega$ and $\delta > 0$. Then, provided Γ is an arbitrary curve surrounding the spectrum of A within $D_\varepsilon \coloneqq (S(z_0, \pi, \varphi) \cup B_\delta(z_0)) \setminus \overline{S(z_0, \pi, \omega + \varepsilon)}$ for some $\varepsilon \in (0, \varphi - \omega)$, in words Γ lies inside an extended subsector and outside some closed subsector slightly bigger than a sector in which the spectrum of the operator A is contained (see fig. 1.3),*

$$\begin{aligned} \psi_{\delta,\varphi} : H_\delta(S(z_0, \pi, \varphi)) &\to \mathcal{L}(X) \\ f &\mapsto \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A)\, dz \end{aligned}$$

is a well-defined homomorphism of algebras.

Proof. Let $f \in H_\delta(S(z_0, \pi, \varphi))$. We start by showing that the value of the integral is independent of the chosen curve. Therefore let Γ, Γ' be two curves as described above. Denote with γ_R (resp. with γ'_R) the subcurves of Γ (resp. Γ') connecting two points on Γ (resp. on Γ') on the opposite sides of the real axis with real parts $-R$. These points can be joined by vertical curves γ_V, γ'_V within the domain of holomorphy of f and $z \mapsto R(z, A)$ (see fig. 1.4). Thus the Cauchy integral theorem yields

$$0 = \frac{1}{2\pi i} \int_{\gamma_R} f(z) R(z, A)\, dz + \frac{1}{2\pi i} \int_{\gamma_V} f(z) R(z, A)\, dz$$

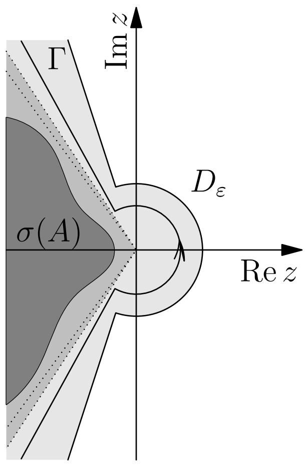

Figure 1.3: Sketch of the domains and paths involved in Lemma 1.1.16. The dotted lines indicate the boundaries of the sectors $S(\pi, \omega)$ and $S(\pi, \omega + \varepsilon)$.

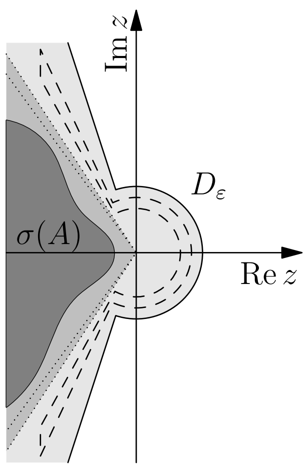

Figure 1.4: The Cauchy integral theorem shows the independence of the path of integration

$$- \frac{1}{2\pi i} \int_{\gamma_R'} f(z)R(z,A)\,dz - \frac{1}{2\pi i} \int_{\gamma_V'} f(z)R(z,A)\,dz. \quad (1.1.3)$$

Since f is an elementary function, there exist constants $M, R_0, s > 0$ such that $|f(z)| \leq M\,|z-z_0|^{-s}$ for $|z-z_0| > R_0$. Moreover, as A is sectorial, the resolvent fulfills the estimate $\|R(z,A)\| \leq \frac{C}{|z-z_0|}$ for some positive constant C on γ_V and γ_V'. Observe that

$$\begin{aligned} \left\| \frac{1}{2\pi i} \int_{\gamma_V} f(z)R(z,A)\,dz \right\| &\leq \frac{1}{2\pi} \int_{\gamma_V} \|f(z)R(z,A)\|\;|dz| \leq \frac{MC}{2\pi} \int_{\gamma_V} |z-z_0|^{-(1+s)}\;|dz| \\ &\leq \frac{MC}{2\pi} \cdot \frac{L(\gamma_V)}{|R - \operatorname{Re} z_0|^{1+s}} \leq \frac{MC}{2\pi} \cdot \frac{\tan\varphi}{|R-\operatorname{Re} z_0|^{s}} \quad \text{for } R > R_0, \end{aligned} \quad (1.1.4)$$

where we have used the the fact that the length of γ_V is dominated by $\tan\varphi \cdot |R - \operatorname{Re} z_0|$. Obviously, the same estimate holds for γ_V'. So as R tends to infinity, (1.1.4) vanishes. Taking limits on both sides of (1.1.3) yields

$$\frac{1}{2\pi i} \int_\Gamma f(z)R(z,A)\,dz = \frac{1}{2\pi i} \int_{\Gamma'} f(z)R(z,A)\,dz.$$

From now on let Γ be the positively oriented boundary of $S(z_0,\pi,\vartheta) \cup B_{\delta/2}(z_0)$ for fixed $\vartheta \in (\omega+\varepsilon,\varphi)$. Since f is an elementary function, it is bounded on compact subsets of the trace of Γ and vanishes as $|z|$ goes to infinity. Hence, f is bounded on Γ by a positive constant D. Using the notations from the other estimates above, we get for $\tilde{\vartheta} = \pi - \frac{\vartheta}{2}$

$$\begin{aligned} \left\| \frac{1}{2\pi i} \int_\Gamma f(z)R(z,A)\,dz \right\| &\leq CD + \frac{1}{\pi} \int_\delta^\infty \left\| f(z_0 + re^{i\tilde{\vartheta}}) R(z_0 + re^{i\tilde{\vartheta}}, A) \right\|\,dr \\ &\leq CD + \frac{1}{\pi} \int_\delta^{R_0} \left\| f(z_0 + re^{i\tilde{\vartheta}}) R(z_0 + re^{i\tilde{\vartheta}}, A) \right\|\,dr \\ &\quad + \frac{MC}{\pi} \int_{R_0}^\infty r^{-(1+s)}\,dr < \infty. \end{aligned}$$

So $\psi_{\delta,\varphi}(f)$ is a bounded linear operator and therefore well-defined.

From now on we will write ψ instead of $\psi_{\delta,\varphi}$ in order to shorten notations. Further, we may assume without loss of generality that $z_0 = 0$: if $z_0 \neq 0$, we replace A by $A - z_0$. Obviously, ψ is a linear mapping between the underlying vector spaces. It remains to show that ψ is compatible with the inner multiplication of the algebra. Let $f, g \in H_\delta(S(\pi,\varphi))$. We choose Γ' to be the positively oriented boundary of $S(\pi,\vartheta') \cup B_{3\delta/4}(0)$ for some

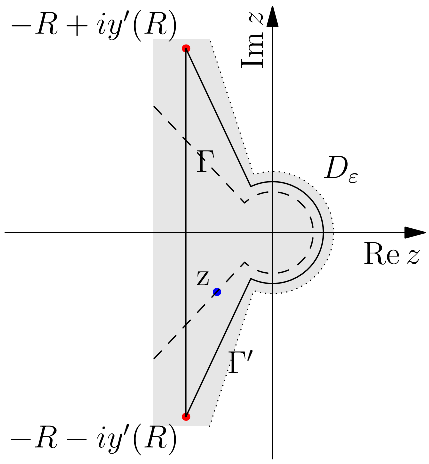

Figure 1.5: The Cauchy integral theorem and Cauchy's integral formula for elementary functions

$\vartheta' \in (\vartheta, \varphi)$, so Γ' lies to the right of Γ. A direct calculation shows that

$$
\begin{aligned}
\psi(f)\cdot\psi(g) &= \left(\frac{1}{2\pi i}\right)^2 \int_\Gamma f(z)R(z,A)\,dz \int_{\Gamma'} g(w)R(w,A)\,dw \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_\Gamma\int_{\Gamma'} f(z)g(w)R(z,A)R(w,A)\,dw\,dz \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_\Gamma\int_{\Gamma'} f(z)g(w)\frac{R(z,A)-R(w,A)}{w-z}\,dw\,dz \qquad (1.1.5)\\
&\overset{\text{Fubini}}{=} \frac{1}{2\pi i}\int_\Gamma \frac{1}{2\pi i}\int_{\Gamma'} \frac{g(w)}{w-z}\,dw f(z)R(z,A)\,dz \\
&\quad + \frac{1}{2\pi i}\int_{\Gamma'} \frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)}{z-w}\,dz g(w)R(w,A)\,dw,
\end{aligned}
$$

where we have used the resolvent identity $R(z,A)-R(w,A) = (w-z)R(z,A)R(w,A)$ in the third line. We continue by evaluating the inner integrals in the two summands. Let $w \in \Gamma'$ respectivley $z \in \Gamma$ be fixed points on the two curves. Choose R bigger than the absolute values of the real parts of z and w. Let γ_R respectively γ_R' be the subcurves which consist of the points whose absolute values of the real parts are smaller than R. Denote the absolute values of the imaginary parts of the end points of γ_R respectively γ_R' with $y(R)$ respectively $y'(R)$. Since f and g are holomorphic, we obtain by Cauchy's integral theorem and Cauchy's integral formula (see fig. 1.5)

$$
g(z) = \frac{1}{2\pi i}\int_{\gamma_R'} \frac{g(w)}{w-z}\,dw + \frac{1}{2\pi i}\int_{-R+iy'(R)}^{-R-iy'(R)} \frac{g(w)}{w-z}\,dw \qquad (1.1.6a)
$$

$$
0 = \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(z)}{z-w}\,dz + \frac{1}{2\pi i}\int_{-R+iy(R)}^{-R-iy(R)} \frac{f(z)}{z-w}\,dz. \qquad (1.1.6b)
$$

By elementary trigonometry, we have $y(R) = R\tan\vartheta$. Thus for sufficiently large R we have

$$\begin{aligned}
\left|\frac{1}{2\pi i}\int_{-R+iy(R)}^{-R-iy(R)} \frac{f(z)}{z-w}\,dz\right| &\leq \frac{1}{2\pi}\int_{-R\tan\vartheta}^{R\tan\vartheta}\left|\frac{f(-R+it)}{-R+it-w}\right| dt\\
&\leq \frac{M}{2\pi}\int_{-R\tan\vartheta}^{R\tan\vartheta} |R-it|^{-s}\,(R-|w|)^{-1}\,dt\\
&\leq \frac{MR\tan\vartheta}{\pi R^s(R-|w|)} \underset{R\text{ large}}{\overset{\frac{R}{R-|w|}\leq\pi}{\leq}} MR^{-s}\tan\vartheta.
\end{aligned}$$

Therefore the second term in (1.1.6b) vanishes as R tends to infinity. Note that the same argument holds for (1.1.6a). So as R tends to infinity, equation (1.1.6) yields

$$g(z) = \frac{1}{2\pi i}\int_{\Gamma'} \frac{g(w)}{w-z}\,dw \tag{1.1.7a}$$

$$0 = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)}{z-w}\,dz. \tag{1.1.7b}$$

Using these results, we can finish our calculations in (1.1.5):

$$\begin{aligned}
\psi(f)\cdot\psi(g) &= \frac{1}{2\pi i}\int_\Gamma \frac{1}{2\pi i}\int_{\Gamma'}\frac{g(w)}{w-z}\,dw f(z)R(z,A)\,dz\\
&\quad + \frac{1}{2\pi i}\int_{\Gamma'} \frac{1}{2\pi i}\int_{\Gamma}\frac{f(z)}{z-w}\,dz g(w)R(w,A)\,dw\\
&= \frac{1}{2\pi i}\int_\Gamma g(z)f(z)R(z,A)\,dz = \psi(f\cdot g).
\end{aligned}$$

Hence, ψ is a homomorphism of algebras as desired. □

In equation (1.1.7) we gave a weak generalization of the Cauchy integral theorem and Cauchy's integral formula for elementary functions which could be generalized to a larger class of curves.

Corollary 1.1.17 (CIT/CIF for Elementary Functions)**.** *Let $f \in H_\delta(S(z_0, d, \varphi))$ be an elementary function for some $\varphi < \pi$ and let Γ be the boundary of some extended sector within the domain of f. Then the Cauchy integral theorem*

$$\int_\Gamma f(z)\,dz = 0$$

holds. Similiarly, if $z_0 \in \mathbb{C}$ lies to the left of Γ, we have Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}\,dz.$$

We have seen that we can vary the path of integration without changing the value of the integral. Therefore the family $(\psi_{\delta,\varphi})_{\substack{\delta>0,\\ \pi>\varphi>\omega}}$ gives naturally rise to a homomorphism of algebras from $H(A)$ to the Banach algebra of bounded operators on X.

Theorem 1.1.18. *Let $A \in \operatorname{Sect}(z_0, \omega)$ for some $\omega < \pi$. Then for an arbitrary curve Γ as described in Lemma 1.1.16*

$$\begin{aligned} \psi : H(A) &\to \mathcal{L}(X) \\ f &\mapsto \frac{1}{2\pi i}\int_\Gamma f(z)R(z,A)\,dz \end{aligned}$$

is a well-defined homomorphism of algebras. As a shorthand notation, we will also write

$$f(A) \coloneqq \psi(f).$$

1.1.3 Characterizations of Holomorphic Semigroups

We return to the situation in the last but one section. Let $(T(t))$ be a bounded strongly continuous semigroup whose infinitesimal generator A is in $\operatorname{Sect}(\omega)$ for some $\omega < \pi$. Observe that for $\omega < \omega' < \pi$ and $z \in S(\pi, \omega')$ we have

$$\left|e^{tz}\right| = e^{t \operatorname{Re} z} = e^{-t|z|\cos(\pi - \arg z)} \leq e^{-t|z|\cos(\omega'/2)} \quad \text{for } t > 0.$$

This shows that $z \mapsto e^{tz}$ is in $H_1(S(\pi,\omega')) \subset H(A)$. Therefore we can use the functional calculus for sectorial operators developed in the last section and we obtain a family of bounded operators

$$U(t) \coloneqq \frac{1}{2\pi i}\int_\Gamma e^{tz}R(z,A)\,dz \in \mathcal{L}(X). \tag{1.1.8}$$

By Theorem 1.1.18, $(U(t))$ obeys the semigroup law $U(t+s) = U(t)U(s)$ for $t, s > 0$. Now we would expect $(U(t))$ to be the strongly continuous semigroup generated by A. Indeed, this is true, but one can even get more from the representation of $(U(t))$ as a Cauchy integral. Observe that formula (1.1.8) does make sense for certain complex numbers $t = re^{i\varphi}$ as well. Indeed,

$$\left|e^{tz}\right| = e^{\operatorname{Re}(tz)} = e^{-r|z|\cos(\pi - (\varphi + \arg z))}$$

and so $z \mapsto e^{tz}$ is an element of $H_1(S(\pi,\omega'))$ for some $\omega < \omega' < \pi$ if and only if there exists some $\varepsilon > 0$ such that

$$|\varphi + \arg z - \pi| < \frac{\pi}{2} - \varepsilon \; \forall z \in S(\pi, \omega').$$

This in turn holds if and only if

$$\left|\varphi \pm \frac{\omega'}{2}\right| < \frac{\pi}{2} \iff |\varphi| < \frac{\pi - \omega'}{2}. \tag{1.1.9}$$

Hence, $z \mapsto e^{tz}$ lies in $H(A)$ if and only if φ is chosen such that $t \in S(0, \pi - \omega)$. Therefore $(U(t))$ can be extended to a semigroup on a sector arround the non-negative real axis.

In the next lemma we will show that $(U(t))$ is indeed a strongly continuous semigroup and even more.

Lemma 1.1.19. *Let $A \in \operatorname{Sect}(\omega)$ for $\omega \in (0, \pi)$. We write for $z \in S(0, \pi - \omega)$*

$$U(z) := e^{zA} = \frac{1}{2\pi i} \int_\Gamma e^{\mu z} R(\mu, A)\, d\mu. \tag{1.1.10}$$

Then the maps $U(z)$ are bounded linear operators satisfying the following properties.

(a) *$\|U(z)\|$ is uniformly bounded in $S(0, \delta)$ for $\delta < \pi - \omega$. Moreover, the bound only depends on δ, ω and on the upper bound of $\mu \mapsto \mu R(\mu, A)$ on $S(0, \delta)$.*

(b) *The map $z \mapsto U(z)$ is holomorphic in $S(0, \pi - \omega)$.*

(c) *$U(z_1 + z_2) = U(z_1)U(z_2)$ for all $z_1, z_2 \in S(0, \pi - \omega)$.*

(d) *$\lim\limits_{\substack{z \to 0 \\ z \in S(0,\delta)}} U(z)x = x$ for all $x \in \overline{D(A)}$ and $0 < \delta < \pi - \omega$.*

Proof. We begin with (a): Choose $\varepsilon > 0$ such that $\delta < \delta + 2\varepsilon < \pi - \omega$. Consequently, $\omega < \omega + \delta + 2\varepsilon < \pi$ and for all $z \in S(0, \delta)$ we see using the calculations finished in (1.1.9) that by our choice

$$|\arg z| < \frac{\delta}{2} < \frac{\pi - (\omega + 2\varepsilon)}{2}.$$

Hence, $\mu \mapsto e^{\mu z}$ is (for $r > 0$ arbitrary) an elementary function in $H_r(S(\pi, \omega + 2\varepsilon))$ and is a fortiori contained in $H(A)$ for all $z \in S(0, \delta)$. Therefore Γ can be chosen to be the positively oriented boundary of $S(\pi, \omega + \varepsilon) \cup B_{1/|z|}(0)$. Let $\vartheta = \frac{\omega}{2} + \frac{\varepsilon}{2}$. We decompose Γ in three parts given by the arc A formed by the boundary of the disk and the two rays going to infinity:

$$\begin{aligned} \frac{1}{2\pi i} \int_\Gamma e^{\mu z} R(\mu, A)\, d\mu &= \frac{1}{2\pi i} \int_A e^{\mu z} R(\mu, A)\, d\mu \\ &+ \frac{1}{2\pi i} \int_{1/|z|}^\infty e^{re^{i(\pi+\vartheta)} z} R(re^{i(\pi+\vartheta)}, A) e^{i(\pi+\vartheta)}\, dr \\ &+ \frac{1}{2\pi i} \int_{1/|z|}^\infty e^{re^{i(\pi-\vartheta)} z} R(re^{i(\pi-\vartheta)}, A) e^{i(\pi-\vartheta)}\, dr. \end{aligned} \tag{1.1.11}$$

Observe that for $z \in S(0, \delta)$ we have $|\arg z| < \delta/2$. Thus,

$$\arg(re^{i(\pi \pm \vartheta)} z) \in \left(\pi - \frac{\omega + \varepsilon + \delta}{2}, \pi + \frac{\omega + \varepsilon + \delta}{2}\right).$$

Hence,

$$\begin{aligned} \operatorname{Re}\left(re^{i(\pi\pm\vartheta)} z\right) &= r\,|z| \cos\left(\arg(re^{i(\pi\pm\vartheta)} z)\right) \le r\,|z| \cos\left(\pi \pm \frac{\omega + \varepsilon + \delta}{2}\right) \\ &= -r\,|z| \cos\left(\frac{\omega + \varepsilon + \delta}{2}\right). \end{aligned}$$

By our choice, $\frac{\omega+\varepsilon+\delta}{2} < \frac{\pi}{2}$. Therefore there exists a positive constant $\varepsilon' > 0$ such that

$$\operatorname{Re}\left(re^{i(\pi\pm\vartheta)}z\right) \leq -\varepsilon' r\,|z| \qquad \text{for all } z \in S(0,\delta).$$

Since A is a sectorial operator, $\|\mu R(\mu, A)\|$ is bounded by a positive constant M on Γ. By this and (1.1.11), we get

$$\begin{aligned}
\left\|\frac{1}{2\pi i}\int_\Gamma e^{\mu z} R(\mu, A)\, d\mu\right\| &\leq \frac{M}{2\pi}\int_A e^{\operatorname{Re}(\mu z)}\,|\mu|^{-1}\,|d\mu| + \frac{M}{\pi}\int_{1/|z|}^{\infty} e^{-\varepsilon' r|z|} r^{-1}\, dr \\
&\overset{s=r|z|}{\leq} \frac{M\,|z|}{2\pi}\int_A e^{|z^{-1}z|}\,|d\mu| + \frac{M}{\pi}\int_1^\infty e^{-\varepsilon' s}\frac{|z|}{s}\frac{1}{|z|}\, ds \\
&\leq eM + \frac{M}{\pi}\int_1^\infty e^{-\varepsilon' s} s^{-1}\, ds
\end{aligned}$$

for all $z \in S(0,\delta)$. This shows that $(U(z))$ is uniformly bounded and converges absolutely on $S(0,\delta)$ and that the bound only depends on M and ε', which in turn only depends on ω and δ. Clearly, for every connected subcurve Γ_f of finite length

$$U_{\Gamma_f} : z \mapsto \frac{1}{2\pi i}\int_{\Gamma_f} e^{\mu z} R(\mu, A)\, d\mu$$

is a holomorphic function. The absolute convergence shown above implies that (U_{Γ_f}) converges to U uniformly on $S(0,\delta)$ as the length of Γ_f goes to infinity. Being the uniform limit of holomorphic functions, $z \mapsto U(z)$ is holomorphic on $S(0,\delta)$ for every $\delta < \pi - \omega$. Consequently, since $S(0, \pi-\omega)$ can be covered by such sectors, $z \mapsto U(z)$ is holomorphic on $S(0, \pi - \omega)$. So we have shown (a) and (b).

Property (c) is a direct consequence of Theorem 1.1.18. Alternatively, it follows directly from the identity theorem for holomorphic functions.

It remains to show that $(U(z))$ satisfies (d). Fix $0 < \delta < \pi - \omega$. We now choose Γ to be the positively oriented boundary of $S(\pi, \omega+\varepsilon) \cup B_1(0)$ with ε as above. This means that we have chosen a fixed radius for the ball around the origin. By Cauchy's integral formula for elementary functions (see Corollary 1.1.17), we have

$$\frac{1}{2\pi i}\int_\Gamma \frac{e^{\mu z}}{\mu}\, d\mu = 1$$

for all $z \in S(0,\delta)$. Let $x \in D(A)$. We use

$$R(\mu, A)Ax = R(\mu, A)(-\mu + A + \mu)x = \mu R(\mu, A)x - x$$

to obtain

$$\begin{aligned}
U(z)x - x &= \frac{1}{2\pi i}\int_\Gamma e^{\mu z}\left(R(\mu, A) - \frac{1}{\mu}\right) x\, d\mu \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{e^{\mu z}}{\mu} R(\mu, A)Ax\, d\mu.
\end{aligned}$$

Taking limits on both sides yields

$$\lim_{\substack{z \to 0 \\ z \in S(0,\delta)}} U(z)x - x = \frac{1}{2\pi i} \lim_{\substack{z \to 0 \\ z \in S(0,\delta)}} \int_\Gamma \frac{e^{\mu z}}{\mu} R(\mu, A)Ax \, d\mu = \frac{1}{2\pi i} \int_\Gamma \frac{1}{\mu} R(\mu, A)Ax \, d\mu.$$

Here the interchange of the integration and the limiting process can be justified by Lebesgue's dominated convergence theorem because the familiy of maps $(f_z)_{z \in S(0,\delta)}$ defined by $f_z(\mu) = \frac{e^{\mu z}}{\mu} R(\mu, A)Ax$ is dominated by

$$\frac{M}{|\mu|^2} e^{\operatorname{Re}(\mu z)} \|Ax\| \le \frac{M}{|\mu|^2} e^{-\varepsilon' |\mu z|} \|Ax\| \le \frac{M}{|\mu|^2} \|Ax\|$$

on the the two rays going to infinity and by

$$\frac{M}{|\mu|^2} e^{\operatorname{Re}(\mu z)} \|Ax\| \le \frac{M}{|\mu|^2} e^{|\mu z|} \|Ax\| = \frac{M}{|\mu|^2} e^{|z|} \|Ax\|$$

on the arc. Therefore the family of functions $(f_z)_{z \in S(0,\delta), |z| \le 1}$ is dominated by an integrable function in the following way:

$$\|f_z(\mu)\| \le \frac{M}{|\mu|^2} (1 + e^{|z|}) \|Ax\| \le \frac{M(1+e)}{|\mu|^2} \|Ax\| \quad \text{for all } \mu \in \Gamma \text{ and } z \in S(0,\delta) \cap B_1(0).$$

Finally, we have to calculate the value of

$$\frac{1}{2\pi i} \int_\Gamma \frac{1}{\mu} R(\mu, A)Ax \, d\mu.$$

This can again be done with Cauchy's integral theorem. Denote γ_R the subcurve of Γ consisting of the points whose absolute values of the imaginary parts are smaller than R. We close γ_R to its right within the domain of holomorphy of the integrand with the arc A_R of a negatively oriented circle of radius R centered in the origin (see fig. 1.6). For this closed curve the Cauchy integral theorem applies and we obtain

$$0 = \frac{1}{2\pi i} \int_{\gamma_R} \frac{1}{\mu} R(\mu, A)Ax \, d\mu + \frac{1}{2\pi i} \int_{A_R} \frac{1}{\mu} R(\mu, A)Ax \, d\mu.$$

We see that

$$\left\| \frac{1}{2\pi i} \int_{A_R} \frac{1}{\mu} R(\mu, A)Ax \, d\mu \right\| \le \frac{M}{R} \|Ax\|.$$

Hence, as $R \to \infty$, the second term vanishes and we have therefore shown that

$$\lim_{\substack{z \to 0 \\ z \in S(0,\delta)}} U(z)x - x = 0 \qquad \text{for all } x \in D(A).$$

Since $(U(z))$ is uniformly bounded on $S(0,\delta)$, a standard 3ε-argument shows that the above identity holds for all $x \in \overline{D(A)}$. This proves (d) and the proof is complete. □

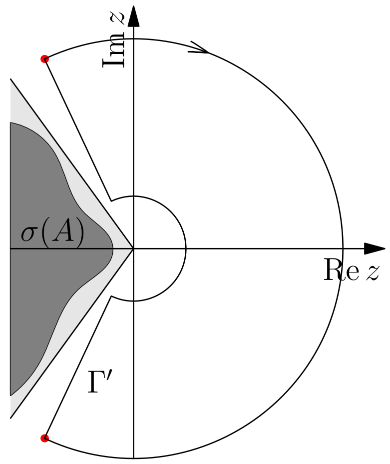

Figure 1.6: Closing γ_R with the arc A_R

Remark 1.1.20. If we additionally require that A is the infinitesimal generator of a bounded strongly continuous semigroup, Lemma 1.1.19(c) and (d) show - as promised - that $(U(t))_{t\geq 0}$ is a strongly continuous semigroup as well because the Hille-Yosida Generation theorem 1.1.8 shows that $\overline{D(A)} = X$.

Now given a densely defined sectorial operator $A \in \mathrm{Sect}(\omega)$ for some $\omega < \pi$, it remains to show that the infinitesimal generator of the strongly continuous semigroup we obtain by restricting $(U(z)) = (e^{zA})$ - as defined in (1.1.10) - to non-negative real numbers is A. Since the generator determines the C_0-semigroup uniquely (see [EN00, Theorem II.1.4] or [Paz83, Theorem I.2.6]), this would finally imply that $(U(z))$ is indeed a holomorphic extension of the strongly continuous semigroup generated by A.

Lemma 1.1.21. *Let A be the infinitesimal generator of a bounded strongly continuous semigroup $(T(t))$. Assume further that $A \in \mathrm{Sect}(\omega)$ for some $\omega < \pi$. Then the generator of the strongly continuous semigroup defined by* (1.1.10) *is A.*

Proof. Let B be the infinitesimal generator of $(U(t))$. Notice that it suffices to show that

$$R(\lambda, A) = R(\lambda, B)$$

for some $\lambda > 0$. We know that the resolvent of B is given by the Laplace transform (see [EN00, Theorem I.1.10]), more precisely that

$$R(\lambda, B)x = \int_0^\infty e^{-\lambda t} U(t)x\, dt \qquad \text{for all } x \in X \text{ and } \operatorname{Re}\lambda > 0.$$

We choose the same path Γ as in the proof of part (d) of Lemma 1.1.19. Then by (1.1.10) we can write

$$R(\lambda, B)x = \lim_{T\to\infty} \frac{1}{2\pi i} \int_0^T \int_\Gamma e^{-\lambda t} e^{tz} R(z, A)x\, dz\, dt.$$

Further, for $T > 0$ we have

$$\begin{aligned} \frac{1}{2\pi i}\int_0^T \int_\Gamma e^{-\lambda t}e^{tz}R(z,A)x\,dz\,dt &\overset{\text{Fubini}}{=} \frac{1}{2\pi i}\int_\Gamma R(z,A)x\,dz\int_0^T e^{(z-\lambda)t}\,dt \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{e^{(z-\lambda)T}-1}{z-\lambda}R(z,A)x\,dz \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{R(z,A)}{\lambda - z}x\,dz + \frac{1}{2\pi i}\int_\Gamma \frac{e^{(z-\lambda)T}}{z-\lambda}R(z,A)x\,dz. \end{aligned} \tag{1.1.12}$$

Now choose $\lambda = 2$. We can again use Cauchy's integral formula to obtain the value of the first integral. As in the last proof, let γ_R be the subcurve formed by the points on Γ whose imaginary parts' absolute values are smaller than R. Close γ_R to its right with the negatively oriented arc A_R of the circle with radius R centered in the origin (see again fig. 1.6). Then for $R > \lambda$ we apply Cauchy's integral formula

$$-R(\lambda,A)x = \frac{1}{2\pi i}\int_{\gamma_r} \frac{R(z,A)}{z-\lambda}x\,dz + \frac{1}{2\pi i}\int_{A_R} \frac{R(z,A)}{z-\lambda}x\,dz.$$

Again, the second integral vanishes as $R \to \infty$ because

$$\left\| \frac{1}{2\pi i}\int_{A_R} \frac{R(z,A)}{\lambda - z}x\,dz \right\| \le \frac{M}{2\pi}\int_{A_R} \frac{\|x\|}{|z(\lambda - z)|}\,|dz| \le \frac{M\,\|x\|}{R - |\lambda|}.$$

Therefore

$$R(\lambda,A)x = \frac{1}{2\pi i}\int_\Gamma \frac{R(z,A)}{\lambda - z}x\,dz.$$

The second term in (1.1.12) can be controlled by a smiliar estimate. Indeed, since $\operatorname{Re} z \le 1$, we have because of $\operatorname{Re}(z-\lambda) = \operatorname{Re}(z-2) \le -1$

$$\left\| \int_\Gamma \frac{e^{(z-\lambda)T}}{z-\lambda}R(z,A)x\,dz \right\| \le Me^{-T}\int_\Gamma \frac{\|x\|}{|z-\lambda|\,|z|}\,|dz| \le M'\,\|x\|\,e^{-T}\int_\Gamma \frac{1}{|z|^2}\,|dz|$$

for a positive constant M'. Hence letting $T \to \infty$ on both sides of (1.1.12), we have shown the desired result

$$R(\lambda,A)x = \int_0^\infty e^{-\lambda T}T(t)x\,dt = R(\lambda,B)x. \qquad \square$$

Recall that the infinitesimal generator of a bounded strongly continuous semigroup is sectorial of angle π. We have now finally shown that, if A fulfills the additional assumption of being sectorial of angle $\omega < \pi$, the semigroup generated by A can be extended to a holomorphic function around the non-negative real axis. Such a semigroup is called *holomorphic*. The exact definition is given next.

Definition 1.1.22 (Holomorphic Semigroup)**.** A family of bounded linear operators $(T(z))_{z\in S(0,\delta)\cup\{0\}}$ is called a *holomorphic semigroup (of angle δ)* if

(i) $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in S(0,\delta)$.

(ii) The map $z \mapsto T(z)$ is holomorphic in $S(0,\delta)$.

(iii) $\lim\limits_{\substack{z \to 0 \\ z \in S(0,\delta')}} T(z)x = x$ for all $x \in X$ and $0 < \delta' < \delta$. If, in addition,

(iv) $\|T(z)\|$ is bounded in $S(0,\delta')$ for every $0 < \delta' < \delta$, we call $(T(z))_{z\in S(0,\delta)\cup\{0\}}$ a *bounded holomorphic semigroup.*

We now want to give different characterizations of holomorphic semigroups like it was given for strongly continuous semigroups by the Hille-Yosida theorem. Observe that the following estimates are much more simple than the ones in the Hille-Yoshida theorem: While one needs only one single estimate on the resolvent for the case of a holomorphic semigroup, estimates on all powers of the resolvent are needed for the general (non-contractive) case of a strongly continuous semigroup.

Theorem 1.1.23. *Let $(A, D(A))$ be a linear operator on X. The following statements are equivalent.*

(a) A generates a bounded holomorphic semigroup $(T(z))$ in a sector $S(0, \pi-\omega)$ on X.

(b) A generates a bounded strongly continuous semigroup $(T(t))$ with $\|T(t)\| \le M$ on X and there exists a constant $C > 0$ such that

$$\|R(r+is, A)\| \le \frac{C}{|s|} \tag{1.1.13}$$

for all $r > 0$ and $0 \ne s \in \mathbb{R}$.

(c) A is densely defined and $A \in \operatorname{Sect}(\omega)$ for some $\omega < \pi$.

Moreover, if one and therefore all of the above statements hold, $(T(z))$ can be written as the Cauchy integral

$$T(z) = \frac{1}{2\pi i}\int_\Gamma e^{\mu z} R(\mu, A)\, d\mu, \tag{1.1.14}$$

where Γ is a curve as described in Lemma 1.1.16.

Further, if (b) holds, we can choose $\pi - \omega = 2\arctan C^{-1}$ and the upper bound of $\|T(z)\|$ on a smaller subsector $S(0,\delta)$ only depends on C, M and δ.

Proof. First, we show that (a) implies (b). Let $0 < 2\delta' < \delta := \pi - \omega$. Then $\|T(z)\| \le M$ for a positive constant M on the closed sector $\overline{S(0, 2\delta')}$. For $r > 0$ we can write the resolvent in terms of the Laplace transform

$$R(r+is, A)x = \int_0^\infty e^{-(r+is)t} T(t)x\, dt.$$

Fix $r, s > 0$ and let $R > 0$. We apply the Cauchy integral theorem to the positively oriented boundary of the triangle formed by the points 0, R and $R - iR\tan\delta'$ in the complex plane (see fig. 1.7):

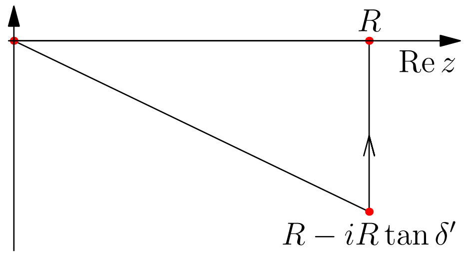

Figure 1.7: We shift the path of integration from the non-negative real axis to the ray $\{\rho e^{-i\delta'} : \rho > 0\}$

$$\begin{aligned}0 = &-\frac{1}{2\pi i}\int_0^R e^{-(r+is)t}T(t)\,dt - \frac{1}{2\pi i}\int_0^{R\tan\delta'} e^{-(r+is)(R-ih)}T(R-ih)(-i)\,dh\\ &+\frac{1}{2\pi i}\int_0^{R/\cos\delta'} e^{-(r+is)\rho e^{-i\delta'}}T(\rho e^{-i\delta'})e^{-i\delta'}\,d\rho.\end{aligned}$$

The second term can be estimated by

$$\frac{M}{2\pi}\int_0^{R\tan\delta'} e^{-rR-sh}\,dh \le \frac{M\tan\delta'}{2\pi}Re^{-rR}$$

and vanishes as R goes to infinity. Hence, we can shift the path of integration from the non-negative real axis to the ray $\{\rho e^{-i\delta'} : \rho > 0\}$. Further,

$$\begin{aligned}\|R(r+is,A)x\| &\le M\,\|x\|\int_0^\infty e^{-\operatorname{Re}\left((r+is)\rho e^{-i\delta'}\right)}\,d\rho = M\,\|x\|\int_0^\infty e^{-\rho(r\cos\delta'+s\sin\delta')}\,d\rho\\ &= \frac{M\,\|x\|}{r\cos\delta'+s\sin\delta'} \le \frac{M}{\sin\delta'}\frac{1}{s}\cdot\|x\|\,.\end{aligned}$$

Now let $s < 0$. Then by the same argument, we can shift the path of integration to the ray $\{\rho e^{i\delta} : \rho > 0\}$. Again, the same estimate shows

$$\|R(r+is,A)x\| \le \frac{M}{\sin\delta'}\frac{1}{-s}\cdot\|x\|\,.$$

Consequently, we have shown for $r > 0$ and $s \neq 0$ that

$$\|R(r+is,A)\| \le \frac{M}{\sin\delta'}\frac{1}{|s|}.$$

Thus (a) implies (b).

Suppose (b) holds. By assumption, A is the infinitesimal generator of a bounded strongly continuous semigroup and therefore densely defined. Therefore the half-plane right to the imaginary axis lies in the resolvent set of A by the Hille-Yosida theorem (Theorem 1.1.8). As shown in [EN00, Proposition IV.1.3], the Taylor expansion of the resolvent map in λ_0 is

$$R(\lambda,A) = \sum_{k=0}^{\infty}(\lambda_0-\lambda)^k R(\lambda_0,A)^{k+1} \qquad \text{for } \operatorname{Re}\lambda_0 > 0.$$

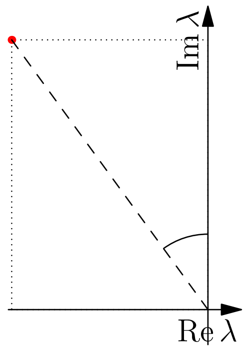

Figure 1.8: The Taylor series of the resolvent map converges in the sector $S(0, \pi + 2\delta)$ with $\delta = \arctan C^{-1}$

The series converges uniformly for $\|R(\lambda_0, A)\| \, |\lambda_0 - \lambda| \leq \rho < 1$, where $\rho \in (0,1)$ is arbitrary. Fix λ with non-positive real part. Now choose $\lambda_0 = r + i \operatorname{Im} \lambda$ with arbitrary $r > 0$. Let $|r - \operatorname{Re} \lambda| = |\lambda_0 - \lambda| \leq \rho \frac{|\operatorname{Im} \lambda_0|}{C}$. Then the series converges because

$$\|R(\lambda_0, A)\| \, |\lambda_0 - \lambda| \leq \rho \, \|R(\lambda_0, A)\| \, \frac{|\operatorname{Im} \lambda_0|}{C} \leq \rho$$

by assumption (1.1.13). Since $r > 0$ and $0 < \rho < 1$ are arbitrary, we conclude that

$$\left\{ \lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq 0 \text{ and } \frac{|\operatorname{Re} \lambda|}{|\operatorname{Im} \lambda|} < \frac{1}{C} \right\} \subset \rho(A).$$

Hence, as can be seen in fig. 1.8, one has $S(0, \pi + 2\delta) \subset \rho(A)$ for $\delta = \arctan C^{-1}$ (or equivalently $\sigma(A) \subset \overline{S(\pi, \pi - 2\delta)}$).

It remains to show that $\|\lambda R(\lambda, A)\|$ is bounded on strictly smaller subsectors $S(0, \pi + 2\delta')$ for $0 < \delta' < \delta$. Let us begin with the case of $\operatorname{Re} \lambda > 0$. By the Hille-Yosida theorem (Theorem 1.1.8), we have

$$\|R(\lambda, A)\| \leq \frac{M}{\operatorname{Re} \lambda} \qquad \text{for } \operatorname{Re} \lambda > 0.$$

Moreover, by assumption (1.1.13) we conclude that

$$\|R(\lambda, A)\| \leq (C + M) \min \left\{ \frac{1}{\operatorname{Re} \lambda}, \frac{1}{|\operatorname{Im} \lambda|} \right\}.$$

Observe that either $\operatorname{Re} \lambda \geq 1/\sqrt{2} \, |\lambda|$ or $|\operatorname{Im} \lambda| \geq 1/\sqrt{2} \, |\lambda|$ holds and therefore

$$\|R(\lambda, A)\| \leq \frac{\sqrt{2}(C + M)}{|\lambda|} \qquad \text{for } \operatorname{Re} \lambda > 0.$$

We continue by showing a similiar estimate for the second case $\operatorname{Re} \lambda \leq 0$. There exists a unique $q \in (0,1)$ such that $\delta' = \arctan (q/C)$. Thus for all λ under consideration

$$\frac{|\operatorname{Re} \lambda|}{|\operatorname{Im} \lambda|} \leq \frac{q}{C}$$

holds. Choose $0 < q < q' < 1$, for example one can choose the arithmetic mean. Then we can choose independently of λ a sufficiently small positive number r_0 such that

$$|r_0 - \operatorname{Re}\lambda| \leq q' \frac{|\operatorname{Im}\lambda|}{C}.$$

As we have seen above, the Taylor series around $\lambda_0 = r_0 + i \operatorname{Im}\lambda$ converges in λ and we obtain

$$\begin{aligned}\|R(\lambda, A)\| &\leq \sum_{k=0}^{\infty} |r_0 - \operatorname{Re}\lambda|^k \, \|R(r_0 + i \operatorname{Im}\lambda, A)\|^{k+1} \leq \sum_{k=0}^{\infty} q'^k \frac{|\operatorname{Im}\lambda|^k}{C^k} \frac{C^{k+1}}{|\operatorname{Im}\lambda|^{k+1}} \\ &= \frac{C}{1-q'} \frac{1}{|\operatorname{Im}\lambda|}.\end{aligned}$$

Further,

$$\begin{aligned}|\lambda|^2 = |\operatorname{Re}\lambda|^2 + |\operatorname{Im}\lambda|^2 &= |\operatorname{Im}\lambda|^2 \left(\frac{|\operatorname{Re}\lambda|^2}{|\operatorname{Im}\lambda|^2} + 1 \right) \\ &\leq |\operatorname{Im}\lambda|^2 \left(\frac{q^2}{C^2} + 1 \right) \leq |\operatorname{Im}\lambda|^2 \left(\frac{C^2+1}{C^2} \right).\end{aligned}$$

This shows that

$$\|R(\lambda, A)\| \leq \frac{1}{1-q'} \frac{C^2}{\sqrt{C^2+1}} \frac{1}{|\lambda|} \leq \frac{C}{1-q'} \frac{1}{|\lambda|} \qquad \text{for } \operatorname{Re}\lambda \leq 0.$$

Therefore the upper bound only depends on C, M and q' which in turn only depends on q and therefore on δ'.

Finally, we have already shown in Lemma 1.1.19 that (c) implies (a) together with the qualitative statements at the end of the theorem. □

We can generalize Theorem 1.1.23 to arbitrary semigroups by rescaling. Let $(T(t))$ be a strongly continuous semigroup with infinitesimal generator A. Then there exist $M \geq 1, \omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$. Observe that $S(t) = e^{-\omega t}T(t)$ is a semigroup as well and that $(T(t))$ is holomorphic if and only if $(S(t))$ is holomorphic because one can be obtained by multiplication of the other with the holomorphic function $e^{\pm \omega t}$. The rescaled semigroup $(S(t))$ is bounded and generated by $A - \omega$. So we can apply Theorem 1.1.23 to obtain the general result.

Theorem 1.1.24. *Let $(A, D(A))$ be a linear operator on X. The following statements are equivalent.*

(a) A generates a holomorphic semigroup $(T(z))$ in a sector $S(0, \pi - \vartheta)$ on X such that for each $0 < \delta < \pi - \vartheta$ there exists a positive constant $\tilde{M}$ such that

$$\|T(z)\| \leq \tilde{M} e^{\omega \operatorname{Re} z} \qquad \textit{for all } z \in S(0, \delta).$$

(b) *A generates a strongly continuous semigroup $(T(t))$ with $\|T(t)\| \le Me^{\omega t}$ on X and there exists a constant $C > 0$ such that*

$$\|R(r+\omega+is, A)\| \le \frac{C}{|s|} \tag{1.1.15}$$

for all $r > 0$ and $0 \ne s \in \mathbb{R}$.

(c) *A is densely defined and $A \in \mathrm{Sect}(\omega, \vartheta)$ for $\vartheta < \pi$.*

Moreover, if one and therefore all of the above statements hold, $(T(z))$ can be written as the Cauchy integral

$$T(z) = \frac{1}{2\pi i} \int_\Gamma e^{\mu z} R(\mu, A)\, d\mu, \tag{1.1.16}$$

where Γ is a curve as described Lemma 1.1.16.

Further, if (b) holds, we can choose $\pi - \vartheta = 2\arctan C^{-1}$ and for a smaller subsector $S(0,\delta)$ the constant $\tilde{M}$ only depends on C, M and δ.

Proof. As described above, all properties follow directly from Theorem 1.1.23 applied to the bounded semigroup $(S(t)) = (e^{-\omega t}T(t))$ and its infinitesimal generator $A - \omega$, where ω is the growth bound of the semigroup $(T(t))$. For part (b) notice that

$$R(r+is, A-\omega) = R(r+\omega+is, A). \qquad \square$$

A useful sufficient condition for holomorphy can be given if the infinitesimal generator is a normal operator on a Hilbert space. For the definition of not necessarily bounded operators and their properties see Appendix A.2.

Theorem 1.1.25. *Let $A : H \supset D(A) \to H$ be a normal operator on some Hilbert space H satisfying*

$$\sigma(A) \subset \overline{S(\omega, \pi, \delta)}$$

for some $\omega \in \mathbb{R}$ and $\delta \in [0, \pi)$. Then $A \in \mathrm{Sect}(\omega, \delta)$ and generates a holomorphic semigroup on $S(0, \pi - \delta)$.

