Gaussian estimates for elliptic operators with unbounded drift

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Abstract

We consider a strictly elliptic operator
\[ Au = \sum_{i,j} D_i(a_{ij} D_j u) - b \cdot \nabla u + \text{div}(c \cdot u) - Vu, \]
where \( 0 \leq V \in L^\infty_{\text{loc}} \), \( a_{ij} \in C^1_b(\mathbb{R}^N) \), \( b, c \in C^1(\mathbb{R}^N, \mathbb{R}^N) \). If \( \text{div} b \leq \beta V \), \( \text{div} c \leq \beta V \), \( 0 < \beta < 1 \), then a natural realization of \( A \) generates a positive \( C_0 \)-semigroup \( T \) in \( L^2(\mathbb{R}^N) \). The semigroup satisfies pseudo-Gaussian estimates if
\[ |b| \leq k_1 V^\alpha + k_2, \quad |c| \leq k_1 V^\alpha + k_2, \]
where \( \frac{1}{2} \leq \alpha < 1 \). If \( \alpha = \frac{1}{2} \), then Gaussian estimates are valid. The constant \( \alpha = \frac{1}{2} \) is optimal with respect to this property.

Keywords: Gaussian estimates; Pseudo-Gaussian estimates; Strictly elliptic operator

0. Introduction

We consider a strictly elliptic operator of the form
\[ Au = \sum_{i,j=1}^N D_i(a_{ij} D_j u) - b \cdot \nabla u + \text{div}(c u) - Vu \]
on \( L^2(\mathbb{R}^N) \) where \( a_{ij} \in C^1_b(\mathbb{R}^N) \), \( b, c \in C^1(\mathbb{R}^N, \mathbb{R}^N) \) and \( V \in L^\infty_{\text{loc}}(\mathbb{R}^N) \) are real coefficients. If \( b, c, V \) are bounded, then this is a classical elliptic operator and semigroup properties have been studied extensively. In particular, it is known that the canonical realization of \( A \) in \( L^2(\mathbb{R}^N) \) generates a positive \( C_0 \)-semigroup satisfying Gaussian estimates (see e.g. [4,7,16] and the survey [3]). Here we are interested in the case where the drift terms \( b \) and \( c \) are...
unbounded. Then one still obtains a semigroup satisfying various regularity properties if the potential $V$ compensates the unbounded drift. We consider the assumption
\[ \text{div} b \leq \beta V, \quad \text{div} c \leq \beta V \tag{H_1} \]
where $0 < \beta < 1$. Then we show that there is a natural unique realization $A$ of the differential operator $A$ which generates a minimal positive semigroup $T$ on $L^2(\mathbb{R}^N)$. This semigroup as well as its adjoint are submarkovian. We say that $T$ satisfies pseudo-Gaussian estimates of order $m \geq 2$ if $T(t)$ has a kernel $k_t$ satisfying
\[
0 \leq k_t(x, y) \leq c_1 e^{\omega t} t^{-N/2} \exp\left\{-c_2(|x - y|^m/t)^{1/m-1}\right\}
\]
for all $x, y \in \mathbb{R}^N$, $t > 0$ and some constants $c_1, c_2 > 0$, $\omega \in \mathbb{R}$. In the case where $m = 2$ we say that $T$ satisfies Gaussian estimates. In order to obtain such pseudo-Gaussian estimates we impose an additional growth condition on the drift terms $b$ and $c$, namely,
\[ |b| \leq k_1 V^{\alpha} + k_2, \quad |c| \leq k_1 V^{\alpha} + k_2 \tag{H_2} \]
where $\frac{1}{2} \leq \alpha < 1$, $k_1, k_2 \geq 0$. If $\alpha = \frac{1}{2}$, then it was proved in [2] that $T$ has Gaussian estimates. The purpose of this paper is to show on one hand that $\alpha = \frac{1}{2}$ is optimal for this property (Section 3). On the other hand, if $\frac{1}{2} < \alpha < 1$, then we show that $T$ still satisfies pseudo-Gaussian estimates even though $T$ need not be holomorphic in that case. Pseudo-Gaussian estimates of order $m > 2$ are still of interest. For instance, they imply that the realizations $A_p$ of $A$ in $\mathbb{L}^p(\mathbb{R}^N)$ have all the same spectrum, $1 \leq p \leq \infty$, at least if $m < \frac{2N}{N-2}$. For elliptic operators with moderately growing drift terms but no compensating $V$ such pseudo-Gaussian estimates had been obtained before by Karrmann [9]. Here we do not study regularity properties of the operator $A$. For this we refer to [2,14,15]. We also mention the works by Liskevich, Sobol and Vogt [12,13,18] where a different approximation is used and spectral properties are studied.