Proof. Choose $\lambda \in \rho(A)$. By Lemma A.2.6, $R(\lambda, A) \in \mathcal{L}(H)$ is normal as well. Given a bounded normal operator, its operator norm is given by the spectral radius (see [Wer09, Satz VII.2.16]), so

$$\|R(\lambda, A)\| = r(R(\lambda, A)).$$

We now determine the spectrum of the resolvent. We have for $\mu \ne 0$

$$(\mu - R(\lambda, A)) = (\mu(\lambda - A) - I)R(\lambda, A) = \mu((\lambda - \mu^{-1}) - A)R(\lambda, A).$$

Since the operators commute, μ lies in the resolvent set of $R(\lambda, A)$ if and only if $\lambda - \mu^{-1}$ lies in the resolvent set of A, or equivalently $\frac{1}{\lambda-\mu} \in \rho(R(\lambda, A))$ if and only if $\mu \in \rho(A)$. This shows

$$\sigma(R(\lambda, A)) \setminus \{0\} = \left\{ \frac{1}{\lambda - \mu} : \mu \in \sigma(A) \right\}.$$

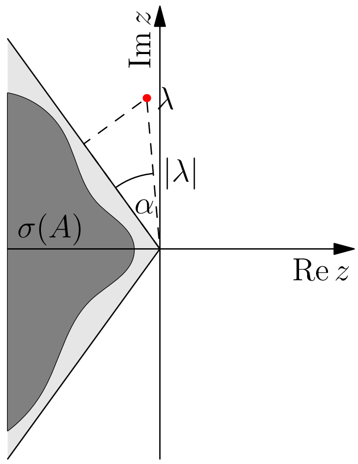

Figure 1.9: Proximum for $\omega = 0$ and some λ in the upper half-plane

Further,

$$r(R(\lambda, A)) = \sup_{\mu \in \sigma(R(\lambda,A))} |\mu| = \sup_{\mu \in \sigma(A)} \frac{1}{|\lambda - \mu|} = \frac{1}{\operatorname{dist}(\lambda, \sigma(A))} \leq \frac{1}{\operatorname{dist}(\lambda, S(\omega, \pi, \delta))}.$$

By abuse of notation, denote S the closure of $S(\omega, \pi, \delta)$. Observe that for a given λ the proximum in S is $\omega + i0$ if $|\arg(\lambda - \omega)| \leq \frac{\pi-\delta}{2}$ and the foot of the perpendicular from λ and the line that forms the part of the boundary on the same (upper or lower) half-plane otherwise. Suppose $|\arg(\lambda - \omega)| > \frac{\pi-\delta}{2}$ and let α be the angle in ω of the triangle formed by the proximum, λ and ω in the complex plane (see fig. 1.9). Then if λ is in the open upper half-plane, we have

$$\alpha = \pi - \frac{\delta}{2} - \arg(\lambda - \omega).$$

Elementary trigonometry shows that the distance of λ to the sector is given by

$$|\lambda - \omega| \sin\left(\pi - \frac{\delta}{2} - \arg(\lambda - \omega)\right) = |\lambda - \omega| \sin\left(\frac{\delta}{2} + |\arg(\lambda - \omega)|\right).$$

Notice that the last estimate also holds for λ in the open lower half-plane and that if $|\arg(\lambda - \omega)| \leq \frac{\pi-\delta}{2}$, the distance is $|\lambda - \omega|$. We now show that $A \in \operatorname{Sect}(\omega, \delta)$. For this purpose let $\varepsilon > 0$ and $\lambda \in S(\omega, 0, 2\pi - \delta - \varepsilon)$. Thus $|\arg(\lambda - \omega)| < \pi - \frac{\delta+\varepsilon}{2}$ and

$$|\lambda - \omega| \, \|R(\lambda, A)\| \leq \begin{cases} 1 & \text{if } |\arg(\lambda - \omega)| \leq \frac{\pi-\delta}{2} \\ \frac{1}{\sin\left(\frac{\delta}{2} + |\arg(\lambda-\omega)|\right)} & \text{else} \end{cases} \leq \frac{1}{\sin(\pi - \frac{\varepsilon}{2})} = \frac{1}{\sin(\frac{\varepsilon}{2})}.$$

Hence, A is sectorial as claimed and generates a holomorphic semigroup on $S(0, \pi - \delta)$ by Theorem 1.1.24(c). ☐

Corollary 1.1.26. *Let A be a self-adjoint generator of a strongly continuous semigroup. Then A generates a holomorphic semigroup on $S(0, \pi)$.*

Proof. The infinitesimal generator A is a fortiori normal and has real spectrum (see [Wer09, Satz VII.2.16]), so $\sigma(A) \subset (-\infty, d]$ for some $d \in \mathbb{R}$ by the Hille-Yosida theorem (Theorem 1.1.8). Hence, Theorem 1.1.25 applies with $\delta = 0$. $\square$

Example 1.1.27 (Heat Semigroup)**.** We have mentioned in Examples 1.1.1 and 1.1.10 that the infinitesimal generator of the heat semigroup on $L^2(\mathbb{R};\mathbb{C})$ is self-adjoint. Hence, the heat semigroup possesses a holomorphic extension on $S(0,\pi)$ by Corollary 1.1.26.

We also check the holomorphy directly. We have already seen that A is unitary equivalent to a multiplication operator, more precisely that

$$A = \mathcal{F}_2^{-1} \circ [f \mapsto -x^2 f] \circ \mathcal{F}_2.$$

We see that the spectrum of the multiplication operator and therefore the spectrum of A is the non-positive real axis $(-\infty, 0]$ and that for $\lambda \in \rho(A)$ the resolvent map is given by

$$(\lambda - A)^{-1} = \mathcal{F}_2^{-1} \circ \left[g \mapsto \frac{g}{\lambda + x^2} \right] \circ \mathcal{F}_2.$$

Moreover, for $\varepsilon > 0$ the map $(\lambda, x) \mapsto \frac{|\lambda|}{|\lambda + x^2|}$ is bounded on $S(0, 2\pi - \varepsilon) \times \mathbb{R}$ by some constant C_ε. Since $\mathcal{F}_2$ is unitary, we obtain

$$\left\| (\lambda - A)^{-1} \right\| \le \left\| \left[g \mapsto \frac{g}{\lambda + x^2} \right] \right\| \le \frac{C_\varepsilon}{|\lambda|} \qquad \forall \lambda \in S(0, 2\pi - \varepsilon),$$

which again shows that A is a sectorial operator and therefore generates a holomorphic semigroup on $S(0,\pi)$ by Theorem 1.1.24(c).

The above two arguments only yield the holomorphy of the heat semigroup on $L^p(\mathbb{R};\mathbb{C})$ for the case $p = 2$. We now show the holomorphy for all $1 \le p < \infty$. Observe that $T(t)f$ is given by the convolution of f with the kernel

$$k_t(x) := \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}.$$

Let $g \in L^\infty(\mathbb{R};\mathbb{C})$. Then the mapping

$$z \mapsto \langle g, k_z \rangle = \int_{\mathbb{R}} k_z(x) g(x) \, dx$$

is holomorphic in the right half-plane because the integral converges absolutely. Hence, $z \mapsto k_z$ from the right half-plane into $L^1(\mathbb{R};\mathbb{C})$ is weakly holomorphic which is equivalent to the holomorphy of the map by Theorem C.2.12. Further, for $0 < \delta < \pi$ one sees using the well-known identities for Gaussian integrals that

$$\sup_{z \in S(0,\delta)} \|k_z\|_{L^1} = \sup_{z \in S(0,\delta)} \sqrt{\frac{|z|}{\operatorname{Re} z}} = \frac{1}{\cos(\frac{\delta}{2})} < \infty.$$

Hence, $G(z)f := k_z * f$ defines a holomorphic mapping for fixed $f \in L^p(\mathbb{R};\mathbb{C})$ because

$$\|G(z)f\|_p = \|k_z * f\|_p \le \|k_z\|_1 \, \|f\|_p \,.$$

Since the point evaluations separate points in $\mathcal{L}(X)$ for an arbitrary Banach space X, $z \mapsto G(z)$ is holomorphic. Here we once more used the equivalence of holomorphy and weak holomorphy. Further, the above estimate shows that G is a bounded holomorphic extension of the heat semigroup. Observe that the validity of the semigroup law for G follows directly from the identity theorem and the fact that the semigroup law holds for real arguments. The strong continuity of the holomorphic extension can be verified as in the real case. We conclude that the heat semigroup on L^p can be extended to a bounded holomorphic semigroup on $S(0,\pi)$ for all $1 \le p < \infty$.

If we know that A generates a strongly continuous semigroup, inequaltiy (1.1.15) can be replaced by an estimate for the resolvent on the imaginary axis. This will be useful in the proof of Kato's characterization of holomorphic semigroups and is therefore content of the next lemma.

Lemma 1.1.28. *Let A be the infinitesimal generator of a strongly continuous semigroup $(T(t))$ with $\|T(t)\| \le Me^{\omega t}$. Suppose that there exist positive constants s_0, C such that $is \in \rho(A)$ for $|s| > s_0 > 0$ and*

$$\|R(is, A)\| \le \frac{C}{|s|}. \tag{1.1.17}$$

Then A generates a holomorphic semigroup on a sector $S(0,\delta)$. Moreover, δ and the upper bounds for smaller subsectors only depend on C, s_0, ω and M.

Proof. By the Hille-Yosida theorem (Theorem 1.1.8), every complex number with real part bigger than ω lies in the resolvent set of A and fulfills

$$\|R(\lambda, A)\| \le \frac{M}{\operatorname{Re}\lambda - \omega}. \tag{1.1.18}$$

Choose $\beta > \omega$ and let $r > 0, s \neq 0$. By the resolvent equation (Theorem A.1.5),

$$\begin{aligned} R(r+\beta+is, A) = (i(s+\operatorname{sgn}(s)s_0) - (r+\beta+is))R(r+\beta+is, A)& \\ R(i(s+\operatorname{sgn}(s)s_0), A) + R(i(s+\operatorname{sgn}(s)s_0), A).& \end{aligned}$$

Taking the operator norm on both sides and (1.1.17) yield

$$\begin{aligned} \|R(r+\beta+is, A)\| &\le C\,|r+\beta - i\operatorname{sgn}(s)s_0|\,\|R(r+\beta+is, A)\|\,|s+\operatorname{sgn}(s)s_0|^{-1} \\ &\qquad + C\,|s+\operatorname{sgn}(s)s_0|^{-1} \\ &\le C\,|r+\beta - i\operatorname{sgn}(s)s_0|\,\|R(r+\beta+is, A)\|\,|s|^{-1} + C\,|s|^{-1}. \end{aligned}$$

Since $r \mapsto \frac{|r+\beta \pm is_0|}{r+\beta-\omega}$ is bounded on the positive axis by a positive constant D, we can use (1.1.18) to obtain

$$\begin{aligned} \|R(r+\beta+is, A)\| &\le CD(r+\beta-\omega)\,\|R(r+\beta+is, A)\|\,|s|^{-1} + C\,|s|^{-1} \\ &\le CDM\,|s|^{-1} + C\,|s|^{-1} \\ &= C(DM+1)\,|s|^{-1}. \end{aligned} \tag{1.1.19}$$

Observe that the constants in the last estimate only depend on s_0, C, ω and M. Therefore A generates a holomorphic semigroup with the stated properties by Theorem 1.1.24. $\square$

Remark 1.1.29. Conversely, if $(T(z))$ is holomorphic, the sectoriality of the infinitesimal generator A implies that there exists a positive constant s_0 such that (1.1.17) holds for $|s| > s_0$.

Remark 1.1.30. We note that if $(T(t))$ is bounded and if we can choose $s_0 = 0$ in Lemma 1.1.28, $(T(t))$ extends to a bounded holomorphic semigroup because with a careful second look at the proof we see that in this case we can choose $\beta = \omega = 0$.

1.2 Approximation of the Identity Operator and Holomorphic Semigroups

We are now interested in the regularity of a semigroup which is, roughly spoken, induced by the quality of approximation of the identity operator as the parameter goes to zero. A well-known result in this direction states that if $(T(t))$ is an arbitrary semigroup of bounded linear operators on some Banach space (we only require $(T(t))$ to obey the semigroup law!) such that

$$\limsup_{t\downarrow 0} \|T(t) - I\| < 1,$$

then $(T(t))$ is even a uniformly continuous semigroup (see [LR04, p. 71, Proposition 2.1]). We want to show that the following similiar result holds.

Theorem 1.2.1 (Kato (1969))**.** *Let $(T(t))$ be a strongly continuous semigroup such that*

$$\limsup_{t\downarrow 0} \|T(t) - I\| < 2.$$

Then $(T(t))$ can be extended to a holomorphic semigroup.

A weaker form of the above theorem was proven by J.W. Neuberger [Neu70, Theorem A] in 1969: Under the assumptions of Theorem 1.2.1, $(T(t))$ is an *immediately differentiable semigroup*, that is $T(t)X \subset D(A)$ and $AT(t)$ is a bounded operator for all $t > 0$. This implies that for all $x \in X$ the trajectories $t \mapsto T(t)x$ lie in $\cap_{n\in\mathbb{N}} D(A^n)$ and therefore are infinitely often differentiable for $t > 0$. By Kato's result, the trajectories are even holomorphic.

We now present Kato's proof of Theorem 1.2.1 given in [Kat70]. Let $(T(t))$ be a strongly continuous semigroup with

$$\|T(t)\| \leq Me^{\omega t},$$

where $M \geq 0$ and $\omega \in \mathbb{R}$ are constants. Moreover, the spectral radius of $T(t)$ is given by Beurling's formula:

$$\rho(T(t)) = \inf_{n\in\mathbb{N}} \|T(t)^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|T(nt)\|^{1/n} \leq \inf_{n\in\mathbb{N}} M^{1/n} e^{\omega t} = e^{\omega t}.$$

Hence,

$$\limsup_{t\downarrow 0} \rho(T(t)) \leq 1.$$

In other words, each complex number ζ with $\zeta > 1$ belongs to $\rho(T(t))$ for sufficiently small t. Now it is natural to consider the situation in which some ζ in the unit sphere belongs to $\rho(T(t))$ for sufficiently small t. The next lemma shows that this happens in a certain sense for some $\zeta \neq 1$ if and only if $(T(t))$ can be extended to a holomorphic semigroup.

Lemma 1.2.2. *Let $(T(t))$ be a strongly continuous semigroup. The following three conditions are equivalent.*

(a) $(T(t))$ can be extended to a holomorphic semigroup.

(b) For each complex number ζ with $|\zeta| \geq 1$, $\zeta \neq 1$, there exist positive constants δ and K such that

$$\zeta \in \rho(T(t)), \quad \|R(\zeta, T(t))\| \leq K \qquad \text{for } 0 < t < \delta.$$

(c) There exists a complex number ζ with $|\zeta| = 1$ and a positive number δ such that

$$\|(\zeta - T(t))x\| \geq \|x\| / K \qquad \text{for } x \in X \text{ and } 0 < t < \delta.$$

Moreover, if $(T(t))$ is bounded, then so is its holomorphic extension.

Proof. We first show that (a) implies (b). Note that there exist two constants $M \geq 0$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leq M e^{\omega t}. \tag{1.2.1}$$

Suppose $(T(t))$ can be extended to a holomorphic semigroup which we again denote $(T(t))$. Then by Theorem 1.1.24, its infinitesimal generator A satisfies $A \in \operatorname{Sect}(\omega, \vartheta)$ for some $\vartheta < \pi$. Choose $\pi < \alpha < 2\pi - \vartheta$. Then there exists a positive constant C such that

$$\|R(z, A)\| \leq \frac{C}{|z - \omega|} \qquad \text{for } z \in S(\omega, 0, \alpha). \tag{1.2.2}$$

Moreover, $T(t)$ can be written as the Cauchy integral

$$T(t) = \frac{1}{2\pi i} \int_\Gamma e^{tz} R(z, A)\, dz = \frac{1}{2\pi i} \int_\Gamma e^{tz} (z - A)^{-1}\, dz \qquad \text{for } t > 0, \tag{1.2.3}$$

where we choose Γ to be the positively oriented boundary of $S(\pi, \tilde{\vartheta}) \cup B_1(\omega)$ for some $\tilde{\vartheta} \in (\vartheta, \pi)$.

By a change of variable to $w = tz$, (1.2.3) can be rewritten as

$$T(t) = \frac{1}{2\pi i} \int_{\Gamma_t} e^w (w - tA)^{-1}\, dw \qquad \text{for } t > 0,$$

where $\Gamma_t = \{tz : z \in \Gamma\}$. Let $|\zeta| \geq 1$, $\zeta \neq 1$. As $e^w = \zeta$ would imply $w = \ln|\zeta| + 2\pi i m \arg(\zeta)$ for some $m \in \mathbb{Z}$, we can ensure by either choosing t smaller than $\ln|\zeta|$ for $|\zeta| > 1$ or t smaller than $2\pi\, |\arg(\zeta)|$ (if we choose $\arg(\zeta) \in (-\pi, \pi]$) for $|\zeta| = 1$ that $e^z \neq \zeta$

for all z lying on Γ_t or to the left of Γ_t. Fix such a sufficiently small t_0. Remember that, as a consequence of the Cauchy integral theorem (Corollary 1.1.17), we can modify the path of integration from Γ_t to Γ_{t_0} without changing the value of the integral by Theorem 1.1.18. In order to simplify notations, we will from now on write Γ instead of Γ_{t_0}. So

$$T(t) = \frac{1}{2\pi i}\int_\Gamma e^z(z-tA)^{-1}\,dz \qquad \text{for } t>0. \tag{1.2.4}$$

As $e^z \neq \zeta$ for all $z \in \Gamma$, we define

$$B(t) := \frac{1}{2\pi i}\int_\Gamma \frac{e^z}{e^z-\zeta}(z-tA)^{-1}\,dz \qquad \text{for } 0<t<t_0.$$

Since $z \mapsto e^z - \zeta$ is bounded from below on Γ by a constant B_L, the integral converges absolutely and defines a bounded linear operator. If necessary, choose a smaller t_0 such that $(t,z) \mapsto \frac{|z|}{|z-t\omega|}$ is bounded from above by a positive constant B_U for all $(t,z) \in [0,t_0]\times\Gamma$. Hence, by (1.2.2)

$$\begin{aligned}\|B(t)\| &\le \frac{1}{2\pi}\int_\Gamma \left|\frac{e^z}{e^z-\zeta}t^{-1}(z/t-A)^{-1}\,dz\right| \le \frac{C}{2\pi}\int_\Gamma \left|t^{-1}(z/t-\omega)^{-1}\frac{e^z}{e^z-\zeta}\,dz\right| \\ &= \frac{C}{2\pi}\int_\Gamma \left|(z-t\omega)^{-1}e^z(e^z-\zeta)^{-1}\,dz\right| \le \frac{CB_UB_L^{-1}}{2\pi}\int_\Gamma \left|z^{-1}e^z\,dz\right| =: L\end{aligned}$$

is uniformly bounded for $0<t<t_0$ by a constant L.

We now see that

$$e^z(e^z-\zeta)^{-1}e^z = e^z(e^z-\zeta+\zeta)(e^z-\zeta)^{-1} = e^z + \zeta e^z(e^z-\zeta)^{-1}.$$

So the functional calculus for sectorial operators (Theorem 1.1.18) yields

$$T(t)B(t) = B(t)T(t) = T(t) + \zeta B(t).$$

Hence,

$$(\mathrm{Id}-B(t))(\zeta-T(t)) = (\zeta-T(t))(\mathrm{Id}-B(t)) = \zeta - \zeta B(t) - T(t) + T(t)B(t) = \zeta.$$

Therefore $\zeta \in \rho(T(t))$ for $0<t<t_0$ and

$$R(\zeta, T(t)) = (\zeta - T(t))^{-1} = \zeta^{-1}(\mathrm{Id}-B(t)),$$

which shows that

$$\|R(\zeta,T(t))\| \le |\zeta|^{-1}(1+L) \qquad \text{for } 0<t<t_0.$$

Now suppose (b) holds. Fix $|\zeta|=1, \zeta\neq 1$. Then there are $\delta, K>0$ such that $\zeta\in\rho(T(t))$ and

$$\left\|(\zeta-T(t))^{-1}\right\| = \|R(\zeta,T(t))\| \le K \qquad \text{for } 0<t<\delta.$$

Therefore

$$\|x\| = \left\|(\zeta - T(t))^{-1}(\zeta - T(t))x\right\| \leq K \left\|(\zeta - T(t))x\right\| \qquad \text{for } 0 < t < \delta.$$

Hence,

$$\|(\zeta - T(t))x\| \geq \frac{\|x\|}{K} \qquad \text{for } 0 < t < \delta.$$

Finally, we show that (c) implies (a). Since for any real number α the closed operator $A - i\alpha$ generates the semigroup $(e^{-it\alpha}T(t))$, we see that for $x \in D(A)$

$$e^{-it\alpha}T(t)x - x = \int_0^t \frac{d}{ds}\left(e^{-is\alpha}T(s)x\right)\,ds = \int_0^t e^{-is\alpha}T(s)(A - i\alpha)x\,ds.$$

Therefore

$$\begin{aligned}\left\|(T(t) - e^{it\alpha})x\right\| &\leq \int_0^t \|T(s)\|\,\|(A - i\alpha)x\|\,ds \leq M\,\|(A - i\alpha)x\| \int_0^t e^{\omega s}\,ds \\ &= M\frac{e^{\omega t} - 1}{\omega}\,\|(A - i\alpha)x\|\,.\end{aligned}$$

Observe that for $\omega t < 1$

$$\frac{e^{\omega t} - 1}{\omega} = \sum_{k=1}^{\infty} \frac{\omega^{k-1}t^k}{k!} \leq t\sum_{k=0}^{\infty}(\omega t)^k = \frac{t}{1 - \omega t}.$$

This yields

$$\left\|(T(t) - e^{it\alpha})x\right\| \leq \frac{Mt}{1 - \omega t}\,\|(A - i\alpha)x\| \qquad \text{for } \omega t < 1. \tag{1.2.5}$$

Let ζ be as in (c). Choose two positive numbers ϑ_1, ϑ_2 such that $e^{i\vartheta_1} = e^{-i\vartheta_2} = \zeta$.

Let $\alpha > \max\{\vartheta_1/\delta, \omega\vartheta_1\}$ and set $t = \vartheta_1/\alpha$. Then $0 < t < \delta$ and $\omega t < 1$. Since (c) is satisfied and $e^{it\alpha} = e^{i\vartheta_1} = \zeta$, we have

$$\left\|(T(t) - e^{it\alpha})x\right\| \geq \frac{\|x\|}{K}.$$

It follows from above and (1.2.5) that

$$\begin{aligned}\|(A - i\alpha)x\| &\geq \frac{1 - \omega t}{MKt}\,\|x\| \\ &= \frac{\alpha - \omega\vartheta_1}{Mk\vartheta_1}\,\|x\| \qquad \forall x \in D(A).\end{aligned} \tag{1.2.6}$$

We define for $\varepsilon > 0$

$$\begin{aligned}F_\varepsilon &:= \{w \in \mathbb{C} : \|(A - w)x\| > \varepsilon\,\|x\| \;\; \forall x \in D(A)\},\\ G_\varepsilon &:= \rho(A) \cap F_\varepsilon.\end{aligned}$$

Observe that G_ε is open in F_ε. We will show that G_ε is also closed in F_ε. Let (z_n) be a sequence in G_ε with $z_n \to z \in F_\varepsilon$. We want to show that $z \in G_\varepsilon$. Observe that for $w \in F_\varepsilon$ the closed linear operator $A - w$ is injective and will be invertible as soon as it is surjective. So we can rewrite

$$G_\varepsilon = \{w \in F_\varepsilon : A - w \text{ is surjective}\}.$$

Thus it remains to show that $A - z$ is surjective. Let $y \in X$. Since $z_n \in G_\varepsilon$, there are unique $x_n \in D(A)$ such that

$$(A - z_n)x_n = y. \tag{1.2.7}$$

Since $z_n \in F_\varepsilon$, the sequence (x_n) is bounded by $\varepsilon^{-1}\|y\|$. Therefore we have

$$\begin{aligned} \|(A-z)x_n - y\| &= \|(A-z_n)x_n - y + (z_n - z)x_n\| \\ &= |z - z_n|\,\|x_n\| \le \varepsilon^{-1}|z-z_n|\,\|y\|. \end{aligned} \tag{1.2.8}$$

Additionally, we get

$$\begin{aligned} \|x_n - x_m\| &\le \varepsilon^{-1}\|(A-z)(x_n - x_m)\| \\ &\le \varepsilon^{-1}\|(A-z)x_n - y\| + \varepsilon^{-1}\|(A-z)x_m - y\|. \end{aligned} \tag{1.2.9}$$

Combining (1.2.8) and (1.2.9), we see that

$$\|x_n - x_m\| \le \varepsilon^{-2}\|y\|\left(|z-z_n| + |z-z_m|\right),$$

which shows that (x_n) is a Cauchy sequence. Let x be its limit point. Taking limits on both sides of (1.2.7), we see that $Ax_n \to y + zx$ as n tends to infinity. Since A is closed, we conclude that $x \in D(A)$ and $(A-z)x = y$. So $z \in G_\varepsilon$ which is therefore closed.

We have seen that G_ε is a closed-open set. Now fix $\varepsilon > 0$. Since A is the infinitesimal generator of a C_0-semigroup, $i\alpha + \xi \in \rho(A)$ for $\xi > \omega$ by the Hille-Yosida theorem (Theorem 1.1.8). Choose $\xi_0 > \omega$ and α_1 such that

$$\frac{\alpha_1 - \omega\vartheta_1}{MK\vartheta_1} - \xi_0 > \varepsilon.$$

Thus by (1.2.6), $i\alpha + \xi_0 \in G_\varepsilon$ for $\alpha > \alpha_1$. Moreover, for $\alpha > \alpha_1$ the straight line joining $i\alpha$ and $i\alpha + \xi_0$ lies completely in F_ε, so $i\alpha$ and $i\alpha + \xi_0$ can be joined by a path in F_ε. Consequently, both lie in the same connected component. Since G_ε is closed-open, it is the union of connected components including the one of $i\alpha+\xi_0$. But $i\alpha$ is also a member of this component. This proves that $i\alpha \in G_\varepsilon$, or equivalently $i\alpha \in \rho(A)$ for $\alpha > \alpha_1$. Now (1.2.6) implies that $\|R(i\alpha, A)\| \le \frac{MK\vartheta_1}{\alpha - \omega\vartheta_1}$.

We can repeat the same argument if we replace ϑ_1 by ϑ_2. Indeed, we get $e^{-it\alpha} = e^{-i\vartheta_2} = \zeta$. Thus we know that there exists $\alpha_2 > 0$ such that $-i\alpha \in \rho(A)$ for $\alpha > \alpha_2$ and that $\|R(-i\alpha, A)\| \le \frac{MK\vartheta_2}{\alpha - \omega\vartheta_2}$ holds. We have therefore shown that there exist constants β, C and $\alpha_0 > \beta$ such that $|\alpha| > \alpha_0$ implies

$$\|R(i\alpha, A)\| \le \frac{C}{|\alpha| - \beta}.$$

Since $\frac{|\alpha|}{|\alpha|-\beta}$ is bounded for $|\alpha| > \alpha_0$, for a bigger constant C' we have

$$\|R(i\alpha, A)\| \leq \frac{C'}{|\alpha|} \qquad \text{for } |\alpha| > \alpha_0.$$

Taking a second look at the constants involved in the above estimate, we see that (for fixed ε!) these constants only depend on M, ω, K and ζ (because α_i depends on M, K, ω and ϑ_i which in turn only depends on ζ). Hence by Lemma 1.1.28, $(T(t))$ can be extended to a holomorphic semigroup with the stated properties. Taking a further (third) look at the calculations above, we see that if $(T(t))$ is bounded, we can even choose $\alpha_0 = 0$. Thus in this case the holomorphic extension is bounded as well by Remark 1.1.30. □

Example 1.2.3. Let $(T(t))$ be the left shift semigroup defined in Example 1.1.4. We have

$$r(T(t)) = \inf_{n\in\mathbb{N}} \|T(nt)\|^{1/n} = \inf_{n\in\mathbb{N}} 1 = 1,$$

so every complex number outside the closed unit ball lies in the resolvent set of $T(t)$ for all $t \geq 0$. Now let $|\lambda| \leq 1$ and $t > 0$. We show that λ lies in the spectrum of A. For this we show that

$$f \mapsto (\lambda - T(t))f = \lambda f - f(t + \cdot)$$

is not injective. Choose a non-zero continuous function $f : [0, t] \to \mathbb{C}$ such that $\lambda f(0) = f(t)$ and now define recursively $f(s) := \lambda f(s - t)$ (it should be clear what this sloppy definition means). Then $f : [0, \infty) \to \mathbb{C}$ is well-defined. Since $\lambda \leq 1$, f is uniformly continuous and bounded. Hence, $(\lambda - T(t))f = 0$ and $\lambda \in \sigma(A)$. We infer from Lemma 1.2.2 that $(T(t))$ cannot be extended to a holomorphic semigroup. In Remark 1.2.5 we give an easier argument for this assertion.

Remark 1.2.4. Remember that we have motivated the above proof with the observation that every complex number outside the closed unit ball lies in the resolvent set of $T(t)$ for sufficiently small t. One is therefore tempted to change the assumption in Lemma 1.2.2(c) in the following way: suppose there exists $|\zeta| < 1$ and $\delta, K > 0$ such that

$$\|(\zeta - T(t))x\| \geq \|x\| / K \qquad \text{for } x \in X \text{ and } 0 < t < \delta.$$

Does this imply the holomorphy of $(T(t))$? The answer is: No! Let $m : \mathbb{R} \to \mathbb{R}$ be a real-valued surjective measurable function. Then

$$T(t)f := e^{imt} f$$

defines a strongly continuous (not uniformly continuous) multiplication semigroup on $L^2(\mathbb{R}; \mathbb{C})$ with infinitesimal generator

$$\begin{aligned} D(A) &:= \{f \in L^2(\mathbb{R}; \mathbb{C}) : imf \in L^2(\mathbb{R}; \mathbb{C})\} \\ Af &:= imf. \end{aligned}$$

For more details on multiplication semigroups see [EN00, II.2.9]. Further, notice that $(T(t))$ is even a unitary group because $T(t)^* f = e^{-imt} f$. The spectrum of A is the essential range of im, that is the whole imaginary axis. Therefore A is not sectorial. We infer from Theorem 1.1.24 that $(T(t))$ cannot be extended to a holomorphic semigroup. However, for $|\zeta| < 1$ we see that the above weaker assumption is fulfilled for all $t > 0$ as

$$\|(\zeta - T(t))x\| \geq \|T(t)x\| - |\zeta| \, \|x\| = (1 - |\zeta|) \, \|x\| \, .$$

From what has already been shown, it is now easy to prove Kato's theorem.

Proof of Theorem 1.2.1. There exist $\delta > 0$ and $0 < \varepsilon \leq 2$ such that

$$\|T(t) - \mathrm{Id}\| \leq 2 - \varepsilon \qquad \text{for } 0 < t < \delta.$$

We choose $\zeta = -1$. Since

$$\|(-\mathrm{Id} - T(t))x\| = \|-2x + (\mathrm{Id} - T(t))x\| \geq 2 \, \|x\| - \|(\mathrm{Id} - T(t))x\| \geq \varepsilon \, \|x\| \, ,$$

we deduce from Lemma 1.2.2(c) that $(T(t))$ extends to a holomorphic semigroup. □

Remark 1.2.5. Notice that the left shift semigroup $(T(t))$ considered in Example 1.1.9 shows that the constant 2 is optimal. Since the resolvent set of the infinitesimal generator A of $(T(t))$ is exactly the open right half-plane, Theorem 1.1.23 shows that $(T(t))$ cannot be extended to a holomorphic semigroup. Since $(T(t))$ is a semigroup of contractions, we obviously have

$$\limsup_{t \downarrow 0} \|T(t) - \mathrm{Id}\| \leq 2.$$

Kato's theorem (Theorem 1.2.1) now immediately implies

$$\limsup_{t \downarrow 0} \|T(t) - \mathrm{Id}\| = 2. \tag{1.2.10}$$

This can also be shown directly. Let $f_n(t) := \cos(\pi n t)$. Then $f_n \in UC_b[0, \infty)$ and $\|f_n\|_\infty = 1$ for all $n \in \mathbb{N}$. Further,

$$\begin{aligned} 2 &= \left| f_n\left(\frac{1}{n}\right) - f_n(0) \right| = \left| \left(T\left(\frac{1}{n}\right) f_n \right)(0) - f_n(0) \right| = \left\| T\left(\frac{1}{n}\right) f_n - f_n \right\|_\infty \\ &= \left\| T\left(\frac{1}{n}\right) - \mathrm{Id} \right\|, \end{aligned}$$

which again implies (1.2.10).

We now present a generalization of Kato's theorem that was proven by A. Beurling in the same year [Beu70]. The idea of the following proof is taken from [Pis80a].

Theorem 1.2.6 (Kato-Beurling (1969))**.** *Let $(T(t))$ be a strongly continuous semigroup on some Banach space X. Suppose that there exists a natural number N such that*

$$\limsup_{t \downarrow 0} \left\| (T(t) - \mathrm{Id})^N \right\|^{1/N} < 2.$$

Then $(T(t))$ extends to a holomorphic semigroup. Moreover, if we have $\|T(t)\| \le Me^{\omega t}$ and

$$\limsup_{t \downarrow 0} \left\|(T(t) - \mathrm{Id})^N\right\|^{1/N} = \rho < 2,$$

then $(T(t))$ extends to a holomorphic semigroup on a sector $S(0, \varphi)$, where φ and the bounds on smaller subsectors only depend on ρ, N and on the growth bound constants M and ω. Further, if $(T(t))$ is bounded, then so is its holomorphic extension.

Proof. There exist constants $0 < \rho < 2$ and $\delta > 0$ such that

$$\left\|(\mathrm{Id} - T(t))^N\right\| \le \rho^N \qquad \text{for } 0 < t < \delta.$$

Let $V(t) \coloneqq \frac{1}{2}(\mathrm{Id} - T(t))$. Then

$$\left\|V(t)^N\right\| \le \left(\frac{\rho}{2}\right)^N < 1 \qquad \text{for } 0 < t < \delta.$$

Thus for $0 < t < \delta$ the Neumann series shows that $\mathrm{Id} - V(t)^N$ is invertible and that its inverse can be estimated by

$$\left(\mathrm{Id} - V(t)^N\right)^{-1} \le \frac{2}{2-\rho}.$$

Moreover, we see that for $0 < t < \delta$

$$\left(\mathrm{Id} - V(t)^N\right)^{-1} \left(\sum_{k=0}^{N-1} V(t)^k\right) (\mathrm{Id} - V(t)) = \left(\mathrm{Id} - V(t)^N\right)^{-1} (\mathrm{Id} - V(t)^N) = \mathrm{Id}.$$

Since the above operators commute, this shows that $\mathrm{Id} - V(t)$ is invertible for $0 < t < \delta$. Hence for $0 < t < \min\{\delta, 1\}$,

$$\begin{aligned}
\left\|(\mathrm{Id} + T(t))^{-1}\right\| &= \left\|(2\,\mathrm{Id} - \mathrm{Id} + T(t))^{-1}\right\| = \frac{1}{2} \left\|\left(\mathrm{Id} - \frac{\mathrm{Id} - T(t)}{2}\right)^{-1}\right\| = \frac{1}{2}\left\|(\mathrm{Id} - V(t))^{-1}\right\| \\
&\le \frac{1}{2}\left\|\left(\mathrm{Id} - V(t)^N\right)^{-1}\right\| \sum_{k=0}^{N-1} \|V(t)\|^k \le \frac{1}{2-\rho} \cdot \sum_{k=0}^{N-1} \frac{1}{2^k}(1 + \|T(t)\|)^k \\
&\le \frac{1}{2-\rho} \cdot \sum_{k=0}^{N-1} \frac{1}{2^k}\left(1 + Me^{\omega t}\right)^k \le \frac{1}{2-\rho} \cdot \sum_{k=0}^{N-1} \frac{1}{2^k}\left(1 + M\max\{1, e^{\omega}\}\right)^k.
\end{aligned}$$

So we have shown that $\mathrm{Id} + T(t)$ is invertible for $0 < t < \min\{\delta, 1\}$ and that for $x \in X$ we have $\|(\mathrm{Id} + T(t))x\| \ge \frac{\|x\|}{K}$ for some non-negative constant K that only depends on N, ρ, ω and M. Now Lemma 1.2.2(c) applied to $\zeta = -1$ yields the holomorphy of the semigroup and the other parts of the statement. $\square$

Remark 1.2.7. Again, Example 1.1.9 shows that the constant 2 is optimal. Since the left shift semigroup $(T(t))$ is contractive, we have

$$\limsup_{t\downarrow 0} \left\|(T(t)-\mathrm{Id})^N\right\|^{1/N} \leq 2.$$

Since $(T(t))$ does not extend to a holomorphic semigroup, the Kato-Beurling theorem (Theorem 1.2.6) implies

$$\limsup_{t\downarrow 0} \left\|(T(t)-\mathrm{Id})^N\right\|^{1/N} = 2.$$

As above, this can be shown directly. Expanding the inner term shows

$$(\mathrm{Id} - T(t))^N = \sum_{k=0}^{N} \binom{N}{k} (-T(t))^k = \sum_{k=0}^{N} \binom{N}{k} (-1)^k T(kt).$$

We again let $f_n(t) := \cos(\pi n t)$. Then

$$\begin{aligned}\left(\left(\mathrm{Id} - T\left(\frac{1}{n}\right)\right)^N f_n\right)(0) &= \sum_{k=0}^{N} \binom{N}{k} (-1)^k f_n\left(\frac{k}{n}\right) = \sum_{k=0}^{N} \binom{N}{k} (-1)^k \cos(\pi k) \\ &= \sum_{k=0}^{N} \binom{N}{k} (-1)^k (-1)^k = \sum_{k=0}^{N} \binom{N}{k} = 2^N.\end{aligned}$$

Hence,

$$\left\|\left(T\left(\frac{1}{n}\right) - \mathrm{Id}\right)^N\right\| = \left\|\left(\mathrm{Id} - T\left(\frac{1}{n}\right)\right)^N f_n\right\|_\infty = 2^N.$$

This once more shows

$$\limsup_{t\downarrow 0} \left\|(T\,(t)-\mathrm{Id})^N\right\|^{1/N} = 2.$$

The Kato-Beurling theorem 1.2.6 can easily be generalized to a certain class of polynomials.

Theorem 1.2.8. *Let P be a non-constant polynomial having at least one zero in $\mathbb{S} := \{z \in \mathbb{C} : |z| = 1\}$. Suppose further that $(T(t))$ is a strongly continuous semigroup such that*

$$\limsup_{t\downarrow 0} \|P(T(t)) + \mathrm{Id}\| < 1.$$

Then $(T(t))$ extends to a holomorphic semigroup.

Proof. Let ζ be one zero of P in $\mathbb{S}$ and $Q(x) := P(2x+1)+1$. Observe that for $x_0 := \frac{1}{2}(\zeta - 1)$ we have $Q(x_0) = 1$. By assumption, there exist $\rho, \delta > 0$ such that

$$\left\|Q\left(\frac{T(t)-\mathrm{Id}}{2}\right)\right\| = \|P(T(t)) + \mathrm{Id}\| = \rho < 1 \qquad \text{for } 0 < t < \delta.$$

Hence, $\operatorname{Id} - Q\left(\frac{T(t)-\operatorname{Id}}{2}\right)$ is invertible for $0 < t < \delta$. Since x_0 is a zero of $1 - Q$, we have $1 - Q(x) = (x_0 - x)R(x)$ for some polynomial R. This implies that for $V(t) \coloneqq \frac{1}{2}(T(t) - \operatorname{Id})$

$$(x_0 - V(t))\, R(V(t))(\operatorname{Id} - Q(V(t)))^{-1} = (\operatorname{Id} - Q(V(t)))(\operatorname{Id} - Q(V(t)))^{-1} = \operatorname{Id}.$$

Hence, $x_0 - V(t)$ is invertible for $0 < t < \delta$. Further,

$$x_0 - V(t) = x_0 - \frac{T(t) - \operatorname{Id}}{2} = \frac{1}{2}((2x_0 + 1) - T(t)) = \frac{1}{2}(\zeta - T(t)).$$

Moreover, similiar as in the proof of the Kato-Beurling theorem 1.2.6, one sees that

$$(\zeta - T(t))^{-1} = \frac{1}{2}(x_0 - V(t))^{-1} \leq K \qquad \text{for } 0 < t < \delta$$

for some constant K. Hence, Theorem 1.2.2(c) implies the holomorphy of $(T(t))$. □

Remark 1.2.9. One could formulate a qualitative version of the above theorem as in the Kato-Beurling theorem 1.2.6. Then one obtains the Kato-Beurling theorem as the special case $P(x) \coloneqq \left(\frac{1-x}{2}\right)^n - 1$ with $P(-1) = 0$.

1.3 Approximation Properties as Necessary Conditions for Holomorphy

In light of Kato's theorem and its generalizations due to Beurling, it is now natural to ask whether, given a strongly continuous semigroup $(T(t))$, an approximation property of the form

$$\limsup_{t \downarrow 0} \|T(t) - \operatorname{Id}\| < 2$$

is even necessary for the holomorphy of the semigroup. A partial positive result to this question is known if we make additional assumptions on the underlying Banach space.

Definition 1.3.1 (Uniformly Convex Space)**.** A normed vector space E is called *uniformly convex* if for every $\varepsilon > 0$ there is some $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$

$$\left\| \frac{x+y}{2} \right\| > 1 - \delta \qquad \text{implies} \qquad \|x - y\| < \varepsilon.$$

Remark 1.3.2. One often only requires the above property for $\|x\| = \|y\| = 1$ in the definition of a uniformly convex space. However, one can show that this weaker definition implies our definition (see [LT96, II, p. 60]).

The following result for uniformly convex Banach spaces can be found in [Paz83, Corollary II.5.8].

Theorem 1.3.3. *Let $(T(z))$ be a holomorphic semigroup of contractions on a uniformly convex Banach space X. Then*

$$\limsup_{t \downarrow 0} \|T(t) - I\| < 2.$$

Proof. Since $(T(t))$ is a semigroup of contractions, we have

$$\|T(t) - \mathrm{Id}\| \leq \|T(t)\| + \|\mathrm{Id}\| = 2.$$

Assume that

$$\limsup_{t \downarrow 0} \|T(t) - I\| = 2. \tag{1.3.1}$$

Then we can choose sequences (t_n) and (x_n) such that $t_n \downarrow 0$, $\|x_n\| = 1$ and

$$\|(-\mathrm{Id} + T(t_n))x_n\| \to 2 \qquad \text{as } n \to \infty.$$

Since $(T(t))$ is a semigroup of contractions, we see that $\|T(t_n)x_n\| \leq 1$. Since X is uniformly convex, we conclude that

$$\|(-\mathrm{Id} - T(t_n))x_n\| \to 0 \qquad \text{as } n \to \infty.$$

By Lemma 1.2.2(b) for $\zeta = -1$, there exists a positive constant K such that for all but finitely many n we have $-1 \in \rho(T(t_n))$ and

$$1 = \left\|(-\mathrm{Id} - T(t_n))^{-1}(-\mathrm{Id} - T(t_n))x_n\right\| \leq K \left\|(-\mathrm{Id} - T(t_n))x_n\right\|.$$

Taking limits on both sides, we conclude that $1 \leq 0$. Contradiction! So (1.3.1) cannot be true and therefore we have

$$\limsup_{t \downarrow 0} \|T(t) - I\| < 2. \qquad \square$$

The next counterexample shows that we need some restriction on the Banach space in the above theorem.

Example 1.3.4 (Heat Semigroup on L^1)**.** We have seen in Example 1.1.27 that the heat semigroup is holomorphic for $1 \leq p < \infty$. We are now interested in the case $p = 1$. As $T(t)$ is contractive, we clearly have $\|T(t) - I\|_{\mathcal{L}(L^1(\mathbb{R};\mathbb{C}))} \leq 2$. Since further $T(t)L^1(\mathbb{R}) \subset L^1(\mathbb{R})$, we know that

$$\|T(t) - I\|_{\mathcal{L}(L^1(\mathbb{R};\mathbb{C}))} \geq \|T(t) - I\|_{\mathcal{L}(L^1(\mathbb{R}))}.$$

We now want to estimate the right hand side. By abuse of notation, let $T(t)$ denote the restriction of $T(t)$ on $L^1(\mathbb{R})$ as well. In what follows we need some results from the theory of Banach lattices which are summarized in Appendix A.3. Since $T(t)$ is a positive kernel operator with continuous kernel, $|I - T(t)| = I + T(t)$ by Corollary A.3.11 and we obtain

$$\|T(t) - I\|_{\mathcal{L}(L^1(\mathbb{R}))} \overset{\text{Theorem A.3.9}}{=} \||T(t) - I|\|_{\mathcal{L}(L^1(\mathbb{R}))} = \|T(t) + I\|_{\mathcal{L}(L^1(\mathbb{R}))}.$$

Moreover, for $f \geq 0$ with $\|f\|_{L^1} = 1$ we get

$$\|T(t)f\|_{L^1} = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-(x-y)^2/4t} f(y)\, dy\, dx \overset{\text{Tonelli}}{=} \int_{\mathbb{R}} f(y) \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-(x-y)^2/4t}\, dx\, dy$$

$$= \int_{\mathbb{R}} f(y)\, dy = \|f\|_{L^1} .$$

Thus
$$\|T(t) - I\|_{\mathcal{L}(L^1(\mathbb{R};\mathbb{C}))} \geq \|(T(t) + I)f\|_{L^1} = \|T(t)f\|_{L^1} + \|f\|_{L^1} = 2.$$

We conclude that $\|T(t) - I\|_{\mathcal{L}(L^1(\mathbb{R};\mathbb{C}))} = 2$. Therefore the heat semigroup on $L^1(\mathbb{R};\mathbb{C})$ is holomorphic with $\limsup_{t\downarrow 0} \|T(t) - I\|_{\mathcal{L}(L^1(\mathbb{R};\mathbb{C}))} = 2$.