1. Elliptic operators with unbounded drift

In this section we define the realization of an elliptic operator with unbounded drift in $L^2(\mathbb{R}^N)$. The construction is similar to the one in [2] but we ask for less regularity. Moreover, we establish an additional coerciveness property which is used later to prove quasi-Gaussian estimates. We assume throughout this section that $a_{ij} \in L^\infty(\mathbb{R})$ and
\[ \sum_{i,j=1}^N a_{ij}(x)\xi_i \xi_j \geq \nu |\xi|^2 \tag{1.1} \]
for all $x \in \mathbb{R}^N$, $\xi \in \mathbb{R}^N$, where $\nu > 0$ is a fixed constant. Let $b = (b_1, \ldots, b_N)$, $c = (c_1, \ldots, c_N) \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, and let $V \in L^\infty_{\text{loc}}(\mathbb{R}^N)$. We assume in this section that
\[ \text{div} b \leq V, \quad \text{div} c \leq V. \tag{H_0} \]
Later in Section 2 we will replace (H_0) by a stronger assumption (H_1) and require more regularity on the diffusion coefficients $a_{ij}$ and positivity of the potential. Define the elliptic operator
\[ A : H^1_{\text{loc}}(\mathbb{R}^N) \to \mathcal{D}(\mathbb{R}^N)', \]
\[ Au = \sum_{i,j=1}^N D_i(a_{ij} D_j u) - b \cdot \nabla u + \text{div}(cu) - Vu, \]
i.e., for $u \in H^1_{\text{loc}}(\mathbb{R}^N)$ and $v \in \mathcal{D}(\mathbb{R}^N)$ we have
\[ -\langle Au, v \rangle = \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij} D_i u D_j v \, dx + \int_{\mathbb{R}^N} \left\{ \sum_{j=1}^N (b_j D_j u v + c_j u D_j v) + Vu v \right\} \, dx. \]
We define the maximal operator $A_{\text{max}}$ in $L^2(\mathbb{R}^N)$ by
\[ D(A_{\text{max}}) = \{ u \in L^2(\mathbb{R}^N) \cap H^1_{\text{loc}}(\mathbb{R}^N), \, Au \in L^2(\mathbb{R}^N) \}, \]
\[ A_{\text{max}} u = Au. \]
Now we describe the minimal realization of $A$ in $L^2(\mathbb{R}^N)$ as follows.

**Theorem 1.1.** There exists a unique operator $A$ on $L^2(\mathbb{R}^N)$ such that

(a) $A \subset A_{\text{max}}$;
(b) $A$ generates a positive $C_0$-semigroup $T$ on $L^2(\mathbb{R}^N)$;
(c) if $B \subset A_{\text{max}}$ generates a positive $C_0$-semigroup $S$, then $T(t) \leq S(t)$ for all $t \geq 0$.

We call $A$ the minimal realization of $A$ in $L^2(\mathbb{R}^N)$.

When giving the proof we also establish important properties of $A$ and of $T$.

**Proposition 1.2 (Coerciveness).** One has $D(A) \subset H^1(\mathbb{R}^N)$ and

$$ - (Au, u) \geq v \|u\|^2_{H^1} \tag{1.2} $$

for all $u \in D(A)$.

**Proposition 1.3 (Ultracontractivity).** The semigroup $T$ and its adjoint are submarkovian. Moreover $T$ is ultracontractive, namely

$$ \|T(t)\|_{L^1(L^\infty)} \leq c_v t^{-N/2} \quad (t > 0), \tag{1.3} $$

where $c_v > 0$ depends only on the space dimension and the ellipticity constant $v$.

Recall that a $C_0$-semigroup $S$ on $L^2(\mathbb{R}^N)$ is called submarkovian if $S$ is positive and

$$ \|S(t)f\|_{\infty} \leq \|f\|_{\infty} \quad (t > 0), $$

for all $f \in L^\infty \cap L^2$. If $B$ is an operator on $L^2(\mathbb{R}^N)$ we let

$$ \|B\|_{L^p(L^q)} := \sup_{\|f\|_p \leq 1} \|Bf\|_q. $$

Since $T$ and $T^*$ are submarkovian, it follows from the Riesz–Thorin Theorem that

$$ \|T(t)\|_{L^p} \leq 1 \quad (t \geq 0), $$

for all $1 \leq p \leq \infty$.

The remainder of this section is devoted to the proofs of Theorem 1.1 and Propositions 1.2, 1.3. As in [2] we approximate the operator $A$ by realizations of $A$ on balls whose radii go to $\infty$. However, here we do not study regularity properties of $A$ and we restrict ourselves to the Hilbert space case $L^2(\mathbb{R}^N)$ (whereas $L^p(\mathbb{R}^N)$ was considered in [2]). Our assumptions on $V$ and $a_{ij}$ are more general than in [2]. Denote by $B_r = \{x \in \mathbb{R}^N : |x| < r\}$ the ball of radius $r > 0$. The bilinear form

$$ a_r(u, v) := \int_{B_r} \sum_{i,j=1}^N a_{ij} D_j u D_i v \, dx + \int_{B_r} \left\{ \sum_{j=1}^N (b_j D_j uv + c_j u D_j v) + Vu v \right\} \, dx $$

is continuous on $H^1_0(B_r)$. We show that

$$ a_r(u, u) \geq v \int_{B_r} |\nabla u|^2 \, dx \tag{1.4} $$

for all $u \in H^1_0(B_r)$. In fact, let $u \in H^1_0(B_r)$. Then
\[ a_r(u, u) \geq v \int_{B_r} |\nabla u|^2 \, dx + \int_{B_r} \left\{ \sum_{j=1}^N (b_j + c_j) \frac{1}{2} D_j u^2 +Vu^2 \right\} \, dx \]
\[ = v \int_{B_r} |\nabla u|^2 \, dx + \int_{B_r} \left( - \text{div} \frac{b+c}{2} + V \right) u^2 \, dx \geq v \int_{B_r} |\nabla u|^2 \, dx. \]

In view of Poincaré’s inequality, (1.4) implies that \( a_r \) is coercive. Denote by \( -A_r \) the associated operator on \( L^2(B_r) \). Then \( A_r \) generates a \( C_0 \)-semigroup \( T_r \) on \( L^2(B_r) \). Since \( u \in H_0^1(B_r) \) implies that \( u^+, u^- \in H_0^1(B_r) \) and \( a(u^+, u^-) = 0 \) the semigroup \( T_r \) is positive by the first Beurling–Deny criterion on forms [16, Theorem 2.6]. Since \( a_r \) is coercive, \( T_r \) is contractive [16, Chapter 1]. Next we show that for \( 0 < r_1 < r_2 \)
\[ T_{r_1}(t) \leq T_{r_2}(t), \quad (1.5) \]
or, equivalently,
\[ R(\lambda, A_{r_1}) \leq R(\lambda, A_{r_2}) \quad (\lambda > 0). \quad (1.6) \]

Here we identify \( L^2(B_r) \) with a subspace of \( L^2(\mathbb{R}^N) \) and extend an operator \( B \) on \( L^2(B_r) \) to \( L^2(\mathbb{R}^N) \) by defining it as 0 on \( L^2(B_r) \). Similarly, we may identify \( H_0^1(B_{r_1}) \) with a subspace of \( H_0^1(B_{r_2}) \), see [5, Proposition IX.18].

**Proof of (1.6).** Let \( 0 \leq f \in L^2(\mathbb{R}^N), \lambda > 0, u_1 = R(\lambda, A_{r_1}) f, u_2 = R(\lambda, A_{r_2}) f \). We want to show that \( u_1 \leq u_2 \). One has by definition of \( A_{r_1}, A_{r_2}, \)
\[ \lambda \int_{B_{r_1}} u_k v + \int_{B_{r_1}} \sum_{i,j=1}^N a_{ij} D_i u_k D_j v + \int_{B_{r_1}} \sum_{i=1}^N b_i D_i u_k v + \int_{B_{r_1}} \sum_{i=1}^N c_i D_i v u_k + \int_{B_{r_1}} V u_k v = \int_{B_{r_1}} f v \]
for all \( v \in H_0^1(B_{r_1}), k = 1, 2 \). Since \( u_2 \geq 0 \) one has \( (u_1 - u_2)^+ \leq u_1 \), hence \( (u_1 - u_2)^+ \in H_0^1(B_{r_1}) \). Taking \( v = (u_1 - u_2)^+ \) and subtracting the two identities we obtain
\[ \lambda \int_{B_{r_1}} (u_1 - u_2)(u_1 - u_2)^+ + \int_{B_{r_1}} \sum_{i,j=1}^N a_{ij} D_i (u_1 - u_2) \cdot D_j (u_1 - u_2)^+ + \int_{B_{r_1}} \sum_{i=1}^N b_i D_i (u_1 - u_2)(u_1 - u_2)^+ \]
\[ + \int_{B_{r_1}} \sum_{i=1}^N c_i D_i (u_1 - u_2)^+(u_1 - u_2) + \int_{B_{r_1}} V (u_1 - u_2)(u_1 - u_2)^+ = 0. \]