The Kato-Beurling theorem 1.2.6 motivates the following question. For a strongly continuous semigroup $(T(t))$, is the condition

$$\limsup_{t\downarrow 0} r(T(t) - I) < 2 \tag{1.3.2}$$

necessary or even sufficient for $(T(t))$ to be holomorphic? Remember that we have shown as a motivation for Kato's lemma 1.2.2 that $\limsup_{t\downarrow 0} r(T(t)) = 1$ for an arbitrary exponentially bounded semigroup $(T(t))$, so the estimate

$$\limsup_{t\downarrow 0} r(T(t) - I) \leq \limsup_{t\downarrow 0} r(T(t)) + 1 = 2$$

is trivial. Notice that in comparison to Theorem 1.3.3 we have replaced the operator norm by the spectral radius which is dominated by the former.

Theorem 1.3.5. *Let $(T(z))$ be a holomorphic semigroup on an arbitrary Banach space. Then*
$$\limsup_{t\downarrow 0} r(T(t) - I) < 2.$$

Proof. Let $(T(z))$ be a holomorphic semigroup with infinitesimal generator A. We infer from the spectral mapping theorem for holomorphic semigroups (see [EN00, Corollary IV.3.12(iii)]) that

$$\sigma(T(t) - \mathrm{Id}) = \sigma(T(t)) - 1 = e^{t\sigma(A)} - 1.$$

We can assume without loss of generality that $(T(z))$ is bounded: if $(T(z))$ is unbounded, we can multiply $T(z)$ with a suitable factor of the form $e^{-\omega z}$ such that the rescaled semigroup $S(z) \coloneqq e^{-\omega z} T(z)$ is a bounded holomorphic semigroup. Then

$$r(S(t) - I) = \sup_{\lambda \in \sigma(A)} \left| e^{t(\lambda - \omega)} - 1 \right| .$$

Notice that $\left| e^{-\omega t} \right| \to 1$ as t tends to zero. Hence,

$$\limsup_{t\downarrow 0} r(S(t) - I) = \limsup_{t\downarrow 0} r(T(t) - I),$$

which shows that the rescaling does not affect the result. Now observe that $\left| e^{t\lambda} - 1 \right|$ equals 2 if and only if $e^{t\lambda} = -1$ because in the case of a bounded semigroup $\sigma(A)$ only contains complex numbers with non-positive real part and therefore $\left|e^{t\lambda}\right| = e^{t \operatorname{Re}\lambda} \le 1$ for all $\lambda \in \sigma(A)$. In order to show estimate (1.3.2) it is therefore sufficient and necessary to show that there exists $\varepsilon, \delta > 0$ such that

$$\operatorname{dist}(e^{t\sigma(A)}, -1) \ge \varepsilon \qquad \text{for all } 0 < t < \delta.$$

Assume that this is wrong. Then there exist $t_n > 0$ with $t_n \to 0$ and $\lambda_n \in \sigma(A)$ such that $e^{t_n \lambda_n} \to -1$ as $n \to \infty$. This implies

$$t_n \operatorname{Re}\lambda_n \xrightarrow[n\to\infty]{} 0 \quad \text{and} \quad \liminf_{n\to\infty} t_n \left|\operatorname{Im}\lambda_n\right| \ge \pi.$$

Since $(T(t))$ is holomorphic, $\sigma(A)$ is contained in a sector $S(\pi, \omega)$ for some $\omega < \pi$ by Theorem 1.1.23(c). Hence, $\frac{|\operatorname{Im}\lambda|}{|\operatorname{Re}\lambda|} \le \tan(\frac{\omega}{2})$ for all $\lambda \in \sigma(A)$. But

$$\frac{|\operatorname{Im}\lambda_n|}{|\operatorname{Re}\lambda_n|} = \frac{t_n\,|\operatorname{Im}\lambda_n|}{t_n\,|\operatorname{Re}\lambda_n|} \xrightarrow[n\to\infty]{} \infty.$$

This contradiction shows that indeed $\limsup_{t\downarrow 0} r(T(t) - I) < 2$ holds. □

Remark 1.3.6. The above proof also indicates how to show that (1.3.2) is not sufficient for a semigroup to be holomorphic. Notice that we have only used the spectral mapping theorem and the fact that the spectrum of A is contained in a sector, but not the sectoriality of A. Now consider the nilpotent semigroup

$$T(t)f(s) := \begin{cases} f(s+t) & \text{for } s+t \le 1, \\ 0 & \text{for } s+t > 1 \end{cases}$$

on $L^2(0,1)$. It is *eventually differentiable*, that is $t \mapsto T(t)f$ is differentiable for $t > 1$ for every $f \in L^2(0,1)$, but not immediately differentiable (see [EN00, II.4.31]) and therefore a fortiori not holomorphic. Further, the spectrum of its infinitesimal generator A is empty, i.e. $\sigma(A) = \emptyset$ (see [EN00, Corollary IV.2.5]). Moreover, for eventually differentiable semigroups the spectral mapping theorem applies [EN00, Corollary IV.3.12] and we obtain $\sigma(T(t)) = \{0\}$ for all $t > 0$. Hence, $(T(t))$ is not holomorphic but

$$\limsup_{t\downarrow 0} r(T(t) - I) = 1.$$

1.4 Holomorphic Semigroups in Interpolation Spaces

Two of our proofs of the holomorphy of the heat semigroup on L^2 in Example 1.1.26 relied heavily on Hilbert space techniques. So they cannot be generalized to other L^p-spaces for $p \ne 2$. This situation is typical. But once the holomorphy on L^2 is known, one could try to deduce the holomorphy on other L^p-spaces from the special case $p = 2$. More precisely, we are interested in interpolation results for holomorphic semigroups. One way to do this is given by Kato's theorem and its counterpart for uniformly convex spaces. We now want to study this application more systematically on general interpolation spaces. For a short overview of interpolation spaces see Appendix B.

Theorem 1.4.1. *Let F be an exact interpolation functor of type $0 \leq \theta \leq 1$ and (X, Y) an interpolation couple of Banach spaces. Suppose that $T(t) \in \mathcal{L}((X, Y))$ is a family of bounded operators such that $T(t)_{|X}$ and $T(t)_{|Y}$ obey the semigroup laws. Then $(F(T(t)))$ obeys the semigroup laws. Further, if both restrictions are contractive and one of the following conditions holds*

(a) $0 \leq \theta < 1$, X is uniformly convex and $(T(t)_{|X})$ is a holomorphic semigroup,

(b) $0 < \theta \leq 1$, Y is uniformly convex and $(T(t)_{|Y})$ is a holomorphic semigroup

and $(F(T(t)))$ is strongly continuous, $(F(T(t)))$ is a holomorphic semigroup of contractions as well.

Proof. The semigroup laws can be verified directly. For $x + y \in X + Y$ we obtain

$$\begin{aligned} F(T(t+s))(x+y) &= T(t)_{|X}(t+s)x + T(t)_{|Y}(t+s)y = T(t)_{|X}T(s)_{|X}x + T(t)_{|Y}T(s)_{|Y}y \\ &= F(T(t))F((T(s))(x+y). \end{aligned}$$

It remains to verify the holomorphy. We will only consider assumption (a) because the proof in the other case is completely similiar. Since $(T(t)_{|X})$ satisfies the assumptions of Theorem 1.3.3, we infer that for some $\delta > 0$ and $0 < \rho < 2$

$$\left\| T(t)_{|X} - \mathrm{Id}_X \right\|_{\mathcal{L}(X)} \leq \rho \qquad \text{for } 0 < t < \delta.$$

Since F is an exact interpolation fucntor of type $0 \leq \theta < 1$, we see that for $0 < t < \delta$

$$\begin{aligned} \|F(T(t)) - I\|_{\mathcal{L}(F((X,Y)))} &= \|F(T(t) - I)\|_{\mathcal{L}(F((X,Y)))} \\ &\leq \left\| T(t)_{|X} - \mathrm{Id}_X \right\|_{\mathcal{L}(X)}^{1-\theta} \left\| T(t)_{|Y} - \mathrm{Id}_Y \right\|_{\mathcal{L}(Y)}^{\theta} \\ &\leq \rho^{1-\theta} \cdot 2^{\theta} < 2. \end{aligned}$$

Now the Kato theorem (Theorem 1.2.1) yields directly the holomorphy of $(F(T(t)))$. □

We will now apply the above theorem to the two most important interpolation functors which are given by the real and complex interpolation method. Notice that these interpolation functors are exact of some type θ.

Theorem 1.4.2 (Stein Interpolation on Interpolation Spaces)**.** *Let $1 \leq q < \infty$, $0 < \theta < 1$ and X, Y be two complex Banach spaces. Suppose we have given two compatible semigroups $(T_X(t))$ and $(T_Y(t))$ on X respectively Y such that at least one of them is strongly continuous and such that for the other the following holds: for all $z \in X$ (or all $z \in Y$) the trajectories $t \mapsto T(t)z$ are bounded in some small neighbourhood of $t = 0$. Then by real or complex interpolation one obtains a strongly continuous semigroup $(S(t))$ on $(X, Y)_{\theta,q}$ or $(X, Y)_{[\theta]}$. If moreover both semigroups are contractive(!), one of the semigroups is holomorphic and its underlying Banach space is uniformly convex, then $(S(t))$ is holomorphic as well.*

Proof. Denote $\overline{A}$ the interpolation space. Then $X \cap Y$ is dense in $\overline{A}$ for both interpolation methods. We assume without loss of generality that $(T_X(t))$ is strongly continuous. Then for $z \in X \cap Y$ there exist constants $M, \delta > 0$ such that

$$\|T_Y(t)z\|_Y \leq M \qquad \text{for } 0 < t < \delta.$$

Now Lemma B.2.3 or Lemma B.2.6 yield for $0 < t < \delta$

$$\begin{aligned}\|S(t)z - z\|_{(X,Y)} &\leq c\,\|T_X(t)z - z\|_X^{1-\theta}\,\|T_Y(t)z - z\|_Y^{\theta} \\ &\leq c\,\|T_X(t)z - z\|_X^{1-\theta}\,(M + \|z\|_Y)^{\theta} \xrightarrow[t\to 0]{} 0\end{aligned}$$

for some constant c, where $\|\cdot\|_{(X,Y)}$ is the respective norm in the interpolation space. Further, interpolation shows that $(S(t))$ is a contractive semigroup and together with the density of $X \cap Y$ in $\overline{A}$ we infer that the above identity is valid for all $z \in \overline{A}$. This shows the strong continuity of $(S(t))$. Finally, the holomorphy follows directly from Theorem 1.4.1. □

As a special case of the above result we obtain a weak form of the *Stein interpolation theorem* for holomorphic semigroups.

Theorem 1.4.3 (Stein Interpolation for Holomorphic Semigroups). *Let $1 \leq p_1 < p_2 \leq \infty$ and suppose two compatible contractive semigroups $(T_{p_1}(t))$ and $(T_{p_2}(t))$ are given on L^{p_1} respectively L^{p_2} over some fixed measure space. Further, assume that they fulfill the assumptions of Theorem 1.4.2. Then for every $p_1 < p < p_2$ one obtains a contractive strongly continuous semigroup $(T_p(t))$ on L^p. Further, assume that either $(T_{p_1}(t))$ for $p_1 > 1$ or $(T_{p_2}(t))$ for $p_2 < \infty$ is holomorphic. Then $(T_p(t))$ is even holomorphic.*

Proof. We first notice that L^{p_i} is uniformly convex for $p_i \in (1, \infty)$. Choose $\theta \in (0,1)$ such that $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. The complex interpolation method yields $(L^{p_1}, L^{p_2})_{[\theta]} = L^p$ (Theorem B.2.8). So the assertion follows from Theorem 1.4.2. □

Example 1.4.4. We have seen in Example 1.1.5 that the heat semigroup $(T(t))$ is strongly continuous on L^p for all $1 \leq p < \infty$. Moreover, the holomorphy for $p = 2$ was shown in Example 1.1.27. Now the Stein interpolation theorem (Theorem 1.4.3) yields the holomorphy of the heat semigroup for $1 < p < \infty$. Notice that the holomorphy for $p = 1$ cannot be obtained with the same method because $\limsup_{t\downarrow 0} \|T(t) - I\| = 2$ by Example 1.3.4.

2 Applications of Semigroups: B-convexity and K-convexity

In this chapter we apply the theory of holomorphic semigroups to present a result in the geometry of Banach spaces being called the most beautiful result in this area by B. Maurey [Mau03, p. 7]: Pisier's proof of the equivalence of B-convexity and K-convexity. The notion of B-convexity first appeared in A. Beck's study of the strong law of large numbers for vector-valued random variables. He was the first to define B-convexity and showed that a certain form of the strong law of large numbers holds if and only if the underying Banach space is B-convex. K-convexity was introduced by G. Pisier and B. Maurey as a reaction on the failure of a general duality theory for type and cotype. K-convexity characterizes these spaces for which we get a complete duality theory.

In the first three sections we introduce the concepts mentioned above and thereafter we present Pisier's proof of the equivalence. Next we apply Pisier's result to present a powerful duality theorem for type and cotype and show König's astonishing characterization of B/K-convex spaces in terms of absolutely convergent Fourier series. The last section is devoted to the long open question of whether all B-convex spaces are reflexive. We present a beautiful construction of non-reflexive B-convex spaces using interpolation spaces that was given by G. Pisier and Q. Xu.

2.1 Type and Cotype

Given independent and identically distributed random variables $\varepsilon_1, \ldots, \varepsilon_n$ with $\mathbb{P}(\varepsilon_i = \pm -1) = \frac{1}{2}$ for every i, $p \in [1, \infty)$ and complex numbers $x_1, \ldots, x_n$, the *Khintchine inequality* (see also Corollary D.2.2) states that there exist positive constants A_p and B_p depending only on p such that

$$A_p \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \leq \left(\mathbb{E} \left| \sum_{i=1}^{n} \varepsilon_i x_i \right|^p \right)^{1/p} \leq B_p \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}.$$

However, the inequality does not generalize to arbitrary Banach spaces as we will see soon. The properties *type* and *cotype* of a Banach space were introduced by Hoffmann-Jørgensen in [HJ74]. They describe, roughly spoken, to which extent (possibly) weaker forms of the Khintchine inequality stay true in a Banach space. To get a precise definition and for later use we introduce some probabilistic terminology.

We set

$$D_n \coloneqq \{-1, 1\}^n, \qquad D \coloneqq \{-1, 1\}^{\mathbb{N}}.$$

We can easily give $\{-1,1\}$ the structure of a probability space by setting

$$\mathbb{P}(\{1\}) = \frac{1}{2}, \qquad \mathbb{P}(\{-1\}) = \frac{1}{2}.$$

It is well known that there exists a unique product probability measure μ_n on $(D_n, \mathcal{P}(D_n))$ such that

$$\mu_n\left(\prod_{i=1}^n A_i\right) = \prod_{i=1}^n \mathbb{P}(A_i) \qquad \text{for all } A_i \in \{\emptyset, \{1\}, \{-1\}, \{-1,1\}\}.$$

Similiarly, there exists a unique probability measure μ on the measurable space D whose σ-algebra $\mathcal{F}$ is generated by elements of the form $\prod_{i=1}^\infty A_i$ for which $A_i \neq \{-1,1\}$ for only finitely many i - these sets are often called *cylindrical sets* - (see Appendix D.1) such that for every such cylindrical set

$$\mu\left(\prod_{i=1}^\infty A_i\right) = \prod_{i=1}^\infty \mathbb{P}(A_i). \tag{2.1.1}$$

Observe that the product is finite and is therefore well defined.

Remark 2.1.1. We give a second construction for μ as described in [Pis80a] or [Pis82]. The multiplicative group $\{-1,1\}$ together with the discrete topology is a topological group. Then $D = \prod_{i=1}^\infty \{-1,1\}$ is a topological group as well. By Tychonoff's theorem (see [Mun00]), D is a compact topological group. Hence, there exists a unique normalized Haar measure $\tilde{\mu}$ on $(D, \mathcal{B}(D))$ (see [DE08, Thm. 1.3.4]), where $\mathcal{B}(D)$ is the Borel σ-algebra on D, that is the smallest σ-algebra containing all open sets of D.

It remains to show that the two measures μ and $\tilde{\mu}$ coincide. Observe first that the cylindrical sets are exactly the open sets in D. Hence, $\mathcal{B}(D) = \mathcal{F}$. Let $\prod_{i=1}^\infty A_i$ be an arbitrary cylindrical set. There exists a natural number n such that $A_i \in \{\{1\}, \{-1\}\}$ and therefore $\mathbb{P}(A_i) = 1/2$ holds for n different A_is and $A_i = \{-1,1\}$ and $\mathbb{P}(A_i) = 1$ otherwise. As $\tilde{\mu}$ is invariant under multiplication, we get

$$\tilde{\mu}\left(\prod_{i=1}^\infty A_i\right) = \frac{1}{2^n} = \prod_{i=1}^\infty \mathbb{P}(A_i) = \mu\left(\prod_{i=1}^\infty A_i\right).$$

So $\tilde{\mu}$ is a second measure fulfilling (2.1.1). Since the extension is unique, we conclude $\mu = \tilde{\mu}$.

From now on D and D_n will always implicitly be seen as the measure spaces described above. Define the *coordinate functions* or *Rademacher functions*

$$\begin{aligned} \varepsilon_n : D &\to \{-1,1\} \\ (\omega_i)_{i\in\mathbb{N}} &\mapsto \omega_n. \end{aligned}$$

In the same way, one defines the first n coordinates ε_i on D_n. Obviously, the coordinate functions are measurable. So the coordinate functions can be naturally seen as random variables describing the outcome of a fair coin toss. Observe that the coordinates are independent random variables as a direct consequence of the definition of the probability measures μ and μ_n. Hence, the coordinates simply model an experiment of infinitely many independent fair coin tosses. Moreover, seen as elements of $L^2(D, \mathcal{B}(D), \mu)$ resp. $L^2(D_n, \mathcal{P}(D_n), \mu_n)$, the coordinates fulfill

$$\begin{aligned} \int \varepsilon_n^2(\omega)\, d\mu(\omega) &= 1 \qquad \text{for all } n \in \mathbb{N}, \\ \int \varepsilon_n(\omega)\varepsilon_m(\omega)\, d\mu(\omega) &= 0 \qquad \text{for } n \neq m. \end{aligned}$$

The first identity is trivial, whereas the second is a direct consequence of the independence of the coordinate functions. Hence, they form an orthonormal system.

Definition 2.1.2. We call the orthogonal projection in $L^2(D, \mathcal{B}(D), \mu)$ onto the subspace spanned by the first n Rademacher functions the n-th *Rademacher projection* R_n. It is given by

$$\begin{aligned} L^2(D, \mathcal{B}(D), \mu) &\to L^2(D, \mathcal{B}(D), \mu) \\ R_n : f &\mapsto \sum_{i=1}^{n} \varepsilon_i \int_D \varepsilon_i(\omega) f(\omega)\, d\mu(\omega). \end{aligned}$$

More generally, for each finite subset A of $\mathbb{N}$ set $\varepsilon_A = \prod_{i \in A} \varepsilon_i$ (with the obvious restrictions on A for D_n), where we set by convention $\varepsilon_\emptyset = 1$. Again, we have

$$\begin{aligned} \int \varepsilon_A^2(\omega)\, d\mu(\omega) &= 1 \qquad \text{for all } A, \\ \int \varepsilon_A(\omega)\varepsilon_B(\omega)\, d\mu(\omega) &= 0 \qquad \text{for } A \neq B. \end{aligned}$$

This is a direct consequence of the independence of the coordinate functions. Indeed, for $A \neq B$ we have $A \bigtriangleup B \neq \emptyset$ and so

$$\int \varepsilon_A(\omega)\varepsilon_B(\omega)\, d\mu(\omega) = \int \varepsilon_{A \bigtriangleup B}(\omega)\, d\mu(\omega) = \prod_{i \in A \bigtriangleup B} \int \varepsilon_i(\omega)\, d\mu(\omega) = 0.$$

These functions are called *Walsh functions*. Since $L^2(D_n, \mathcal{P}(D_n), \mu_n)$ is a 2^n-dimensional vector space and $|\mathcal{P}(\{1, \ldots, n\})| = 2^n$, the Walsh functions form an orthonormal basis of $L^2(D_n, \mathcal{P}(D_n), \mu_n)$.

Remark 2.1.3. The Walsh functions also form an orthonormal basis for $L^2(D, \mathcal{B}(D), \mu)$. For this purpose we have to show that the Walsh functions are total. Even more generally, we will show that for some Banach space X the functions of the form $\omega \mapsto \varepsilon_A(\omega)x$ for some finite subset A and some $x \in X$ are total in $L^p(D, \mathcal{B}(D), \mu; X)$ for $1 \leq p < \infty$.

We observe that every element $f \in L^p(D_n, \mathcal{P}(D_n), \mu_n; X)$ for some $n \in \mathbb{N}$ is a simple function and can be written as

$$f = \sum_{\omega \in D_n} \mathbb{1}_{\{\omega\}} x_\omega.$$

Since $\mathbb{1}_{\{\omega\}}$ is an element of $L^p(D_n, \mathcal{P}(D_n), \mu_n)$, it can be written as a finite linear combination of the Walsh functions. Hence,

$$f = \sum_{A \in \mathcal{P}(D_n)} \varepsilon_A x_A$$

for some x_A in X. This shows that $L^p(D_n, \mathcal{P}(D_n), \mu_n; X)$ is canonically isometrically isomorphic to some subspace of $L^p(D, \mathcal{B}(D), \mu; X)$ and to some subspace of the Bochner space $L^p(D_m, \mathcal{P}(D_m); \mu_m; X)$ for every $m \geq n$: we identify naturally the Walsh functions in these spaces. Under these identifications $\cup_{n\in\mathbb{N}} L^p(D_n, \mathcal{P}(D_n), \mu_n; X)$ is a subspace of $L^p(D, \mathcal{B}(D), \mu; X)$ spanned by the Walsh functions. Moreover, it is dense in $L^p(D, \mathcal{B}(D), \mu; X)$:

Since the simple functions are dense by Theorem C.2.9, it is sufficient to show that an arbitrary simple function

$$f = \sum_{i=1}^{n} x_i \mathbb{1}_{A_i}$$

with A_i measurable and $f \neq 0$ can be approximated by elements in $L^p(D_n, \mathcal{P}(D_n), \mu_n; X)$ arbitrarily well. For this recall that we have seen in Remark 2.1.1 that μ is the unique normalized Haar measure on the compact group D. We say that a Borel measure λ defined on a σ-algebra containing all open sets is *outer regular* if for every measurable set A one has

$$\lambda(A) = \inf\{\lambda(U) : A \subset U \text{ open}\}.$$

Let $\varepsilon > 0$. We are going to show in a moment that μ is outer regular. Then there exists for each A_i an open set U_i such that $\mu(U_i \setminus A_i) = \mu(U_i) - \mu(A_i) \leq \frac{\varepsilon^p}{n}(\max_{i=1,\ldots,n} \|x_i\|)^{-p}$. Set $\tilde{f} \coloneqq \sum_{i=1}^{n} x_i \mathbb{1}_{U_i}$. We see that

$$\tilde{f} - f = \sum_{i=1}^{n} x_i \mathbb{1}_{U_i \setminus A_i}.$$

Hence,

$$\begin{aligned}\left(\int_D \left\|\tilde{f}(\omega) - f(\omega)\right\|^p \, d\mu(\omega)\right)^{1/p} &= \left(\sum_{i=1}^{n} \|x_i\|^p \mu(U_i \setminus A_i)\right)^{1/p} \\ &\leq \max_{i=1,\ldots,n} \|x_i\| \cdot \left(\sum_{i=1}^{n} \mu(U_i \setminus A_i)\right)^{1/p} \leq \varepsilon.\end{aligned}$$

Since U_i is open, $\varepsilon_j(U_i) = \{-1, 1\}$ does hold for all but finitely many j. This means that under the chosen identification $\mathbb{1}_{U_i}$ is an element of $L^p(D_n, \mathcal{P}(D_n), \mu_n; X)$ for n sufficiently large. Since f is a finite linear combination of indicator functions, the same holds for f. This shows that $\cup_{n\in\mathbb{N}} L^p(D_n, \mathcal{P}(D_n), \mu_n; X)$ is dense.

We still have to keep our promise: μ is outer regular. Since μ is the Haar measure on D, it is a fortiori outer regular (see [DE08, Theorem 1.3.4]). But we also give a direct proof.

For an arbitrary measurable set $A \subset D$ set U_i to be the set whose first i coordinates coincide with $\varepsilon_i(A)$ and with $\{-1, 1\}$ otherwise. Clearly, U_i is open and $U_i \supset U_j \supset A$ for $i \leq j$. Since μ is continuous from above, we have established the outer regularity:

$$\lim_{i \to \infty} \mu(U_i) = \mu\left(\bigcap_{i=1}^{\infty} U_i\right) = \mu(A).$$

Observe that

$$\int_D \left\| \sum_{i=1}^n \varepsilon_i(\omega) x_i \right\|^p d\mu(\omega) = 2^{-n} \sum_{(s_i) \in D_n} \left\| \sum_{i=1}^n s_i x_i \right\|^p$$

is just the p-average over all permutations of signs.

We now give the definition of type and cotype and state their elementary properties following [AK06, Section 6.2].

Definition 2.1.4. A Banach space X has *Rademacher type p* (in short, *type p*) for some $1 \leq p \leq 2$ if there is a constant C such that for $x_1, \dots, x_n$ in X

$$\left(\int_D \left\| \sum_{i=1}^n \varepsilon_i(\omega) x_i \right\|^2 d\mu(\omega) \right)^{1/2} \leq C \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p}. \tag{2.1.2}$$

The smallest constant for which (2.1.2) holds is called the *type p constant of X* and is denoted by $T_p(X)$.

Analogously, a Banach space X is said to have *Rademacher cotype q* (in short, *cotype q*) for some $2 \leq q < \infty$ if there is a constant C such that for $x_1, \dots, x_n$ in X

$$\left(\sum_{i=1}^n \|x_i\|^q \right)^{1/q} \leq C \left(\int_D \left\| \sum_{i=1}^n \varepsilon_i(\omega) x_i \right\|^2 d\mu(\omega) \right)^{1/2} \tag{2.1.3a}$$

and for $q = \infty$ if

$$\max_{i=1,\dots,n} \|x_i\| \leq C \left(\int_D \left\| \sum_{i=1}^n \varepsilon_i(\omega) x_i \right\|^2 d\mu(\omega) \right)^{1/2}. \tag{2.1.3b}$$

The smallest constant for which (2.1.3) holds is called the *cotype q constant of X* and is denoted by $C_q(X)$.

Remark 2.1.5. By Remark 2.1.3, we can freely replace D with D_n and μ with μ_n if desired. Note that for some sequence $y = (y_n)$ we have $\|y\|_s \leq \|y\|_r$ if $r \leq s$. So if X has type p and cotype q, X has type p' for $p' < p$ and cotype q' for $q' > q$. By the triangle inequality, every Banach space X has type 1 with $T_1(X) = 1$. Moreover, X has cotype ∞ with $C_\infty(X) = 1$: Let $\|x_j\| = \max_{i=1,\dots,n} \|x_i\|$. By the Hahn-Banach theorem we can choose $x' \in X'$ in the unit sphere such that $\|x_j\| = \langle x', x_j \rangle$. Then by the orthogonality of the coordinate functions

$$\begin{aligned}\max_{i=1,\dots,n} \|x_i\| &= \int_D \left\langle \varepsilon_j(\omega)x', \sum_{i=1}^n \varepsilon_i(\omega)x_i \right\rangle d\mu(\omega) \leq \int_D \left\| \sum_{i=1}^n \varepsilon_i(\omega)x_i \right\| d\mu(\omega) \\ &\leq \left(\int_D \left\| \sum_{i=1}^n \varepsilon_i(\omega)x_i \right\|^2 d\mu(\omega) \right)^{1/2}.\end{aligned}$$

Moreover, the restriction on the range of type and cotype is natural because every non-zero Banach space has type lesser than or equal to 2 and cotype greater than or equal to 2. To see this choose $x \in X$ with $\|x\| = 1$ and $x_1 = \ldots = x_n = x$. Then

$$\begin{aligned}\left(\int_D \left\| \sum_{i=1}^n \varepsilon_i(\omega)x_i \right\|^2 d\mu(\omega) \right)^{1/2} &= \left(\int_D \left| \sum_{i=1}^n \varepsilon_i(\omega) \right|^2 d\mu(\omega) \right)^{1/2} \\ &= \left(\sum_{i=1}^n \int_D |\varepsilon_i(\omega)|^2 \, d\mu(\omega) \right)^{1/2} = n^{1/2}.\end{aligned}$$

The Kahane-Khintchine inequality (Theorem D.2.2) shows that one would obtain an equivalent definition - of course with different type and cotype constants - of type and cotype if one would replace the L^2-average $\left(\int_D \|\sum_{i=1}^n \varepsilon_i(\omega)x_i\|^2 \, d\mu(\omega) \right)^{1/2}$ by any other L^p-average for $1 \leq p < \infty$.

Remark 2.1.6. Observe that as a consequence of the Kahane-Khintchine inequality the vector-valued analogue of the Khintchine inequality holds in a Banach space X for all $1 \leq p < \infty$ if and only if X has both type 2 and cotype 2.

Clearly, type and cotype are inherited by subspaces and are invariant under isomorphisms.

Lemma 2.1.7. *Let Y be a closed subspace of a Banach space X. If X has type p and cotype q, then Y has type p and cotype q as well. Moreover,*

$$T_p(Y) \leq T_p(X), \qquad C_q(Y) \leq C_q(X).$$

Lemma 2.1.8. *Let $T : X \to Y$ be an isomorphism between two Banach spaces X and Y. Then X is of type p (resp. of cotype q) if and only if Y is of type p (resp. of cotype q).*

2.1.1 Type and Cotype of certain Banach Spaces

In this section we will calculate the type and cotype of some important Banach spaces.

Theorem 2.1.9. *A Hilbert space H has type 2 and cotype 2 with $T(H) = C(H) = 1$.*

Proof. Let $x_1, \dots, x_n$ be a finite sequence in H. By the orthogonality of the Walsh functions, we have

$$\begin{aligned}\int_D \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^2 d\mu &= \int_D \left\langle \sum_{i=1}^n \varepsilon_i x_i, \sum_{j=1}^n \varepsilon_j x_j \right\rangle d\mu \\ &= \sum_{i,j=1}^n \langle x_i, x_j \rangle \int_D \varepsilon_i \varepsilon_j \, d\mu = \sum_{i=1}^n \|x_i\|^2 . \qquad \square\end{aligned}$$

Remark 2.1.10. The identity shown in the proof of Theorem 2.1.9 is called the *generalized Parallelogram Law.*

Remark 2.1.11. Kwapień proved in [Kwa72] - even before the definitions of type and cotype were given - the converse statement: if X has both type 2 and cotype 2, then X is isomorphic to a Hilbert space.

Next we want to determine the type and cotype of the vector-valued Lebesgue spaces $L^r(X)$ for Banach spaces with given type p and cotype q. From this we can immediately deduce the type and cotype of many important Banach spaces. We start with some preparatory lemmata.

Lemma 2.1.12. *Let $\alpha_1, \dots, \alpha_n$ be non-negative real numbers. For $0 < s \leq 1$, we have*

$$(\alpha_1 + \cdots + \alpha_n)^s \leq \alpha_1^s + \cdots + \alpha_n^s.$$

For $s \geq 1$, we have

$$(\alpha_1 + \cdots + \alpha_n)^s \geq \alpha_1^s + \cdots + \alpha_n^s.$$

Proof. We will only prove the case $0 < s \leq 1$ because the second case is almost identical. If $\sum_{i=1}^n \alpha_i = 1$, we have $0 \leq \alpha_i \leq 1$ for all i and therefore

$$\sum_{i=1}^n \alpha_i^s \geq \sum_{i=1}^n \alpha_i = 1 = \left(\sum_{i=1}^n \alpha_i \right)^s .$$

For an arbitrary sequence set $S := \sum_{i=1}^n \alpha_i$. If $S = 0$, the inequality holds trivially. Otherwise, by the first case we have

$$\sum_{i=1}^n \left(\frac{\alpha_i}{S} \right)^s \geq \left(\sum_{i=1}^n \frac{\alpha_i}{S} \right)^s . \tag{2.1.4}$$

Multiplying both sides of (2.1.4) with S^s yields the desired inequality. $\square$

Lemma 2.1.13 (Reverse Minkowski Inequality)**.** *Let $0 < r < 1$ and let f, g be non-negative functions in $L^r(\Omega, \Sigma, \lambda)$, where $(\Omega, \Sigma, \lambda)$ is a σ-finite measure space. Then*

$$\|f + g\|_r \geq \|f\|_r + \|g\|_r .$$

Proof. We first assume that $\|f+g\|_r = 1$. We introduce a new measure ν on (Ω, Σ)

$$\nu(A) := \int_A (f+g)^r d\lambda.$$

Notice that ν is a probability measure and that $(f+g)^r$ is the Radon-Nikodym derivative of ν with respect to λ. Thus we can rewrite

$$\begin{aligned}\|f\|_r &= \left(\int_\Omega f^r \, d\lambda\right)^{1/r} = \left(\int_{\{f+g>0\}} f^r \, d\lambda\right)^{1/r} = \left(\int_{\{f+g>0\}} \frac{f^r}{(f+g)^r}(f+g)^r \, d\lambda\right)^{1/r} \\ &= \left(\int_{\{f+g>0\}} \frac{f^r}{(f+g)^r} \frac{d\nu}{d\lambda} \, d\lambda\right)^{1/r} = \left(\int_{\{f+g>0\}} \frac{f^r}{(f+g)^r} \, d\nu\right)^{1/r}.\end{aligned}$$

Since $t \mapsto t^{1/r}\mathbb{1}_{(0,\infty)}(t)$ is convex for $0 < r < 1$, we can apply Jensen's inequality (Theorem D.2.1):

$$\left(\int_{\{f+g>0\}} \frac{f^r}{(f+g)^r} \, d\nu\right)^{1/r} \leq \int_{\{f+g>0\}} \frac{f}{f+g} \, d\nu = \int_{\{f+g>0\}} \frac{f}{f+g}(f+g)^r \, d\lambda.$$

Analogously, we get

$$\|g\|_r \leq \int_{\{f+g>0\}} \frac{g}{f+g}(f+g)^r \, d\lambda.$$

Combining the two cases, we have

$$\|f\|_r + \|g\|_r \leq \int_{\{f+g>0\}} (f+g)^r \, d\lambda = 1 = \|f+g\|_r \, .$$

Now, if $\|f+g\|_r = 0$, the inequality holds trivially and otherwise the general statement follows from applying the above special case to $\frac{f}{\|f+g\|_r}$ and $\frac{g}{\|f+g\|_r}$. □

Theorem 2.1.14. *Let* $(\Omega, \Sigma, \lambda)$ *be a* σ*-finite measure space and* X *a Banach space with type* p *and cotype* $q < \infty$*. Then for* $1 \leq r < \infty$*,* $L^r(\Omega, \Sigma, \lambda; X)$ *has type* $\min\{p, r\}$ *and cotype* $\max\{q, r\}$*.*

Proof. To simplify notations, we write $L^r(X)$ instead of $L^r(\Omega, \Sigma, \lambda; X)$. Let $f_1, \ldots, f_n$ be vectors in $L^r(X)$. We first show the type estimate. We have

$$\begin{aligned}\left(\int_D \left\|\sum_{i=1}^n \varepsilon_i f_i\right\|_{L^r(X)}^2 d\mu\right)^{1/2} &\underset{\text{Cor. D.2.3}}{\overset{\text{Kah.-Khin. Ineq.}}{\leq}} C_2 \left(\int_D \left\|\sum_{i=1}^n \varepsilon_i f_i\right\|_{L^r(X)}^r d\mu\right)^{1/r} \\ &= C_2 \left(\int_D \int_\Omega \left\|\sum_{i=1}^n \varepsilon_i(\omega) f_i(t)\right\|_X^r d\lambda(t) \, d\mu(\omega)\right)^{1/r}\end{aligned}$$

$$\overset{\text{Fubini}}{=} C_2 \left(\int_\Omega \int_D \left\| \sum_{i=1}^n \varepsilon_i(\omega) f_i(t) \right\|_X^r d\mu(\omega)\, d\lambda(t) \right)^{1/r}$$

$$\underset{\text{Cor. D.2.3}}{\overset{\text{Kah.-Khin. Ineq.}}{\leq}} C_2 C_r \left(\int_\Omega \left(\int_D \left\| \sum_{i=1}^n \varepsilon_i(\omega) f_i(t) \right\|_X^2 d\mu(\omega) \right)^{r/2} d\lambda(t) \right)^{1/r}$$

$$\overset{X \text{ type } p}{\leq} C_2 C_r T_p(X) \left(\int_\Omega \left(\sum_{i=1}^n \|f_i(t)\|_X^p \right)^{r/p} d\lambda(t) \right)^{1/r}.$$

Now we must look at two different cases.

Case 1: $r \geq p$. Observe that

$$\left(\int_\Omega \left(\sum_{i=1}^n \|f_i(t)\|_X^p \right)^{r/p} d\lambda(t) \right)^{1/r} = \left\| \sum_{i=1}^n \|f_i\|_X^p \right\|_{L^{r/p}}^{1/p}.$$

As $\frac{r}{p} \geq 1$, the Minkowski inequality on $L^{r/p}$ yields

$$\left\| \sum_{i=1}^n \|f_i\|_X^p \right\|_{L^{r/p}}^{1/p} \leq \left(\sum_{i=1}^n \left\| \|f_i\|_X^p \right\|_{L^{r/p}} \right)^{1/p}$$

$$= \left(\sum_{i=1}^n \left(\int_\Omega \|f_i(t)\|_X^r \, d\lambda(t) \right)^{p/r} \right)^{1/p} = \left(\sum_{i=1}^n \|f_i\|_{L^r(X)}^p \right)^{1/p}.$$

Thus $L^r(X)$ has type $p = \min\{r, p\}$ as desired.

Case 2: $p > r$. We have $\frac{r}{p} < 1$ and apply Lemma 2.1.12 to obtain

$$\left(\int_\Omega \left(\sum_{i=1}^n \|f_i(t)\|_X^p \right)^{r/p} d\lambda(t) \right)^{1/r} \leq \left(\sum_{i=1}^n \int_\Omega \|f_i(t)\|_X^r \, d\lambda(t) \right)^{1/r}$$

$$= \left(\sum_{i=1}^n \|f_i\|_{L^r(X)}^r \right)^{1/r}.$$

Thus $L^r(X)$ has type $r = \min\{r, p\}$. Hence, in both cases $L^r(X)$ has type $\min\{r, p\}$.

We continue with the cotype estimate for $L^r(X)$, which is similiar to the one above. We have

$$\left(\int_D \left\| \sum_{i=1}^n \varepsilon_i f_i \right\|_{L^r(X)}^2 d\mu \right)^{1/2} \underset{\text{Cor. D.2.3}}{\overset{\text{Kah.-Khin. Ineq.}}{\geq}} C_r^{-1} \left(\int_D \left\| \sum_{i=1}^n \varepsilon_i f_i \right\|_{L^r(X)}^r d\mu \right)^{1/r}$$

$$= C_r^{-1} \left(\int_D \int_\Omega \left\| \sum_{i=1}^n \varepsilon_i(\omega) f_i(t) \right\|_X^r d\lambda(t)\, d\mu(\omega) \right)^{1/r}$$

$$\overset{\text{Fubini}}{=} C_r^{-1}\left(\int_\Omega\int_D\left\|\sum_{i=1}^n \varepsilon_i(\omega)f_i(t)\right\|_X^r\,d\mu(\omega)\,d\lambda(t)\right)^{1/r}$$

$$\underset{\text{Cor. D.2.3}}{\overset{\text{Kah.-Khin. Ineq.}}{\geq}} C_2^{-1}C_r^{-1}\left(\int_\Omega\left(\int_D\left\|\sum_{i=1}^n \varepsilon_i(\omega)f_i(t)\right\|_X^2\,d\mu(\omega)\right)^{r/2}\,d\lambda(t)\right)^{1/r}$$

$$\overset{X\text{ cotype }q}{\geq} C_2^{-1}C_r^{-1}C_q(X)\left(\int_\Omega\left(\sum_{i=1}^n\|f_i(t)\|_X^q\right)^{r/q}\,d\lambda(t)\right)^{1/r}.$$

Again, there are two different cases.

Case 1: $r \geq q$. In this case we have $\frac{r}{q} \geq 1$ and we can again use Lemma 2.1.12:

$$\begin{aligned}\left(\int_\Omega\left(\sum_{i=1}^n\|f_i(t)\|_X^q\right)^{r/q}\,d\lambda(t)\right)^{1/r} &\geq \left(\sum_{i=1}^n\int_\Omega\|f_i(t)\|_X^r\,d\lambda(t)\right)^{1/r}\\ &= \left(\sum_{i=1}^n\|f_i\|_{L^r(X)}^r\right)^{1/r}.\end{aligned}$$

This shows that $L^r(X)$ has cotype $r = \max\{r,q\}$.

Case 2: $r < q$. Notice that

$$\left(\int_\Omega\left(\sum_{i=1}^n\|f_i(t)\|_X^q\right)^{r/q}\,d\lambda(t)\right)^{1/r} = \left(\left\|\sum_{i=1}^n\|f_i\|_X^q\right\|_{L^{r/q}}\right)^{1/q}.$$

Since $\frac{r}{q} < 1$, we can use the reverse Minkowski inequality (Lemma 2.1.13) to finish the proof. Indeed,

$$\begin{aligned}\left(\left\|\sum_{i=1}^n\|f_i\|_X^q\right\|_{L^{r/q}}\right)^{1/q} &\geq \left(\sum_{i=1}^n\left\|\|f_i\|_X^q\right\|_{L^{r/q}}\right)^{1/q}\\ &= \left(\sum_{i=1}^n\left(\int_\Omega\|f_i(t)\|^r\,d\lambda(t)\right)^{q/r}\right)^{1/q} = \left(\sum_{i=1}^n\|f_i\|_{L^r(X)}^q\right)^{1/q}.\end{aligned}$$

Thus $L^r(X)$ has cotype $q = \max\{r,q\}$. Therefore the cotype of $L^r(X)$ in both cases is $\max\{r,q\}$. □

The types and cotypes of many important Banach spaces now follow immediately.

Corollary 2.1.15. *Let $1 \leq p < \infty$ and let λ be the Lebesgue measure. Then $L^p(\lambda)$ has type* $\min\{p,2\}$ *and cotype* $\max\{p,2\}$.

Proof. This follows from Theorem 2.1.9 and Theorem 2.1.14. □

Corollary 2.1.16. *Let $1 \le p < \infty$. Then ℓ_p has type* $\min\{p, 2\}$ *and cotype* $\max\{p, 2\}$. *Moreover, the type result is optimal for $1 \le p \le 2$, whereas the cotype result is optimal for $p \le 2 < \infty$.*

Proof. Let ν be the counting measure. Notice that $\ell_p = L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu)$. Again, the first claim follows from Theorem 2.1.9 and Theorem 2.1.14. For the second claim let (e_n) be the canonical Schauder basis in ℓ_p for $1 \le p < \infty$. Then for any signs (ε_i) we have

$$\left\| \sum_{i=1}^{n} \varepsilon_i e_i \right\|_p = n^{1/p}.$$

So ℓ_p cannot be of type greater than p for $1 \le p \le 2$ and cannot be of cotype smaller than p for $2 \le p < \infty$. □

The finite n-fold direct sum $\bigoplus_{i=1}^n X$ of some Banach space X can be endowed with the norms $\|(x_1, \ldots, x_n)\|_r = (\sum_{i=1}^n \|x_i\|^r)^{1/r}$ for $1 \le r < \infty$ (and the usual modification for $r = \infty$) which we will denote $\ell_r^n(X)$. Recall that these norms are all equivalent. Moreover, $\bigoplus_{i=1}^n X$ is naturally isomorphic to $L^r(\{1, \ldots, n\}, \mathcal{P}(\{1, \ldots, n\}), \nu; X)$, where ν is the counting measure on $\{1, \ldots, n\}$.

Corollary 2.1.17. *Let X be of type p and cotype q. Then $\bigoplus_{i=1}^n X$ is of type p and of cotype q. More precisely, $\ell_n^r(X)$ is of type p and cotype q for every $1 \le r \le \infty$.*

Proof. By what have been said above, $\ell_n^r(X)$ is isomorphic to $\ell_n^2(X)$ for every $1 \le r \le \infty$. Since type and cotype are invariant under isomorphisms by Lemma 2.1.8, Theorem 2.1.14 applied to $L^2(\{1, \ldots, n\}, \mathcal{P}(\{1, \ldots, n\}), \nu; X)$ shows that $\bigoplus_{i=1}^n X$ is of type p and of cotype q. □

Last but not least, we take a look at c_0 and ℓ_∞.