Since \( D_i (u_1 - u_2)(u_1 - u_2)^+ = D_i (u_1 - u_2)^+(u_1 - u_2)^+ \) this gives
\[ \lambda \int_{B_{r_1}} (u_1 - u_2)^+ + \int_{B_{r_1}} v |\nabla (u_1 - u_2)^+|^2 \, dx + \int_{B_{r_1}} \left\{ \sum_{j=1}^N \frac{b_j + c_j}{2} D_i (u_1 - u_2)^+ + V (u_1 - u_2)^+ \right\} \leq 0. \]

The third term equals
\[ \int_{B_{r_1}} \left( - \text{div} \frac{b+c}{2} + V \right) (u_1 - u_2)^+ \, dx \]
which is \( \geq 0 \) by the hypothesis \((H_0). \) Thus \((u_1 - u_2)^+ \leq 0 \), hence \( u_1 \leq u_2 \) on \( B_{r_1}. \) \( \square \)

Next we show that
\[ \lim_{r \to \infty} T_r(t) f = : T(t) f \quad \text{exists in } L^2(\mathbb{R}^N) \quad \text{for all } f \in L^2(\mathbb{R}^N) \]
and defines a positive contraction \( C_0 \)-semigroup whose generator we denote by \( A. \)
**Proof of (1.7).** (a) Let $0 \leq f \in L^2(\mathbb{R}^N)$. Since $T_{r_1}(t)f \leq T_{r_2}(t)f$ for $0 < r_1 \leq r_2$ and $\|T_r(t)f\|_2 \leq \|f\|_2$, the limit in (1.7) exists in $L^2(\mathbb{R}^N)$. It follows that $T(t)$ is a positive contraction and $T(t+s) = T(t)T(s)$ for $s, t \geq 0$.

In order to show that $T$ is strongly continuous, let $0 \leq f \in \mathcal{D}(\mathbb{R}^N)$. Let $t_n \downarrow 0$, $f_n = T(t_n)f$. We have to show that $f_n \rightarrow f$ in $L^2(\mathbb{R}^N)$ as $n \rightarrow \infty$. Let $r > 0$ such that $\text{supp} f \subset B_r$. Observe that $0 \leq g_n := T_r(t_n)f \leq f_n$.

Since $T_r$ is strongly continuous, $\lim_{n \rightarrow \infty} g_n = f$. Moreover, $\|f_n\|_2 \leq \|f\|_2$. Hence $\limsup_{n \rightarrow \infty} \|g_n - f_n\|_2^2 = \limsup_{n \rightarrow \infty} \{\|g_n\|^2_2 + \|f_n\|^2_2 - 2(g_n, f_n)\} \leq \limsup_{n \rightarrow \infty} \{\|f\|^2_2 - 2(g_n, f_n)\} = 0$. □

We mention that, by dominated convergence as in [1, Section 3.6], property (1.7) implies that

$$R(\lambda, A)f = \lim_{r \uparrow \infty} R(\lambda, A_r)f$$

for all $\lambda > 0$, $f \in L^2(\mathbb{R}^N)$. Next we show that

$$D(A) \subset H^1(\mathbb{R}^N) \quad \text{and} \quad v \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \leq (-Au|u)$$

for all $u \in D(A)$. Moreover,

$$A \subset A_{\max}.$$ (1.10)

(a) We prove (1.9). Let $f \in L^2(\mathbb{R}^N)$, $u_n = R(1, A_{r_n})f$, $u = R(1, A)f$ where $r_n \uparrow \infty$. Then $u_n \rightarrow u$ in $L^2(\mathbb{R}^N)$ by (1.8). Since $u_n - A_{r_n}u_n = f$ and $u - Au = f$ in $L^2(B_{r_n})$, it follows that

$$A_{r_n}u_n \rightarrow Au \quad \text{in} \quad L^2(\mathbb{R}^N).$$

By (1.4) we have

$$v \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \leq -(A_{r_n}u_n|u_n).$$