Theorem 2.1.18. *The Banach spaces c_0 and ℓ_∞ have neither non-trivial type nor non-trivial cotype.*

Proof. It is sufficient to show the claim for c_0 because of Lemma 2.1.7. Denote (e_n) the unit vectors in c_0. We see that $\|\sum_{i=1}^n \varepsilon_i(\omega) e_i\|_\infty = 1$ for all $\omega \in D_n$. Assume that c_0 has non-trivial cotype $q < \infty$. Then

$$n^{1/q} = \left(\sum_{i=1}^{n} \|e_i\|_\infty^q \right)^{1/q} \le C_q(c_0) \left(\int_{D_n} \left\| \sum_{i=1}^{n} \varepsilon_i(\omega) e_i \right\|_\infty^2 d\mu_n(\omega) \right)^{1/2} = C_q(c_0)$$

for all $n \in \mathbb{N}$ which is impossible.

Now assume that c_0 has non-trivial type $p > 1$. For $n \in \mathbb{N}$ we introduce the *Rademacher sequences* $r^1, \ldots, r^n$, whose definition depends on n, given by

$$r_k^n = \begin{cases} 1 & \text{if } k \in \cup_{m=1}^{2^{k-1}} [2(m-1)2^{n-k} + 1, (2m-1)2^{n-k}] \\ -1 & \text{if } k \in \cup_{m=1}^{2^{k-1}} [(2m-1)2^{n-k} + 1, (2m)2^{n-k}] \\ 0 & \text{if } k > 2^n. \end{cases}$$

For example, we get for $n = 3$

$$\begin{aligned} r^1 &= (1, 1, 1, 1, -1, -1, -1, -1, 0, \ldots) \\ r^2 &= (1, 1, -1, -1, 1, 1, -1, -1, 0, \ldots) \\ r^3 &= (1, -1, 1, -1, 1, -1, 1, -1, 0, \ldots). \end{aligned}$$

Writing the r^i's in a matrix scheme as above, we see that for each possible combination of signs ε_i, there is exactly one column such that the entries in this column multiplied with the chosen signs sum up to n. Thus $\left\|\sum_{i=1}^n \varepsilon_i(\omega) r^i\right\|_\infty = n$ for all $\omega \in D_n$. Then

$$n = \left(\int_{D_n} \left\| \sum_{i=1}^n \varepsilon_i(\omega) r^i \right\|_\infty^2 \, d\mu_n(\omega) \right)^{1/2} \leq T_p(c_0) \left(\sum_{i=1}^n \left\| r^i \right\|_\infty^p \right)^{1/p} = T_p(c_0) n^{1/p}$$

for all $n \in \mathbb{N}$. Hence, c_0 cannot have non-trivial type. □

2.2 B-convexity

B-convexity was introduced by A. Beck in [Bec62]. He proved in his previous paper [Bec58] that the following strong law of large numbers holds for random variables which take values in a uniformly convex Banach space.

Theorem 2.2.1 (Beck's Strong Law of Large Numbers)**.** *Let X be a uniformly convex Banach space and let (X_i) be a sequence of independent random variables (taking values in X) with $\mathbb{E}(X_i) = 0$ for all i. Assume additionally that* $\operatorname{Var}(X_i) = \mathbb{E}(\|X_i\|^2)$ *are uniformly bounded. Then*

$$\frac{1}{n} \sum_{i=1}^n X_i \to 0 \qquad \textit{strongly almost surely.}$$

He also gave a counterexample on a non-uniformly convex space in which the above theorem fails. This showed that certain restrictions on the underlying Banach space were necessary. However, uniform convexity was not optimal. Beck showed in [Bec62] that the strong law of large numbers as above holds if and only if X has a certain cancellation property for large enough sums which is now known as *B-convexity* (Beck used the term 'condition (B)'). We now give an exact definition.

Definition 2.2.2. A Banach space X is called *B-convex* if there exist a $\delta > 0$ and an integer $n \geq 2$ such that for any $x_1, \ldots, x_n \in X$ we can choose signs $(\varepsilon_i)_{i=1}^n \in \{-1, 1\}^n$ such that

$$\left\| \frac{1}{n} \sum_{i=1}^n \varepsilon_i x_i \right\| \leq (1 - \delta) \max_{i=1,\ldots,n} \|x_i\| .$$

Remark 2.2.3. Observe that by the triangle inequality,

$$\left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i x_i \right\| \leq \max_{i=1,\dots,n} \|x_i\| ,$$

so B-convexity guarantees that for well-chosen signs we get a non-trivial estimate due to some known amount of cancellation in the summation process.

It follows directly from the definition of B-convexity that a closed subspace of a B-convex space is again B-convex.

Lemma 2.2.4. *Let $Y \subset X$ be a closed subspace of a Banach space X. If X is B-convex, then so is Y.*

We already know that all uniformly convex Banach spaces are B-convex. Indeed, Beck showed that for uniformly convex spaces the law of large numbers holds and therefore these spaces are B-convex. However, we will not use the probabilistic characterization of B-convexity and will only work with Definition 2.2.2. Luckily, it is easy to check the definition directly.

Theorem 2.2.5. *All uniformly convex Banach spaces are B-convex.*

Proof. Suppose X is uniformly convex. Then there exists a $\delta(1) > 0$ such that whenever for x, y in the unit ball $\|x - y\| > 1$ holds, we have $\|x + y\| < 2(1 - \delta(1))$. We show that X is B-convex for $n = 2$ and $\delta = \min\{\frac{1}{2}, \delta(1)\}$. Let x_1, x_2 be in the unit ball. We look at two cases: $\|x_1 - x_2\| \leq 1$ and $\|x_1 - x_2\| > 1$. In the first case we get $\|x_1 - x_2\| \leq 1 = 2(1 - \frac{1}{2}) \leq 2(1 - \delta)$, whereas in the second case we have $\|x + y\| \leq 2(1 - \delta(1)) \leq 2(1 - \delta)$. For general vectors $x_1, x_2 \in X$ rescaling shows that for at least one choice of $\varepsilon \in \{-1, 1\}$

$$\frac{1}{2} \|x_1 + \varepsilon x_2\| \leq (1 - \delta) \max_{i=1,2} \|x_i\| . \qquad \square$$

Theorem 2.2.5 implies the B-convexity of some important Banach spaces.

Corollary 2.2.6. *Every Hilbert space and the spaces ℓ_p and L^p for $1 < p < \infty$ are B-convex.*

For a complex Banach space it is natural to ask the following question: what happens if one replaces $\{-1, 1\}$ by $\mathbb{T} \coloneqq \{z \in \mathbb{C} : |z| = 1\}$? Even more generally, one could ask the same question for an arbitrary subset of $\mathbb{T}$. This leads to the following definition.

Definition 2.2.7. Let $\mathbb{K}$ be the field of real or complex numbers. Let A be a subset of $\mathbb{T} \coloneqq \{z \in \mathbb{K} : |z| = 1\}$. A Banach space X is called *A-convex* if there exist a $\delta > 0$ and an integer $n \geq 2$ such that for any $x_1, \dots, x_n \in X$ we can choose $(\lambda_i)_{i=1}^n \in A^n$ such that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} \lambda_i x_i \right\| \leq (1 - \delta) \max_{i=1,\dots,n} \|x_i\| .$$

By convention, $B \coloneqq \{-1, 1\}$. Therefore our new definition of B-convexity coincides with the old one given in Definition 2.2.2.

Obviously, a non-zero Banach space can never be A-convex if A is a singleton set. To see this, simply choose $x_1, \dots, x_n = x$, where $x \neq 0$ is arbitrary. It is also clear that if $A_2 \supset A_1$ and X is A_1-convex, X is A_2-convex as well. But a lot more is true. The following theorem was proven by D.P. Giesy [Gie66, Theorem 5].

Theorem 2.2.8. *If both A_1 and A_2 contain at least two points, X is A_1-convex if and only if X is A_2-convex.*

Its proof needs some preparations. We introduce the moduli $\beta_n^A(X)$ of a Banach space X.

Definition 2.2.9. Let X be a Banach space and $A \subset \mathbb{T}$. We define

$$\beta_n^A(X) := \sup_{\substack{x_1,\dots,x_n \\ \|x_i\| \le 1}} \inf_{(\lambda_i) \in A^n} \left\| \frac{1}{n} \sum_{i=1}^n \lambda_i x_i \right\|.$$

We will use $\beta_n(X)$ as a synonym for $\beta_n^A(X)$.

Remark 2.2.10. If A is a closed (and therefore compact) set, the infimum can be replaced by a minimum.

Observe that $0 \le \beta_n^A(X) \le 1$ for all n. Moreover, the $\beta_n^A(X)$s are directly connected with A-convexity.

Lemma 2.2.11. *A Banach space X fails to be A-convex if and only if $\beta_n^A(X) = 1$ for all $n \ge 2$.*

Proof. Assume X fails to be A-convex. Then for any given $\delta > 0$ and $n \ge 2$ there are $y_1, \dots, y_n$ such that for any $(\lambda_i)_{i=1}^n \in A^n$ we have

$$\left\| \frac{1}{n} \sum_{i=1}^n \lambda_i y_i \right\| > (1-\delta) \max_{i=1,\dots,n} \|y_i\|.$$

Now set $x_i := y_i (\max_{j=1,\dots,n} \|y_j\|)^{-1}$. Observe that $x_1, \dots, x_n$ are vectors in the unit ball satisfying

$$\inf_{(\lambda_i) \in A^n} \left\| \frac{1}{n} \sum_{i=1}^n \lambda_i x_i \right\| \ge (1-\delta).$$

Since δ is arbitrary, we see that $\beta_n^A(X) = 1$ for all $n \ge 2$.

Conversely, let $\beta_n^A(X) = 1$ for all $n \ge 2$. Then for any $\delta > 0$ and any $n \ge 2$ there are vectors $x_1, \dots, x_n$ in the unit ball of X such that no matter how we choose $(\lambda_i)_{i=1}^n \in A^n$ we have

$$\left\| \frac{1}{n} \sum_{i=1}^n \lambda_i x_i \right\| > 1 - \delta \ge (1-\delta) \max_{i=1,\dots,n} \|x_i\|.$$

Thus X cannot be A-convex. $\square$

Example 2.2.12. Choosing $x_i = e_i$, where (e_n) is the canonical Schauder basis of ℓ_1, we see that $\beta_n(\ell_1) = 1$ for all $n \in \mathbb{N}$. Hence, ℓ_1 is not B-convex. An almost identical argument as in Theorem 2.1.18 can be used to show that neither ℓ_∞ nor c_0 are B-convex.

The next lemma shows that the $\beta_n^A(X)$s are submultiplicative if A is closed under multiplication.

Lemma 2.2.13. *Let $A \subset \mathbb{T}$ be a multiplicative closed subset. For natural numbers n, k we have*

$$\beta_{nk}^A(X) \leq \beta_n^A(X) \cdot \beta_k^A(X).$$

Proof. Fix vectors $x_1, \ldots, x_{nk}$ in the unit ball. Let $\varepsilon > 0$. We split these vectors into n blocks of k vectors size per block. For each block we choose $\tilde{\lambda}^i \in A^k$ such that for each $0 \leq i < n$ we have

$$\left\| \sum_{j=1}^{k} \tilde{\lambda}_j^i x_{ik+j} \right\| \leq \inf_{(\lambda_j) \in A^k} \left\| \sum_{j=1}^{k} \lambda_j x_{ik+j} \right\| + \varepsilon.$$

For $0 \leq i < n$ we set

$$y_i := \sum_{j=1}^{k} \tilde{\lambda}_j^i x_{ik+j}.$$

Then $\|y_i\| \leq k\beta_k^A(X) + \varepsilon$. Using the definition of $\beta_n^A(X)$ we see that

$$\inf_{\eta \in A^n} \left\| \sum_{i=0}^{n-1} \eta_i y_i \right\| \leq n \cdot \beta_n^A(X) \cdot \max_{i=0,\ldots,n-1} \|y_i\| \leq n \cdot \beta_n^A(X) \cdot (k\beta_k^A(X) + \varepsilon).$$

Notice that each sum $\sum_{i=0}^{n-1} \eta_i y_i$ can be written as a sum of the form $\sum_{l=1}^{nk} \lambda_l x_l$ for an appropriate chosen $(\lambda_l) \in A^{kn}$ because A is closed under multiplication. Hence,

$$\inf_{(\lambda_l) \in A^{kn}} \left\| \sum_{l=1}^{nk} \lambda_l x_l \right\| \leq \inf_{\eta \in A^n} \left\| \sum_{i=0}^{n-1} \eta_i y_i \right\| \leq n \cdot \beta_n^A(X) \cdot (k\beta_k^A(X) + \varepsilon).$$

Moreover $\varepsilon > 0$ is arbitrary, so we see that

$$\inf_{(\lambda_l) \in A^{kn}} \left\| \sum_{l=1}^{nk} \lambda_l x_l \right\| \leq nk \cdot \beta_n^A(X) \cdot \beta_k^A(X).$$

Since $x_1, \ldots, x_{nk}$ are arbitrary, we have shown that $\beta_{nk}^A(X) \leq \beta_n^A(X) \cdot \beta_k^A(X)$. □

Lemma 2.2.13 says that the more vectors one takes the more cancellation one gets on a A-convex space.

Corollary 2.2.14. *Let A be multiplicative closed and X be an A-convex Banach space. Then for every $\varepsilon > 0$ there exists a natural number M such that $\beta_M^A(X) \leq \varepsilon$.*

Proof. Let $\varepsilon > 0$. Lemma 2.2.11 shows that there exists a natural number N such that $\beta_N^A(X) \leq \delta < 1$. By Lemma 2.2.13, $\beta_{N^m}^A(X) \leq \delta^m$. This shows $\lim_{m\to\infty} \beta_{N^m}^A(X) = 0$. $\square$

The submultiplicativity is one essential key to the proof of Theorem 2.2.8.

Proof of Theorem 2.2.8. If X is a real Banach space, the conditions imply $A_1 = A_2 = B$ and the assertion is trivial. Suppose X is a complex Banach space. We will proceed in four steps.

(a) If X is A_1-convex, X is $\mathbb{T}$-convex.
(b) If X is $\mathbb{T}$-convex, X is μ_5-convex, where $\mu_5 := \{\zeta_5^0, \dots, \zeta_5^4\}$ is the multiplicative group of the fifth roots of unity.
(c) If X is μ_5-convex, X is B-convex.
(d) If X is B-convex, X is A_2-convex.

(a) follows directly because $A \subset \mathbb{T}$.

(b): For each $\lambda \in \mathbb{T}$ choose $\mu(\lambda)$ to be a fifth root of unity closest to λ (see fig. 2.1). Then the short arc of the unit circle joining λ and $\mu(\lambda)$ is at most $\pi/5$ long, so $|\lambda - \mu(\lambda)| \leq \frac{\pi}{5} < \varepsilon < 1$ for some $0 < \varepsilon < 1$. Choose $\delta > 0$ such that $1 - \delta - \varepsilon > 0$. By Corollary 2.2.14,

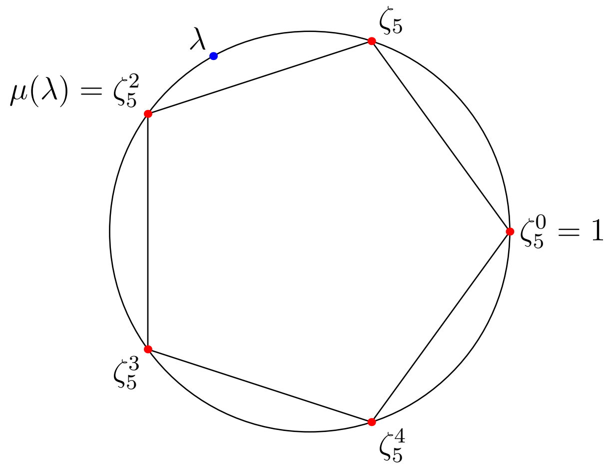

Figure 2.1: Approximating λ with fifth roots of unity

there exists a n such that $\beta_n^A(X) < 1 - \delta - \varepsilon$. Given arbitrary $x_1, \dots, x_n$, we may therefore choose $\lambda_1, \dots, \lambda_n \in A$ such that

$$\frac{1}{n} \left\| \sum_{i=1}^n \lambda_i x_i \right\| \leq (1 - \delta - \varepsilon) \max_{i=1,\dots,n} \|x_i\| .$$

Approximating the λ_is with fifth roots of unity, we still get non-trivial cancellation. Indeed,

$$\frac{1}{n} \left\| \sum_{i=1}^n \mu(\lambda_i) x_i \right\| \leq \frac{1}{n} \left\| \sum_{i=1}^n \lambda_i x_i \right\| + \frac{1}{n} \left\| \sum_{i=1}^n (\mu(\lambda_i) - \lambda_i) x_i \right\|$$

$$\leq (1 - \delta - \varepsilon + \varepsilon) \max_{i=1,\ldots,n} \|x_i\| = (1 - \delta) \max_{i=1,\ldots,n} \|x_i\| .$$

Thus, we have shown that X is μ_5-convex.

(c): Again, we want to approximate the fifth roots of unity with ± 1 in the hope that this approximation is still good enough to guarantee non-trivial cancellation. However,

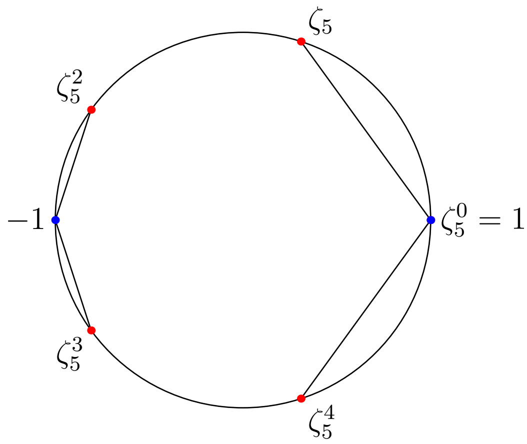

Figure 2.2: Approximating the fifth roots of unity with ± 1. The best approximation is indicated with a path.

this time one has besides $\left|\zeta_5^0 - 1\right| = 0$ the following for the best approximations (compare fig. 2.2):

$$\left|\zeta_5 - 1\right| = \left|\zeta_5^4 - 1\right| =$$
$$\sqrt{\left(\cos\left(\frac{2\pi}{5}\right) - 1\right)^2 + \sin^2\left(\frac{2\pi}{5}\right)} = \sqrt{2\left(1 - \cos\left(\frac{2\pi}{5}\right)\right)} = 2\sin\left(\frac{\pi}{5}\right) > 1$$

and analogously

$$\left|\zeta_5^2 - 1\right| = \left|\zeta_5^3 - 1\right| = 2\sin\left(\frac{\pi}{10}\right) < 1.$$

Therefore one has to estimate more carefully. We are going to show that we can obtain every estimate (choosing different coefficients if necessary) in such a way that at most $2/5$ of the coefficients are ζ_5 or ζ_5^4. Then the remaining $3/5$ are at distance at most $2\sin(\pi/10)$ from their best approximations. We set

$$\varepsilon := \frac{2}{5} \cdot 2\sin\left(\frac{\pi}{5}\right) + \frac{3}{5} \cdot 2\sin\left(\frac{\pi}{10}\right) \approx 0.85 < 1.$$

Choose $\delta > 0$ such that $\varepsilon + \delta < 1$. Again by Corollary 2.2.14, there exists a n such that $\beta_n^{\mu_5}(X) < 1 - \delta - \varepsilon$. Given arbitrary $x_1, \ldots, x_n$, we may choose $\mu_1', \ldots, \mu_n' \in \mu_5$ such that

$$\frac{1}{n}\left\|\sum_{i=1}^{n} \mu_i' x_i\right\| \leq (1 - \delta - \varepsilon) \max_{i=1,\ldots,n} \|x_i\| .$$

Not necessarily at most $2n/5$ of the μ_i' are ζ_5 or ζ_5^4, but there is a k such that at most $2n/5$ of the μ_i' are ζ_5^{k-1} and ζ_5^{k+1}:

assume this claim is wrong. Denote $W_j := \{m : \mu_m = \zeta_n^j\}$. Then $|W_{j-1} \cup W_{j+1}| > 2n/5$ for all j. By the inclusion-exclusion principle,

$$n = \left| \bigcup_{j=1}^{5} (W_{j-1} \cup W_{j+1}) \right| \geq \sum_{j=1}^{5} |W_{j-1} \cup W_{j+1}| - \sum_{j=1}^{5} |W_j| > 5 \cdot 2n/5 - n = n.$$

Contradiction! Now set $\mu_i := \zeta_5^{-k} \mu_i'$. Then at most $2n/5$ of the μ_is are ζ_5 or ζ_5^4. Moreover,

$$\frac{1}{n} \left\| \sum_{i=1}^{n} \mu_i x_i \right\| \leq (1 - \delta - \varepsilon) \max_{i=1,\ldots,n} \|x_i\| .$$

If $\mu_1 = 1, \zeta_5$ or ζ_5^4, let $\varepsilon_i = 1$, while if $\mu_i = \zeta_5^2$ or ζ_5^3, let $\varepsilon_i = -1$. Then for at most $2n/5$ of the coefficients μ_i, $|\varepsilon_i - \mu_i| = 2\sin(\pi/5)$, while for at least $3n/5$ of the i, $|\varepsilon_i - \mu_i| \leq 2\sin(\pi/10)$. Thus

$$\frac{1}{n} \sum_{i=1}^{n} |\varepsilon_i - \mu_i| \leq \frac{2}{5} \cdot 2 \sin\left(\frac{\pi}{5}\right) + \frac{3}{5} \cdot 2 \sin\left(\frac{\pi}{10}\right) = \varepsilon.$$

Hence,

$$\begin{aligned} \frac{1}{n} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| &\leq \frac{1}{n} \left\| \sum_{i=1}^{n} (\varepsilon_i - \mu_i) x_i \right\| + \frac{1}{n} \left\| \sum_{i=1}^{n} \mu_i x_i \right\| \\ &\leq \left(\frac{1}{n} \sum_{i=1}^{n} |\varepsilon_i - \mu_i| + 1 - \varepsilon - \delta \right) \max_{i=1,\ldots,n} \|x_i\| = (1 - \delta) \max_{i=1,\ldots,n} \|x_i\| . \end{aligned}$$

Therefore X is B-convex.

(d): By assumption, we can choose two different elements $a, b \in A_2$. Let $\alpha := \frac{a+b}{2}$, $\beta := \frac{a-b}{2}$. Notice that $\alpha \pm \beta \in A_2$. Since the euclidean norm is strictly convex, we have $|\alpha| < 1$. Thus it is possible to choose $0 < \varepsilon < 1$ and $0 < \delta < 1$ such that $|\alpha| + |\beta|(1 - \varepsilon) = 1 - \delta < 1$. Now since X is B-convex, there exists a n such that $\beta_n(X) < 1 - \varepsilon$. Hence, given arbitrary $x_1, \ldots, x_n$, there are $\varepsilon_i \in \{-1, 1\}$ such that

$$\frac{1}{n} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| < (1 - \varepsilon) \max_{i=1,\ldots,n} \|x_i\| .$$

As noticed above, we have $\alpha + \varepsilon_i \beta \in A_2$ for all i and

$$\begin{aligned} \frac{1}{n} \left\| \sum_{i=1}^{n} (\alpha + \varepsilon_i \beta) x_i \right\| &\leq \frac{1}{n} |\alpha| \sum_{i=1}^{n} \|x_i\| + \frac{1}{n} |\beta| \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| \\ &\leq (|\alpha| + |\beta|(1 - \varepsilon)) \max_{i=1,\ldots,n} \|x_i\| = (1 - \delta) \max_{i=1,\ldots,n} \|x_i\| . \end{aligned}$$

Thus X is A_2-convex. □

The submultiplicativity can also be used to show that B-convexity is invariant under isomorphisms.

Lemma 2.2.15. *Let $T : X \to Y$ be an isomorphism between two Banach spaces X and Y. Then X is B-convex if and only if Y is B-convex.*

Proof. If $X = Y = 0$, there is nothing to show. Assume $X \neq 0$ is B-convex and let $0 < \delta < 1$. Corollary 2.2.14 shows that we can choose a natural number n such that $\beta_n(X) < (\|T\| \|T^{-1}\|)^{-1} \cdot (1-\delta)$. Let $y_1, \ldots, y_n \in Y$. Choose signs $\varepsilon_1, \ldots, \varepsilon_n$ such that

$$\frac{1}{n} \left\| \sum_{i=1}^{n} \varepsilon_i T^{-1} y_i \right\|_X \leq (\|T\| \|T^{-1}\|)^{-1}(1-\delta) \cdot \max_{i=1,\ldots,n} \left\| T^{-1} y_i \right\|_X .$$

Hence,

$$\begin{aligned}\frac{1}{n} \left\| \sum_{i=1}^{n} \varepsilon_i y_i \right\|_Y &\leq \frac{1}{n} \|T\| \left\| \sum_{i=1}^{n} \varepsilon_i T^{-1} y_i \right\|_X \\ &\leq \|T\| (\|T\| \|T^{-1}\|)^{-1}(1-\delta) \cdot \max_{i=1,\ldots,n} \left\| T^{-1} y_i \right\|_X \leq (1-\delta) \max_{i=1,\ldots,n} \|y_i\|_Y .\end{aligned}$$

Since $T^{-1} : Y \to X$ is an isomorphism as well, the converse follows directly. $\square$

One knows from probability theory that Beck's strong law of large numbers holds for random variables with values in $\mathbb{K}^n$. So every finite dimensional vector space must be B-convex. With the invariance under isomorphisms in our hands we are now able to verify this only using our working definition of B-convexity.

Theorem 2.2.16. *Every finite dimensional vector space is B-convex.*

Proof. Every finite dimensional vector space is isomorphic to $(\mathbb{K}^n, \|\cdot\|_2)$ for some $n \in \mathbb{N}$. As Hilbert spaces are B-convex, the claim follows from Lemma 2.2.15. $\square$

D.P. Giesy studied various further aspects of B-convex spaces and gave a geometrical characterization of B-convexity [Gie66]. In order to state Giesy's result we need some additional terminology.

Definition 2.2.17. Let $1 \leq p \leq \infty$ and let $\mathbb{K}$ be the field of real or complex numbers. We write ℓ_p^n for the vector space $\mathbb{K}^n$ endowed with the $\|\cdot\|_p$ norm. For $\lambda > 1$ we say that X contains a λ-*isomorphic* subspace to ℓ_p^n if there exist $x_1, \ldots, x_n$ in X such that for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{K}^n$

$$\lambda^{-1} \left(\sum_{i=1}^{n} |\alpha_i|^p \right)^{1/p} \leq \left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \leq \left(\sum_{i=1}^{n} |\alpha_i|^p \right)^{1/p} .$$

We say that X contains ℓ_p^n*'s uniformly* if for some fixed $\lambda > 1$ the space X contains subspaces $X_n = \operatorname{span}\{x_{1n}, \ldots, x_{nn}\}$ λ-isomorphic to ℓ_p^n for all $n \in \mathbb{N}$.

Remark 2.2.18. The last part of Definition 2.2.17 can be restated in terms of operators. Using the same notation as above, we define the isomorphisms

$$\begin{aligned} T_n : X_n &\to \ell_p^n \\ \sum_{i=1}^n \alpha_i x_{in} &\mapsto (\alpha_1, \ldots, \alpha_n). \end{aligned}$$

Since X_n is λ-isomorphic to ℓ_p^n, we have $\|T_n\| \leq \lambda$ and $\left\|T_n^{-1}\right\| \leq 1$. The *multiplicative Banach-Mazur distance* $d(X_n, \ell_n^p)$ is defined as

$$d(X_n, \ell_n^p) \coloneqq \inf \left\{ \|S\| \left\|S^{-1}\right\| : S \text{ is an isomorphism between } X_n \text{ and } \ell_n^p \right\}.$$

One directly sees from the submultiplicativity of the operator norm that the Banach-Mazur distance of two non-zero Banach spaces is always greater than or equal to 1 and that for three Banach spaces X, Y, Z the multiplicative triangle inequality $d(X, Z) \leq d(X, Y) \cdot d(Y, Z)$ holds.

Notice that for every isomorphism S the rescaled isomorphism $S' = \|S\|^{-1} S$ fulfills $\|S\| \left\|S^{-1}\right\| = \|S'\| \left\|S'^{-1}\right\|$. Therefore we can restrict our attention to isomorphisms for which the mapping or its inverse is normed to 1. Therefore X contains ℓ_p^n's uniformly if and only if

$$\sup_{n \in \mathbb{N}} d(X_n, \ell_n^p) \leq \lambda$$

for some sequence (X_n) of n-dimensional subspaces of X.

Now we can give the promised geometrical characterization of B-convexity.

Theorem 2.2.19. *A Banach space X is B-convex if and only if X does not contain ℓ_1^n's uniformly.*

Proof. Assume that X contains λ-uniform copies of all ℓ_1^n's for some $\lambda > 1$. Then for all $n \in \mathbb{N}$ there is a n-dimensional subspace X_n of X together with an isomorphism $u_n : X_n \to \ell_1^n$ such that $\|u_n\| \leq \lambda$ and $\left\|u_n^{-1}\right\| \leq 1$. Hence,

$$\begin{aligned} n = \min_{(\varepsilon_i) \in D_n} \left\| \sum_{i=1}^n \varepsilon_i e_i \right\|_{\ell_1^n} &\leq \lambda \min_{(\varepsilon_i) \in D_n} \left\| \sum_{i=1}^n \varepsilon_i u_n^{-1}(e_i) \right\|_X \\ &\leq \lambda \cdot n \cdot \beta_n(X) \cdot \max_{i=1,\ldots,n} \left\|u_n^{-1}(e_i)\right\|_X \leq \lambda \cdot n \cdot \beta_n(X). \end{aligned}$$

This shows $\beta_n(X) \geq \lambda^{-1}$ for all n.

Suppose that X is B-convex. Then by Corollary 2.2.14 there exists a natural number n_0 such that $\beta_{n_0}(X) < \lambda^{-1}$, which is in contradiction to what has just been shown. So X is not B-convex. We have shown that if X is B-convex, X does not contain ℓ_1^n's uniformly.

Now assume that X is not B-convex. Theorem 2.2.8 shows that X is not $\mathbb{T}$-convex as well (these two notions only differ in the case of a complex Banach space). By

Lemma 2.2.11, this is equivalent to $\beta_n^{\mathbb{T}}(X) = 1$ for all $n \geq 2$. We will show that for all $\lambda > 1$ the space X contains ℓ_1^n's λ-uniformly. For $n = 1$ the assertion is trivial. So we may assume $n \geq 2$. Fix $\lambda > 1$. Define $0 < \delta < 1$ by $1 - \delta = \lambda^{-1}$. Since $\beta_n^{\mathbb{T}}(X) = 1$, we can find $x_1, \dots x_n$ in the unit ball such that no matter how we choose $(\lambda_i)_{i=1}^n \in \mathbb{T}$ we have

$$n - \delta \leq \left\| \sum_{i=1}^n \lambda_i x_i \right\| .$$

We want to show that $X_n \coloneqq \operatorname{span}\{x_1, \dots, x_n\}$ is λ-isomorphic to ℓ_1^n, more precisely that for all scalars $\alpha_1, \dots, \alpha_n$

$$\lambda^{-1} \sum_{i=1}^n |\alpha_i| \leq \left\| \sum_{i=1}^n \alpha_i x_i \right\| \leq \sum_{i=1}^n |\alpha_i|$$

holds. Notice that the second inequality is trivial, so it remains to show the first one. Define $\operatorname{sign} z \coloneqq z/|z|$ for $z \neq 0$ and $\operatorname{sign} 0 \coloneqq 1$. We let $S \coloneqq \sum_{i=1}^n |\alpha_i|$ and $\lambda_i \coloneqq \operatorname{sign} \alpha_i$. Further,

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha_i x_i \right\| &= \left\| \sum_{i=1}^n \lambda_i (|\alpha_i| - S + S) x_i \right\| \geq \left\| \sum_{i=1}^n \lambda_i S x_i \right\| - \left\| \sum_{i=1}^n \lambda_i (|\alpha_i| - S) x_i \right\| \\ &\geq S(n - \delta) - \sum_{i=1}^n (S - |\alpha_i|) = nS - \delta S - nS + S = (1 - \delta) S = \lambda^{-1} \sum_{i=1}^n |\alpha_i| . \end{aligned}$$

Hence, X contains ℓ_1^n's uniformly. □

Remark 2.2.20. Note that we have even proven that if X is not B-convex, X contains ℓ_1^n's λ-uniformly for all $\lambda > 1$! So X is B-convex if and only if X does not contain ℓ_n^1's λ-uniformly for some $\lambda > 1$ or equivalently, X fails to be B-convex if and only if X contains ℓ_n^1's λ-uniformly for all $\lambda > 1$.

A further geometrical characterization of B-convexity is given by the Rademacher type. Note that if X is not B-convex, X contains ℓ_1^n's uniformly by Theorem 2.2.19. Therefore X is not of type p for any $p > 1$ because ℓ_1 is not of type p for any $p > 1$ either (see Corollary 2.1.16). So if X is of non-trivial type, X is B-convex. The converse statement holds as well.

Theorem 2.2.21. *A Banach space is B-convex if and only if it is of non-trivial type.*

Proof. See [DJT95, Theorem 13.10]. □

2.3 K-convexity

The notion of K-convexity arises naturally in the study of type and cotype. It is natural to ask whether there is some connection between these two notions. We will see that there is some kind of duality between the type of a Banach space X and the cotype of its dual (and respectively between the type of the dual and the cotype of X). However, for general Banach spaces this duality fails partially. Therefore we are interested in finding an additional assumption on X such that we get a complete duality theory. This assumption will be the property of being K-convex. In order to motivate its rather abstract definition, we will try to prove the duality theorem and see where we need an additional assumption to complete our proof.

With this goal in our minds, we begin with a partial positive result.

Lemma 2.3.1. *Let X be a Banach space of type p. Then X' has cotype q, where $\frac{1}{p}+\frac{1}{q}=1$ and $C_q(X') \leq T_p(X)$.*

Proof. Let $x_1',\dots,x_n' \in X'$. For a given $\varepsilon > 0$ we can find $x_1,\dots,x_n \in X$ such that $\|x_i\| = 1$ and $|x_i'(x_i)| \geq (1-\varepsilon)\,\|x_i'\|$. Therefore we have

$$\left(\sum_{i=1}^n \left|x_i'(x_i)\right|^q\right)^{1/q} \geq (1-\varepsilon)\left(\sum_{i=1}^n \left\|x_i'\right\|^q\right)^{1/q}. \tag{2.3.1}$$

Let $(a_i)_{i=1}^n$ be an arbitrary sequence of scalars with $\sum_{i=1}^n |a_i|^p \leq 1$. Then, by the orthogonality of the Walsh functions

$$\begin{aligned}
\sum_{i=1}^n a_i x_i'(x_i) &= \int_D \left\langle \sum_{i=1}^n \varepsilon_i x_i', \sum_{j=1}^n \varepsilon_j a_j x_j \right\rangle d\mu \leq \int_D \left\|\sum_{i=1}^n \varepsilon_i x_i'\right\| \left\|\sum_{j=1}^n \varepsilon_j a_j x_j\right\| d\mu \\
&\leq \left(\int_D \left\|\sum_{i=1}^n \varepsilon_i x_i'\right\|^2 d\mu\right)^{1/2} \left(\int_D \left\|\sum_{j=1}^n \varepsilon_j a_j x_j\right\|^2 d\mu\right)^{1/2} \\
&\leq \left(\int_D \left\|\sum_{i=1}^n \varepsilon_i x_i'\right\|^2 d\mu\right)^{1/2} T_p(X) \left(\sum_{j=1}^n |a_j|^p \, \|x_i\|^p\right)^{1/p}. \\
&\leq T_p(X) \left(\int_D \left\|\sum_{i=1}^n \varepsilon_i x_i'\right\|^2 d\mu\right)^{1/2}.
\end{aligned} \tag{2.3.2}$$

Since

$$\left(\sum_{i=1}^n \left|x_i'(x_i)\right|^q\right)^{1/q} = \sup\left\{\left|\sum_{i=1}^n a_i x_i'(x_i)\right| : \sum_{i=1}^n |a_i|^p \leq 1\right\},$$

we get by taking the supremum on both sides of (2.3.2)

$$\left(\sum_{i=1}^n \left|x_i'(x_i)\right|^q\right)^{1/q} \leq T_p(X) \left(\int_D \left\|\sum_{i=1}^n \varepsilon_i x_i'\right\|^2 d\mu\right)^{1/2}.$$

Further by (2.3.1),

$$\left(\sum_{i=1}^n \left\|x_i'\right\|^q\right)^{1/q} \leq (1-\varepsilon)^{-1} \left(\sum_{i=1}^n \left|x_i'(x_i)\right|^q\right)^{1/q}$$

$$\leq (1-\varepsilon)^{-1} T_p(X) \left(\int_D \left\| \sum_{i=1}^n \varepsilon_i x_i' \right\|^2 \, d\mu \right)^{1/2}.$$

Since $\varepsilon > 0$ was arbitrary, this shows $C_q(X') \leq T_p(X)$. $\square$

Remark 2.3.2. Note that if X' is of type p, then X is of cotype q: By Lemma 2.3.1, X'' is of cotype q. Since X is canonically isometrically isomorphic to a closed subspace of X'' and the cotype of a space is inherited by its subspaces and is invariant under isomorphisms by Lemmata 2.1.7 and 2.1.8, X is of cotype q as well.

However, we cannot in general deduce in the spirit of Lemma 2.3.1 the type of a Banach space from the cotype of its dual or predual. We give a counter-example: ℓ_1 is of cotype 2 by Corollary 2.1.16, whereas $\ell_\infty = (\ell_1)'$ is not of non-trivial type by Theorem 2.1.18. Moreover, $\ell_1 = (c_0)'$ is of cotype 2, but c_0 is not of non-trivial type by Theorem 2.1.18.

Nevertheless we try to prove that X' has type p if X has cotype q. Of course, we could also try to prove that X has type p if X' has cotype q. But we will see later that the first effort leads to a restriction on X, whereas the second leads to the same restriction on X'. So it will be more useful to take the first approach. Before we can start with the proof, we need to identify the dual of $L^2(D_n, \mathcal{P}(D_n), \mu_n; X)$. This will be done in the next two lemmata.

Lemma 2.3.3. *Let $X_1, X_2, X_3, \dots$ be Banach spaces. Then the direct sum $\bigoplus_{i=1}^\infty X_i$ endowed with the p-norm is again a Banach space and its dual is isometrically isomorphic to $\bigoplus_{i=1}^\infty X_i'$ endowed with the q-norm, where $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. $\bigoplus_{i=1}^\infty X_i$ is a normed space with $\|(x_i)_{i=1}^\infty\|_p = \left(\sum_{i=1}^\infty \|x_i\|_{X_i}^p \right)^{1/p}$. It is routine to show that this space is complete. Choose $\tilde{x}' \in (\bigoplus_{i=1}^\infty X_i)'$. Let $j_i : X_i \hookrightarrow \bigoplus_{i=1}^\infty X_i$ be the canonical isometric embedding. Then

$$\begin{aligned} x_i' : X_i &\to \mathbb{K} \\ x_i &\mapsto \langle \tilde{x}', j_i(x) \rangle \end{aligned}$$

lies in the (topological) dual of X_i. Let

$$\begin{aligned} i : \left(\bigoplus_{i=1}^\infty X_i \right)' &\to \bigoplus_{i=1}^\infty X_i' \\ \tilde{x}' &\mapsto (x_1', x_2', x_3', \dots). \end{aligned}$$

We have to show that i is well-defined. For this let $\varepsilon > 0$ and choose x_i in the unit sphere such that $\langle x_i', x_i \rangle \geq (1-\varepsilon)\|x_i'\|$. Further, let $(a_i)_{i=1}^\infty$ with $\sum_{i=1}^\infty |a_i|^p \leq 1$. Then

$$\sum_{i=1}^\infty |a_i| \|x_i'\| \leq (1-\varepsilon)^{-1} \sum_{i=1}^\infty |a_i| \langle x_i', x_i \rangle = (1-\varepsilon)^{-1} \sum_{i=1}^\infty |a_i| \langle \tilde{x}', j_i(x_i) \rangle$$

$$= (1-\varepsilon)^{-1} \left\langle \tilde{x}', \sum_{i=1}^{\infty} |a_i|\, j_i(x_i) \right\rangle \le (1-\varepsilon)^{-1} \left\|\tilde{x}'\right\| \left(\sum_{i=1}^{\infty} \|a_i x_i\|^p \right)^{1/p}$$
$$= (1-\varepsilon)^{-1} \left\|\tilde{x}'\right\| \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{1/p} \le (1-\varepsilon)^{-1} \left\|\tilde{x}'\right\|.$$

Since $\varepsilon > 0$ and the sequence $(a_i)_{i=1}^{\infty}$ with $\sum_{i=1}^{\infty} |a_i|^p \le 1$ can be chosen arbitrarily, we conclude

$$\left(\sum_{i=1}^{\infty} \left\|x_i'\right\|^q \right)^{1/q} = \sup \left\{ \sum_{i=1}^{\infty} |a_i| \left\|x_i'\right\| : \sum_{i=1}^{\infty} |a_i|^p \le 1 \right\} \le \left\|\tilde{x}'\right\|.$$

This shows that i is well-defined. Conversely, one sees easily that its inverse is given by

$$\begin{aligned} \tilde{i} : \bigoplus_{i=1}^{\infty} X_i' &\to \left(\bigoplus_{i=1}^{\infty} X_i \right)' \\ (x_1', x_2', x_3', \ldots) &\mapsto \left[\tilde{x}' : (x_1, x_2, x_3, \ldots) \mapsto \sum_{i=1}^{\infty} \langle x_i', x_i \rangle \right]. \end{aligned}$$

Moreover, we observe that by the Hölder inequality

$$\left| \sum_{i=1}^{\infty} \langle x_i', x_i \rangle \right| \le \sum_{i=1}^{\infty} \left\|x_i'\right\| \|x_i\| \le \left(\sum_{i=1}^{\infty} \left\|x_i'\right\|^q \right)^{1/q} \left(\sum_{i=1}^{\infty} \|x_i\|^p \right)^{1/p}.$$

Thus, $\|\tilde{x}'\| \le (\sum_{i=1}^{\infty} \|x_i'\|^q)^{1/q}$. We have shown that both i and $\tilde{i}$ are contractive, which implies that i is an isometric isomorphism. □

Lemma 2.3.4. *The dual space of $L^2(D_n, \mathcal{P}(D_n), \mu_n; X)$ is isometrically isomorphic to $L^2(D_n, \mathcal{P}(D_n), \mu_n; X')$. Moreover, each functional is of the form*

$$f \mapsto \int_{D_n} \langle g(\omega), f(\omega) \rangle \, d\mu_n(\omega)$$

for some $g \in L^2(D_n, \mathcal{P}(D_n), \mu_n; X')$.

Proof. Observe that every function in $L^2(D_n, \mathcal{P}(D_n), \mu_n; X)$ is a step function. Therefore $L^2(D_n, \mathcal{P}(D_n), \mu_n; X)$ is isometrically isomorphic to $\bigoplus_{i=1}^{2^n} X$ with weighted ℓ_2-norm. Hence, Lemma 2.3.3 implies that its dual is isomorphic to $L^2(D_n, \mathcal{P}(D_n), \mu_n; X')$. Moreover, one sees that the duality is obtained by associating with $g \in L^2(D_n, \mathcal{P}(D_n), \mu_n; X')$ the functional

$$\begin{aligned} \varphi_g : L^2(D_n, \mathcal{P}(D_n), \mu_n; X) &\to \mathbb{K} \\ f &\mapsto \frac{1}{2^n} \sum_{\omega \in D_n} \langle g(\omega), f(\omega) \rangle = \int_{D_n} \langle g(\omega), f(\omega) \rangle \, d\mu_n(\omega). \end{aligned}$$

Further, the bijective linear operator $g \mapsto \varphi_g$ is isometric:

$$\begin{aligned}
\|\varphi_g\| &= \sup_{\|f\|_{L^2(\mu_n;X)}\leq 1} \frac{1}{2^n}\sum_{\omega\in D_n}\langle g(\omega), f(\omega)\rangle = \sup_{2^{-n/2}\cdot\|(f(\omega))\|_{\ell_2(X)}\leq 1} \frac{1}{2^n}\sum_{\omega\in D_n}\langle g(\omega), f(\omega)\rangle \\
&\overset{\text{Lemma 2.3.3}}{=} \frac{1}{2^{n/2}}\left(\sum_{\omega\in D_n}\|g(\omega)\|^2\right)^{1/2} = \|g\|_{L^2(\mu_n;X')}\,. \qquad \square
\end{aligned}$$

We now assume that X has cotype q and let $x_1', \ldots, x_n' \in X'$. Let $\delta > 0$. As a consequence of Lemma 2.3.4 there exists a function $f \in L^2(D_n, \mathcal{P}(D_n), \mu_n; X)$ in the unit sphere such that

$$(1+\delta)\int_{D_n}\left\langle \sum_{i=1}^n \varepsilon_i(\omega)x_i', f(\omega)\right\rangle d\mu_n(\omega) \geq \left(\int_{D_n}\left\|\sum_{i=1}^n \varepsilon_i(\omega)x_i'\right\|^2 d\mu_n(\omega)\right)^{1/2}. \tag{2.3.3}$$

Further,

$$\begin{aligned}
\int_{D_n}\left\langle \sum_{i=1}^n \varepsilon_i(\omega)x_i', f(\omega)\right\rangle d\mu_n(\omega) &\overset{\text{Theorem C.2.6}}{=} \sum_{i=1}^n \left\langle x_i', \int_{D_n}\varepsilon_i(\omega)f(\omega)\,d\mu_n(\omega)\right\rangle \\
&\leq \sum_{i=1}^n \left\|\int_{D_n}\varepsilon_i(\omega)f(\omega)\,d\mu_n(\omega)\right\| \|x_i'\| \\
&\overset{\text{Hölder ineq.}}{\leq} \left(\sum_{i=1}^n \left\|\int_{D_n}\varepsilon_i(\omega)f(\omega)\,d\mu_n(\omega)\right\|^q\right)^{1/q}\left(\sum_{i=1}^n\|x_i'\|^p\right)^{1/p} \\
&\overset{\text{cot. } q}{\leq} C_q(X)\left(\int_{D_n}\left\|\sum_{i=1}^n\varepsilon_i(\omega)\int_{D_n}\varepsilon_i(\tilde\omega)f(\tilde\omega)d\mu_n(\tilde\omega)\right\|^2 d\mu_n(\omega)\right)^{1/2}\left(\sum_{i=1}^n\|x_i'\|^p\right)^{1/p}.
\end{aligned} \tag{2.3.4}$$

If we want to show that X' is of type p, we must therefore control the middle term. Notice that this term is just the formal vector-valued analogue of the scalar Rademacher projection.