Since $-(A_{r_n}u_n|u_n) \rightarrow (-Au|u)$ as $n \rightarrow \infty$, it follows that

$$v \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \leq (-Au|u).$$ (1.11)

Thus $(u_n)_{n \in \mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$. Considering a subsequence, we may assume that $u_n \rightarrow u$ weakly in $H^1(\mathbb{R}^N)$. Let $h = (h_1, \ldots, h_N) \in L^2(\mathbb{R}^N)^N$ such that $\|h\|_2 \leq 1$. Then by (1.11),

$$\int_{\mathbb{R}^N} \nabla u \cdot h \, dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \nabla u_n \cdot h \, dx \leq \lim_{n \rightarrow \infty} \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 \right)^{1/2} \leq \left[ (-Au|u)/v \right]^{1/2}.$$  

Hence

$$\left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{1/2} = \sup_{h \in L^2(\mathbb{R}^N)^N, \|h\|_2 \leq 1} \int_{\mathbb{R}^N} \nabla u \cdot h \, dx \leq \left[ (-Au|u)/v \right]^{1/2}.$$  

Thus (1.9) is proved.

(b) In order to prove (1.10) we keep the notations of (a) and have to show that $u \in D(A_{\max})$ and $Au = A_{\max}u$. Let $v \in D(\mathbb{R}^N)$. Then

$$(-A_{r_n}u_n|v) = \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij} D_j u_n D_i v \, dx + \int_{\mathbb{R}^N} \left\{ \sum_{j=1}^N (b_j D_j u_n v + c_j u_n D_j v) + Vu_n \right\} \, dx.$$  

Since $u_n \rightarrow u$ weakly in $H^1(\mathbb{R}^N)$ and $A_{r_n}u_n \rightarrow Au$ in $L^2(\mathbb{R}^N)$, it follows that $(-Au|v) = (Au|v)$. 


Next we show the minimality property in Theorem 1.1. Assume that $S$ is a positive semigroup whose generator $B$ satisfies $B \subset A_{\text{max}}$. Then

$$0 \leq T(t) \leq S(t) \quad (t \geq 0). \quad (1.12)$$

**Proof of (1.12).** We have to show that

$$R(\lambda, A) \leq R(\lambda, B) \quad (1.13)$$

for $\lambda > 0$ sufficiently large. Let $r > 0$; because of (1.8) it suffices to show that

$$R(\lambda, A_r) \leq R(\lambda, B). \quad (1.14)$$

Let $f \in L^2(\mathbb{R}^N), f \geq 0, u_1 = R(\lambda, A_r)f, u_2 = R(\lambda, B)f$. Then $0 \leq u_1 \in H^1_0(B_r), 0 \leq u_2 \in H^1_{\text{loc}}(\mathbb{R}^N)$. We have to show that $u_1 \leq u_2$. Since $B \subset A_{\text{max}}$ we have $\lambda u_2 - \lambda u_2 = f$ in $\mathcal{D}(B_r)'$, and also $\lambda u_1 - \lambda u_1 = f$ in $\mathcal{D}(B_r)'$ by the definition of $A_r$. Hence

\[
\begin{align*}
\lambda \int_{B_r} (u_1 - u_2) v \, dx + \sum_{i,j=1}^N a_{ij} D_j(u_1 - u_2) D_i v \, dx + \sum_{j=1}^N (b_j D_j(u_1 - u_2) + c_j (u_1 - u_2) D_j v) \, dx \\
+ \int_{B_r} V(u_1 - u_2) v \, dx = 0
\end{align*}
\]

for all $v \in \mathcal{D}(B_r)$. This identity remains true for $v \in H^1_0(B_r)$ by passing to the limit. Since $u_2 \geq 0$ one has $(u_1 - u_2)^+ \leq u_1$, hence $(u_1 - u_2)^+ \in H^1_0(B_r)$. Choosing $v = (u_1 - u_2)^+$ in the identity above we obtain

\[
\begin{align*}
\lambda \int_{B_r} (u_1 - u_2)^+ \, dx + \sum_{i,j=1}^N a_{ij} D_j(u_1 - u_2)^+ D_i v \, dx \\
+ \sum_{j=1}^N (b_j D_j(u_1 - u_2)^+ + c_j D_j(u_1 - u_2)^+ (u_1 - u_2)^+) \, dx + \int_{B_r} V(u_1 - u_2)^+ \, dx \\
= 0.
\end{align*}
\]

Consequently

\[
\begin{align*}
\lambda \int_{B_r} (u_1 - u_2)^+ \, dx + \int_{B_r} |\nabla(u_1 - u_2)^+|^2 \, dx + \int_{B_r} \left( - \text{div} \left( \frac{b + c}{2} \right) + V \right) (u_1 - u_1)^+ \, dx \leq 0.
\end{align*}
\]

Since $- \text{div}(\frac{b + c}{2} + V) \geq 0$ this implies that $(u_1 - u_2)^+ = 0$; i.e., $u_1 \leq u_2$. \hfill \Box

The proofs of Theorem 1.1 and Proposition 1.2 are complete.