Definition 2.3.5. Let X be a Banach space. The vector-valued *Rademacher projections* R_n^X are given by

$$\begin{aligned}
R_n^X : L^2(D_n, \mathcal{B}(D_n), \mu_n; X) &\to L^2(D_n, \mathcal{B}(D_n), \mu_n; X) \\
f &\mapsto \sum_{i=1}^n \varepsilon_i \int_{D_n}\varepsilon_i(\omega)f(\omega)\,d\mu_n(\omega).
\end{aligned}$$

Lemma 2.3.6. *The Rademacher projections R_n^X are bounded linear operators. Moreover, $\left\|R_n^X\right\| \leq n$.*

Proof. It is clear that the Rademacher projections are linear. Observe that

$$\begin{aligned}\left\|R_n^X f\right\|_{L^2(X)} &\le \sum_{i=1}^n \left(\int_{D_n} \left\| \varepsilon_i(\omega) \int_{D_n} \varepsilon_i(\tilde{\omega}) f(\tilde{\omega})\, d\mu_n(\tilde{\omega}) \right\|^2 d\mu_n(\omega) \right)^{1/2} \\ &= \sum_{i=1}^n \left\| \int_{D_n} \varepsilon_i(\tilde{\omega}) f(\tilde{\omega})\, d\mu_n(\tilde{\omega}) \right\| \le \sum_{i=1}^n \left(\int_{D_n} \|f(\tilde{\omega})\|^2 \, d\mu_n(\tilde{\omega}) \right)^{1/2} \\ &= n \|f\|_{L^2(X)} .\end{aligned}$$

□

We now make the additional assumption that $\sup_{n\in\mathbb{N}} \left\|R_n^X\right\| < \infty$. Since $\delta > 0$ is arbitrary, (2.3.3) and (2.3.4) yield

$$\left(\int_{D_n} \left\| \sum_{i=1}^n \varepsilon_i(\omega) x_i' \right\|^2 d\mu_n(\omega) \right)^{1/2} \le C_q(X) \sup_{n\in\mathbb{N}} \left\|R_n^X\right\| \left(\sum_{i=1}^n \left\|x_i'\right\|^p \right)^{1/p} .$$

So under this additional assumption we can prove in a natural way that X' has type p if X has cotype q. This is exactly why the above assumption is the definition of *K-convexity* which was introduced by B. Maurey and G. Pisier [MP76]. However, it is not clear why they chose the letter K!

Definition 2.3.7. A Banach space X is called *K-convex* if $\sup_{n\in\mathbb{N}} \left\|R_n^X\right\| < \infty$. We call $K(X) := \sup_{n\in\mathbb{N}} \left\|R_n^X\right\|$ the *K-convexity constant* of X.

Remark 2.3.8. Note that by essentially redoing the above calculations, one sees that if X' is K-convex, then X has type p if X' has cotype q. We will prove later in Theorem 2.5.2 that X is K-convex if and only if X' is K-konvex.

We present some obvious properties of K-convexity.

Lemma 2.3.9. *Let Y be a closed subspace of a K-convex Banach space X. If X is K-convex, then so is Y.*

Lemma 2.3.10. *Let $T : X \to Y$ be an isomorphism of two Banach spaces X and Y. Then X is K-convex if and only if Y is K-convex.*

An important class of examples of K-convex spaces are Hilbert spaces.

Example 2.3.11. Let H be a Hilbert space. Recall that we have seen in Remark 2.1.3 that $f \in L^2(D_n, \mathcal{P}(D_n), \mu_n; H)$ can be written as $f = \sum_{A\in\mathcal{P}(D_n)} \varepsilon_A x_A$ for some x_A in H. The L^2-norm is then given by

$$\begin{aligned}\|f\|^2 &= \int_{D_n} \left\langle \sum_A \varepsilon_A(\omega) x_A, \sum_B \varepsilon_B(\omega) x_B \right\rangle d\mu_n(\omega) \\ &= \sum_A \sum_B \langle x_A, x_B \rangle \int_{D_n} \varepsilon_A(\omega) \varepsilon_B(\omega)\, d\mu_n(\omega) = \sum_A \|x_A\|^2 .\end{aligned}$$

Moreover,

$$\left\|R_n^X f\right\|^2 = \int_{D_n} \left\langle R_n^X f(\omega), R_n^X f(\omega)\right\rangle \, d\mu_n(\omega) = \int_{D_n} \left\langle \sum_{i=1}^n \varepsilon_i(\omega)x_i, \sum_{j=1}^n \varepsilon_j(\omega)x_j \right\rangle \, d\mu_n(\omega)$$
$$= \sum_{i,j=1}^n \langle x_i, x_j\rangle \int_{D_n} \varepsilon_i(\omega)\varepsilon_j(\omega)\, d\mu_n(\omega) = \sum_{i=1}^n \|x_i\|^2 \leq \|f\|^2 .$$

This shows that R_n^X is a contraction for all $n \in \mathbb{N}$. Hence, H is K-convex.

Remark 2.3.12. Note that we have not given an example of a non K-convex Banach space yet. This will be done later in the proof of Pisier's Theorem.

Remark 2.3.13. The density of $\cup_{n\in\mathbb{N}} L^2(D_n, \mathcal{P}(D_n), \mu_n; X)$ (see Remark 2.1.3) gives us an equivalent characterization of K-convexity: a Banach space X is K-convex if and only if the Rademacher projections R_n^X extend to a bounded linear operator R^X on $L^2(D, \mathcal{B}(D), \mu; X)$. In this case we see that

$$\left\|R^X\right\| = \sup_{n\in\mathbb{N}} \left\|R_n^X\right\| = K(X).$$

This is much closer to the definition given in [Pis80a] and [Pis82]. They called a Banach space X K-convex if and only if for the scalar-valued Rademacher projection $R^{\mathbb{K}}$ the bounded linear operator

$$R^{\mathbb{K}} \otimes_\pi \mathrm{Id}_X : L^2(D, \mathcal{B}(D), \mu) \otimes_\pi X \to L^2(D, \mathcal{B}(D), \mu; X)$$

extends to a bounded linear operator on $L^2(D, \mathcal{B}(D), \mu; X)$ (we can naturally identify $L^2(D, \mathcal{B}(D), \mu) \otimes_\pi X$ with a dense subspace of $L^2(D, \mathcal{B}(D), \mu; X)$; for more details see C.2.3). A careful look at both extension methods shows that they yield the same result for the dense set $\cup_{n\in\mathbb{N}} L^2(D_n, \mathcal{P}(D_n), \mu_n; X)$. Since there is only one single way to extend a bounded operator on a dense subset to a bounded operator on the closure, our definition of K-convexity is equivalent to Pisier's and Maurey's definition.

Remark 2.3.14. Let X be K-convex. Observe that the Kahane-Khintchine inequality implies for $2 \leq p < \infty$, $f \in L^p(D_n, \mathcal{P}(D_n), \mu_n; X)$ and $x_i := \int_{D_n} \tilde{\varepsilon_i}(\tilde{\omega}) f(\tilde{\omega})\, d\mu_n(\tilde{\omega})$ that

$$\begin{aligned}
\left\|R_n^X f\right\|_{L^p(\mu_n;X)} &= \left(\int_{D_n} \left\|\sum_{i=1}^n \varepsilon_i(\omega)x_i\right\|^p \, d\mu_n(\omega)\right)^{1/p} \\
&\leq C_p \left(\int_{D_n} \left\|\sum_{i=1}^n \varepsilon_i(\omega)x_i\right\|^2 \, d\mu_n(\omega)\right)^{1/2} \\
&\leq C_p K(X) \|f\|_{L^2(\mu_n;X)} \overset{\text{Hölder ineq.}}{\leq} C_p K(X) \|f\|_{L^p(\mu_n;X)} .
\end{aligned}$$

Thus if X is K-convex, the Rademacher projections R_n^X extend to a bounded linear operator from $L^p(D, \mathcal{B}(D), \mu; X)$ into itself.

We can now summarize our achievements in the following theorem.

Theorem 2.3.15. *Let X be a K-convex Banach space. Then X is of cotype q if and only if X' is of type p, where $\frac{1}{p} + \frac{1}{q} = 1$. Moreover,*

$$C_q(X) \leq T_p(X') \leq K(X) C_q(X)$$

and the implication from type to cotype stays true without the additional assumption of K-convexity.

Proof. Let X' be of type p. Then X'' is of cotype q and $C_q(X'') \leq T_p(X')$ by Lemma 2.3.1. Since X is canonically isometrically isomorphic to a closed subspace of X'', X is of cotype q and $C_q(X) \leq C_q(X'') \leq T_p(X')$ by Lemmata 2.1.7 and 2.1.8. That X' is of type p and that $T_p(X') \leq K(X) C_q(X)$ holds if X is of cotype q has already been shown in the motivation of the definition of K-convexity. □

2.4 Pisier's Theorem

Pisier's theorem states that a Banach space is K-convex if and only if it is B-convex. We will see that it can be shown by elementary (yet tricky!) arguments that K-convexity implies B-convexity. The converse however is a very deep result whose proof uses the theory of holomorphic semigroups. The extremely beautiful proof was given by Pisier [Pis80a], [Pis82]. Before, there was no evidence that the converse holds: two years before the proof was given, Pisier put a lot of effort in showing the existence of a B-convex space that is not K-convex [Pis80b]. It was even unknown that every uniformly convex space is K-convex.

We will follow the presentations in [Pis80a] and [DJT95].

2.4.1 K-convexity implies B-convexity

As a first step, we show that ℓ_1 is not K-convex. After that we will extend this result to Banach spaces not containing ℓ_1^n's uniformly.

Lemma 2.4.1. *ℓ_1 is not K-convex.*

Proof. Denote $(e_n)_{n=1}^{\infty}$ the canonical Schauder basis of ℓ_1. Fix $n \in \mathbb{N}$. We define $I_n := 2^{-n} \sum_{i=1}^{2^n} e_n$ and the *Rademacher sequences* $r^1, \ldots, r^n$ (which depend on n!) given by

$$r_k^n = \begin{cases} 2^{-n} & \text{if } k \in \cup_{m=1}^{2^{k-1}} [2(m-1)2^{n-k} + 1, (2m-1)2^{n-k}] \\ -2^{-n} & \text{if } k \in \cup_{m=1}^{2^{k-1}} [(2m-1)2^{n-k} + 1, (2m)2^{n-k}] \\ 0 & \text{if } k > 2^n. \end{cases}$$

For example, we have for $n = 3$

$$r^1 = \frac{1}{8}(1, 1, 1, 1, -1, -1, -1, -1, 0, \ldots)$$

$$r^2 = \frac{1}{8}(1, 1, -1, -1, 1, 1, -1, -1, 0, \ldots)$$
$$r^3 = \frac{1}{8}(1, -1, 1, -1, 1, -1, 1, -1, 0, \ldots).$$

Finally, let $f_n : D_n \to \ell_1$ be the vector-valued function

$$f_n : \omega \mapsto \prod_{i=1}^{n} (I_n + \varepsilon_i(\omega) r^i),$$

where multiplication means componentwise multiplication. The k-th component ($1 \leq k \leq 2^n$) of $I_n + \varepsilon_i(\omega) r^i$ is zero if r^i_k and $\varepsilon_i(\omega)$ have different signs and $2 \cdot 2^{-n}$ otherwise. Thus the k-th component of the product does not vanish if and only if $r^i_k = \varepsilon_i(\omega)$ for all $1 \leq i \leq n$. As k varies, $(r^i_k)_{i=1}^n$ gives us precisely each possible combination of signs. Therefore exactly one component does not vanish and we have $f_n(\omega) = (\delta_{km})_{m=1}^{\infty}$ for some $1 \leq k \leq 2^n$. Hence, $\|f_n(\omega)\|_1 = 1$ for all $\omega \in D_n$ and $\|f_n\|_{L^2(D_n, \mathcal{P}(D_n), \mu_n; \ell_1)} = 1$. Moreover,

$$\begin{aligned}
&\int_{D_n} \varepsilon_i(\omega) \prod_{k=1}^{n} (I_n + \varepsilon_k(\omega) r^k) \, d\mu_n(\omega) \\
&\quad = \int_{D_n} \varepsilon_i(\omega) I_n \, d\mu_n(\omega) + \sum_{\substack{A \in \mathcal{P}(\{1,\ldots,n\}) \\ A \neq \emptyset}} \int_{D_n} \varepsilon_i(\omega) \varepsilon_A(\omega) \prod_{k \in A} r^k \, d\mu_n(\omega) \\
&\quad = \int_{D_n} \varepsilon_i^2(\omega) r^i \, d\mu_n(\omega) = r^i
\end{aligned}$$

because the Walsh functions form an orthonormal system. Thus $R^X_n f_n = \sum_{i=1}^n \varepsilon_i r^i$. Since ℓ_1 has cotype 2 by Corollary 2.1.16, we have

$$\begin{aligned}
\left\| R^X_n f_n \right\|_{L^2(X)} &= \left(\int_{D_n} \left\| \sum_{i=1}^{n} \varepsilon_i(\omega) r^i \right\|_1^2 d\mu_n(\omega) \right)^{1/2} \\
&\geq C(\ell_1)^{-1} \left(\sum_{i=1}^{n} \left\| r^i \right\|_1^2 \right)^{1/2} = C(\ell_1)^{-1} n^{1/2}.
\end{aligned}$$

So $\left\| R^X_n \right\| \geq C(\ell_1)^{-1} n^{1/2}$ is not bounded and therefore ℓ_1 is not K-convex. □

Notice that in the above proof we actually only worked in $\ell_1^{2^n}$. So if X contains ℓ_1^n's uniformly, we can just redo the calculations. This will be done in the next proof.

Theorem 2.4.2. *Every K-convex Banach space is B-convex.*

Proof. Suppose X is a Banach space that is not B-convex. Then X contains ℓ_1^n's λ-uniformly for some $\lambda > 1$. Hence for $n \in \mathbb{N}$, we can find vectors $x_1, \ldots, x_n$ such that the canonical isomorphism $T_n : X_n := \operatorname{span}\{x_1, \dots, x_n\} \to \ell_n^1$ has multiplicative Banach-Mazur distance smaller than λ. Let $r^1, \dots, r^n$ be the Rademacher sequences as in the proof of Lemma 2.4.1 (we will use the notation from there in this proof without further notice), this time seen as elements of $\ell_1^{2^n}$. Define $\tilde{f}_n : D_n \ni \omega \mapsto T_{2^n}^{-1} f_n(\omega)$. We have $\left\|\tilde{f}_n(\omega)\right\| \le \left\|T_{2^n}^{-1}\right\|$ for all $\omega \in D_n$. Further, we have

$$\int_{D_n} \varepsilon_i(\omega) T_{2^n}^{-1} \left(\prod_{k=1}^n (I_n + \varepsilon_k(\omega) r^k) \right) d\mu_n(\omega)$$
$$= T_{2^n}^{-1} \left(\int_{D_n} \varepsilon_i(\omega) \prod_{k=1}^n (I_n + \varepsilon_k(\omega) r^k)\, d\mu_n(\omega) \right) = T_{2^n}^{-1}(r^i).$$

Thus, $R_n^X \tilde{f}_n = \sum_{i=1}^n \varepsilon_i T_{2^n}^{-1}(r^i)$. Moreover, we see that for all $\omega \in D_n$

$$\left\| \sum_{i=1}^n \varepsilon_i(\omega) r^i \right\|_1 = \left\| T_{2^n} T_{2^n}^{-1} \left(\sum_{i=1}^n \varepsilon_i(\omega) r^i \right) \right\|_1 \le \|T_{2^n}\| \left\| T_{2^n}^{-1} \left(\sum_{i=1}^n \varepsilon_i(\omega) r^i \right) \right\|$$

and therefore

$$\left\| R_n^X \tilde{f}_n \right\|_{L^2(X)} \ge \|T_{2^n}\|^{-1} \left(\int_{D_n} \left\| \sum_{i=1}^n \varepsilon_i(\omega) r^i \right\|_1^2 d\mu_n(\omega) \right)^{1/2} \ge \|T_{2^n}\|^{-1} C(\ell_1)^{-1} n^{1/2}.$$

Since $\left\| \left\|T_{2^n}^{-1}\right\|^{-1} \tilde{f}_n \right\|_{L^2(X)} \le 1$, we conclude that

$$\left\|R_n^X\right\| \ge \left(\left\|T_{2^n}^{-1}\right\| \|T_{2^n}\| \right)^{-1} C(\ell_1)^{-1} n^{1/2} \ge \lambda^{-1} C(\ell_1)^{-1} n^{1/2}$$

for all $n \in \mathbb{N}$. This shows that X is not K-convex. □

2.4.2 Pisier's Proof of B-convexity implies K-convexity

As Pisier's proof is quite involved, we will first sketch its main ideas. Thereafter we will give a complete proof divided into several steps.

Sketch of the Proof

We will show that the Rademacher projections R_m^X can be written as sums of conditional expectation operators, more precisely that

$$R_m^X = \sum_{i=1}^m \left(\mathbb{E}[\cdot|\sigma(\varepsilon_i)] - \mathbb{E}[\cdot] \right).$$

Instead of analyzing the right hand side directly, one considers the strongly continuous semigroup

$$S(t) := \prod_{i=1}^m (P_i + e^{-t}(\mathrm{Id} - P_i)),$$

where P_i is the conditional expectation with respect to all but the i-th coordinate. Expanding the above product and writing it as a finite power series in e^{-t}, that is $S(t) = \sum_{k=0}^{m} e^{-tk} Q_k$, we will see that its second coefficient Q_1 is exactly R_m^X. We will show with the help of the Kato-Beurling theorem that the B-convexity of X implies that $(S(t))$ can be extended to a bounded holomorphic semigroup on some sector. The crucial fact in this step is that the size of the sector und the upper bound can be chosen independently of X and the semigroup (and therefore of the projections and m!). The holomorphy of $(S(t))$ allows us to express R_m^X, completely analogous to Cauchy's integral formula, as a curve integral in the complex plane. So the uniform bounds on $(S(z))$ in the sector yield uniform bounds for the Rademacher projections!

Step 1: Obtaining the Holomorphy of the Semigroup

Our goal for this step is the proof of the following theorem.

Theorem 2.4.3. *Let X be a complex B-convex Banach space. Then there are constants $\delta > 0$ and $M > 0$ such that for any finite number of commuting norm-one projections $P_1, \ldots, P_m$*

$$S(t) \coloneqq \prod_{i=1}^{m} (P_i + e^{-t}(\operatorname{Id} - P_i))$$

is a holomorphic semigroup on $S(0, \delta)$ bounded by M.

First of all, we show that $(S(t))$ is a strongly continuous semigroup.

Lemma 2.4.4. *The family of mappings $(S(t))$ defined in Theorem 2.4.3 forms a strongly continuous semigroup of contractions.*

Proof. We see that

$$S(0) = \prod_{i=1}^{m} (P_i + \operatorname{Id} - P_i) = \operatorname{Id}.$$

The semigroup law can be verified by a direct calculation as well:

$$\begin{aligned} S(t)S(s) &= \left(\prod_{i=1}^{m} (P_i + e^{-t}(\operatorname{Id} - P_i)) \right) \left(\prod_{i=1}^{m} (P_i + e^{-s}(\operatorname{Id} - P_i)) \right) \\ &\overset{P_i \text{ commute}}{=} \prod_{i=1}^{m} (P_i + e^{-t}(\operatorname{Id} - P_i))(P_i + e^{-s}(\operatorname{Id} - P_i)) \\ &= \prod_{i=1}^{m} (P_i^2 + (e^{-t} + e^{-s})(P_i - P_i^2) + e^{-(t+s)}(\operatorname{Id} - P_i)^2) \\ &\overset{P_i^2 = P_i}{=} \prod_{i=1}^{m} (P_i + e^{-(t+s)}(\operatorname{Id} - P_i)) = S(t+s). \end{aligned}$$

$S(t)$ is the composition of linear mappings and therefore a linear mapping. For a subset of $\{1,\ldots,m\}$ set $P_A \coloneqq \prod_{i\in A} P_i$, with the convention that $P_\emptyset = \mathrm{Id}$. Then

$$S(t) = \prod_{i=1}^{m}(e^{-t}\,\mathrm{Id} + (1-e^{-t})P_i) = \sum_{i=0}^{m} e^{-(m-i)t}(1-e^{-t})^i \sum_{|A|=i} P_A.$$

Hence,

$$\|S(t)\| \le \sum_{i=0}^{m} e^{-(m-i)t}(1-e^{-t})^i \binom{m}{i} = (e^{-t}+1-e^{-t})^m = 1.$$

Therefore $(S(t))$ is a semigroup of contractions. It remains to show that the semigroup is strongly continuous. For every $x \in X$ we have

$$\begin{aligned}
\|S(t)x - x\| &= \left\| \left(\sum_{i=0}^{m} e^{-(m-i)t}(1-e^{-t})^i \sum_{|A|=i} P_A \right) x - x \right\| \\
&= \left\| \left(\sum_{i=1}^{m} e^{-(m-i)t}(1-e^{-t})^i \sum_{|A|=i} P_A \right) x - (1-e^{-mt})x \right\| \\
&\le \left(\sum_{i=1}^{m} e^{-(m-i)t}(1-e^{-t})^i \binom{m}{i} + 1 - e^{-mt} \right) \|x\| \\
&= ((e^{-t}+1-e^{-t})^m - e^{-mt} + 1 - e^{-mt})\,\|x\| \\
&= 2(1-e^{-mt})\,\|x\|\,.
\end{aligned}$$

So $\lim_{t\to 0}\|S(t)x - x\| = 0$ as desired. ☐

Lemma 2.4.5. *Suppose that X is a B-convex Banach space. Then there exist a real number $0 < \rho < 2$ and a natural number N such that any N commuting norm-one projections $P_1,\ldots,P_N \in \mathcal{L}(X)$ satisfy*

$$\left\| \prod_{i=1}^{N} (\mathrm{Id} - P_i) \right\| \le \rho^N.$$

Proof. Assume that the conclusion is false. Then we can find for every $\varepsilon > 0$ and every natural number N commuting norm-one projections $P_1,\ldots,P_N \in \mathcal{L}(X)$ such that

$$\left\| \prod_{i=1}^{N} (\mathrm{Id} - P_i) \right\| > 2^N - \frac{\varepsilon}{2}.$$

Hence, there exists a unit vector $x \in X$ such that

$$\left\| \prod_{i=1}^{N} (\mathrm{Id} - P_i)x \right\| > 2^N - \varepsilon.$$

We are going to show that X is not B-convex, more precisely that $\{x, P_1 x, \ldots, P_N x\}$ is $(1-2^N\varepsilon)^{-1}$-isomorphic to ℓ^1_{N+1}, that is

$$(1-2^N\varepsilon)\sum_{i=0}^{N}|\alpha_i| \le \left\|\sum_{i=0}^{N}\alpha_i P_i x\right\| \le \sum_{i=0}^{N}|\alpha_i| \tag{2.4.1}$$

for arbitrary $(\alpha_i) \in \ell_1^{N+1}$, where we have set $P_0 \coloneqq \mathrm{Id}$ to simplify notations. Notice that the second inequality in (2.4.1) is trivial, so it remains to show the first one. As a first step, we show that for an arbitrary choice of signs $(\varepsilon_i) \in D_N$ the following inequality holds:

$$\left\|x + \sum_{i=1}^{N}\varepsilon_i P_i x\right\| \ge N + 1 - 2^N\varepsilon. \tag{2.4.2}$$

After that we will use inequality (2.4.2) to show the general estimate (2.4.1). We set $P \coloneqq \{i : \varepsilon_i = 1\}$ and $M \coloneqq \{i : \varepsilon_i = -1\}$. For a subset A of $\{1, \ldots, N\}$, let $P_C :$ $= \prod_{i\in A} P_i$, where by convention $P_\emptyset$ is the identity operator. Seperating the natural numbers belonging to P and respectively to M and expanding the first product yield (remember that the projections commute)

$$\begin{aligned}\prod_{i=1}^{N}(\mathrm{Id}-P_i) = \prod_{i\in P}(\mathrm{Id}-P_i)\prod_{i\in M}(\mathrm{Id}-P_i) &= \left(\sum_{A\subset P}(-1)^{|A|}P_A\right)\prod_{i\in M}(\mathrm{Id}-P_i)\\ &= \left(\sum_{A\subsetneq P}(-1)^{|A|}P_A + (-1)^{|P|}P_P\right)\prod_{i\in M}(\mathrm{Id}-P_i).\end{aligned}$$

Rearranging yields

$$(-1)^{|P|}P_P\prod_{i\in M}(\mathrm{Id}-P_i) = \prod_{i=1}^{N}(\mathrm{Id}-P_i) - \left(\sum_{A\subsetneq P}(-1)^{|A|}P_A\right)\prod_{i\in M}(\mathrm{Id}-P_i).$$

Since the second sum is over $2^{|P|}-1$ operators, each of them having operator norm at most $2^{|M|}$, we have

$$\begin{aligned}\left\|P_P\prod_{i\in M}(\mathrm{Id}-P_i)x\right\| &\ge \left\|\prod_{i=1}^{N}(\mathrm{Id}-P_i)x\right\| - \left\|\left(\sum_{A\subsetneq P}(-1)^{|A|}P_A\right)\prod_{i\in M}(\mathrm{Id}-P_i)x\right\|\\ &> 2^N - \varepsilon - (2^{|P|}-1)2^{|M|} = 2^N - \varepsilon - 2^N + 2^{|M|} = 2^{|M|} - \varepsilon.\end{aligned}$$

Since the projections P_i commute, we see that

$$P_P\prod_{i\in P}(\mathrm{Id}+P_i) = \prod_{i\in P}P_i(\mathrm{Id}+P_i) = \prod_{i\in P}2P_i = 2^{|P|}P_P.$$

Further, since the projections P_i are contractive, we conclude

$$\left\|\prod_{i=1}^{N}(\mathrm{Id}+\varepsilon_i P_i)x\right\| = \left\|\prod_{i\in P}(\mathrm{Id}+P_i)\prod_{i\in M}(\mathrm{Id}-P_i)x\right\| \ge \left\|P_P\prod_{i\in P}(\mathrm{Id}+P_i)\prod_{i\in M}(\mathrm{Id}-P_i)x\right\|$$

$$= \left\| 2^{|P|} P_P \prod_{i \in M} (\mathrm{Id} - P_i) x \right\| \geq 2^{|P|} (2^{|M|} - \varepsilon) \geq 2^N (1 - \varepsilon).$$

Expanding the product $\prod_{i=1}^N (\mathrm{Id} + \varepsilon_i P_i) x$ yields

$$x + \sum_{i=1}^N \varepsilon_i P_i x = \prod_{i=1}^N (\mathrm{Id} + \varepsilon_i P_i) x - \Sigma',$$

where Σ' is the sum of $2^N - (N+1)$ vectors in the unit ball of X. Now we can finally establish inequality (2.4.2):

$$\left\| x + \sum_{i=1}^N \varepsilon_i P_i x \right\| \geq 2^N - 2^N \varepsilon - 2^N + N + 1 = N + 1 - 2^N \varepsilon.$$

Now we essentially repeat the last part of the proof of Theorem 2.2.19. Choose $(\alpha_i) \in \ell_1^{N+1}(\mathbb{R})$. Set $S = \sum_{i=0}^N |\alpha_i|$ and choose $\varepsilon_i \coloneqq \operatorname{sign} \alpha_i$. Then by (2.4.2)

$$\begin{aligned}
\left\| \sum_{i=0}^N \alpha_i P_i x \right\| = \left\| \sum_{i=0}^N \varepsilon_i \left|\alpha_i\right| P_i x \right\| &\geq \left\| \sum_{i=0}^N \varepsilon_i S P_i x \right\| - \left\| \sum_{i=0}^N \varepsilon_i (S - |\alpha_i|) P_i x \right\| \\
&\geq S(N + 1 - 2^N \varepsilon) - \sum_{i=0}^N (S - |\alpha_i|) \\
&= S(N + 1 - 2^N \varepsilon) - (N+1)S + S = S(1 - 2^N \varepsilon) \\
&= (1 - 2^N \varepsilon) \sum_{i=0}^N |\alpha_i| .
\end{aligned}$$

This shows (2.4.1). Hence, X or its underlying real Banach space in the complex case contains $\ell_1^N(\mathbb{R})$'s λ-uniformly for all $\lambda > 1$. In the case of a real Banach space Lemma 2.2.19 shows directly that X is not B-convex. By the same argument, in the second case of a complex Banach space its underlying real Banach space is not B-convex either. Since $B \subset \mathbb{R}$, B-convexity is a real notion and therefore X is not B-convex either (notice that Theorem 2.2.8 even shows that X contains $\ell_1^N(\mathbb{C})$'s λ-uniformly for all $\lambda > 1$). ☐

Proof of Theorem 2.4.3. By Lemma 2.4.5, there exist a real number $0 < \rho < 2$ and a natural number N such that any N commuting norm-one projections $P_1, \ldots, P_N \in \mathcal{L}(X)$ satisfy

$$\left\| \prod_{i=1}^N (\mathrm{Id} - P_i) \right\| \leq \rho^N.$$

We will show that for every $t \geq 0$ one has

$$\left\| (\mathrm{Id} - S(t))^N \right\| \leq \rho^N. \tag{2.4.3}$$

From this the holomorphy of $(S(t))$ follows directly from the Kato-Beurling criterion (Theorem 1.2.6). More precisely, there exists a $\delta > 0$ such that all those strongly continuous semigroups extend to holomorphic semigroups on $S(0,\delta)$. Moreover, choosing δ a little bit smaller if necessary, there exists a positive constant M such that all these extensions are bounded by M (on the new potentially smaller sector $S(0,\delta)$).

To show (2.4.3) we introduce a probabilistic model: an unfair coin toss which is given on $\{0,1\}$ by the probabilities

$$\tilde{\mathbb{P}}(\{1\}) = e^{-t}, \quad \tilde{\mathbb{P}}(\{0\}) = 1 - e^{-t}.$$

Now consider the m-fold product probability space

$$(\Omega, \Sigma, \mathbb{P}) \coloneqq \left(\{0,1\}^m, \mathcal{P}\left(\{0,1\}^m\right), \bigotimes_{i=1}^m \tilde{\mathbb{P}} \right).$$

The coordinate functions $X_i : \{0,1\}^m \to \{0,1\}$ are random variables modelling m independent equally distributed unfair coin tosses as described above. Now set

$$\pi(\omega) \coloneqq \prod_{i=1}^m (P_i + X_i(\omega)(\operatorname{Id} - P_i)) = \prod_{i:\, X_i(\omega)=0} P_i.$$

This shows that $\pi(\omega)$ is a norm-one projection for every $\omega \in \Omega$. Further, the expectation of π is given by

$$\begin{aligned} \int_\Omega \pi(\omega)\, d\,\mathbb{P}(\omega) &\overset{\text{ind.}}{=} \prod_{i=1}^m \int_\Omega P_i + X_i(\omega)(\operatorname{Id} - P_i)\, d\,\mathbb{P}(\omega) \\ &= \prod_{i=1}^m (e^{-t} \operatorname{Id} + (1 - e^{-t}) P_i) = \prod_{i=1}^m (P_i + e^{-t}(\operatorname{Id} - P_i)) = S(t). \end{aligned}$$

Moreover, we have

$$(\operatorname{Id} - S(t))^N = \prod_{i=1}^N \int_\Omega (\operatorname{Id} - \pi(\omega))\, d\,\mathbb{P}(\omega) = \int \cdots \int \prod_{i=1}^N (\operatorname{Id} - \pi(\omega_i))\, d\,\mathbb{P}(\omega_1) \,\ldots\, d\,\mathbb{P}(\omega_N).$$

Since for every ω_i the norm-one projection $\pi(\omega_i)$ is a finite product of the P_is, they commute with each other and as a consequence of Lemma 2.4.5 we have

$$\left\| \prod_{i=1}^N (\operatorname{Id} - \pi(\omega_i)) \right\| \le \rho^N.$$

Hence,

$$\begin{aligned} \left\| (\operatorname{Id} - S(t))^N \right\| &\le \int \cdots \int \left\| \prod_{i=1}^N (\operatorname{Id} - \pi(\omega_i)) \right\| d\,\mathbb{P}(\omega_1) \,\ldots\, d\,\mathbb{P}(\omega_N) \\ &\le \int \cdots \int \rho^N d\,\mathbb{P}(\omega_1) \,\ldots\, d\,\mathbb{P}(\omega_N) = \rho^N. \end{aligned}$$

This shows (2.4.3) and the properties of the holomorphic extension of $(S(t))$. $\square$

Step 2: Constructing Projections: Conditional Expectations

In order to use the holomorphy of the semigroups of projections shown in the previous step, it is necessary to construct projections systematically and to understand their behaviour. More precisely, we want to use the uniform boundedness of the semigroups in a sector independent of the projections to show the existence of a uniform bound for the Rademacher projections. Therefore one wants to construct the Rademacher projections out of projections on $L^p(X)$. Natural candidates are the conditional expectation operators with respect to the σ-algebras generated by some of the ε_is. Remember that in general the conditional expectation operators do not commute with each other as needed in Theorem 2.4.3. However, this is true if we additionally require the independence of the given σ-algebras. For the theory of vector-valued conditional expectations we refer to Appendix D.3.

Lemma 2.4.6. *Let $(\mathcal{F}_i)_{i\in I}$ be a family of independent sub-σ-algebras of some probability space $(\Omega, \Sigma, \mathbb{P})$ and $A, B \subset I$. Then*

$$\mathbb{E}\left[\mathbb{E}\left[\cdot\middle|\sigma\left(\bigcup_{i\in A}\mathcal{F}_i\right)\right]\middle|\sigma\left(\bigcup_{i\in B}\mathcal{F}_i\right)\right] = \mathbb{E}\left[\cdot\middle|\sigma\left(\bigcup_{i\in A\cap B}\mathcal{F}_i\right)\right].$$

In particular, the conditional expectation operators with respect to some union $\sigma(\cup_{i\in P}\mathcal{F}_i)$ of independent σ-algebras for some $P \in \mathcal{P}(I)$ form a family of commuting norm-one projections.

Proof. Let $f \in L^1(\Omega, \Sigma, \mathbb{P}; X)$. We are going to show that $\mathbb{E}[f|\sigma(\cup_{i\in A\cap B}\mathcal{F}_i)]$ is a version of $\mathbb{E}[\mathbb{E}[f|\sigma(\bigcup_{i\in A}\mathcal{F}_i)]|\sigma(\bigcup_{i\in B}\mathcal{F}_i)]$. Clearly, $\mathbb{E}[f|\sigma(\cup_{i\in A\cap B}\mathcal{F}_i)]$ is $\sigma(\cup_{i\in B}\mathcal{F}_i)$- measurable. It remains to verify that for all $A \in \sigma(\cup_{i\in B}\mathcal{F}_i)$ we have

$$\int_A \mathbb{E}\left[f\middle|\sigma\left(\bigcup_{i\in A\cap B}\mathcal{F}_i\right)\right] d\,\mathbb{P} = \int_A \mathbb{E}\left[f\middle|\sigma\left(\bigcup_{i\in A}\mathcal{F}_i\right)\right] d\,\mathbb{P}. \tag{2.4.4}$$

We first observe that the set $\mathcal{D}$ of all $A \in \sigma(\cup_{i\in B}\mathcal{F}_i)$ satisfying the above identity is a Dynkin system (see Definition C.1.1). Moreover, let $\mathcal{E}$ be the set of all A of the form $A = \cap_{i=1}^n A_i$ where $A_i \in \mathcal{F}_{j_i}$ for some $j_i \in B$, i.e. which are finite intersections of some sets, each belonging to one σ-algebra of the family $(\mathcal{F}_i)_{i\in B}$. Then $\mathcal{E}$ is stable under finite intersections and the σ-algebra generated by $\mathcal{E}$ coincides with $\sigma(\cup_{i\in B}\mathcal{F}_i)$ because of $\mathcal{F}_i \subset \mathcal{E}$. Next we want to show that $\mathcal{E} \subset \mathcal{D}$. For this purpose let $A = (\cap_{i=1}^n B_i) \cap (\cap_{i=1}^m A_i) \in \mathcal{E}$ for some $A_i \in \cup_{j\in A\cap B}\mathcal{F}_j$ and some $B_i \in \cup_{j\in B\setminus A}\mathcal{F}_j$. Then

$$\begin{aligned}\int_A \mathbb{E}\left[f\middle|\sigma\left(\bigcup_{i\in A\cap B}\mathcal{F}_i\right)\right] d\,\mathbb{P} &= \int_{\cap_{i=1}^m A_i} \mathbb{E}\left[f\middle|\sigma\left(\bigcup_{i\in A\cap B}\mathcal{F}_i\right)\right] \prod_{i=1}^n \mathbb{1}_{B_i}\, d\,\mathbb{P} \\ &\overset{\text{ind.}}{=} \prod_{i=1}^n \mathbb{P}(B_i) \cdot \int_{\cap_{i=1}^m A_i} \mathbb{E}\left[f\middle|\sigma\left(\bigcup_{i\in A\cap B}\mathcal{F}_i\right)\right] d\,\mathbb{P}\end{aligned}$$

$$= \prod_{i=1}^{n} \mathbb{P}(B_i) \cdot \int_{\cap_{i=1}^m A_i} f \, d\mathbb{P} \, .$$

For the right hand side of (2.4.4) we have analogously

$$\int_A \mathbb{E}\left[f \middle| \sigma\left(\bigcup_{i\in A} \mathcal{F}_i \right) \right] d\mathbb{P} = \int_{\cap_{i=1}^m A_i} \mathbb{E}\left[f \middle| \sigma\left(\bigcup_{i\in A} \mathcal{F}_i \right) \right] \prod_{i=1}^{n} \mathbb{1}_{B_i} \, d\mathbb{P}$$
$$= \prod_{i=1}^{n} \mathbb{P}(B_i) \cdot \int_{\cap_{i=1}^m A_i} f \, d\mathbb{P} \, .$$

Thus $A \in \mathcal{D}$ as desired. Now Dynkin's theorem C.1.3 shows that $\sigma(\cup_{i\in B}\mathcal{F}_i) = \sigma(\mathcal{E}) = \mathcal{D}(\mathcal{E}) \subset \mathcal{D}$. Since one has trivially $\mathcal{D} \subset \sigma(\cup_{i\in B}\mathcal{F}_i)$, we have shown that (2.4.4) holds for all $A \in \sigma(\cup_{i\in B}\mathcal{F}_i)$. Notice that we can interchange the roles of A and B. Hence, the commutativity of the conditional expectation operators follows directly. □

Step 3: Finishing the proof

We can now finally establish the main result of Pisier's theorem. We repeat the short description of the proof: the conditional expectation operators will yield a representation of the Rademacher projections as products of commuting norm-one projections. But for such projections we can use the holomorphy of the corresponding semigroup on a uniform sector to obtain the uniform boundedness of the Rademacher projections.

Theorem 2.4.7. *A complex B-convex Banach X space is K-convex.*

Proof. Since X is B-convex, X has non-trivial type by Theorem 2.2.21. Further, Theorem 2.1.14 shows that $L^2(D, \mathcal{B}(D), \mu; X)$ has non-trivial type, too. A second application of Theorem 2.2.21 shows that $L^2(D, \mathcal{B}(D), \mu; X)$ is B-convex.

We now construct the promised commuting norm-one projections out of conditional expectation operators. Fix $m \in \mathbb{N}$. For $1 \le i \le m$ set $\mathcal{F}_i := \sigma(\varepsilon_i)$ and let $A_i := \{1, \ldots, m\} \setminus \{i\}$. Further choose

$$P_i := \mathbb{E}\left[\cdot \middle| \sigma\left(\bigcup_{j\in A_i} \mathcal{F}_j \right) \right],$$

that is the conditional expectation with respect to the first m but one coordinates. By Lemma 2.4.6, the P_is form a family of commuting norm-one projections. Since the Bochner space $L^2(D, \mathcal{B}(D), \mu; X)$ is B-convex, Theorem 2.4.3 applies to $P_1, \ldots, P_m$: there exist a $\delta > 0$ and a $M > 0$ such that for all m and independently of the chosen projections the semigroup

$$S(t) := \prod_{i=1}^{m} (P_i + e^{-t}(\mathrm{Id} - P_i)) \tag{2.4.5}$$

is holomorphic on $S(0,\delta)$ and uniformly bounded by M. Expanding the product (2.4.5) yields

$$S(t) = \sum_{k=0}^{m} e^{-kt} Q_k,$$

where

$$Q_k \coloneqq \sum_{\substack{A \in \mathcal{P}(1,\ldots,m) \\ |A|=k}} \prod_{j \in A} (\operatorname{Id} - P_j) \prod_{j \notin A} P_j.$$

For the proof of Pisier's theorem we are only interested in the restriction $\tilde{Q}_1$ of

$$Q_1 = \sum_{i=1}^{m} (\operatorname{Id} - P_i) \prod_{\substack{j=1 \\ j \neq i}}^{m} P_j$$

to $L^2(D_m, \mathcal{P}(D_m), \mu_m; X)$ (we agree that tilde always indicates restriction to this subspace). This restriction is R_m^X. Indeed by Lemma 2.4.6, we have

$$Q_1 = \sum_{i=1}^{m} \left(\prod_{\substack{j=1 \\ j \neq i}}^{m} P_j - \prod_{j=1}^{m} P_j \right) = \sum_{i=1}^{m} \left(\mathbb{E}[\cdot|\mathcal{F}_i] - \mathbb{E}[\cdot|\sigma(\emptyset)] \right) = \sum_{i=1}^{m} \left(\mathbb{E}[\cdot|\sigma(\varepsilon_i)] - \mathbb{E}[\cdot] \right).$$

In order to show that $\tilde{Q}_1$ and R_m^X agree, it is sufficient to show that the image of the vector-valued Walsh functions $\omega \mapsto \varepsilon_A x_A$ for some $x_A \in X$ agree because each function can be written as a finite linear combination of these Walsh functions by Remark 2.1.3. On the one hand by the orthogonality of the Walsh functions, we have

$$R_m^X(\varepsilon_A x_A) = \sum_{i=1}^{m} \varepsilon_i x_A \int_{D_m} \varepsilon_i(\omega) \varepsilon_A(\omega) \, d\mu_m = \begin{cases} \varepsilon_i x_{\{i\}} & \text{if } A = \{i\} \text{ for some } i \in [1,m] \\ 0 & \text{else} \end{cases}.$$

On the other hand by the independence of the coordinate functions, one has

$$\begin{aligned} \tilde{Q}_1(\varepsilon_A x_A) &= \sum_{i=1}^{m} \left(\mathbb{E}[\varepsilon_A x_A | \sigma(\varepsilon_i)] - \mathbb{E}[\varepsilon_A x_A] \right) \\ &= \begin{cases} \sum_{i=1}^{m} \mathbb{E}\left[\prod_{j \in A} \varepsilon_j x_A \middle| \sigma(\varepsilon_i) \right] & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset. \end{cases} \\ &= \begin{cases} \sum_{i=1}^{m} \prod_{j \in A \setminus \{i\}} \mathbb{E}(\varepsilon_j) \cdot \mathbb{E}[\varepsilon_i x_A | \sigma(\varepsilon_i)] & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset. \end{cases} \\ &= \begin{cases} \varepsilon_i x_{\{i\}} & \text{if } A = \{i\} \text{ for some } i \in [1,m] \\ 0 & \text{else} \end{cases}. \end{aligned}$$

Hence, it is sufficient to establish an upper bound for $\tilde{Q}_1$. Here the holomorphy of $(S(t))$ comes into play. Notice that

$$z \mapsto \sum_{k=0}^{m} e^{-kz} \tilde{Q}_k \qquad \text{or equivalently} \qquad z \mapsto \prod_{i=1}^{m} (\tilde{P}_i + e^{-z}(\tilde{\mathrm{Id}} - \tilde{P}_i))$$

defines a holomorphic semigroup of operators - simply check that the calculations in the proof of Lemma 2.4.4 remain valid for complex arguments - on $L^2(D_m, \mathcal{P}(D_m), \mu_m; X)$ whose restriction to the non-negative real axis is $(\tilde{S}(t))$. Since a holomorphic function defined on a domain containing the positive real axis is uniquely determined by its values on this axis by the identity theorem for holomorphic functions, we know that for all $z \in S(0, \delta)$

$$\tilde{S}(z) = \sum_{k=0}^{m} e^{-kz} \tilde{Q}_k.$$

Now choose $a > \pi / \tan(\delta/2)$. Then for any $-\pi \le b \le \pi$ we see that $a + bi \in S(0, \delta)$ because of $\left|\frac{b}{a}\right| < \tan(\delta/2)$. Thus we can calculate

$$\frac{e^a}{2\pi} \int_{-\pi}^{\pi} \tilde{S}(a + ib) e^{ib} \, db = \frac{1}{2\pi} \sum_{k=0}^{m} e^{(1-k)a} \tilde{Q}_k \int_{-\pi}^{\pi} e^{i(1-k)b} \, db = \tilde{Q}_1.$$

Since $(\tilde{S}(t))$ is uniformly bounded on $S(0, \delta)$, we have

$$\left\| R_m^X \right\| = \left\| \tilde{Q}_1 \right\| \le e^a \sup_{-\pi \le b \le \pi} \left\| \tilde{S}(a + ib) \right\| \le M e^a.$$

Remember that this estimate is independent of m, therefore we have shown that

$$\sup_{m \in \mathbb{N}} \left\| R_m^X \right\| \le M e^{\pi / \tan(\delta/2)}. \qquad \square$$

Remark 2.4.8. Exactly the same argument as in the proof above works more generally for arbitrary $\tilde{Q}_l$:

$$\frac{e^{la}}{2\pi} \int_{-\pi}^{\pi} \tilde{S}(a + ib) e^{ilb} \, db = \frac{1}{2\pi} \sum_{k=0}^{m} e^{(l-k)a} \tilde{Q}_k \int_{-\pi}^{\pi} e^{i(l-k)b} \, db = \tilde{Q}_l.$$

Hence, for all $l \in \mathbb{N}$

$$\left\| \tilde{Q}_l \right\| \le M e^{l\pi / \tan(\delta/2)}. \tag{2.4.6}$$

We want to get an explicit formula for $\tilde{Q}_l$. So let us look at the images of the Walsh functions. We see that

$$P_j \left(\sum_{B \in \mathcal{P}(D_m)} \varepsilon_B x_B \right) = \mathbb{E} \left[\sum_{B \in \mathcal{P}(D_m)} \varepsilon_B x_B \middle| \sigma \left(\bigcup_{k \in D_m \setminus \{j\}} \mathcal{F}_j \right) \right]$$

$$
\begin{aligned}
&= \sum_{j\notin B} \varepsilon_B x_B + \sum_{j\in B} \mathbb{E}\left[\prod_{k\in B} \varepsilon_k x_B \middle| \sigma\left(\bigcup_{k\in D_m\setminus\{j\}} \mathcal{F}_j\right)\right] \\
&= \sum_{j\notin B} \varepsilon_B x_B + \sum_{j\in B} \mathbb{E}(\varepsilon_j)\cdot \mathbb{E}\left[\prod_{k\in B\setminus\{j\}} \varepsilon_k x_B \middle| \sigma\left(\bigcup_{k\in D_m\setminus\{j\}} \mathcal{F}_j\right)\right] \\
&= \sum_{j\notin B} \varepsilon_B x_B.
\end{aligned}
$$

Hence, for $A \in \mathcal{P}(D_m)$ we have

$$
\prod_{j\notin A} P_j \left(\sum_{B\in\mathcal{P}(D_m)} \varepsilon_B x_B\right) = \sum_{B\subset A} \varepsilon_B x_B.
$$

Moreover,

$$
\prod_{j\in A} (\mathrm{Id} - P_j) \left(\sum_{B\in\mathcal{P}(D_m)} \varepsilon_B x_B\right) = \sum_{A\subset B} \varepsilon_B x_B.
$$

The two calculations above finally show that

$$
\begin{aligned}
\tilde{Q}_l \left(\sum_{B\in\mathcal{P}(D_m)} \varepsilon_B x_B\right) &= \sum_{\substack{A\in\mathcal{P}(1,\ldots,m)\\ |A|=l}} \prod_{j\in A} (\tilde{\mathrm{Id}} - \tilde{P}_j) \prod_{j\notin A} \tilde{P}_j \left(\sum_{B\in\mathcal{P}(D_m)} \varepsilon_B x_B\right) \\
&= \sum_{\substack{A\in\mathcal{P}(1,\ldots,m)\\ |A|=l}} \varepsilon_A x_A.
\end{aligned}
$$

Hence, $\tilde{Q}_l$ is the projection onto the subspace spanned by the Walsh functions having support of cardinality l. Therefore we call $\tilde{Q}_l$ the *l-th generalized Rademacher projection* and we write $R^X_{m,l}$. With this definition in our hands (2.4.6) shows

$$
\sup_{m\in\mathbb{N}} \left\| R^X_{m,l} \right\| \le M e^{l\pi/\tan(\delta/2)}.
$$

The above remark gives us together with Remark 2.1.3 an extended characterization of K-convexity.

Corollary 2.4.9. *A complex Banach space X is K-convex if and only if for all $l \in \mathbb{N}$ the bounded operators $R^X_{m,l}$ extend to a bounded operator R^X_l on $L^2(D, \mathcal{B}(D), \mu; X)$ such that*

$$
\sup_{l\in\mathbb{N}} \left\| R^X_l \right\|^{1/l} < \infty.
$$

The work done in this section can now be summarized.

Corollary 2.4.10. *A complex Banach space is B-convex if and only if it is K-convex.*

We want to extend the above corollary to real Banach spaces. This can be done by complexification of the real Banach space.

Theorem 2.4.11 (Pisier's Theorem). *A Banach space is B-convex if and only if it is K-convex.*

Proof. Notice first that Theorem 2.4.2 remains valid for real Banach spaces. So K-convexity implies B-convexity independently of the scalar field. However, for proving the converse we relied heavily on the holomorphy of some semigroup of projections and therefore on the complex structure of the Banach space. So some more work is needed here. Let X be a real B-convex Banach space. Let $X^{\mathbb{C}}$ be its complexification (for proofs / references of the used facts about complexification see A.4.1) which is isomorphic to $X \oplus X$ as real Banach spaces. Since B-convexity is invariant under isomorphisms by Lemma 2.2.15, for the underlying real Banach space of $X^{\mathbb{C}}$ to be B-convex, it is sufficient to show that $X \oplus X$ is B-convex. For this observe that X has non-trivial type by Theorem 2.2.21. Hence, Corollary 2.1.17 shows that $X \oplus X$ has non-trivial type as well and therefore applying Theorem 2.2.21 once again yields the B-convexity of $X \oplus X$. As argued earlier, B-convexity is a real notion and therefore $X^{\mathbb{C}}$ is B-convex as complex Banach space. Now we can apply Corollary 2.4.10: $X^{\mathbb{C}}$ is K-convex! A fortiori, the underlying real space of $X^{\mathbb{C}}$ is K-convex. Since X is isometrically isomorphic to some closed subspace of the underlying real Banach space of $X^{\mathbb{C}}$, Lemmata 2.3.10 and 2.3.9 show that X is K-convex. $\square$

2.5 Applications of Pisier's Theorem

With the help of Pisier's theorem we can prove some surprising properties or characterizations of B/K-convex spaces. We first present a complete duality theory for type and cotype for spaces with non-trivial type.

2.5.1 Direct Consequences of Pisier's Theorem

Along the way we have proved several characterizations of B-convexity and K-convexity. Together with Pisier's theorem they lead to deep geometric characterizations of K-convexity.

Theorem 2.5.1 (Pisier's Equivalence). *Let X be a Banach space. The following statements are equivalent.*

(a) X is K-convex.

(b) X is B-convex.

(c) X has non-trivial type.

(d) X does not contain ℓ_1^n's uniformly.

Proof. Luckily, all the work is already done. The theorem just summarizes the statements of Theorems 2.4.11, 2.2.21 and 2.2.19. □

We have promised to establish a complete duality theory for K-convex Banach spaces and we are almost done! We want to deduce the type of X from the cotype of its dual: it would be natural to apply Theorem 2.3.15 to X'. However, therefore we need X' to be K-convex. Luckily, this is true and rather easy to prove.

Theorem 2.5.2. *A Banach space X is K-convex if and only if its dual X' is K-convex. In this case we have $K(X) = K(X')$.*

The main point is that the adjoints of the Rademacher projections coincide with the Rademacher projections on X'.

Lemma 2.5.3. *Let X be a Banach space. Under the natural identifications the adjoints of R_n^X are given by the Rademacher projections $R_n^{X'}$ on X'.*

Proof. Remember that by Lemma 2.3.4 $L^2(D_n, \mathcal{P}(D_n), \mu_n; X')$ is isometrically isomorphic to the dual of $L^2(D_n, \mathcal{P}(D_n), \mu_n; X)$. Moreover, every functional is of the form

$$L^2(D_n, \mathcal{P}(D_n), \mu_n; X) \ni f \mapsto \int_{D_n} \big\langle g'(\omega), f(\omega) \big\rangle \, d\mu_n(\omega)$$

for some $g' \in L^2(D_n, \mathcal{P}(D_n), \mu_n; X')$. The rest of the proof consists of standard manipulations of the Bochner integral. Let f, g' be as above. Then

$$\begin{aligned}
\Big\langle \big(R_n^X\big)'(g'), f \Big\rangle &= \left\langle g', \sum_{i=1}^n \varepsilon_i \int_{D_n} \varepsilon_i(\omega) f(\omega) \, d\mu_n(\omega) \right\rangle \\
&= \int_{D_n} \left\langle g'(\omega'), \sum_{i=1}^n \varepsilon_i(\omega') \int_{D_n} \varepsilon_i(\omega) f(\omega) \, d\mu_n(\omega) \right\rangle d\mu_n(\omega') \\
&= \sum_{i=1}^n \int_{D_n} \varepsilon_i(\omega') \left\langle g'(\omega'), \int_{D_n} \varepsilon_i(\omega) f(\omega) \, d\mu_n(\omega) \right\rangle d\mu_n(\omega') \\
&\overset{\text{Theorem C.2.6}}{=} \sum_{i=1}^n \int_{D_n} \int_{D_n} \varepsilon_i(\omega') \varepsilon_i(\omega) \langle g'(\omega'), f(\omega) \rangle \, d\mu_n(\omega) \, d\mu_n(\omega') \\
&\overset{\text{Fubini}}{=} \sum_{i=1}^n \int_{D_n} \int_{D_n} \varepsilon_i(\omega') \varepsilon_i(\omega) \langle g'(\omega'), f(\omega) \rangle \, d\mu_n(\omega') \, d\mu_n(\omega) \\
&= \sum_{i=1}^n \int_{D_n} \varepsilon_i(\omega) \int_{D_n} \varepsilon_i(\omega') \langle g'(\omega), f(\omega) \rangle \, d\mu_n(\omega') \, d\mu_n(\omega) \\
&\overset{\text{Theorem C.2.6}}{=} \int_{D_n} \sum_{i=1}^n \varepsilon_i(\omega) \left\langle \int_{D_n} \varepsilon_i(\omega') g'(\omega') \, d\mu_n(\omega'), f(\omega) \right\rangle d\mu_n(\omega) \\
&= \int_{D_n} \left\langle \sum_{i=1}^n \varepsilon_i(\omega) \int_{D_n} \varepsilon_i(\omega') g'(\omega') \, d\mu_n(\omega'), f(\omega) \right\rangle d\mu_n(\omega)
\end{aligned}$$

$$= \left\langle \sum_{i=1}^{n} \varepsilon_i \int_{D_n} \varepsilon_i(\omega') g'(\omega')\, d\mu_n(\omega'), f \right\rangle = \left\langle R_n^{X'}(g'), f \right\rangle. \qquad \square$$

Proof of Theorem 2.5.2. Let X be a K-convex Banach space. Lemma 2.5.3 yields

$$\sup_{n\in\mathbb{N}} \left\| R_n^{X'} \right\| = \sup_{n\in\mathbb{N}} \left\| \left(R_n^X\right)' \right\| = \sup_{n\in\mathbb{N}} \left\| R_n^X \right\| = K(X).$$

Thus X' is K-convex with $K(X') = K(X)$. Now assume that X' is K-convex. The first part applied to X' shows that X'' is K-convex with $K(X'') = K(X')$. Since X is canonically isometrically isomorphic to a closed subspace of X'', X is K-convex with $K(X) \le K(X'') = K(X') = K(X)$ by Lemmata 2.3.10 and 2.3.9. Hence, X is K-convex if and only if X' is K-convex. $\square$

Corollary 2.5.4. *Let X be a Banach space. Then X fulfills one of the statements in Theorem 2.5.1 if and only if X' fulfills one of these statements. In this case both X and X' fulfill all statements.*

Remark 2.5.5. Let X be K-convex. We have seen in Remark 2.3.14 that the R_n^Xs extend to bounded linear operators on $L^p(D, \mathcal{B}(D), \mu; X)$ for $2 \le p < \infty$. This result can now be extended to $1 < p < 2$ in the following way. By what have already been shown and the argument given in the proof of Lemma 2.5.3, we know that (modulo isometric isomorphisms) the $R_n^{X'}$s - which in turn can be identified with the $(R_n^X)'s$ - can be extended to a bounded linear operator

$$R^{X'} : L^q(D, \mathcal{B}(D), \mu; X') \to L^q(D, \mathcal{B}(D), \mu; X'),$$

where $\frac{1}{p} + \frac{1}{q} = 1$. This follows from the facts that X' is K-convex as well by Theorem 2.5.2 and that $q > 2$. Since a linear operator between two Banach spaces is continuous if and only if its adjoint operator is continuous and the norms of both operators agree in this case, we conclude that the R_n^Xs can indeed be extended to a bounded linear operator

$$R^X : L^p(D, \mathcal{B}(D), \mu; X) \to L^p(D, \mathcal{B}(D), \mu; X).$$

Remember that we have seen in Remark 2.1.5 that a Banach space X is of type $p' < p$ (resp. of cotype $q' > q$) if X is of type p (resp. of cotype q). This leads naturally to the following definition.

Definition 2.5.6. Let X be a Banach space. We define

$$\begin{aligned} p(X) &\coloneqq \sup\{p : X \text{ is of type } p\}, \\ q(X) &\coloneqq \inf\{q : X \text{ is of cotype } q\}. \end{aligned}$$

If X is not of cotype q for any $q < \infty$, we set $q(X) \coloneqq \infty$.

By Remark 2.1.5, we always have $1 \le p(X) \le 2 \le q(X) \le \infty$. Finally, we present the long promised duality result.

Corollary 2.5.7 (Duality for Type and Cotype). *Let $p \in (1,2]$ and X be a K-convex Banach space. Then X is of type p (resp. of cotype q) if and only if X' is of cotype q (resp. of type p) with $\frac{1}{p} + \frac{1}{q} = 1$. In particular, if $p(X) > 1$, we have*

$$\frac{1}{p(X)} + \frac{1}{q(X')} = \frac{1}{p(X')} + \frac{1}{q(X)} = 1.$$

Proof. All the work is already done. By Theorem 2.3.15, X is of cotype q if and only if X' is of type p. Lemma 2.3.1 shows that X' is of cotype q if X is of type p. By Theorem 2.5.2, X' is K-convex as well. A further application of Theorem 2.3.15 shows that X'' is of type p if X' of cotype q. Since X is canonically isometrically isomorphic to a closed subspace of X'' and the type is inherited by closed subspaces and is invariant under isomorphisms by Lemmata 2.1.7 and 2.1.8, we see that X is of type p as well.

If $p(X) > 1$, X is of type p' for some $p' > 1$. Therefore X is K-convex by Theorem 2.5.1. Now the rest of the assertion follows directly from what has already been shown. □

2.5.2 Absolutely Summable Fourier Coefficients

In the theory of Banach spaces one central object of interest is the question to which extend certain theorems can be generalized from the scalar-valued to the vector-valued case. In this spirit we are interested in the decay of the Fourier coefficients

$$\hat{f}(n) := \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} \, dx$$

of 2π-periodic vector-valued functions. For this purpose we agree that in this section X always denotes a complex Banach space. The probably most famous result of this kind is the *Riemann-Lebesgue lemma* [Kat04, Theorem I.2.8] which states that for $f \in L^1(0, 2\pi)$

$$\lim_{n\to\infty} \left| \hat{f}(n) \right| = 0.$$

This remains true in the vector-valued case.

Theorem 2.5.8 (Riemann-Lebesgue Lemma, vector-valued). *Let X be a Banach space and $f \in L^1((0, 2\pi); X)$. Then*

$$\lim_{n\to\infty} \left\| \hat{f}(n) \right\| = 0.$$

Proof. We define the linear operator

$$\begin{aligned} \mathcal{F} : L^1((0, 2\pi); X) &\to \ell_\infty(X) \\ f &\mapsto (\hat{f}(0), \hat{f}(1), \hat{f}(-1), \hat{f}(2), \ldots). \end{aligned}$$

Since

$$\left\| \hat{f}(n) \right\| \leq \frac{1}{2\pi} \int_0^{2\pi} \left\| f(x) \right\| \, dx = \frac{1}{2\pi} \left\| f \right\|_{L^1(X)},$$

$\mathcal{F}$ is bounded. Moreover for $f(t) := \sum_{i=1}^{m} g_i(t)x_i$ with $g_i \in L^1(0, 2\pi)$ and $x_i \in X$, it follows from the scalar-valued case that $\mathcal{F}(f) \in c_0(X)$. Since $c_0(X)$ is a closed subspace of $\ell_\infty(X)$ and functions of the above form are dense in $L^1((0, 2\pi); X)$ because even the simple functions are dense by Theorem C.2.9, the continuity of $\mathcal{F}$ yields

$$\mathcal{F}(L^1((0, 2\pi); X)) \subset c_0(X). \qquad \square$$

Remark 2.5.9. The scalar-valued case can be proven with the same argument. Indeed, for trigonometric polynomials the claim is clearly true and the general case follows from the density of the trigonometric polynomials in $L^1(0, 2\pi)$.

Further, in the scalar-valued case a result of Bernstein [Kat04, Theorem I.6.3] shows that if f is α-Hölder continuous with $\alpha > \frac{1}{2}$, the Fourier coefficients of f are absolutely summable. Remember that a function between two normed spaces is called α-Hölder continuous if there exist non-negative constants C and α such that

$$\|f(x) - f(y)\| \le C \|x - y\|^\alpha$$

for all x, y in the domain of f. However as the next example by H. König [Kön91] shows, this is in general even false for Lipschitz continuous or continuously differentiable functions in the vector-valued case.

Theorem 2.5.10. *There exists a 2π-periodic continuously differentiable function $f : [0, 2\pi] \to L^1(0, 2\pi)$ such that its Fourier coefficients are not absolutely summable.*

Proof. Let $(a_n)_{n\in\mathbb{N}_0}$ be a monotonically decreasing sequence of non-negative numbers with $\lim_{n\to\infty} a_n = 0$. Moreover, we require (a_n) to be convex, that is

$$a_{n-1} + a_{n+1} - 2a_n \ge 0 \tag{2.5.1}$$

and $(\frac{a_n}{n})_{n\in\mathbb{N}}$ not to be summable, so $\sum_{n=1}^{\infty} \frac{a_n}{n} = \infty$. For example, a popular choice would be $a_n = \frac{1}{\log(n+2)}$. By telescoping, we have

$$\sum_{n=0}^{\infty}(a_n - a_{n+1}) = \lim_{N\to\infty} a_0 - a_{N+1} = a_0.$$

The convexity condition (2.5.1) shows that the sequence $(b_n) := (a_n - a_{n+1})$ is monotonically decreasing. Moreover, it is non-negative because (a_n) is monotonically decreasing. Hence,

$$0 \le nb_n = 2 \cdot \frac{n}{2} b_n \le 2 \sum_{k=\lfloor \frac{n}{2} \rfloor}^{n} b_n \le 2 \sum_{k=\lfloor \frac{n}{2} \rfloor}^{n} b_k \le 2 \sum_{k=\lfloor \frac{n}{2} \rfloor}^{\infty} b_k \xrightarrow[n\to\infty]{} 0.$$

This shows $\lim_{n\to\infty} n(a_n - a_{n+1}) = 0$. Further,

$$\begin{aligned}
\sum_{n=1}^{\infty} n(a_{n-1} + a_{n+1} - 2a_n) &= \lim_{N\to\infty} \left(\sum_{n=1}^{N} na_{n-1} + \sum_{n=1}^{N} na_{n+1} - 2\sum_{n=1}^{N} na_n \right) \\
&= \lim_{N\to\infty} \left(\sum_{n=0}^{N-1} (n+1)a_n + \sum_{n=2}^{N+1} (n-1)a_n - 2\sum_{n=1}^{N} na_n \right) \\
&= \lim_{N\to\infty} (a_0 + 2a_1 + (N-1)a_N + Na_{N+1} - 2a_1 - 2Na_N) \\
&= \lim_{N\to\infty} (a_0 - a_N - N(a_N - a_{N+1})) = a_0.
\end{aligned} \tag{2.5.2}$$

Now set

$$\begin{aligned}
g : [0, 2\pi] &\to L^1(0, 2\pi) \\
x &\mapsto \sum_{n=1}^{\infty} n(a_{n-1} + a_{n+1} - 2a_n) F_{n-1}(x - \cdot),
\end{aligned}$$

where F_{n-1} denotes the $(n-1)$-th Féjer kernel (see [Kat04, p. 12]). We observe that $g(x)$ is measurable for each fixed x. Since the Féjer kernel is positive, the monotone convergence theorem shows

$$\begin{aligned}
\|g(x)\|_{L^1} &= \sum_{n=1}^{\infty} n(a_{n-1} + a_{n+1} - 2a_n) \int_0^{2\pi} F_{n-1}(x-t)\,dt \\
&= \sum_{n=1}^{\infty} n(a_{n-1} + a_{n+1} - 2a_n) \int_0^{2\pi} F_{n-1}(t)\,dt \\
&= \sum_{n=1}^{\infty} n(a_{n-1} + a_{n+1} - 2a_n) \, \|F_{n-1}\|_{L^1} \\
&= 2\pi \sum_{n=1}^{\infty} n(a_{n-1} + a_{n+1} - 2a_n) = 2\pi a_0.
\end{aligned}$$

Hence, g is well-defined and $\|g(x)\|_{L^1} = 2\pi a_0$ for all x. Moreover, g is even continuous. Let $\varepsilon > 0$. Then there exists a N such that $\sum_{n=N+1}^{\infty} n(a_{n-1} + a_{n+1} - 2a_n) \le \frac{\varepsilon}{8\pi}$. Since F_i is uniformly continuous on $[0, 2\pi]$ for all i, there exists a $\delta > 0$ such that $|u - w| < \delta$ implies $|F_i(u) - F_i(w)| \le \frac{\varepsilon}{4\pi a_0}$ for all $i = 0, \ldots, N-1$. Hence,

$$\|F_i(x - \cdot) - F_i(y - \cdot)\|_{L^1} = \int_0^{2\pi} |F_i(x-t) - F_i(y-t)| \; dt \le \frac{\varepsilon}{2a_0}$$

for $|x - y| < \delta$. Then for $|x - y| < \delta$ we see that

$$\|g(x) - g(y)\|_{L^1} \le \sum_{n=1}^{N} n(a_{n-1} + a_{n+1} - 2a_n) \, \|F_{n-1}(x - \cdot) - F_{n-1}(y - \cdot)\|_{L^1}$$

$$
\begin{aligned}
&+ \sum_{n=N+1}^{\infty} n(a_{n-1}+a_{n+1}-2a_n) \left\| F_{n-1}(x-\cdot) - F_{n-1}(y-\cdot) \right\|_{L^1} \\
&\leq \frac{\varepsilon}{2a_0}(a_0 - a_N - N(a_N - a_{N+1})) + 4\pi \cdot \frac{\varepsilon}{8\pi} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

This shows $g \in C_p([0,2\pi]; L^1(0,2\pi))$. Let us calculate its Fourier coefficients.

$$
\begin{aligned}
\hat{g}(m) &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=1}^{\infty} n(a_{n-1}+a_{n+1}-2a_n) F_{n-1}(x-\cdot)e^{-imx}\,dx \\
&\overset{\text{dom. conv.}}{=} \sum_{n=1}^{\infty} n(a_{n-1}+a_{n+1}-2a_n)\frac{1}{2\pi}\int_0^{2\pi} F_{n-1}(x-\cdot)e^{-imx}\,dx.
\end{aligned}
$$

In order to evaluate the integral we notice that the Féjer kernel is uniformly continuous on the compact interval $[0,2\pi]$ and therefore the integrand is a continuous function with values in $C[0,2\pi]$! Since every continuous function on a compact interval is integrable, the integral exists. Let $\varphi_t : C[0,2\pi] \to \mathbb{K}$ be the point evaluation at t for $0 \leq t \leq 2\pi$. Since φ_t is a continuous linear functional on $C[0,2\pi]$, we have

$$
\begin{aligned}
&\varphi_t\left(\int_0^{2\pi} F_{n-1}(x-\cdot)e^{-imx}\,dx\right) = \int_0^{2\pi} \varphi_t(F_{n-1}(x-\cdot)e^{-imx})\,dx \\
&= \int_0^{2\pi} F_{n-1}(x-t)e^{-imx}\,dx \overset{y=x-t}{=} \int_{-t}^{2\pi-t} F_{n-1}(y)e^{-im(y+t)}\,dy \\
&= e^{-imt}\int_0^{2\pi} F_{n-1}(y)e^{-imy}\,dy = e^{-imt}\hat{F_{n-1}}(m) = \varphi_t\left(e^{-im\cdot}\hat{F_{n-1}}(m)\right).
\end{aligned}
$$

As the point evaluations separate points in $C[0,2\pi]$, we conclude that

$$
\int_0^{2\pi} F_{n-1}(x-\cdot)e^{-imx}\,dx = e^{-im\cdot}\hat{F_{n-1}}(m).
$$

Since $C[0,2\pi]$ is continuously embedded into $L^1(0,2\pi)$, the above result remains valid in $L^1(0,2\pi)$. Now we can continue our calculations and we obtain

$$
\begin{aligned}
\hat{g}(m) &= e^{-im\cdot}\sum_{n=1}^{\infty} n(a_{n-1}+a_{n+1}-2a_n)\hat{F_{n-1}}(m) \\
&= e^{-im\cdot}\sum_{n=|m|+1}^{\infty} n(a_{n-1}+a_{n+1}-2a_n)\left(1-\frac{|m|}{n}\right) \\
&= e^{-im\cdot}\sum_{n=|m|+1}^{\infty} (n-|m|)(a_{n-1}+a_{n+1}-2a_n) \\
&= e^{-im\cdot}\sum_{n=1}^{\infty} n(a_{n+|m|-1}+a_{n+|m|+1}-2a_{n+|m|}) = e^{-im\cdot}a_{|m|},
\end{aligned}
$$

where we have used the exactly same argument as in (2.5.2) in the last step. Now let

$$\tilde{g}(x) := g(x) - \hat{g}(0) = g(x) - a_0.$$

Of course, $\tilde{g}$ is a continuous $L^1(0,2\pi)$-valued function with the same Fourier coefficients as g, except for $\hat{\tilde{g}}(0) = 0$. Therefore $\tilde{g}$ is integrable and its antiderivative $f(x) := \int_0^x \tilde{g}(y)\,dy$ is a continuously differentiable function. Moreover, it is periodic because

$$f(2\pi) = \int_0^{2\pi} \tilde{g}(y)\,dy = 2\pi\hat{\tilde{g}}(0) = 0 = f(0).$$

Hence, $f \in C_p^1([0,2\pi]; L^1(0,2\pi))$ with

$$\begin{aligned}\hat{f}(m) &= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-imx}\,dx \overset{\text{int. by parts}}{=} -\frac{i}{m}\frac{1}{2\pi}\int_0^{2\pi} f'(x)e^{-imx}\,dx \\ &= -\frac{i}{m}\hat{\tilde{g}}(m) = -\frac{i}{m}e^{-im\cdot}a_{|m|} \qquad \text{for } m \neq 0.\end{aligned}$$

Since

$$\sum_{m\in\mathbb{Z}} \left\|\hat{f}(m)\right\|_{L^1} \geq 4\pi\sum_{m=1}^{\infty}\frac{a_m}{m} = \infty,$$

f is such a desired function. $\square$

The above example shows that there exists a 2π-periodic continuously differentiable $L^1(0,2\pi)$-valued function whose Fourier coefficients are not absolutely summable. Luckily, one can nevertheless exactly classify those Banach spaces for which there exist such badly behaved continuously differentiable functions. We will now present the following theorem due to König [Kön91]: A Banach space X is B/K-convex if and only if for any continuously differentiable function $f : [0,2\pi] \to X$ its Fourier coefficients are absolutely differentiable.

We have already given a negative example for one particular Banach space that is not B/K-convex: $L^1(0,2\pi)$. By Pisier's Equivalence (Theorem 2.5.1), we know that every non-B/K-convex Banach space contains ℓ_1^n's uniformly and so does $L^1(0,2\pi)$. So if we would find a negative example with finite dimensional range, we could use the fact that all non-B/K-convex Banach spaces have a very similiar finite dimensional structure given by the ℓ_1^n's to generalize our negative example to these spaces. Therefore we investigate the finite dimensional nature of our negative example in the next lemma.

Lemma 2.5.11. *There exists a sequence of periodic continuously differentiable functions $f_k : [0,2\pi] \to L^1(0,2\pi)$, each of them having finite dimensional range, such that the f_ks are uniformly bounded in $C_p^1([0,2\pi]; L^1(0,2\pi))$ and $\lim_{k\to\infty}\sum_{m\in\mathbb{Z}}\left\|\hat{f}_k(m)\right\| = \infty$.*

Proof. We are going to show that the sequence of partial sums of the series used in the proof of Theorem 2.5.10 has the desired properties. Let

$$g_j(x) := \sum_{n=1}^{j} n(a_{n+1} + a_{n-1} - 2a_n)F_{n-1}(x - \cdot).$$

Since the Féjer kernels are uniformly continuous, we see that $g_j \in C_p([0,2\pi];C[0,2\pi])$. Moreover, $\|g_j(x)\|_{L_1} \le a_0$ for every x and arbitrary j. So $\|g_j\|_{C([0,2\pi];L^1(0,2\pi))} \le a_0$ for every $j \in \mathbb{N}$. Repeating the calculations shows that their Fourier coefficients are given by

$$\begin{aligned}
\hat{g_j}(m) &= \sum_{n=|m|+1}^{j} (n-|m|)(a_{n+1}-a_{n-1}-2a_n)e^{-im\cdot} \\
&= e^{-im\cdot} \sum_{n=1}^{j-|m|} n(a_{|m|+n+1}+a_{|m|+n-1}-2a_{|m|+n}) \\
&= \begin{cases} e^{-im\cdot}(a_{|m|}-a_j-(j-|m|)(a_j-a_{j+1})) & \text{if } |m|<j \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

For further use, we set $b_m^{(j)} := a_{|m|}-a_j-(j-|m|)(a_j-a_{j+1})$. For fixed $m \in \mathbb{Z}$, we have $\lim_{j\to\infty} a_j + (j-|m|)(a_j-a_{j+1}) = 0$. Hence, for every $\varepsilon > 0$ there exists a sequence (j_k) such that

$$\|\hat{g_{j_k}}(m)\|_{L^1} \ge 2\pi\left(a_{|m|} - \frac{\varepsilon}{m}\right) \qquad \text{for all } |m| \le k.$$

Now let f_k be the antiderivative of $g_{j_k} - \hat{g_{j_k}}(0)$. Then the f_ks are periodic continuously differentiable $L^1(0,2\pi)$-valued functions. Moreover, we have

$$\|f_k(x)\|_{L_1} \le \int_0^x \|g_{j_k}(y) - \hat{g_{j_k}}(0)\|_{L_1}\, dy \le 4\pi a_0.$$

Thus the sequence (f_k) is bounded by $4\pi a_0$. This shows that $\|f_k\|_{C_p^1([0,2\pi];X)} \le (4\pi+1)a_0$. Moreover, the series of their Fourier coefficients can be estimated by

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} \left\|\hat{f}_k(m)\right\|_{L^1} &\ge \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{\|\hat{g_{j_k}}(m)\|_{L^1}}{m} \ge \sum_{0\ne|m|\le k} \frac{\|\hat{g_{j_k}}(m)\|_{L^1}}{m} \ge 4\pi \sum_{m=1}^{k} \frac{a_m - \frac{\varepsilon}{m}}{m} \\
&= 4\pi \sum_{m=1}^{k} \frac{a_m}{m} - 4\pi\varepsilon \sum_{m=1}^{k} \frac{1}{m^2} \ge 4\pi \sum_{m=1}^{k} \frac{a_m}{m} - 4\pi\varepsilon \sum_{m=1}^{\infty} \frac{1}{m^2} \xrightarrow{k\to\infty} \infty.
\end{aligned}$$

It remains to show that the f_ks have finite dimensional range. This can easily be seen once we have written $g_j(x)$ as a trigonometric polynomial. For this purpose we determine the Fourier coefficients of $g_j(x)$

$$\begin{aligned}
\hat{g_j(x)}(m) &= \frac{1}{2\pi}\int_0^{2\pi} g_j(x)(t)e^{-imt}\, dt \\
&= \sum_{n=1}^{j} n(a_{n+1}+a_{n-1}-2a_n)\frac{1}{2\pi}\int_0^{2\pi} F_{n-1}(x-t)e^{-imt}\, dt \\
&\overset{y=x-t}{=} \sum_{n=1}^{j} n(a_{n+1}+a_{n-1}-2a_n)\frac{1}{2\pi}\int_{x-2\pi}^{x} F_{n-1}(y)e^{-im(x-y)}\, dy
\end{aligned}$$

$$= e^{-imx} \sum_{n=1}^{j} n(a_{n+1} + a_{n-1} - 2a_n) \hat{F_{n-1}}(m)$$
$$= \begin{cases} e^{-imx} b_m & \text{for } |m| < j \\ 0 & \text{else.} \end{cases}$$

We can define a second function with the same Fourier coefficients, namely

$$h_j(x) := \sum_{|n|<j} b_n \cos n(x - \cdot).$$

Let us check its Fourier coefficients for fixed x:

$$\hat{h_j(x)}(m) = \sum_{|n|<j} b_n \frac{1}{2\pi} \int_0^{2\pi} \cos n(x-t) e^{-imt}\, dt$$
$$= \sum_{|n|<j} b_n \left(e^{inx} \frac{1}{4\pi} \int_0^{2\pi} e^{-it(m+n)} + \frac{1}{4\pi} e^{-inx} \int_0^{2\pi} e^{it(n-m)} \right)$$
$$= \sum_{|n|<j} b_n \left(e^{inx} \frac{1}{2} \delta_{n,-m} + e^{-inx} \frac{1}{2} \delta_{n,m} \right)$$
$$= \begin{cases} e^{-imx} b_m & \text{for } |m| < j \\ 0 & \text{else.} \end{cases}$$

Since an integrable function is uniquely determined by its Fourier coefficients, we conclude that $g_j = h_j$. By the angle sum and difference identities, we have

$$g_j(x) = \sum_{|n|<j} b_n (\cos(nx)\cos(n\cdot) + \sin(nx)\sin(n\cdot)) \in \operatorname{span}\{\sin(n\cdot), \cos(n\cdot) : |n| < j\} =: E_j.$$

Hence, we see by integrating both sides of the above identity that

$$f_k(x) = \int_0^x g_{j_k}(y) - \hat{g_{j_k}}(0)\, dy$$
$$= \sum_{\substack{|n|<j_k \\ n\neq 0}} \frac{b_n}{n} (\sin(nx)\cos(n\cdot) - \cos(nx)\sin(n\cdot) + \sin(n\cdot)) \in E_{j_k}.$$

Hence, the f_ks have finite dimensional range with $\operatorname{rg} f_k \subset E_{j_k}$. □

We now want to find finite dimensional subspaces of ℓ_1 that are almost isomorphic to the spaces E_j from the last proof. This would allow us to carry over our example to other non-B/K-convex spaces. We prove this in a more general setting in the next lemma.

Lemma 2.5.12. *Let $1 \leq p < \infty$, (Ω, Σ, μ) be a σ-finite measure space and X be a Banach space. Then for every $\lambda > 1$ and for every finite dimensional subspace Z of $L^p(\Omega, \Sigma, \mu; X)$ there exists a natural number N such that Z is λ-isomorphic to some subspace of $\ell_p^N(X)$.*

Proof. Let Z be a n-dimensional subspace of L^p. Choose a normed basis $z_1, \dots, z_n$ of Z. Since the simple functions are dense by Lemma C.2.9, we can choose for every $\varepsilon > 0$ simple functions $y_1, \dots, y_n$ such that

$$\|z_i - y_i\| \leq \varepsilon, \qquad i = 1, \dots, n.$$

Now let Y be the subspace spanned by the vectors $y_1, \dots, y_n$ and let

$$\begin{aligned} T : Z &\to Y \\ \sum_{i=1}^n \alpha_i z_i &\mapsto \sum_{i=1}^n \alpha_i y_i. \end{aligned}$$

Since $(\alpha_i)_{i=1}^n \mapsto \|\sum_{i=1}^n \alpha_i y_i\|$ defines a norm on $\mathbb{K}^n$ and all norms on a finite dimensional space are equivalent, there exists a positive constant C such that

$$\sum_{i=1}^n |\alpha_i| \leq C \left\| \sum_{i=1}^n \alpha_i z_i \right\| \leq C \sum_{i=1}^n |\alpha_i| .$$

Moreover,

$$\begin{aligned} \left\| T\left(\sum_{i=1}^n \alpha_i z_i\right) \right\| &= \left\| \sum_{i=1}^n \alpha_i y_i \right\| \leq \left\| \sum_{i=1}^n \alpha_i z_i \right\| + \left\| \sum_{i=1}^n \alpha_i (y_i - z_i) \right\| \leq \left\| \sum_{i=1}^n \alpha_i z_i \right\| + \varepsilon \sum_{i=1}^n |\alpha_i| \\ &\leq \left\| \sum_{i=1}^n \alpha_i z_i \right\| + C\varepsilon \left\| \sum_{i=1}^n \alpha_i z_i \right\| = (1 + C\varepsilon) \left\| \sum_{i=1}^n \alpha_i z_i \right\| . \end{aligned}$$

Hence, $\|T\| \leq 1 + C\varepsilon$. Similiarly, one gets

$$\begin{aligned} \left\| T\left(\sum_{i=1}^n \alpha_i z_i\right) \right\| &= \left\| \sum_{i=1}^n \alpha_i y_i \right\| \geq \left\| \sum_{i=1}^n \alpha_i z_i \right\| - \left\| \sum_{i=1}^n \alpha_i (y_i - z_i) \right\| \geq \left\| \sum_{i=1}^n \alpha_i z_i \right\| - \varepsilon \sum_{i=1}^n |\alpha_i| \\ &\geq \left\| \sum_{i=1}^n \alpha_i z_i \right\| - \varepsilon C \left\| \sum_{i=1}^n \alpha_i z_i \right\| = (1 - \varepsilon C) \left\| \sum_{i=1}^n \alpha_i z_i \right\| . \end{aligned}$$

Therefore T is injective for sufficiently small ε. Since T is surjective as well, T is an isomorphism between Y and Z with $\left\|T^{-1}\right\| \leq (1 - \varepsilon C)^{-1}$. Hence, the multiplicative Banach-Mazur distance between Y and Z can be estimated by

$$d(Y, Z) \leq \|T\| \left\|T^{-1}\right\| \leq \frac{1 + \varepsilon C}{1 - \varepsilon C}.$$

The functions z_i $(i = 1, \dots, n)$ are simple, so we can find - changing the z_i on a set of measure zero if necessary - disjoint measurable sets $A_1, \dots A_N$ such that $\mu(A_k) > 0$ for all k and such that for each fixed k $z_i|_{A_k}$ is constant for all i. Now let

$$i : \mathcal{T} := \operatorname{span}\{\mathbb{1}_{A_k} : k = 1, \dots, N\} \to \ell_p^N$$

$$\sum_{k=1}^{N} \alpha_k \mathbb{1}_{A_k} \mapsto (\alpha_1 \mu(A_1)^{1/p}, \ldots, \alpha_N \mu(A_N)^{1/p}).$$

Observe that i is an isometry:

$$\left\| \sum_{k=1}^{N} \alpha_k \mathbb{1}_{A_k} \right\|_p = \left(\sum_{k=1}^{N} |\alpha_k|^p \mu(\mathbb{1}_{A_k}) \right)^{1/p} = \left\| (\alpha_1 \mu(A_1)^{1/p}, \ldots, \alpha_N \mu(A_N)^{1/p}) \right\|_{\ell_p^N}.$$

Since Z is a subspace of $\mathcal{T}$, Z is isometrically isomorphic to some closed subspace E of ℓ_p^N. Hence,

$$d(Y, E) \leq \|i\| \left\|i^{-1}\right\| \|T\| \left\|T^{-1}\right\| \leq \frac{1+\varepsilon C}{1-\varepsilon C}.$$

Since ε can be chosen arbitrarily small, this finishes the proof. □

We have already encountered the situation that one can find for every $\lambda > 1$ λ-isomorphic copies of every finite dimensional subspace of a Banach space in another Banach space earlier in the characterization of B-convexity. Indeed, this situation is very common and therefore it is worth of a precise definition.

Definition 2.5.13 (Finitely Representable)**.** A Banach space X is called *finitely representable* in a Banach space Y if for every $\lambda > 0$ and each finite dimensional subspace Z_X of X there exists an isomorphic finite dimensional subspace Z_Y of Y such that

$$d(Z_X, Z_Y) \leq 1 + \lambda.$$

Now we can write very shortly the following corollary that will directly be needed in the proof of König's theorem as described above.

Corollary 2.5.14. *$L^1(0, 2\pi)$ is finitely representable in ℓ_1.*

Remark 2.5.15. We have seen in Remark 2.2.20 that a Banach space X fails to be B-convex if and only if X contains ℓ_1^n's λ-uniformly for all $\lambda > 1$. Moreover, we have seen in Lemma 2.5.12 that for every $\lambda > 1$ and every finite dimensional subspace of ℓ_1, one can find a natural number N such that the subspace is λ-isomorphic to some subspace of ℓ_1^N. Consequently, a Banach space X fails to be B-convex if and only if ℓ_1 is finitely representable in X.

If every continuously differentiable function $f : [0, 2\pi] \to X$ has absolutely summable Fourier coefficients, the closed graph theorem shows that the sum can even be estimated uniformly by the norm of f in $C^1([0, 2\pi]; X)$. We will need this stricter condition later.

Lemma 2.5.16. *Let X be a Banach space such that the Fourier coefficients of every periodic continuously differentiable function $f : [0, 2\pi] \to X$ are absolutely summable. Then there exists a constant $M < \infty$ such that for every f as above one has*

$$\sum_{m \in \mathbb{Z}} \left\| \hat{f}(m) \right\|_X \leq M \|f\|_{C^1([0,2\pi];X)}.$$

Proof. By assumption, the linear operator

$$\begin{aligned} \mathcal{F} : C_p^1([0, 2\pi]; X) &\to \ell_{\mathbb{Z},1}(X) \\ f &\mapsto (\hat{f}(m))_{m\in\mathbb{Z}} \end{aligned}$$

is well-defined. The claim just states that $\mathcal{F}$ is bounded. Therefore let $f_n \to f$ and $(\hat{f}_n(m))_{m\in\mathbb{Z}} \to (x_m)_{m\in\mathbb{Z}}$. Since f_n converges uniformly, we see that

$$x_m \xleftarrow[n\to\infty]{} \hat{f}_n(m) = \frac{1}{2\pi}\int_0^{2\pi} f_n(y)e^{-imy}\,dy \xrightarrow[n\to\infty]{} \frac{1}{2\pi}\int_0^{2\pi} f(y)e^{-imy}\,dy = \hat{f}(m).$$

Therefore $(\hat{f}(m))_{m\in\mathbb{Z}} = (x_m)_{m\in\mathbb{Z}}$. Hence, $\mathcal{F}$ is bounded by the closed graph theorem. □

Until now we only provided tools for giving negative examples for non-B/K-convex Banach spaces. To show that every periodic continuously differentiable function on a B/K-convex space has absolutely convergent Fourier coefficients, we will need the concept of *Fourier type*. Remember that for $1 \le p \le 2$ and $f \in L^p(0, 2\pi)$ the Fourier coefficients of f can be estimated by the *Hausdorff-Young inequality*

$$\left(\sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right|^q\right)^{1/q} \le \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p\,dx\right)^{1/p},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. However, similiar to the situation we encountered with the Khintchine inequality, the Hausdorff-Young inequality does in general not hold in the vector-valued case, for counterexamples see [Pee69, Exemple 2.5]. As mentioned in [Pie07, 6.1.8.1], this was already observed by S. Bochner. This fact motivates the following definition.

Definition 2.5.17 (Fourier Type)**.** Let $1 \le p \le 2$. We say that a Banach space X has *Fourier type* p if there exists a constant C such that for all $f \in L^p((0, 2\pi); X)$ one has

$$\left(\sum_{m\in\mathbb{Z}} \left\|\hat{f}(m)\right\|_X^q\right)^{1/q} \le C\left(\int_0^{2\pi} \|f(x)\|_X^p\,dx\right)^{1/p},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 2.5.18. One often replaces $L^p(0, 2\pi)$ by $L^p(\mathbb{R})$ and the q-norm of the Fourier coefficients by the L^q-norm of the vector-valued Fourier transformation $\mathcal{F}f$ of f in the definition of Fourier type. One can show that these definitions are equivalent (see [Kön91, Proposition 2]).

Remark 2.5.19. Obviously, every Banach space has Fourier type 1. Parseval's identity shows that $X = \mathbb{C}$ has Fourier type 2. Moreover, S. Kwapień showed in his famous paper [Kwa72, Lemma 3.3] that a Banach space X is isomorphic to a Hilbert space if and only if X has Fourier type 2. If $1 < r < p$ and X has Fourier type p, interpolating between $L^1(X)$ and $L^p(X)$ shows that X has Fourier type r as well [Kön91, p. 220].