We now show that $T$ is submarkovian. Because of (1.7), it suffices to show that $T_r$ is submarkovian. By the second criterion of Beurling–Deny–Ouhabaz on forms (see [16]) this is equivalent to

$$a_r(u \wedge 1, (u - 1)^+) \geq 0 \quad (1.15)$$

for all $u \in H^1_0(B_r)$.

**Proof of (1.15).** Since $D_j(u \wedge 1) = D_j u 1_{\{u < 1\}}, D_j((u - 1)^+) = D_j u 1_{\{u > 1\}}$ and $D_j u = 0$ a.e. on $\{u = 1\}$, one has

\[
a_r(u \wedge 1, (u - 1)^+) = \int_{\mathbb{R}^N} \left\{ \sum_{j=1}^N c_j (u \wedge 1) D_j(u - 1)^+ + V(u \wedge 1)(u - 1)^+ \right\} \, dx
\]
\[ \int_{\mathbb{R}^N} \left\{ \sum_{j=1}^{N} c_j D_j (u - 1)^+ + V (u - 1)^+ \right\} \, dx \]
\[ = \int_{\mathbb{R}^N} (- \text{div} c + V) (u - 1)^+ \, dx \geq 0 \]
in view of the hypothesis \((H_1)\).

Next we show that the adjoint semigroup \( T^* = (T(t))^t \geq 0 \) is generated by the minimal realization of the adjoint differential operator \( A^* \) which is defined by replacing \( a_{ij} \) by \( a_{ji} \) and by interchanging \( b \) and \( c \), i.e.
\[ A^* u = \sum_{i,j=1}^{N} D_i (a_{ji} D_j u) + c \nabla u - \text{div} (bu) - Vu \quad (u \in H^1_{\text{loc}}). \]  

\[(1.16)\]

**Lemma 1.4.** The minimal realization in \( L^2(\mathbb{R}^N) \) of \( A^* \) is the adjoint \( A^* \) of \( A \).

**Proof.** The adjoint \(-A^*_r\) of \(-A_r\) is associated with the form \( a^*_r \) defined on \( H^1_0(B_r) \times H^1_0(B_r) \) by
\[ a^*_r(u,v) = a_r(v,u). \]

The semigroup generated by \( A^*_r \) is the adjoint \( T^*_r \) of \( T_r \). Let \( B \) be the minimal realization of \( A^* \) in \( L^2(\mathbb{R}^N) \) and \( S \) the semigroup generated by \( B \). Then
\[ S(t) f = \lim_{r \uparrow \infty} T_r(t)^* f = T(t)^* f \]
for all \( f \in L^2(\mathbb{R}^N) \).

As a consequence, we deduce that also \( T^* \) is submarkovian. Finally, we have to show ultracontractivity. We use the following criterion (cf. [6,19], [3, Section 7], [17]).

**Proposition 1.5.** For each \( \delta > 0 \) there exists a constant \( c_\delta > 0 \) such that the following holds. Let \( S \) be a \( C_0 \)-semigroup on \( L^2(\mathbb{R}^N) \) such that \( S \) and \( S^* \) are submarkovian. Assume that the generator \( B \) of \( S \) satisfies
\[ \begin{align*}
(a) & \quad D(B) \subset H^1(\mathbb{R}^N); \\
(b) & \quad (-Bu|u) \geq \delta \|u\|_{H^1}^2 \quad (u \in D(B)); \\
(c) & \quad (-B^*u|u) \geq \delta \|u\|_{H^1}^2 \quad (u \in D(B^*)).
\end{align*} \]

Then
\[ \|S(t)\|_{L^1(L^\infty)} \leq c_\delta t^{-N/2} \quad (t > 0). \]

The proof of Proposition 1.5 is based on Nash’s inequality
\[ \|u\|_{2^{4/N}}^2 \leq c_N \|u\|_{H^1}^2 \|u\|_{1}^{4/N} \]
for all \( u \in H^1(\mathbb{R}^N) \) and some constant \( c_N > 0 \), and one may choose \( c_\delta = (c_N N)^{N/2} \).

**Proof of Proposition 1.5.** (i) \( D(B) \cap L^1 \) is dense in \( L^1 \cap L^2 \). In fact, the semigroup \( S \) extrapolates to a \( C_0 \)-semigroup on \( L^1 \) (see [8], [3, Section 7.2]). Hence for \( f \in L^1 \cap L^2 \), \( \lambda R(\lambda, B) f \to f \) in \( L^1 \) and in \( L^2 \) as \( \lambda \to \infty \). But \( \lambda R(\lambda, B) f \notin D(B) \).

(ii) Now we modify the proof of [4, Proposition 3.8] to show that
\[ \|S(t) f\|_2 \leq \left( \frac{N c_N}{4 \delta} \right)^{N/4} t^{-N/4} \|f\|_1 \]
\[ (1.19) \]
for all \( f \in D(B) \cap L^1 \). Let \( f \in D(B) \cap L^1 \). Then, by (1.18)
\[
\frac{d}{dt} \|S(t)f\|_2^2 = (BS(t)f|S(t)f) + (S(t)f|B^*S(t)f) \leq -\delta \|S(t)f\|_{H^1}^2 - \frac{2\delta}{c_N} \|S(t)f\|_2^{2+4/N}.
\]
Hence
\[
\frac{d}{dt} \left( \|S(t)f\|_2^{2-2/N} \right) = -\frac{2}{N} \|S(t)f\|_2^{2(1-2/N)} \frac{d}{dt} \|S(t)f\|_2^2 \geq \frac{4\delta}{Nc_N} \|S(t)f\|_1^{4/N} \geq \frac{4\delta}{Nc_N} \|f\|_1^{4/N}.
\]
Integrating, we obtain
\[
\left( \|S(t)f\|_2^{2-2/N} \right) \geq \frac{4\delta}{Nc_N} \frac{1}{\|f\|_1^{4/N}}
\]
which implies (1.19).

It follows from (i) that (1.19) remains true for \( f \in L^1 \cap L^2 \).

(iii) Applying (b) to \( S^* \) instead of \( S \) shows that
\[
\|S^*(t)f\|_2 \leq \left( \frac{Nc_N}{4\delta} \right)^{N/4} t^{-N/4} \|f\|_1
\]
(\( f \in L^1 \cap L^2 \)). Hence
\[
\|S(t)f\|_\infty \leq \left( \frac{Nc_N}{4\delta} \right)^{N/4} t^{-N/4} \|f\|_2
\]
(\( f \in L^2 \cap L^\infty \)). Concluding, for \( f \in L^1 \cap L^2 \),
\[
\|S(t)f\|_\infty = \left\|S \left( \frac{t}{2} \right) S \left( \frac{t}{2} \right) f \right\|_\infty \leq \left( \frac{Nc_N}{4\delta} \right)^{N/4} \left( \frac{t}{2} \right)^{-N/4} \left\|S \left( \frac{t}{2} \right) f \right\|_2 \leq \left[ \left( \frac{Nc_N}{4\delta} \right)^{N/4} \left( \frac{t}{2} \right)^{-N/4} \right]^2 \|f\|_1
\]
and the result follows. \( \square \)

Proposition 1.5 implies the ultracontractivity property (1.3) with \( c_v = (\frac{c\lambda}{\nu})N/2 \) since by (1.9) and Lemma 1.4 the hypotheses (a), (b), (c) in Proposition 1.5 are satisfied for the operator \( B = A \). Thus the proofs of Theorem 1.1 and Propositions 1.2, 1.3 are complete.

2. Pseudo-Gaussian estimates

Let \( T \) be a positive \( C_0 \)-semigroup on \( L^2(\mathbb{R}^N) \). We say that \( T \) satisfies pseudo-Gaussian estimates of type \( m \geq 2 \) if there exist real constants \( c_1 > 0, c_2 > 0, \omega \in \mathbb{R} \) and a measurable kernel \( k_t \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N) \) satisfying
\[
0 \leq k_t(x, y) \leq c_1 e^{\omega t} t^{-N/2} \exp \left(-\frac{c_2 |x-y|^m}{t} \right)^{1/m-1}
\]
x, y-a.e. for all \( t > 0 \) such that
\[
(T(t)f)(x) = \int_{\mathbb{R}^N} k_t(x, y) f(y) \, dy
\]
x-a.e. for all \( t > 0, f \in L^2(\mathbb{R}^N) \). If \( m = 2 \), then we say that \( T \) satisfies Gaussian estimates.