The above remark shows that Fourier type has very similiar properties compared to Rademacher type. So it is natural to ask whether there are any connections between these two notions. One can show that Fourier type p implies Rademacher type p [Kön91, p. 219-220]. However, the converse is not true in general: on the one hand one can show that ℓ_p has Fourier type $\min\{p, q\}$, where $\frac{1}{p} + \frac{1}{q} = 1$, and that these Fourier types are optimal (see [Pee69, p. 20ff.]) and on the other hand we have already observed that $p(\ell_p) = \min\{p, 2\}$. A deep partial converse result was proved by J-P. Bourgain. We will need it in the following form.

Theorem 2.5.20 (Bourgain)**.** *A Banach space X has non-trivial type if and only if it has non-trivial Fourier type.*

Proof. See [Bou82, Proposition 1] and [Bou88, Proposition 3] together with [Kön91, Proposition 2]. □

Remark 2.5.21. Bourgain proved the following stronger statement: if X has type p, then X has Fourier type r, where r is determined by $\frac{1}{r} + \frac{1}{s} = 1$ and $s = 18T_p(X)^q$, where q is again given by $\frac{1}{p} + \frac{1}{q} = 1$.

We can now finally prove König's theorem.

Theorem 2.5.22 (König (1990))**.** *A Banach space X is K-convex if and only if every 2π-periodic continuously differentiable function has absolutely summable Fourier coefficients.*

Proof. Let X be K-convex. Pisier's Equivalence theorem 2.5.1 shows that X has non-trivial type. By Bourgain's theorem 2.5.20, X has Fourier type p for some $p > 1$. Let $f \in C_p^1([0, 2\pi]; X)$. As seen before, we have $\hat{f'}(m) = im\hat{f}(m)$ for all $m \in \mathbb{Z}$. Thus

$$\begin{aligned}
\sum_{m\in\mathbb{N}} \left\|\hat{f}(m)\right\|_X &= \sum_{m\in\mathbb{N}} \frac{1}{m} m \left\|\hat{f}(m)\right\|_X \overset{\text{Hölder ineq.}}{\leq} \left(\sum_{m\in\mathbb{N}} \frac{1}{|m|^p}\right)^{1/p} \left(\sum_{m\in\mathbb{N}} |m|^q \left\|\hat{f}(m)\right\|_X^q\right)^{1/q} \\
&= c_p \left(\sum_{m\in\mathbb{N}} \left\|\hat{f'}(m)\right\|_X^q\right)^{1/q} \overset{\text{Fourier type } p}{\leq} c_p C \left(\int_0^{2\pi} \left\|f'(x)\right\|_X^p \, dx\right)^{1/p} \\
&\leq (2\pi)^{1/p} c_p C \left\|f\right\|_{C_p^1([0,2\pi];X])} < \infty,
\end{aligned}$$

where we have set $c_p := (\sum_{m\in\mathbb{N}} \frac{1}{|m|^p})^{1/p}$ and C is the Fourier type constant. Since the same estimate holds for the negative Fourier coefficients, we have shown that the Fourier coefficients are absolutely summable.

Conversely, let X be not K-convex. Lemma 2.5.11 shows that there exists a sequence of 2π-periodic continuously differentiable functions $f_n : [0, 2\pi] \to L^1(0, 2\pi)$, each of them having finite dimensional range E_n, such that the f_ns are uniformly bounded in $C_p^1([0, 2\pi]; L^1(0, 2\pi))$ by some positive constant M and $\lim_{n\to\infty} \sum_{m\in\mathbb{Z}} \left\|\hat{f}_n(m)\right\| = \infty$. Since $L^1(0, 2\pi)$ is finitely representable in ℓ_1 by Lemma 2.5.14, which is itself finitely representable in X by Pisier's Equivalence (Theorem 2.5.1) and Remark 2.5.15, there exist for every $\lambda > 1$ isomorphisms T_n from E_n onto some finite dimensional subspace of X such that $d(E_n, T(E_n)) \leq \lambda$ for all $n \in \mathbb{N}$, where d denotes the multiplicative Banach-Mazur distance. Now define

$$\begin{aligned} g_n : [0, 2\pi] &\to X \\ t &\mapsto T_n(f_n(t)). \end{aligned}$$

Then (g_n) is again a sequence of 2π-periodic continuously differentiable functions with

$$\|g_n\|_{C^1([0,2\pi];X)} \leq \|T_n\| \, M \leq \lambda M.$$

Since T_n is a bounded linear operator, the Fourier coefficients of g_n are given by $\hat{g_n}(m) = T_n(\hat{f}_n(m))$. Therefore

$$\sum_{m\in\mathbb{Z}} \|\hat{g_n}(m)\| \geq \left\|T_n^{-1}\right\|^{-1} \sum_{m\in\mathbb{Z}} \left\|\hat{f}_n(m)\right\| \geq \lambda^{-1} \sum_{m\in\mathbb{Z}} \left\|\hat{f}_n(m)\right\| \xrightarrow[n\to\infty]{} \infty.$$

Thus Lemma 2.5.16 shows that there exists a 2π-periodic continuously differentiable function whose Fourier coefficients are not absolutely summable. ☐

2.6 B-convexity vs. Reflexivity

In this section we investigate whether there are any connections between B-convexity and reflexivity. This question is motivated by known results for special types of B-convex spaces like uniformly convex spaces. However, will will see that neither reflexivity implies B-convexity nor B-convexity implies reflexivity.

2.6.1 Non-B-convex Reflexive Spaces

We present an example of a reflexive space that is not B-convex. The idea behind our example is very simple: it suffices to construct a reflexive space that is only of trivial type by Theorem 2.2.21. For this we take the infinite sum of reflexive spaces X_n of non-trivial type with the property that $p(X_n) \to 1$ as $n \to \infty$. Since the type of the sum is determined by its worst part, the infinite sum is only of trivial type. However, the reflexivity carries over from the summands to the sum. Examples of this type were already given by A. Beck (see [Gie66, Example I.7 (i)]).

Theorem 2.6.1. *Let $(p_n)_{n=1}^{\infty}$ be a real sequence with $p_n \in (1, 2]$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} p_n = 1$ and let X be the Hilbert sum of the ℓ_{p_n}'s, that is $X \coloneqq \bigoplus_{n=1}^{\infty} \ell_{p_n}$ with $\|(x_n)\|_X \coloneqq \left(\sum_{n=1}^{\infty} \|x_n\|_{\ell_{p_n}}^2\right)^{1/2}$. Then X is a reflexive non-B-convex Banach space.*

Proof. By construction for every $n \in \mathbb{N}$, X contains a closed subspace isometrically isomorphic to ℓ_{p_n}. Since B-convexity is invariant under isomorphisms by Lemma 2.1.8 and $p(\ell_{p_n}) = p_n$ by Corollary 2.1.16, we conclude

$$p(X) \overset{\text{Lemma 2.1.7}}{\leq} \inf_{n\in\mathbb{N}} p(\ell_{p_n}) = \inf_{n\in\mathbb{N}} p_n = 1.$$

Since every Banach space has type 1 (see Remark 2.1.5), we have $p(X) = 1$. Now Theorem 2.2.21 implies that X is not B-convex.

The reflexivity of X follows from the next more general lemma. □

Lemma 2.6.2. *Let $p \in (1, \infty)$ and $X \coloneqq \bigoplus_{i=1}^{\infty} X_i$ with $\|(x_i)\| \coloneqq \left(\sum_{i=1}^{\infty} \|x_i\|_{X_i}^p\right)^{1/p}$ be the sum of reflexive Banach spaces $(X_i)_{i=1}^{\infty}$. Then X is reflexive as well.*

Proof. Let $j : X \hookrightarrow X''$ be the canonical embedding into its bidual. We have to verify that j is surjective. Choose $x'' \in X''$ and let $e_i : X \to X_i$ be the canonical projection onto the i-th coordinate. Clearly, $y_i' \mapsto \langle x'', y_i' \circ e_i \rangle$ lies in the bidual X_i''. The reflexivity of X_i implies that for some $x_i \in X_i$ we have $\langle y_i', x_i \rangle = j_i(x_i)(y_i') = \langle x'', y_i' \circ e_i \rangle$ for all $y_i' \in X_i'$, where $j_i : X_i \hookrightarrow X_i''$ denotes the canonical embedding into the bidual. Hence for all $x' \in X'$, which as shown in Theorem 2.3.3 can be written as $x' = \sum_{i=1}^{\infty} x_i' \circ e_i$ for q-summable $(x_i') \in \bigoplus_{i=1}^{\infty} X_i'$,

$$\langle x'', x' \rangle = \sum_{i=1}^{\infty} \langle x'', x_i' \circ e_i \rangle = \sum_{i=1}^{\infty} \langle x_i', x_i \rangle. \tag{2.6.1}$$

We still must verify that $(x_i)_{i=1}^{\infty}$ is p-summable. By the Hahn-Banach theorem, we can choose $x_i' \in X_i'$ in the unit sphere such that $\langle x_i', x_i \rangle = \|x_i\|$. Let $(a_i)_{i=1}^{\infty}$ be a sequence with $\sum_{i=1}^{\infty} |a_i|^q \leq 1$. Then

$$\begin{aligned}\sum_{i=1}^{\infty} |a_i| \, \|x_i\| &= \sum_{i=1}^{\infty} \left\langle |a_i| \, x_i', x_i \right\rangle \overset{(2.6.1)}{=} \left\langle x'', \sum_{i=1}^{\infty} |a_i| \, x_i' \circ e_i \right\rangle \\ &\leq \|x''\| \left(\sum_{i=1}^{\infty} \|a_i x_i'\|^q \right)^{1/q} \leq \|x''\| .\end{aligned}$$

Hence,

$$\left(\sum_{i=1}^{\infty} \|x_i\|^p \right)^{1/p} = \sup \left\{ \sum_{i=1}^{\infty} |a_i| \, \|x_i\| : \sum_{i=1}^{\infty} |a_i|^q \leq 1 \right\} \leq \|x'\| .$$

This shows that $x \coloneqq (x_i)_{i=1}^{\infty} \in X$ and (2.6.1) implies for all $x' = \sum_{i=1}^{\infty} x_i' \circ e_i \in X'$

$$\langle x'', x' \rangle = \sum_{i=1}^{\infty} \langle x_i', x_i \rangle = \langle x', x \rangle = \langle j(x), x' \rangle.$$

Thus $x'' = j(x)$ and the reflexivity of X is shown. □

2.6.2 Non-Reflexive B-convex Spaces

By the Milman-Pettis theorem (see [Die84, p. 131] or [Rin59]), every uniformly convex Banach space is reflexive. We have seen in Theorem 2.2.5 that every uniformly convex Banach space is B-convex. So it is a natural question to ask whether B-convex Banach spaces are reflexive as well. It will be useful to introduce some more terminology.

Definition 2.6.3. A Banach space X is called *$k_{n,\varepsilon}$-convex* if for every $x_1, \dots, x_n \in X$ we can choose signs $(\varepsilon_i)_{i=1}^n \in \{-1,1\}^n$ such that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i x_i \right\| \le (1-\varepsilon) \max_{i=1,\dots,n} \|x_i\| \, .$$

X is called *uniformly non-square* if it is $k_{2,\varepsilon}$-convex for some $\varepsilon > 0$ and *uniformly non-octahedral* if it is $k_{3,\varepsilon}$-convex for some $\varepsilon > 0$.

Remark 2.6.4. Clearly, a Banach space is B-convex if and only if it is $k_{n,\varepsilon}$-convex for some natural number $n \ge 2$ and some $\varepsilon > 0$.

The problem of reflexivity of B-convex Banach spaces became even more delicate when R.C. James showed that every uniformly non-square Banach space is reflexive and that for every uniformly non-octahedral Banach space X''/X is reflexive [Jam64]. Moreover, D.P. Giesy [Gie66, Theorem 6] and R.C. James [Jam64, Theorem 2.2] could independently prove the following positive statement.

Theorem 2.6.5. *Every B-convex Banach space with an unconditional basis is reflexive.*

Proof. We use the following result by James (see [AK06, Theorem 3.3.3] or [Jam50]): a Banach space with an unconditional basis is reflexive if and only if no closed subspace of X is isomorphic to ℓ_1 or c_0.

Now let Y be a closed subspace of X. Lemma 2.2.4 shows that Y is B-convex as well. We have seen in Example 2.2.12 that neither ℓ_1 nor c_0 are B-convex. Since B-convexity is invariant under isomorphisms by Lemma 2.2.15, Y is not isomorphic to ℓ_1 or c_0. Hence, X is reflexive by the above characterization of reflexivity. $\square$

After some hard struggle R.C. James could finally provide an example of a non-reflexive Banach space which is uniformly non-octahedral in 1974 [Jam74], thereby proving that the conjecture is false. Since B-convexity is equivalent to having non-trivial type by Theorem 2.2.21, one could equivalently ask whether there are non-reflexive Banach spaces with non-trivial type. Later in the same year, W.J. Davies, W.B. Johnson and J. Lindenstrauss proved that for every $p < 2$ there exists a non-reflexive Banach space with type p [DJL76]. After that in 1977, R.C. James gave an example of a non-reflexive Banach space with type 2 [Jam78]. By Kwapień's characterization of Hilbert spaces (see Remark 2.1.11), a non-reflexive Banach space with both type 2 and cotype 2 cannot exist. However, it makes still sense to ask whether there exists a non-reflexive Banach space with type 2 and cotype $q > 2$ or with type $p < 2$ and cotype 2. This question was answered positively by G. Pisier and Q. Xu in 1987 [PX87]. They constructed examples of such spaces with the help of the real interpolation method. They proved the following theorem (the uncommon notations will be explained in the next section).

Theorem 2.6.6 (Pisier, Xu (1987))**.** *Let $\theta \in (0,1) \setminus \{\frac{1}{2}\}$, $1 \le q < \infty$ and $p_\theta = (1-\theta)^{-1}$. Then $(v_1, \ell_\infty)_{\theta,q}$ is a non-reflexive Banach space having type* $\min\{p_\theta, q, 2\}$ *and cotype* $\max\{p_\theta, q, 2\}$.

Corollary 2.6.7. *Let $1 \leq p \leq 2$ and $q > 2$. Then there exists a non-reflexive Banach space with type p and cotype q. Analogously, for each couple (p, q) with $1 \leq p < 2$ and $q \geq 2$ there exists a non-reflexive Banach space with type p and cotype q.*

Proof. Simply choose $q = 2$ and p_θ as close to $\frac{1}{2}$ as necessary in Theorem 2.6.6. $\square$

2.6.3 Pisier's and Xu's Construction: Main Ideas

We will now present the main ideas of Piser's and Xu's proof omitting some of the more technical details. To understand them we introduce the Banach spaces of all sequences of bounded variation.

Definition 2.6.8 (Sequences of Bounded Variation)**.** For $1 \leq p < \infty$ and a Banach space X we denote by $v_p(X)$ the space of all sequences of *p-bounded variation*, that is the space of all sequences $x = (x_n)_{n\in\mathbb{N}}$ in X such that the supremum of

$$\left(\|x_{n_1}\|^p + \sum_{k=2}^{\infty} \left\|x_{n_k} - x_{n_{k-1}}\right\|^p \right)^{1/p}$$

over all increasing sequences $1 \leq n_0 \leq n_1 \leq \ldots$ of integers is finite. We denote this supremum by $\|x\|_{v_p(X)}$. The space $v_1(X)$ is simply called the space of sequences of bounded variation. In the case $X = \mathbb{R}$ we sometimes write v_p instead of $v_p(X)$.

Remark 2.6.9. Observe that for $p = 1$ the supremum is clearly attained for $n_k = k$ by the triangle inequality. Hence,

$$\|x\|_{v_1(B)} = \|x_1\| + \sum_{k=2}^{\infty} \|x_k - x_{k-1}\| .$$

Remark 2.6.10. Taking constant sequences n_k, we see that for $x \in v_p(X)$ one has $\|x_n\| \leq \|x\|_{v_p(X)}$. Hence, $\|x\|_\infty \leq \|x\|_{v_p(X)}$ and therefore $v_p(X)$ is continuously embedded into $\ell_\infty(X)$. Moreover, one can show that $v_p(X)$ is a Banach space.

Pisier's and Xu's idea exploits the following deep characterization of non-reflexive Banach spaces by R.C. James.

Theorem 2.6.11 (James' Criterion)**.** *A Banach space X is non-reflexive if and only if there exist bounded linear operators $S : v_1 \to X$ and $T : X \to \ell_\infty$ such that the following diagram commutes*

$$\begin{tikzcd} v_1 \arrow[rr, "i"] \arrow[dr, "S"'] & & \ell_\infty \\ & X \arrow[ur, "T"'] & \end{tikzcd} ,$$

where $i : v_1 \to \ell_\infty$ denotes the canonical inclusion.

Before giving a sketch of the proof, we recall the definition of a *weakly compact operator*.

Definition 2.6.12 (Weakly Compact Operator)**.** Let X, Y be Banach spaces. A subset $A \subset X$ is called *relatively weakly compact* if the closure of A in the weak topology is compact. A linear operator $T : X \to Y$ is called *weakly compact* if the image of the closed unit ball under T is relatively weakly compact, that is if

$$\overline{T(\{x \in X : \|x\| \leq 1)}^{w}$$

is weakly compact.

Further, we will need that weak compactness characterizes reflexivity.

Theorem 2.6.13. *A Banach space is reflexive if and only if its closed unit ball is weakly compact.*

Proof. See [Meg98, Theorem 2.8.2] or [Die84, p. 18]. □

Proof of Theorem 2.6.11. Define the sum operator

$$\begin{aligned} \sigma : \ell_1 &\to \ell_\infty \\ (x_n)_{n\in\mathbb{N}} &\mapsto \left(\sum_{k=1}^{n} x_k\right)_{n\in\mathbb{N}} . \end{aligned}$$

[LP68, Theorem 8.1] shows that $\mathrm{id} : X \to X$ is not weakly compact if and only if there exist bounded linear operators $\tilde{S} : \ell_1 \to X$ and $T : X \to \ell_\infty$ such that the following diagram commutes

$$\begin{tikzcd} \ell_1 \arrow[r, "\sigma"] \arrow[d, "\tilde{S}"'] & \ell_\infty \\ X \arrow[r, "\mathrm{id}"'] & X \arrow[u, "T"'] \end{tikzcd} .$$

Mapping $(x_n)_{n\in\mathbb{N}} \in \ell_1$ to $(\sum_{k=1}^{n} x_k)_{n\in\mathbb{N}}$ defines an isometric isomorphism j between ℓ_1 and v_1 such that $\sigma \circ j^{-1} = i$. Hence, using the defining equation $j \circ S = \tilde{S}$, the above diagram and the diagram used in the statement of the theorem can be extended to

$$\begin{tikzcd} v_1 \arrow[r, "j^{-1}"] \arrow[rr, bend left, "i"] \arrow[dr, "S"'] & \ell_1 \arrow[r, "\sigma"] \arrow[d, "\tilde{S}"] & \ell_\infty \\ & X \arrow[r, "\mathrm{id}"'] & X \arrow[u, "T"'] \end{tikzcd} .$$

Hence, we can find operators S / $\tilde{S}$ and T such that the above diagram commutes if and only if $\mathrm{id} : X \to X$ is not weakly compact. By Theorem 2.6.13, $\mathrm{id} : X \to X$ is not weakly compact if and only if X is not reflexive. □

Now it is natural to consider some interpolation space between v_1 and ℓ_∞. This is exactly what Pisier and Xu did for the real interpolation spaces $(v_1, \ell_\infty)_{\theta,q}$ obtained by the real interpolation method of Lions-Peetre using the so called K-method. We shortly recall its definition in our concrete case. For $x \in \ell_\infty$ let

$$K(t,x) \coloneqq \inf_{\substack{x=x_1+x_2\\ x_1\in v_1,\, x_2\in\ell_\infty}} \{\|x_1\|_{v_1} + t\,\|x_2\|_{\ell_\infty}\}.$$

Then for $\theta \in (0,1)$ and $q \in [1,\infty)$ we let $(v_1, \ell_\infty)_{\theta,q}$ be the set of all $x \in \ell_\infty$ such that

$$\|x\|_{\theta,q} \coloneqq \left(\int_0^\infty (t^{-\theta} K(t,x))^q \frac{dt}{t}\right)^{1/q} < \infty.$$

One can show that $((v_1, \ell_\infty)_{\theta,q}, \|\cdot\|_{\theta,q})$ is a normed vector space. Moreover, it is complete and therefore a Banach space. For more details and references on interpolation spaces see Appendix B.

Theorem 2.6.14. *The real interpolation spaces $(v_1, \ell_\infty)_{\theta,q}$ are non-reflexive for $\theta \in (0,1)$ and $q \in [1,\infty)$.*

Proof. Let $X = (v_1, \ell_\infty)_{\theta,q}$. Let $i_1 : v_1 \to X$ be the canonical inclusion. Using $\|x\|_{\ell_\infty} \le \|x\|_{v_1}$, we see that for $x \in v_1$

$$K(t,x) = \inf_{\substack{x=x_1+x_2\\ x_1\in v_1,\, x_2\in\ell_\infty}} \{\|x_1\|_{v_1} + t\,\|x_2\|_{\ell_\infty}\} \le \min\{1,t\}\,\|x\|_{v_1}.$$

Hence,

$$\begin{aligned}
\|x\|_{\theta,q} &= \left(\int_0^\infty (t^{-\theta} K(t,x))^q \frac{dt}{t}\right)^{1/q} \le \|x\|_{v_1} \left(\int_0^1 t^{(1-\theta)q-1}\,dt + \int_1^\infty t^{-(1+\theta q)}\,dt\right)^{1/q}\\
&= \left(\frac{1}{(1-\theta)q} + \frac{1}{\theta q}\right)^{1/q} \|x\|_{v_1}.
\end{aligned}$$

Hence, the inclusion is well-defined and v_1 is continuously embedded in $(v_1, \ell_\infty)_{\theta,q}$. Moreover, interpolating between the identity on ℓ_∞ and the canonical contractive embedding of v_1 into ℓ_∞ yields a bounded linear operator $i_2 : (v_1, \ell_\infty)_{\theta,q} \to \ell_\infty$ with

$$\|i_2\| \le 1$$

because $K_{\theta,q}$ is an exact interpolation functor of exponent θ. Therefore the following diagram commutes

$$\begin{tikzcd}
v_1 \arrow[rr, "i"] \arrow[dr, "i_1"'] & & \ell_\infty . \\
& (v_1, \ell_\infty)_{\theta,q} \arrow[ur, "i_2"'] &
\end{tikzcd}$$

By James' Criterion (Theorem 2.6.11), $(v_1, \ell_\infty)_{\theta,q}$ is non-reflexive. □

Type and Cotype of $(v_1, \ell_\infty)_{\theta,q}$

Now that the non-reflexivity of these spaces is shown, it remains to prove the claims on their types and cotypes as given in Theorem 2.6.6.

Theorem 2.6.15. *Let $\theta \in (0,1) \setminus \{\frac{1}{2}\}$, $1 \le q < \infty$ and $p_\theta = (1-\theta)^{-1}$. Then $(v_1, \ell_\infty)_{\theta,q}$ has type $\min\{p_\theta, q, 2\}$ and cotype $\max\{p_\theta, q, 2\}$.*

This is the actual content of Pisier's and Xu's paper [PX87]. In order to shorten notations, we introduce the following convention.

Convention 2.6.16. Until the end of this section, we always let X be a Banach space and $(D, \mathcal{B}(D), \mu)$ the measure space of infinitely many independent coin tosses as described before the definition of Rademacher type and cotype. Moreover, let $\theta \in (0,1)$, $q \in [1,\infty)$ and $\frac{1}{p_\theta} := \frac{1-\theta}{1} + \frac{\theta}{\infty}$. We always use $L_r(X)$ as an abbreviation for $L_r(D, \mathcal{B}(D), \mu; X)$. Moreover, we will write

$$A_{\theta,q}(X) := (v_1(X), \ell_\infty(X))_{\theta,q}.$$

The proof of Theorem 2.6.15 needs some preparations. We will need the further rather technical inclusions whose verifications we omit.

Theorem 2.6.17. *The following continuous embeddings hold.*

(a) For $p < p_\theta$ and $p \le q$ we have $L_p(A_{\theta,q}) \subset A_{\theta,q}(L_p)$.

(b) For $r > p_\theta$ and $r \ge q$ we have $A_{\theta,q}(L_r) \subset L_r(A_{\theta,q})$.

Moreover, the norms of the inclusions only depend on p, q, r and θ.

Proof. See [PX87, Theorem 3]. □

The following lemma guarantees that under certain conditions bounded operators can be extended to operators on some kind of interpolation space. This will be needed in the proof of the next theorem.

Lemma 2.6.18. *Let $p < p_\theta$ and $p \le q$ and $T : L^p \to X$ be a bounded linear operator. Then $T \otimes_\pi \mathrm{Id}_{A_{\theta,q}}$ can be extended to a bounded linear operator*

$$L_p(A_{\theta,q}) \to A_{\theta,q}(X).$$

Analogously, for $r > p_\theta$ and $r \ge q$ and a bounded linear operator $T : X \to L_r$ the operator $T \otimes_\pi Id_{A_{\theta,q}}$ can be extended to a bounded linear operator

$$A_{\theta,q}(X) \to L_r(A_{\theta,q}).$$

Moreover, the norms of these extensions only depend on p, r, q and θ.

Proof. We first recall that $T \otimes_\pi \mathrm{Id}_{A_{\theta,q}}$ is given by

$$L^p \otimes_\pi A_{\theta,q} \ni f \otimes a \mapsto Tf \otimes a.$$

Moreover, recall that for a Banach space Z the bilinear mapping

$$\begin{aligned} Z \times A_{\theta,q} &\to \ell_\infty(Z) \\ (z, (a_n)_{n\in\mathbb{N}}) &\mapsto (a_n \cdot z)_{n\in\mathbb{N}} \end{aligned}$$

induces a bounded linear operator $Z \otimes_\pi A_{\theta,q} \to \ell_\infty(Z)$. Next, we show that the mapping is injective. Therefore let $(\sum_{k=1}^m a_n^{(k)} z_k)_{n\in\mathbb{N}} = 0$. Thus $\sum_{k=1}^m a_n^k z_k = 0$ for all $n \in \mathbb{N}$. Let $e_n' : \ell_\infty \to \mathbb{K}$ be the functional that maps a sequence to its n-th coordinate. Since these coordinate functionals clearly separate points, Lemma A.4.5 shows

$$\sum_{k=1}^m a^{(k)} \otimes z_k = 0.$$

Thus $Z \otimes_\pi A_{\theta,q}$ is continuously embedded into $\ell_\infty(Z)$ and therefore can be identified with a subspace of $\ell_\infty(Z)$. Here, we consider the case $Z = L^p$.

We first show that $T \otimes_\pi \mathrm{Id}_{A_{\theta,q}}$ can be extended to a bounded linear operator $\ell_\infty(L_p) \to \ell_\infty(X)$. For this observe that

$$(f_n)_{n\in\mathbb{N}} \mapsto (Tf_n)_{n\in\mathbb{N}}$$

can be estimated by

$$\|(Tf_n)\|_{\ell_\infty(X)} = \sup_{n\in\mathbb{N}} \|Tf_n\|_X \le \|T\| \sup_{n\in\mathbb{N}} \|f_n\|_{L^p} = \|T\| \, \|(f_n)\|_{\ell_\infty(L^p)} .$$

Moreover, its restriction to $v_1(L_p)$ maps into $v_1(X)$ and is bounded:

$$\begin{aligned} \|(Tf_n)\|_{v_1(X)} &= \|Tf_1\|_X + \sum_{n=2}^\infty \|Tf_n - Tf_{n-1}\|_X \\ &\le \|T\| \left(\|f_1\|_{L^p} + \sum_{n=2}^\infty \|f_n - f_{n-1}\|_{L^p} \right) = \|T\| \, \|(f_n)\|_{v_1(L^p)} . \end{aligned}$$

Now interpolation shows that $T \otimes_\pi \mathrm{Id}_{A_{\theta,q}}$ can be extended to a bounded linear operator (with norm at most $\|T\|$)

$$\tilde{T} : A_{\theta,q}(L_p) = (v_1(L_p), \ell_\infty(L_p))_{\theta,q} \to (v_1(X), \ell_\infty(X))_{\theta,q} = A_{\theta,q}(X).$$

Finally, Theorem 2.6.17(a) implies that $\tilde{T}$ induces a bounded linear operator

$$L_p(A_{\theta,q}) \hookrightarrow A_{\theta,q}(L_p) \xrightarrow{\tilde{T}} A_{\theta,q}(X).$$

The proof of the second part is completely similiar. $\square$

The following theorem is the heart of the proof. It shows that in a certain sense we can interpolate the Khintchine inequality.

Theorem 2.6.19. *There is a constant $C = C(\theta, q)$ such that, for all finite sequences $x_1, \ldots, x_n \in A_{\theta q}$, we have*

$$\frac{1}{C} \left\|(x_1, \ldots, x_n)\right\|_{A_{\theta,q}(\ell_2^n)} \leq \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|_{L_2(A_{\theta,q})} \leq C \left\|(x_1, \ldots, x_n)\right\|_{A_{\theta,q}(\ell_2^n)}.$$

Proof. We only prove the theorem in the case $q > 1$ because we do not need the case $q = 1$. The proof of this case can be found in [PX87, Theorem 8]. For $1 < p < \infty$ and $n \in \mathbb{N}$ let

$$\begin{aligned} T : L^p &\to \ell_2^n \\ f &\mapsto \left(\int_D \varepsilon_i(\omega) f(\omega) d\mu(\omega) \right)_{i=1,\ldots,n}. \end{aligned}$$

The Khintchine inequality (Corollary D.2.4) and the K-convexity of $\mathbb{R}$, that is the uniform boundedness of the Rademacher projections $R_n^{\mathbb{R}}$ (see Remark 2.5.5), imply

$$\begin{aligned} \|Tf\|_{\ell_2^n} &= \left(\sum_{i=1}^n \left| \int_D \varepsilon_i(\omega) f(\omega)\, d\mu(\omega) \right|^2 \right)^{1/2} \leq A_p^{-1} \left\| \sum_{i=1}^n \varepsilon_i \int_D \varepsilon_i(\omega) f(\omega)\, d\mu(\omega) \right\|_{L^p} \\ &= A_p^{-1} \left\| R_n^{\mathbb{R}} f \right\|_{L^p} \leq A_p^{-1} K(\mathbb{R}) \left\| f \right\|_{L^p} = A_p^{-1} \left\| f \right\|_{L^p}, \end{aligned}$$

which is independent of n and f. Now for $p < p_\theta$ and $p \leq q$ Lemma 2.6.18 shows that T can be extended to a bounded linear operator $\tilde{T} : L^p(A_{\theta,q}) \to A_{\theta,q}(\ell_2^n)$. Moreover, since the above estimates are independent of n, $\left\| \tilde{T} \right\|$ is bounded by a positive constant $C = C(p, q, \theta)$ that is independent of n. Now for $p \in (1, \min\{p_\theta, q, 2\})$ and $f = \sum_{i=1}^n \varepsilon_i x_i$ we obtain

$$\begin{aligned} \left\|(x_1, \ldots, x_n)\right\|_{A_{\theta,q}(\ell_2^n)} &= \left\| \left(\int_D \varepsilon_j(\omega) \sum_{i=1}^n \varepsilon_i(\omega) x_i \, d\mu(\omega) \right)_{j=1}^n \right\|_{A_{\theta,q}(\ell_2^n)} \\ &= \left\| \tilde{T} \left(\sum_{i=1}^n \varepsilon_i x_i \right) \right\|_{A_{\theta,q}(\ell_2^n)} \leq C \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|_{L^p(A_{\theta,q})} \\ &\overset{p \leq 2}{\leq} C \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|_{L^2(A_{\theta,q})}. \end{aligned}$$

Since our choice of p only depends on q and θ, the constant C in the last estimate only depends on q and θ. This shows the first inequality of the claim.

The second inequality can be shown using very similiar arguments. One shows with the help of the Khintchine inequality that for $r > p_\theta$ and $r \geq q$ the operator

$$S : \ell_2^n \to L_r$$

$$(\alpha_1, \ldots, \alpha_n) \mapsto \sum_{i=1}^{m} \alpha_i \varepsilon_i$$

extends to a bounded linear operator $A_{\theta,q}(\ell_2^n) \to L^r(A_{\theta,q})$ whose norm is independent of n. As above, this yields the desired inequality. □

Finally, with the help of the above theorem and the following technical embeddings which are closely related to those given in Theorem 2.6.17 and whose verifications we omit again, we can proof Theorem 2.6.15.

Lemma 2.6.20. *The following continuous embeddings hold.*

(a) If $p_\theta < t \leq \infty$ and $s = \min\{p_\theta, q\}$, then

$$\ell_s(A_{\theta,q}) \subset A_{\theta,q}(\ell_t).$$

(b) If $1 \leq t < p_\theta$ and $r = \max\{p_\theta, q\}$, then

$$A_{\theta,q}(\ell_t) \subset \ell_r(A_{\theta,q}).$$

Proof. See [PX87, Lemma 10]. □

Proof of Theorem 2.6.15. Let $x_1, \ldots, x_n \in A_{\theta,q}$. We begin with the case $p_\theta < 2$. By Theorem 2.6.19, we have for a positive constant C that is independent of the chosen x_is

$$\left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|_{L^2(A_{\theta,q})} \leq C \left\| (x_1, \ldots, x_n) \right\|_{A_{\theta,q}(\ell_2^n)}.$$

Since $p_\theta < t \coloneqq 2$, $\ell_s(A_{\theta,q})$ is continuously embedded into $A_{\theta,q}(\ell_2)$ for $s = \min\{p_\theta, q\}$ by Lemma 2.6.20. Therefore

$$\left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|_{L^2(A_{\theta,q})} \leq D \left\| (x_1, \ldots, x_n) \right\|_{\ell_s(A_{\theta,q})} = D \left(\sum_{i=1}^{n} \|x_i\|_{\theta,q}^s \right)^{1/s}$$

for some positive constant $D > 0$. Hence, $A_{\theta,q}$ has type $s = \min\{p_\theta, q\} = \min\{p_\theta, q, 2\}$.

For $r \coloneqq \max\{q, 2\}$ Theorem 2.6.17 shows that $A_{\theta,q}(\ell_r)$ is continuously embedded into $\ell_r(A_{\theta,q})$ with norm $\tilde{C}$. Together with Theorem 2.6.19 this shows

$$\begin{aligned} \left(\sum_{i=1}^{n} \|x_i\|_{\theta,q}^r \right)^{1/r} &= \left\| (x_1, \ldots, x_n) \right\|_{\ell_r(A_{\theta,q})} \leq \tilde{C} \left\| (x_1, \ldots, x_n) \right\|_{A_{\theta,q}(\ell_r)} \\ &\overset{r \geq 2}{\leq} \tilde{C} \left\| (x_1, \ldots, x_n) \right\|_{A_{\theta,q}(\ell_2^n)} \leq \tilde{C} C \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|_{L_2(A_{\theta,q})}. \end{aligned}$$

Thus $A_{\theta,q}$ has cotype $r = \max\{2, q\} = \max\{2, q, p_\theta\}$. The proof of the case $p_\theta > 2$ is completely similiar. □

A Functional Analysis

A.1 Spectral Theory for Closed Operators

We shortly introduce the spectral theory for closed operators, mainly for the reason that the definition of the resolvent differs in the literature. For further details see [EN00, IV.1].

Definition A.1.1 (Closed Operator)**.** Let X, Y be two Banach spaces. A linear operator $A : D(A) \subset X \to Y$ is *closed* if, given a sequence $(x_n) \subset D(A)$ such that $x_n \to x$ and $Ax_n \to y$ for some $x \in X$ and $y \in Y$, one has $x \in D(A)$ and $Ax = y$.

Remark A.1.2. Equivalently, A is closed if and only if the *graph of T* given by $G(T) : = \{(x, Tx) : x \in D(A)\}$ is a closed subset of $X \times Y$.

Clearly, a bounded linear operator is closed. Conversely, the closed graph theorem states that a closed linear operator $T : X \to Y$ between two Banach spaces X and Y is bounded.

Definition A.1.3 (Resolvent)**.** For a Banach space X let $A : D(A) \subset X \to X$ be a linear operator. We call

$$\rho(A) \coloneqq \{\lambda \in \mathbb{C} : \lambda - A \text{ is invertible}\}$$

the *resolvent set* of A. For $\lambda \in A$ the inverse

$$R(\lambda, A) \coloneqq (\lambda - A)^{-1}$$

is called the *resolvent* of A at the point λ.

Remark A.1.4. Let $A : D(A) \subset X \to X$ be as above such that $\rho(A) \neq \emptyset$. Choose $\lambda_0 \in \rho(A)$. Then $(\lambda_0 - A)^{-1}$ is a bounded operator and a fortiori closed. Hence, $\lambda_0 - A$ and therefore A are closed as well. So we may restrict ourselves to closed linear operators.

We conclude with some important facts.

Theorem A.1.5 (Resolvent Equation)**.** *Let A be a closed linear operator. For $\lambda, \mu \in \rho(A)$ one has*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A).$$

Proof. See [EN00, IV.1.2]. $\square$

Theorem A.1.6. *Let A be a closed linear operator on a complex Banach space. Then the resolvent set $\rho(A)$ is open in $\mathbb{C}$ and for $\mu \in \rho(A)$ one has*

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\mu - \lambda)^n R(\mu, A)^{n+1}$$

for all $\lambda \in \mathbb{C}$ satisfying $|\lambda - \mu| < \|R(\mu, A)\|^{-1}$. A fortiori, the resolvent map $\lambda \mapsto R(\lambda, A)$ is holomorphic.

Proof. See [EN00, Proposition IV.1.3]. □

A.2 Normal Operators

We now present some basic facts about not necessarily bounded normal operators.

Definition A.2.1 (Normal Operator)**.** A closed densely defined linear operator $A : H \supset D(A) \to H$ on a Hilbert space H is called *normal* if

$$AA^* = A^*A,$$

where A^* denotes the adjoint of A.

We now prove some fundamental properties of normal operators.

Lemma A.2.2. *Let $A : H \supset D(A) \to H$ be a closed, densely defined operator. Then A is normal if and only if $D(A) = D(A^*)$ and $\|Ax\| = \|A^*x\|$ for all $x \in D(A)$.*

Proof. Suppose that A satisfies the only-if-part. The polarization identity shows that $(Ax|Ay) = (A^*x|A^*y)$ for all $x, y \in D(A)$. Hence, for $x \in D(AA^*)$ and $y \in D(A)$

$$(Ax, Ay) = (A^*x, A^*y) = (AA^*x, y).$$

One sees from the right hand side that $y \mapsto (Ax, Ay)$ can be extended to a continuous linear functional on H. Hence, $x \in D(A^*A)$ and $A^*Ax = AA^*x$. Further, since A^{**} is the closure of A (see [Wer09, Satz VII.2.4c)]) and A is already closed, we have $A^{**} = A$. Hence, the converse inclusion follows from the above calculation if we replace A by A^*.

Conversely, let A be normal. A famous theorem due to J. von Neumann shows that A^*A is self-adjoint (see [Kat95, Theorem V.3.24]) and $D(A^*A) = D(AA^*)$ is a core of A, that is the closure of $\{(x, Tx) : x \in D(A^*A)\}$ in $H \times H$ endowed with the graph norm is the graph of A. Let $x \in D(A)$. Then there exist $x_n \in D(A^*A)$ such that $x_n \to x$ and $Ax_n \to Ax$ as $n \to \infty$. Observe that

$$\begin{aligned} \|A^*x_n - A^*x_m\| &= (A^*(x_n - x_m)|A^*(x_n - x_m))^{1/2} = (x_n - x_m|AA^*(x_n - x_m))^{1/2} \\ &= (x_n - x_m|A^*A(x_n - x_m))^{1/2} = (A(x_n - x_m)|A(x_n - x_m))^{1/2} \\ &= \|Ax_n - Ax_m\|. \end{aligned}$$

Hence, (A^*x_n) is a Cauchy sequence. Therefore $A^*x_n \to y$ for some $y \in H$. Since A^* is closed, we conclude $x \in D(A^*)$ and $A^*x = y$. This shows $D(A) \subset D(A^*)$. Analogously, one shows $D(A^*) \subset D(A)$. This shows $D(A) = D(A^*)$. Moreover, one shows with the same argument that the equality $\|Ax\| = \|A^*x\|$ extends from $x \in D(A^*A)$ to all $x \in D(A)$. □

Lemma A.2.3. *Let $A : H \supset D(A) \to H$ be a closed, densely defined operator such that A^* is densely defined as well. Then A is invertible if and only if A^* is invertible. In this case we have $(A^{-1})^* = (A^*)^{-1}$.*

Proof. Suppose A is invertible. Then $(A^{-1})^* \in \mathcal{L}(H)$ and for all $y \in H$ and $x \in D(A^*)$ we have

$$\big((A^{-1})^*A^*x\big|y\big) = \big(A^*x\big|A^{-1}y\big) = \big(x\big|AA^{-1}y\big) = (x|y).$$

This shows $(A^{-1})^*A^*x = x$ for all $x \in D(A^*)$. Similiarly, for $x \in H$ and $y \in D(A)$ we have

$$\big((A^{-1})^*x\big|Ay\big) = \big(x\big|A^{-1}Ay\big) = (x|y).$$

This implies $(A^{-1})^*x \in D(A^*)$ and $A^*(A^{-1})^*x = x$ for all $x \in H$. Hence, A^* is invertible and $(A^*)^{-1} = (A^{-1})^*$.

Now suppose that A^* is invertible. By the first part, A^{**} is invertible. Since A^{**} is the closure of A (see [Wer09, Satz VII.2.4c)]) and A is already closed, we conclude $A^{**} = A$. □

Remark A.2.4. The above lemma can be shown under weaker assumptions (see [Kat95, Theorem III.5.30]).

Corollary A.2.5. *Let A be as in Lemma A.2.3. Then $\rho(A^*) = \overline{\rho(A)}$.*

Proof. Lemma A.2.3 shows that $\lambda - A$ is invertible if and only if $(\lambda - A)^* = \overline{\lambda} - A^*$ is invertible. □

Lemma A.2.6. *Let $A : H \supset D(A) \to H$ be a normal operator and $\lambda \in \rho(A)$. Then $R(\lambda, A) \in \mathcal{L}(H)$ is normal as well.*

Proof. Let $\lambda \in \rho(A)$. Then $D := D((A^* - \overline{\lambda})(A - \lambda)) = D(A^*A)$: The inclusion $D(A^*A) \subset D$ is obvious. Conversely, let $x \in D$. Then $x \in D(A)$ and $Ax - \lambda x \in D(A^*) = D(A)$. By linearity, we conclude $Ax \in D(A) = D(A^*)$. Hence, $x \in D(A^*A)$. Analogously, one shows $D((A - \lambda)(A^* - \overline{\lambda})) = D(A^*A)$. Now, since A is normal, we have

$$(A^* - \overline{\lambda})(A - \lambda) = (A - \lambda)(A^* - \overline{\lambda}).$$

By Corollary A.2.5, we can take inverses on both sides and get

$$R(\lambda, A)R(\overline{\lambda}, A^*) = R(\overline{\lambda}, A^*)R(\lambda, A).$$

Finally, Lemma A.2.3 yields

$$R(\lambda, A)R(\lambda, A)^* = R(\lambda, A)^*R(\lambda, A).$$ □

A.3 Lattices

We present some basic notions from the theory of lattices as far as they are needed in this thesis. Complete treatments of this topic are [AB06], [MN91] and [Sch74]. We follow the presentation in [Ger10].

Definition A.3.1. Let $(E, \leq)$ be a partial ordered set. E is called a lattice if for every two elements $x, y \in E$ both their *supremum* $\sup(\{x, y\})$ and their *infimum* $\inf(\{x, y\})$ exist (for a subset $F \subset E$ the supremum $\sup(F)$ is the unique element $x \in E$ - if it exists - such that $x \geq z$ for all $z \in F$ and such that for all $y \in E$ with the same property we have $y \geq x$; the infimum is defined completely analogously).

If in addition E is a real vector space, E is called a *vector lattice* if $x \leq y$ implies $x + z \leq y + z$ and $\alpha x \leq \alpha y$ for all $x, y, z \in E$ and $\alpha > 0$. In this case we define

$$x^+ := \sup\{x, 0\}, \quad x^- := (-x)^+ \quad \text{and} \quad |x| := x^+ + x^-.$$

A vector lattice E endowed with a norm such that $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$ for $x, y \in E$ is called a *normed vector lattice*. If in addition E is complete with respect to its norm, E is called a *Banach lattice*.

We will only be interested in Lebesgue spaces.

Example A.3.2. Let (Ω, Σ, μ) be a measure space and $1 \leq p \leq \infty$. Then $L^p(\Omega, \Sigma, \mu)$ is a Banach lattice with respect to its natural ordering

$$f \leq g :\Leftrightarrow f(x) \leq g(x) \text{ for almost every } x \in \Omega.$$

The infimum and supremum of two functions $f, g \in L^p(\Omega, \Sigma, \mu)$ are given pointwise, that is

$$\inf(\{f, g\})(x) = \min\{f(x), g(x)\} \qquad \text{and} \qquad \sup(\{f, g\})(x) = \max\{f(x), g(x)\}.$$

Definition A.3.3 (Positive Operator)**.** A linear operator $T : E \to F$ between two lattices is called *positive* if $x \geq 0$ implies $Tx \geq 0$.

Example A.3.4. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces and denote by (Ω, Σ, μ) the product space. Further, let $k : \Omega \to \mathbb{R}$ be a measurable function such that $k(s, t) \geq 0$ for almost every $(s, t) \in \Omega$ and that for every $f \in L^p(\Omega_1, \Sigma_1, \mu_1)$ one has $k(\cdot, t)f(\cdot) \in L^1(\Omega_1, \Sigma_1, \mu_1)$ for almost every $t \in \Omega_2$. Assume further that for all $f \in L^p(\Omega_1, \Sigma_1, \mu_1)$

$$t \mapsto \int_{\Omega_1} k(s, t) f(s) \, d\mu_1(s) \in L^q(\Omega_2, \Sigma_2, \mu_2).$$

Then a positive integral operator is given by

$$\begin{aligned} T : L^p(\Omega_1, \Sigma_1, \mu_1) &\to L^q(\Omega_2, \Sigma_2, \mu_2) \\ f &\mapsto \int_{\Omega_1} k(s, t) f(s) \, d\mu_1(s). \end{aligned}$$

We now want to endow the span of all positive operators with the structure of a vector lattice.

Definition A.3.5. A linear operator $T : E \to F$ between two vector lattices is called *regular* if it can be written as the difference of two positive operators. The set of all regular operators is a real vector space which we denote by $\mathcal{L}^r(E,F)$. $\mathcal{L}^r(E,F)$ is an partial ordered set with respect to $T \leq S :\Leftrightarrow S - T \geq 0$.

Definition A.3.6 (Dedekind complete)**.** A lattice E is called Dedekind complete if every non-empty bounded subset of E has an infimum and a supremum in E.

Theorem A.3.7. *For $1 \leq p < \infty$ the Lebesgue spaces $L^p(\Omega, \Sigma, \mu)$ are Dedekind complete for an arbitrary measure space (Ω, Σ, μ). The same holds for $p = \infty$ provided the measure is σ-finite.*

Proof. See [Ger10, Example 2.4.6 & Theorem 2.4.8]. $\square$

Theorem A.3.8. *Let E and F be vector lattices and assume that F is Dedekind complete. Then $\mathcal{L}^r(E,F)$ is a vector lattice with*

$$\begin{aligned}
\sup(\{T,S\})(x) &= \sup\{T(x-y) + Sy : y \in E, 0 \leq y \leq x\}, \\
\inf(\{T,S\})(x) &= \inf\{T(x-y) + Sy : y \in E, 0 \leq y \leq x\}, \\
|T|(x) &= \sup\{|Ty| : y \in E, |y| \leq x\} = \sup\{T(2y-x) : y \in E, 0 \leq y \leq x\}
\end{aligned}$$

for all $x \geq 0$ and every $T, S \in \mathcal{L}^r(E,F)$.

Proof. See [Ger10, Theorem 2.2.5]. $\square$

In general, one has $\|T\| \leq \||T|\|$. However, in certain cases one even obtains equality.

Theorem A.3.9. *Let $T \in \mathcal{L}^r(L^1, L^1)$ for some fixed measure space. Then $\||T|\| = \|T\|$.*

Proof. Clearly, L^1 is an abstract AL-space and by the monotone convergence theorem L^1 is a KB-space. Then the assertion follows from [AB06, Theorem 4.75]. $\square$

The following concrete result will be needed in the discussion of holomorphic semigroups.

Lemma A.3.10. *For some $\Omega \subset \mathbb{R}^N$ let $T : L^1(\Omega) \to L^1(\Omega)$ be a positive integral operator with locally bounded kernel $k : \Omega \times \Omega \to \mathbb{R}$ as presented in Example A.3.4. Then* $\inf(\{\mathrm{Id}, T\}) = 0$.

Proof. Recall that for $f \geq 0$ the infimum is given by

$$\inf(\{I,T\})(f) = \inf\{f - g + Tg : 0 \leq g \leq f\}.$$

Clearly, every element of the above set is positive because T is a positive operator. Hence, $\inf(\{I,T\})(f) \geq 0$ for all $f \geq 0$. Conversely, let $\varepsilon > 0$ and $x_0 \in \Omega$. It is sufficient to show that there exists some $0 \leq g \leq f$ and some measurable set A of positive measure with $x_0 \in A$ such that

$$(f - g + Tg)(x) \leq \varepsilon \Leftrightarrow (g - Tg)(x) \geq f(x) - \varepsilon \qquad \text{for almost all } x \in A$$

because this directly implies $\inf(\{I, T\}) \leq \varepsilon$. Let $g_\delta \coloneqq f \mathbb{1}_{(x_0-\delta, x_0+\delta)}$. Then $0 \leq g_\delta \leq f$. There exist $\delta_0 > 0$ and $M \geq 0$ such that $|k(x, y)| \leq M$ for almost all $(x, y) \in [x_0 - \delta_0, x_0 + \delta_0]^2$ because k is locally bounded. Observe that for $\delta < \delta_0$ and $|x - x_0| < \delta_0$

$$|Tg_\delta(x)| = \left| \int_\Omega k(x, y) f(y) \mathbb{1}_{(x_0-\delta, x_0+\delta)}(y)\, dy \right| \leq M \int_\Omega f(y) \mathbb{1}_{(x_0-\delta, x_0+\delta)}(y)\, dy.$$

By Lebesgue's dominated convergence theorem, the last expression tends to 0 as $\delta \to 0$. Thus for a sufficiently small δ_1 we have as desired

$$(g_{\delta_1} - Tg_{\delta_1})(x) \geq f(x) - \varepsilon \qquad \text{for } |x - x_0| < \delta_0. \qquad \square$$

Corollary A.3.11. *Let T be as in Lemma A.3.10. Then $I + T = |I - T|$.*

Proof. This is a direct consequence of Lemma A.3.10 because

$$0 = \inf(\{I, T\}) = \frac{1}{2}(I + T - |I - T|). \qquad \square$$

Remark A.3.12. The above result remains valid in more general settings [Ger10, Theorem 4.3.4].

A.4 The Projective Tensor Product

Recall that the tensor product $U \otimes_K V$ of two K-vector spaces U, V is characterized by the following universal property. Given the natural embedding

$$\begin{aligned} \varphi : U \times V &\to U \otimes_K V \\ (u, v) &\mapsto u \otimes v, \end{aligned}$$

every bilinear map $b : U \times V \to W$ into a K-vector space W factors uniquely through φ, that is there exists a unique linear map $f : U \otimes_K V \to W$ such that $b = f \circ \varphi$. In a diagram this writes as

$$\begin{tikzcd} U \times V \arrow[r, "\varphi"] \arrow[dr, "b"'] & U \otimes V \arrow[d, dashed, "f"] \\ & W \end{tikzcd} .$$

As always, the universal property determines $U \otimes V$ uniquely up to unique isomorphism.

Notice that the universal property makes sense in the categories of normed vector spaces and Banach spaces, both with continuous linear mappings as morphisms, as well. We will see that the so called *projective tensor product* is the tensor product described by the universal property in these categories.

Theorem A.4.1. *Let E, F be normed vector spaces. On $E \otimes F$ we define*

$$\|z\|_\pi := \inf \left\{ \sum_{i=1}^n \|x_i\|_E \|y_i\|_F : n \in \mathbb{N}, x_i \in E, y_i \in F, z = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

This makes $(E \otimes F, \|\cdot\|_\pi)$, the so called projective tensor product, into a normed vector space. One often writes $E \otimes_\pi F$. Moreover, one has $\|x \otimes y\|_\pi = \|x\|_E \cdot \|y\|_F$ for arbitrary $x \in E$, $y \in F$.

Proof. See [Rya02, Proposition 2.1]. □

Theorem A.4.2. *$E \otimes_\pi F$ is the tensor product of the two normed vector spaces E and F in the category of normed vector spaces with continuous linear maps as morphisms.*

Proof. Let $E \times F \to G$ be a bounded bilinear map into a normed vector space G. By the universal property of the tensor product for vector spaces, there exists a unique linear map $B_0 : E \otimes F \to G$ such that $B_0(x \otimes y) = B(x, y)$. A fortiori, this proves the uniqueness of a bounded linear map. It remains to show that B_0 is bounded. Let $z \in E \otimes F$, $\varepsilon > 0$. Choose $z = \sum_{i=1}^n x_i \otimes y_i$ such that $\sum_{i=1}^n \|x_i\|_E \|y_i\|_F \leq \|z\|_\pi + \varepsilon$. Then

$$\begin{aligned} \|B_0(z)\| = \left\| B_0 \left(\sum_{i=1}^n x_i \otimes y_i \right) \right\| &\leq \sum_{i=1}^n \|B(x_i, y_i)\| \\ &\leq \|B\| \sum_{i=1}^n \|x_i\|_E \|y_i\|_F \leq \|B\| \left(\|z\|_\pi + \varepsilon\right). \end{aligned}$$

Since ε is arbitrary, this shows $\|B_0\| \leq \|B\|$. Since $\|x \otimes y\|_\pi = \|x\|_E \cdot \|y\|_F$, we have

$$\begin{aligned} \|B\| &= \sup\{\|B(x, y)\| : \|x\|_E \leq 1, \|y\|_F \leq 1\} \\ &\leq \sup\{\|B_0(x \otimes y)\| : \|x \otimes y\|_\pi \leq 1\} \leq \|B_0\|. \end{aligned}$$

Hence, $\|B\| = \|B_0\|$. □

Notice that in general the tensor product $X \otimes_\pi Y$ of two Banach spaces is not complete and therefore not a Banach space. In the category of Banach spaces we must therefore look at its completion.

Definition A.4.3. Let X, Y be two Banach spaces. The completion of $X \otimes_\pi Y$ with respect to $\|\cdot\|_\pi$ - which we denote by $X \hat{\otimes}_\pi Y$ - is called the *projective tensor product* of X and Y.

Theorem A.4.4. *$X \hat{\otimes}_\pi Y$ is the tensor product in the category of Banach spaces.*

Proof. Let $B : X \times Y \to Z$ be a bounded bilinear map into a Banach space Z. For the dense subset $X \otimes_\pi Y$ existence and uniqueness of a bounded linear map $B_0 : X \otimes_\pi Y \to Z$ is given by the universal property for normed vector spaces. So B factors uniquely through the unique bounded map extending B_0. □

Let E, F, G, H be objects in one of the above categories and $E \xrightarrow{S} G$, $F \xrightarrow{T} H$ two morphisms. Then

$$\begin{aligned} E \times F &\to G \otimes_{(\pi)} H \\ (x, y) &\mapsto Sx \otimes Ty \end{aligned}$$

is bilinear (and bounded with operator norm $\|S\| \, \|T\|$ in the normed cases) and therefore induces a morphism $S \otimes T : E \otimes_{(\pi)} F \to G \otimes_{(\pi)} H$ (with $\|S \otimes T\| = \|T\| \, \|S\|$).

It is often important to know whether two tensors given in different representations are the same. This can obviously be reduced to the problem to determine tensors representing the zero tensor. We call a subset S of the algebraic dual of some vector space U *separating* if it separates points, that is if $\varphi(x) = 0$ for all $\varphi \in S$ implies $x = 0$. For example, the topological dual of some normed vector space always is separating by the Hahn-Banach theorem.

Lemma A.4.5. *Let S, T be separating sets of two vector spaces V, W. The following are equivalent for $u = \sum_{i=1}^{n} x_i \otimes y_i \in V \otimes W$.*

(a) $u = 0$.

(b) $\sum_{i=1}^{n} \varphi(x_i)\psi(y_i) = 0$ *for every* $\varphi \in S$, $\psi \in T$.

(c) $\sum_{i=1}^{n} \varphi(x_i) y_i = 0$ *for every* $\varphi \in S$.

(d) $\sum_{i=1}^{n} \psi(y_i) x_i = 0$ *for every* $\psi \in T$.

Proof. See [Rya02, Proposition 1.2]. □

A.4.1 Complexification of Real Banach Spaces

In this section we want to discuss a natural way to complexify a given real Banach space X. A common algebraic way to do this is to use the tensor product introduced above: In the following let (e_1, e_2) be the canonical basis of $\mathbb{R}^2$ and let $\ell_2^2 = (\mathbb{R}^2, \|\cdot\|_2)$ as usual. Notice that every element of the tensor product $X \otimes_\pi \ell_2^2$ can be written as $x \otimes e_1 + y \otimes e_2$ for some $x, y \in X$. It can be naturally endowed with the structure of a complex vector space by defining

$$(\alpha + i\beta) \cdot (x \otimes e_1 + y \otimes e_2) \coloneqq (\alpha x - \beta y) \otimes e_1 + (\beta x + \alpha y) \otimes e_2.$$

We will write $x + iy$ for $x \otimes e_1 + y \otimes e_2$ and call x the *real part* and y the *imaginary part*. However, it is not obvious that $X \otimes_\pi \ell_2^2$ is a *complex* normed vector space. The ideas of the following proof and a lot more about the complexification of real Banach spaces can be found in [MST99].

Theorem A.4.6. *$X \otimes_\pi \ell_2^2$ is a complex Banach space.*

Proof. Clearly, the definitness and the triangly inequality hold for $\|\cdot\|_\pi$ because these properties do not depend on the scalar field. Since $\|\cdot\|_\pi$ is homogeneous for real numbers, it is sufficient to show that $\left\|e^{it}z\right\|_\pi = \|z\|_\pi$ for all real t and $z = x + iy \in X \otimes_\pi \ell_2^2$. Let $O_t : \ell_2^2 \to \ell_2^2$ be the linear map induced by the matrix

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Then $\|O_t\| = 1$ and therefore

$$\begin{aligned}
\left\|e^{it}(x+iy)\right\|_\pi &= \|x\cos t - y\sin t + i(x\sin t + y\cos t)\|_\pi \\
&= \|x \otimes (e_1\cos t + e_2\sin t) + y \otimes (e_1(-\sin t) + e_2\cos t)\|_\pi \\
&= \|(\mathrm{Id}\otimes O_t)(x\otimes e_1 + y\otimes e_2)\|_\pi \\
&\le \|\mathrm{Id}\otimes O_t\|_{\mathcal{L}(X\otimes_\pi \ell_2^2, X\otimes_\pi \ell_2^2)} \|x\otimes e_1 + y\otimes e_2\|_\pi \\
&= \|x+iy\|_\pi .
\end{aligned}$$

A further application of the above inequality shows

$$\|x+iy\|_\pi = \left\|e^{-it}e^{it}(x+iy)\right\|_\pi \le \left\|e^{it}(x+iy)\right\|_\pi .$$

Hence, $\left\|e^{it}(x+iy)\right\|_\pi = \|x+iy\|_\pi$ as desired. It remains to show that $X \otimes_\pi \ell_2^2$ is complete. Let $x_n + iy_n$ be a Cauchy sequence. We want to show that the real and imaginary parts are themselves Cauchy sequences. For this we show that the norms of the real and imaginary parts are dominated by the π-norm of $x_n + iy_n$. Observe that by the homogenity for complex numbers, we have $\|x+iy\|_\pi = \|x-iy\|_\pi$ (the complex conjugate can be obtained by rotation). Hence,

$$2\|x\| = \|2(x\otimes e_1)\|_\pi = \|(x+iy)+(x-iy)\|_\pi \le \|x+iy\|_\pi + \|x-iy\|_\pi = 2\|x+iy\|_\pi .$$

This shows $\|x\| \le \|x+iy\|_\pi$. Obviously, the same argument works for the imaginary part. Hence, (x_n) and (y_n) are Cauchy sequences in X and they converge, say to x and y. Now, an application of the triangle inequality shows directly that $x_n + iy_n$ converges to $x + iy$ in π-norm. □

Definition A.4.7. Let X be a real Banach space. We will call $X \otimes_\pi \ell_2^2$ the *complexification* of X and will more shortly write $X^{\mathbb{C}}$.

Of course, the procedure described above works for normed vector spaces as well. In this case $X^{\mathbb{C}}$ is complete if and only if X is complete.

Finally, $X^{\mathbb{C}}$ is isomorphic to the direct sum of two copies of X.

Lemma A.4.8. *One has $X^{\mathbb{C}} \simeq X \oplus X$ as real Banach spaces. Moreover, X is isometrically embedded into $X^{\mathbb{C}}$.*

Proof. Let $P_i : \ell_2^2 \to \operatorname{span}\{e_i\}$ be the orthogonal projection onto the linear hull of the i-th coordinate vector. Then $\mathrm{Id}_X \otimes P_i$ maps onto $X \otimes \operatorname{span}\{e_i\} \simeq X$. Let

$$\begin{aligned} f : X^{\mathbb{C}} &\to X \oplus X \\ x + iy &\mapsto ((\mathrm{Id}_X \otimes P_1)(x+iy), (\mathrm{Id}_X \otimes P_2)(x+iy)). \end{aligned}$$

Then f is linear and because of $\|f(x+iy)\| = \|x\| + \|y\| \leq 2\,\|x+iy\|_\pi$ bounded. Moreover, its bounded inverse g is given by

$$\begin{aligned} f : X \oplus X &\to X^{\mathbb{C}} \\ (x, y) &\mapsto x + iy. \end{aligned}$$

Finally, an isometric embedding is given by $x \mapsto x + i0$. □

B Interpolation Spaces

B.1 Interpolation Spaces and Interpolation Functors

We subsume the basic notions and theorems from the theory of interpolation spaces completely restricting our attention to Banach spaces. References on this topic are [BL76] and [Tri98]. We directly follow the presentation in the first reference.

Definition B.1.1 (Compatible Banach Spaces)**.** Let A_0 and A_1 be two Banach spaces. We say that A_0 and A_1 are *compatible* if there is a Hausdorff topological vector space $\mathfrak{A}$ such that both A_0 and A_1 are subspaces of $\mathfrak{A}$.

Given two compatible Banach spaces A_0 and A_1, we can form their sum $A_0 + A_1$ and their intersection $A_1 \cap A_2$.

Lemma B.1.2. *Suppose A_0 and A_1 are compatible Banach spaces. Then both $A_0 \cap A_1$ and $A_0 + A_1$ are Banach spaces with respect to*

$$\begin{aligned} \|a\|_{A_0 \cap A_1} &:= \max(\|a\|_{A_0}, \|a\|_{A_1}) \\ \|a\|_{A_0 + A_1} &:= \inf_{a = a_0 + a_1} (\|a_0\|_{A_0} + \|a_1\|_{A_1}). \end{aligned}$$

Proof. See [BL76, Lemma 2.3.1]. □

We now make the class of all compatible Banach spaces into a category.

Definition B.1.3 (Category of all Compatible Banach Spaces)**.** We denote by $\mathcal{B}_1$ the category of all *compatible couples of Banach spaces* $\overline{A} = (A_0, A_1)$. The morphisms $T : (A_0, A_1) \to (B_0, B_1)$ are given by bounded linear operators from $A_0 + A_1$ to $B_0 + B_1$ such that $T_{|A_0} : A_0 \to B_0$ and $T_{|A_1} : A_1 \to B_1$ are bounded linear operators.

One now easily verifies that $\mathcal{B}_1$ is indeed a category. Further, we denote by $\mathcal{B}$ the category of all Banach spaces with bounded linear operators as morphisms.

Definition B.1.4 (Interpolation Spaces)**.** Let $\overline{A} = (A_0, A_1)$ be a couple in $\mathcal{B}_1$. Then a Banach space A is called an *intermediate space* between A_0 and A_1 (or with respect to $\overline{A}$) if

$$A_0 \cap A_1 \subset A \subset A_0 + A_1$$

with continuous inclusions. The space A is called an *interpolation space* between A_0 and A_1 (or with respect to $\overline{A}$) if in addition

$$T \in \mathrm{Mor}_{\mathcal{B}_1}(\overline{A}, \overline{A}) \quad \text{implies} \quad T_{|A} \in \mathrm{Mor}_{\mathcal{B}}(A, A).$$

More generally, let $\overline{A}$ and $\overline{B}$ be two couples in $\mathcal{B}_1$. Then we say that two Banach spaces A and B are *interpolation spaces* with respect to $\overline{A}$ and $\overline{B}$ if A and B are intermediate spaces with respect to $\overline{A}$ and $\overline{B}$ and if

$$T \in \mathrm{Mor}_{\mathcal{B}_1}(\overline{A}, \overline{B}) \quad \text{implies} \quad T_{|A} \in \mathrm{Mor}_{\mathcal{B}}(A, B).$$

Definition B.1.5 (Interpolation Functor)**.** An *interpolation functor* is a functor F from $\mathcal{B}_1$ into the category of all Banach spaces $\mathcal{B}$ such that if $\overline{A}$ and $\overline{B}$ are couples in $\mathcal{B}_1$, then $F(\overline{A})$ and $F(\overline{B})$ are interpolation spaces with respect to $\overline{A}$ and $\overline{B}$. Moreover, we require

$$F(T) = T_{|F(\overline{A})} \quad \text{for all} \quad T \in \mathrm{Mor}_{\mathcal{B}_1}(\overline{A}, \overline{B}).$$

We say that F is an *exact interpolation functor of exponent* θ if

$$\|F(T)\|_{A,B} \leq \left\|T_{|A_0}\right\|_{A_0,B_0}^{1-\theta} \left\|T_{|A_1}\right\|_{A_1,B_1}^{\theta}.$$

B.2 Interpolation Methods

We now present the two most important explicit interpolation functors obtained by the real interpolation and the complex interpolation method.

B.2.1 The Real Interpolation Method

There are several equivalent ways to obtain the real interpolation method. We now present the so called K-method.

Theorem B.2.1 (K-Method)**.** *Let $q \in [1, \infty)$ and $0 < \theta < 1$ for $1 \leq q < \infty$ and $0 \leq \theta \leq 1$ for $q = \infty$. Moreover, let $\overline{A} = (A_0, A_1) \in \mathcal{B}_1$. One defines*

$$K(t, a) := \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\,\|a_1\|_{A_1}).$$

Now let $\overline{A}_{\theta,q} = K_{\theta,q}(\overline{A})$ be the space of all $a \in A_0 + A_1$ such that

$$\|a\|_{\theta,q} := \left(\int_0^\infty (t^{-\theta} K(t,a))^q \frac{dt}{t}\right)^{1/q} < \infty.$$

Then $(\overline{A}_{\theta,q}, \|\cdot\|_{\theta,q})$ is a Banach space and $K_{\theta,q}$ is an exact interpolation functor of exponent θ.

Proof. See [BL76, Theorem 3.1.2 & Theorem 3.4.2(a)]. □

The following property of the K-method is used in the thesis.

Lemma B.2.2. *Let $\overline{A} = (A_0, A_1)$ be an interpolation couple and $q < \infty$. Then $A_0 \cap A_1$ is dense in $\overline{A}_{\theta,q}$.*

Proof. See [BL76, Theorem 3.4.2(b)]. □

Lemma B.2.3. *Let $\overline{A} = (A_0, A_1)$ be an interpolation couple and $0 < \theta < 1$. There exists a positive number $c_{\theta,q}$ such that for all $a \in A_0 \cap A_1$*

$$\|a\|_{\overline{A}} \le c_{\theta,q} \|a\|_{A_0}^{1-\theta} \|a\|_{A_1}^{\theta}.$$

Proof. See [Tri98, Theorem 1.3.3(g)]. □

B.2.2 The Complex Interpolation Method

For the complex interpolation method we always assume that all Banach spaces are complex.

Lemma B.2.4. *Let $\overline{A} = (A_0, A_1)$ be an interpolation couple. We define the space $\mathcal{F}(\overline{A})$ of all functions f with values in $A_0 + A_1$ which are bounded and continuous on the closed strip $S \coloneqq \{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\}$ and holomorphic on the open strip $S_0 \coloneqq \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ and for which moreover, for $k = 0, 1$ the functions $t \mapsto f(k + it)$ are into A_k, continuous and tend to zero as $|t| \to \infty$. Then $\mathcal{F}(\overline{A})$ is a vector space which endowed with the norm*

$$\|f\|_{\mathcal{F}} \coloneqq \max\left(\sup_{t\in\mathbb{R}} \|f(it)\|_{A_0}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{A_1} \right)$$

becomes a Banach space.

Proof. See [BL76, Lemma 4.1.1]. □

Theorem B.2.5. *Let $0 \le \theta \le 1$ and $\overline{A}_{[\theta]} = C_\theta(\overline{A})$ be the space of all $a \in A_0 + A_1$ such that $a = f(\theta)$ for some $f \in \mathcal{F}(\overline{A})$. We endow $\overline{A}_{[\theta]}$ with the norm*

$$\|a\|_{[\theta]} \coloneqq \inf\{\|f\|_{\mathcal{F}} : f(\theta) = a, f \in \mathcal{F}(\overline{A})\}.$$

Then $\overline{A}_{[\theta]}$ is a Banach space and an interpolation space with respect to $\overline{A}$. The functor C_θ is an exact interpolation functor of exponent θ.

Proof. See [BL76, Theorem 4.1.2]. □

We again state some properties of the complex interpolation method.

Lemma B.2.6. *Let $\overline{A} = (A_0, A_1)$ be an interpolation couple and $0 < \theta < 1$. There exists a positive number c_θ such that for all $a \in A_0 \cap A_1$*

$$\|a\|_{[\theta]} \le c_\theta \|a\|_{A_0}^{1-\theta} \|a\|_{A_1}^{\theta}.$$

Proof. See [Tri98, Theorem 19.3(f)]. □

Lemma B.2.7. *Let $0 \le \theta \le 1$. Then $A_0 \cap A_1$ is dense in $\overline{A}_{[\theta]}$.*

Proof. See [BL76, Theorem 4.2.2]. □

Theorem B.2.8. *Assume that* $p_0 \geq 1$, $p_1 \geq 1$ *and* $0 < \theta < 1$. *Then for* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *one has*

$$(L^{p_0}, L^{p_1})_{[\theta]} = L^p$$

with equal norms.

Proof. See [BL76, Theorem 5.1.1]. □

Remark B.2.9. Observe that one can directly infer the Riesz-Thorin interpolation theorem from the the above theorem and the fact that C_θ is an exact interpolation functor of exponent θ.

C Measure and Integration Theory

C.1 Measure Theory

We begin with some standard material which could be coverd in an introductory course in measure theory.

If one wants to show that a certain property holds for every element in some σ-algebra, it is often much easier to only show this property for a certain subset of the σ-algebra for which the property is more well-behaved and then show that this forces the property to hold for the whole σ-algebra. *Dynkin systems* and *Dynkin's theorem* exactly cover this idea.

Definition C.1.1. A subset $\mathcal{D} \subset \mathcal{P}(X)$ of some set X is called a *Dynkin system* over X if

(i) $X \in \mathcal{D}$.

(ii) $A \in \mathcal{D} \Rightarrow A^C \in \mathcal{D}$.

(iii) If $A_1, A_2, A_3, \ldots \in \mathcal{D}$ are mutually disjoint sets, then $\cup_{n=1}^{\infty} A_n \in \mathcal{D}$.

Clearly, an arbitrary intersection of Dynkin systems is again a Dynkin system. Therefore the following definition makes sense.

Definition C.1.2. Let $\mathcal{A} \subset \mathcal{P}(X)$. Then there exists a smallest Dynkin system containing $\mathcal{A}$. It is called the Dynkin system generated by $\mathcal{A}$ and is denoted by $\mathcal{D}(\mathcal{A})$.

Theorem C.1.3 (Dynkin's Theorem). *Let $\mathcal{E} \subset \mathcal{P}(X)$ be stable under finite intersections. Then $\sigma(\mathcal{E}) = \mathcal{D}(\mathcal{E})$.*

Proof. See [Els09, Satz 6.7]. □

C.2 The Bochner Integral

The *Bochner integral* is the generalization of the Lebesgue integral for vector-valued functions. From now on until the end of this chapter (Ω, Σ, μ) denotes a σ-finite measure space and X denotes a Banach space. We develop its theory only as far as it is needed in this thesis and follow directly the presentation given in [DU77].

C.2.1 Definition and Elementary Properties

Definition C.2.1 (Simple Function)**.** A function $f : \Omega \to X$ is called *simple* if there exist $A_1, \ldots, A_n \in \Sigma$ with $\mu(A_i) < \infty$ such that

$$f = \sum_{i=1}^{n} x_i \mathbb{1}_{A_i}.$$

Definition C.2.2 (Strong Measurability)**.** A function $f : \Omega \to X$ is called *μ-measurable* if there exists a sequence of simple functions (f_n) with $\lim_{n\to\infty} \|f_n - f\| = 0$ μ-almost everywhere.

Definition C.2.3 (Bochner Integrable)**.** A μ-measurable function $f : \Omega \to X$ is called *Bochner integrable* if there exists a sequence of simple functions (f_n) such that

$$\lim_{n\to\infty} \int_\Omega \|f_n - f\| \, d\mu = 0.$$

In this case $\int_E f \, d\mu$ is defined for each $E \in \Sigma$ by

$$\int_E f \, d\mu = \lim_{n\to\infty} \int_E f_n \, d\mu,$$

where $\int_E f_n \, d\mu$ is defined in the obvious way.

A very useful criterion for showing that a given function is Bochner integrable is the following.

Theorem C.2.4. *A μ-measurable function $f : \Omega \to X$ is Bochner integrable if and only if $\int_\Omega \|f\| \, d\mu < \infty$.*

Proof. See [DU77, Theorem II.2]. □

We collect some properties of the Bochner integral.

Theorem C.2.5. *If f is a μ-Bochner integrable function, then*

(a) $\left\| \int_E f \, d\mu \right\| \le \int_E \|f\| \, d\mu$.

(b) If $(E_n) \subset \Sigma$ with $E_i \cap E_j = \emptyset$ for $i \neq j$, then

$$\int_E f \, d\mu = \sum_{n=1}^{\infty} \int_{E_n} f \, d\mu.$$

Proof. See [DU77, Theorem II.4]. □

The following property of the Bochner integral is extremely useful. Note that the integrability assumption on Tf is fulfilled automatically if T is a bounded linear operator.

Theorem C.2.6 (Hille)**.** *Let $T : X \supset D(T) \to Y$ be a closed linear operator. If f and Tf are Bochner integrable with respect to μ, then*

$$T\left(\int_E f\,d\mu\right) = \int_E Tf\,d\mu$$

for all $E \in \Sigma$.

Proof. See [DU77, Theorem II.6]. □

We note that the natural generalizations of Lebesgue's dominated convergence theorem [AE08, Theorem 3.12] and Fubini's theorem [AE08, Theorem 6.16] to the vector-valued case hold. Next we want to introduce the vector-valued analogue of the Lebesgue spaces.

Definition C.2.7 (Lebesgue-Bochner Space)**.** Let $1 \leq p < \infty$. The *Lebesgue-Bochner space* $L^p(\Omega, \Sigma, \mu; X)$, or if there is no ambiguity $L^p(\Omega; X)$ or $L^p(\mu; X)$, is the collection of all (equivalence classes of) μ-Bochner integrable functions $f : \Omega \to X$ such that

$$\|f\|_p := \left(\int_\Omega \|f\|_X^p \; d\mu\right)^{1/p} < \infty.$$

As in the scalar case, $L^p(\Omega; X)$ is a Banach space and the simple functions are dense.

Theorem C.2.8. *$L^p(\Omega; X)$ is a Banach space.*

Proof. See [AE08, Theorem 4.10]. □

Theorem C.2.9. *Let $1 \leq p < \infty$. The simple functions are dense in $L^p(\Omega, \Sigma, \mu; X)$.*

Proof. See [AE08, Satz 4.8]. □

C.2.2 Complex Analysis in Banach Spaces

We now want to generalize the concept of holomorphy to vector-valued functions taking values in a complex Banach space X (for more details see [ABHN01, Appendix A]). In order to adapt notations to common habits, we will not necessarily write scalars in front of vectors - as it is usually done in linear algebra or functional analysis - if no misunderstandings can occur.

Definition C.2.10. Let $U \subset \mathbb{C}$ be open and $f : U \to X$ be a vector-valued function. We say that f is *complex differentiable* in z_0 if the limit

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. We say that f is *holomorphic* in z_0 if there exists a neighbourhood $V \subset U$ of z_0 such that f is complex differentiable for all $z \in V$. Further, we say that f is *holomorphic* if f is holomorphic in all $z \in U$.

Moreover, we call f *weakly holomorphic* (in z_0 / in U) if $x' \circ f$ is holomorphic (in z_0 / in U) for all $x' \in X'$.

One can define the line integral of a vector-valued complex function as in the scalar case. Then the interchangeability of continuous functionals with the integral (Theorem C.2.6) together with the fact that the topological dual X' separates points shows that the Cauchy integral theorem and Cauchy's integral formula generalize from the scalar-valued to the vector-valued case.

Theorem C.2.11. *The Cauchy integral theorem and Cauchy's integral formula remain valid (even for weakly holomorphic functions) in the vector-valued case.*

Cauchy's integral formula is the key tool for proving that the concepts of holomorphy and weak holomorphy are equivalent.

Theorem C.2.12. *A function $f : U \to X$ is holomorphic (in z_0 / in U) if and only if it is weakly holomorphic (in z_0 / in U).*

Proof. See [Bal10, Satz 2.2.5] or [ABHN01, Proposition A.3]. □

Moreover, as in the scalar case the Cauchy integral theorem shows that a holomorphic function can be locally expanded into a power series. From this one deduces the validity of the identity theorem for holomorphic functions in the vector-valued case.

C.2.3 Vector-Valued Extensions of Positive Operators

Let $1 \le p < \infty$. Suppose that a bounded linear operator T on $L^p(\Omega)$ is given. Our aim is to find a natural extension of T on $L^p(\Omega; X)$.

By the universal property of the tensor product, the continuous bilinear map

$$\begin{aligned} L^p(\Omega) \times X &\to L^p(\Omega; X) \\ (f, x) &\mapsto f \cdot x \end{aligned}$$

induces a bounded linear operator $i : L^p(\Omega) \otimes_\pi X \to L^p(\Omega; X)$. We want to show that i is injective. Therefore let $\sum_{i=1}^n f_i \cdot x_i = 0$. Then for every $x' \in X'$ we have $\sum_{i=1}^n f_i \cdot x'(x_i) = 0$. An application of Lemma A.4.5 shows that $\sum_{i=1}^n f_i \otimes x_i = 0$. So i is an embedding and we can identify $L^p(\Omega) \otimes_\pi X$ with a subspace of $L^p(\Omega; X)$. Clearly, this subspace is dense in $L^p(\Omega; X)$ because the simple functions are dense in virtue of Theorem C.2.9. Now the unique extension of $T \otimes \mathrm{Id}_X$ to a bounded linear operator on $L^p(\Omega; X)$ - if it exists - is the natural candidate for the vector-valued extension of T. Sadly, this extension does not exist in general. However, for a *positive operator* T, that is $f \ge 0$ implies $Tf \ge 0$, $T \otimes \mathrm{Id}_X$ can be extended to a bounded linear operator on $L^p(\Omega; X)$.

Theorem C.2.13. *If T is a positive operator on $L^p(\Omega)$, then $T \otimes \mathrm{Id}_X$ extends uniquely to a bounded operator on $L^p(\Omega; X)$ and we have*

$$\|T \otimes \mathrm{Id}_X\| = \|T\| .$$

Proof. See [vN08, Proposition 11.9]. □

D Probability Theory

D.1 Infinite Product of Probability Spaces

Definition D.1.1 (Infinite Product of Measurable Spaces)**.** Let $(\Omega_i, \Sigma_i)_{i\in I}$ be a family of measurable spaces. A set $\prod_{i\in I} A_i \in \prod_{i\in I} \Omega_i$ for which $A_i \neq \Omega_i$ holds for only finitely many i is called a *cylindrical set*. $\bigotimes_{i\in I} \Sigma_i$ is defined to be the σ-algebra created by all cylindrical sets, or in other words the smallest σ-algebra that contains every cylindrical set.

Theorem D.1.2 (Infinite Product of Probability Spaces)**.** *Let $(\Omega_i, \Sigma_i, \mathbb{P}_i)_{i\in I}$ be a family of probability spaces. There exists a unique probability measure $\mathbb{P}$ on $(\prod_{i\in I} \Omega_i, \bigotimes_{i\in I} \Sigma_i)$ such that for every cylindrical set $\prod_{i\in I} A_i \in \bigotimes_{i\in I} \Sigma_i$ one has*

$$\mathbb{P}\left(\prod_{i\in I} A_i\right) = \prod_{i:A_i\neq\Omega_i} \mathbb{P}_i(A_i).$$

Proof. See [Bau02, Satz 9.2]. □

D.2 Probabilistic Inequalities

Theorem D.2.1 (Jensen's Inequality)**.** *Let X be an integrable real random variable on a probability space $(\Omega, \Sigma, \mathbb{P})$ whose values lie in an open interval $I \subset \mathbb{R}$. Then $\mathbb{E}(X) \in I$ and for every convex function $\varphi : I \mapsto \mathbb{R}$, $\varphi \circ X$ is a random variable. If $\varphi \circ X$ is integrable, we have*

$$\varphi(\mathbb{E}(X)) \leq \mathbb{E}(\varphi \circ X).$$

Proof. See [Bau02, Satz 3.9]. □

Theorem D.2.2 (Kahane-Khintchine Inequality (1964))**.** *For each $1 \leq p < \infty$ there exists a constant C_p such that, for every Banach space X and for any finite sequence $x_1, \ldots, x_n$ in X, the following inequality holds:*

$$\int_D \left\|\sum_{i=1}^n \varepsilon_i(\omega)x_i\right\| d\mu(\omega) \leq \left(\int_D \left\|\sum_{i=1}^n \varepsilon_i(\omega)x_i\right\|^p d\mu(\omega)\right)^{1/p} \leq C_p \int_D \left\|\sum_{i=1}^n \varepsilon_i(\omega)x_i\right\| d\mu(\omega).$$

Proof. See [AK06, Theorem 6.2.5]. □

Corollary D.2.3 (Equivalence of p-Averages)**.** *For each $1 \le p_i < \infty$, $i = 1, 2$, there exist constants C_{p_1} and C_{p_2} such that, for every Banach space X and for any finite sequence $x_1, \dots, x_n$ in X, the following inequality holds:*

$$\frac{1}{C_{p_1}} \left(\int_D \left\| \sum_{i=1}^n \varepsilon_i(\omega) x_i \right\|^{p_1} d\mu(\omega) \right)^{1/p_1} \le \left(\int_D \left\| \sum_{i=1}^n \varepsilon_i(\omega) x_i \right\|^{p_2} d\mu(\omega) \right)^{1/p_2}$$
$$\le C_{p_2} \left(\int_D \left\| \sum_{i=1}^n \varepsilon_i(\omega) x_i \right\|^{p_1} d\mu(\omega) \right)^{1/p_1} .$$

Corollary D.2.4 (Khintchine Inequality)**.** *There exist constants A_p, B_p $(1 \le p < \infty)$ such that for any finite sequence of scalars $(a_i)_{i=1}^n$ and any $n \in \mathbb{N}$ we have*

$$A_p \left(\sum_{i=1}^n |a_i|^2 \right)^{1/2} \le \left(\int_D \left| \sum_{i=1}^n \varepsilon_i(\omega) a_i \right|^p d\mu(\omega) \right)^{1/p} \le B_p \left(\sum_{i=1}^n |a_i|^2 \right)^{1/2} .$$

Proof. Choose $p_1 = 2$, $p_2 = p$ and $X = \mathbb{C}$ in Corollary D.2.3. Moreover, the orthogonality of the Walsh functions yields

$$\left(\int_D \left| \sum_{i=1}^n \varepsilon_i(\omega) a_i \right|^2 d\mu(\omega) \right)^{1/2} = \left(\sum_{i=1}^n |a_i|^2 \right)^{1/2} .$$

From this the Khintchine inequality follows directly. □

D.3 Conditional Expectations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$.

D.3.1 Scalar-Valued

Theorem D.3.1 (Conditional Expectation, Scalar Case)**.** *Let $1 \le p \le \infty$. For all $X \in L^p(\Omega)$ there exists a unique element $\mathbb{E}[X|\mathcal{G}]$ in $L^p(\Omega, \mathcal{G})$ such that for all $G \in \mathcal{G}$,*

$$\int_G \mathbb{E}[X|\mathcal{G}] \, d\,\mathbb{P} = \int_G X \, d\,\mathbb{P} .$$

We call $\mathbb{E}[X|\mathcal{G}]$ the conditional expectation *of X with respect to $\mathcal{G}$. Moreover, $\mathbb{E}[\cdot|\mathcal{G}]$ is a contractive positive projection on $L^p(\Omega)$ with range $L^p(\Omega, \mathcal{G})$.*

Proof. See [vN08, Theorem 11.5]. □

We need to keep hold of some properties of conditional expectation operators.

Lemma D.3.2. *The following properties hold.*

(a) If $X \in L^1(\Omega)$ and $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{G}$, then almost surely

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}].$$

(b) If $X \in L^1(\Omega)$ is independent of $\mathcal{G}$, then almost surely

$$\mathbb{E}[X|\mathcal{G}] = \mathbb{E}\,X.$$

(c) If $X \in L^p(\Omega)$ and $Y \in L^q(\Omega, \mathcal{G})$ with $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then almost surely

$$\mathbb{E}[YX|\mathcal{G}] = Y\mathbb{E}[X|\mathcal{G}].$$

Proof. See [vN08, Proposition 11.6]. □

D.3.2 Vector-Valued

In this section we want to extend the conditional expectation operators from $L^p(\Omega)$ to $L^p(\Omega; X)$, where X is an arbitrary Banach space.

Theorem D.3.3. *For $1 \leq p \leq \infty$ the operator $\mathbb{E}[\cdot|\mathcal{G}] \otimes \mathrm{Id}_X$ extends uniquely to a contractive projection on $L^p(\Omega; X)$. The random variable*

$$\mathbb{E}[Y|\mathcal{G}] \coloneqq (\mathbb{E}[\cdot|\mathcal{G}] \otimes I)Y$$

is the unique element of $L^p(\Omega, \mathcal{G}; X)$ with the property that for all $G \in \mathcal{G}$,

$$\int_G \mathbb{E}[Y|\mathcal{G}]\, d\,\mathbb{P} = \int_G Y\, d\,\mathbb{P}\,.$$

Proof. We begin with $1 \leq p < \infty$. We have seen in Theorem D.3.1 that $\mathbb{E}[\cdot|\mathcal{G}]$ is a positive contraction. So Theorem C.2.13 applies. The proofs of the other claims can be found in [vN08, Theorem 11.10]. □

We want to show that the properties of the conditional expectation operators discussed in the previous section hold for the vector-valued case as well. It is convenient to do this by reducing to the scalar case.

Lemma D.3.4. *If $Y \in L^1(\Omega; X)$ and $x' \in X'$, then almost surely*

$$\mathbb{E}[x' \circ Y|\mathcal{G}] = x' \circ \mathbb{E}[Y|\mathcal{G}].$$

Proof. Clearly, one has $x' \circ \mathbb{E}[Y|\mathcal{G}] \in L^1(\Omega, \mathcal{G})$. Moreover, we have for every $A \in \mathcal{G}$

$$\int_A x' \circ \mathbb{E}[Y|\mathcal{G}]\, d\,\mathbb{P} = x'\left(\int_A \mathbb{E}[Y|\mathcal{G}]\, d\,\mathbb{P}\right) = x'\left(\int_A Y\, d\,\mathbb{P}\right) = \int_A x' \circ Y\, d\,\mathbb{P}\,. \qquad \square$$

With the help of the above lemma it is straightforward to deduce the general statement from the scalar-valued one.

Lemma D.3.5. *The following properties hold.*

(a) If $Y \in L^1(\Omega; X)$ and $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{G}$, then almost surely

$$\mathbb{E}[\mathbb{E}[Y|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[Y|\mathcal{H}].$$

(b) If $Y \in L^1(\Omega; X)$ is independent of $\mathcal{G}$, then almost surely

$$\mathbb{E}[Y|\mathcal{G}] = \mathbb{E}\, Y.$$

(c) If $Y \in L^p(\Omega; X)$ and $Z \in L^q(\Omega, \mathcal{G})$ with $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then almost surely

$$\mathbb{E}[ZY|\mathcal{G}] = Z\mathbb{E}[Y|\mathcal{G}].$$

Proof. We only prove (c) because the proofs of the other assertions are almost identical. Clearly, $Z\mathbb{E}[Y|\mathcal{G}]$ is $\mathcal{G}$-measurable. Moreover, for all $A \in \mathcal{G}$ and $x' \in X'$ we have

$$\begin{aligned} x'\left(\int_A \mathbb{E}[ZY|\mathcal{G}]\, d\,\mathbb{P}\right) &= \int_A x' \circ \mathbb{E}[ZY|\mathcal{G}]\, d\,\mathbb{P} = \int_A \mathbb{E}[x' \circ (ZY)|\mathcal{G}]\, d\,\mathbb{P} \\ &= \int_A \mathbb{E}[Z \cdot (x' \circ Y)|\mathcal{G}]\, d\,\mathbb{P} = \int_A Z \cdot \mathbb{E}[x' \circ Y|\mathcal{G}]\, d\,\mathbb{P} \\ &= \int_A Z \cdot x' \circ \mathbb{E}[Y|\mathcal{G}]\, d\,\mathbb{P} = \int_A x' \circ (Z\mathbb{E}[Y|\mathcal{G}])\, d\,\mathbb{P} \\ &= x'\left(\int_A Z\mathbb{E}[Y|\mathcal{G}]\, d\,\mathbb{P}\right). \end{aligned}$$

Since the dual separates points, we have

$$\int_A ZY\, d\,\mathbb{P} = \int_A \mathbb{E}[ZY|\mathcal{G}]\, d\,\mathbb{P} = \int_A Z\mathbb{E}[Y|\mathcal{G}]\, d\,\mathbb{P} \qquad \forall A \in \mathcal{G}. \qquad \square$$

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Ehrenwörtliche Erklärung

Ich erkläre hiermit ehrenwörtlich, dass ich die vorliegende Arbeit selbstständig angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher keiner anderen Prüfungsbehörde vorgelegt und auch noch nicht veröffentlicht.

Ich bin mir bewusst, dass eine unwahre Erklärung rechtliche Folgen haben wird.

Ulm, den 5. April 2011

(Unterschrift)