In fact, the Gaussian semigroup satisfies such an estimate for \( m = 2 \). It is the best case as the following monotonicity property shows.

Proposition 2.1. Let \( b_1, b_2 > 0 \) and let \( m_2 > m_1 \geq 2 \) be real constant. Then there exists \( \omega \geq 0 \) such that
\[
\exp \left(-b_1 \left( \frac{|z|^{m_1}}{t} \right)^{1/(m_1-1)} \right) \leq \exp \left(-b_2 \left( \frac{|z|^{m_2}}{t} \right)^{1/(m_2-1)} \right) e^{\omega t}
\]
for all \( z \in \mathbb{R}^N, t > 0 \).
**Proof.** We have to find a constant $\omega$ such that

$$-b_1 \left( \frac{|z|m_1}{t} \right)^{1/(m_1-1)} \leq -b_2 \frac{|z|m_2}{t^{1/(m_2-1)}} + \omega t.$$ 

Let

$$f_t(x) = b_2 x^{m_2/(m_2-1) t^{-1/(m_2-1)}} - b_1 x^{m_1/(m_1-1) t^{-1/(m_1-1)}} \quad (x \geq 0),$$

where $t > 0$. Since $\frac{m_2}{m_2-1} < \frac{m_1}{m_1-1}$, $f_t(\infty) = -\infty$. Moreover, $f_t(0) \leq 0$. Let $x \geq 0$ such that $f_t'(x) = 0$. Then

$$\frac{b_2 m_2}{m_2 - 1} x^{\frac{1}{m_2-1} t^{-\frac{1}{m_2-1}}} = b_1 \frac{m_1}{m_1 - 1} x^{\frac{1}{m_1-1} t^{-\frac{1}{m_1-1}}}.$$

Hence $a_2(\frac{x}{t})^{\frac{1}{m_2-1}} = a_1(\frac{x}{t})^{\frac{1}{m_1-1}}$. Thus $\frac{a_2}{a_1} = (\frac{x}{t})^{\frac{1}{m_2-1} - \frac{1}{m_1-1}}$. This implies that $x = \beta t$ for some $\beta > 0$ independent of $t > 0$. Thus $\max_{y > 0} f_t(y) = f_t(\beta t) = \hat{b}_2 t - \hat{b}_1 t$ where $\hat{b}_2, \hat{b}_1 \in \mathbb{R}$ are constants. Choose $\omega \geq \hat{b}_2 - \hat{b}_1$. \(\square\)

Pseudo-Gaussian estimates can be established with the help of a version of Davies’ trick which goes as follows. Let

$$\mathcal{W} := \{ \psi \in C^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) : \|D_j \psi\|_\infty \leq 1, \|D_i D_j \psi\|_\infty \leq 1, i, j = 1, \ldots, N \}. $$

Let $S$ be a positive $C_0$-semigroup on $L^2(\mathbb{R}^N)$. For $\varrho \in \mathbb{R}$, $\psi \in \mathcal{W}$ we denote by $S^\varrho$ the $C_0$-semigroup given by

$$S^\varrho(t) f = e^{-\varrho \psi} S(t) \{ e^{\varrho \psi} f \}. \quad (2.4)$$

We keep in mind that $S^\varrho(t)$ also depends on $\psi$, but the estimates should not. In fact, we have the following.

**Proposition 2.2.** Let $m \geq 2$ be a real constant. Assume that there exist $c > 0$, $\omega \in \mathbb{R}$, such that

$$\| S^\varrho(t) \|_{L(L^1, L^\infty)} \leq c t^{-N/2} e^{\omega(1+e^\varrho) t} \quad (2.5)$$

for all $\varrho \in \mathbb{R}$, $\psi \in \mathcal{W}, t > 0$. Then $S$ satisfies pseudo-Gaussian estimates of order $m$.

We recall the Dunford–Pettis criterion which says that an operator $B$ on $L^2(\mathbb{R}^N)$ is given by a measurable kernel $k \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ if and only if $\|B\|_{L(L^1, L^\infty)} < \infty$. In that case,

$$\|k\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^N)} = \|B\|_{L(L^1, L^\infty)}.$$

**Proof of Proposition 2.2.** This is a modification of [4, Proposition 3.3]. It follows from the Dunford–Pettis criterion applied to the operator $S(t)$ that $S(t)$ is given by a measurable kernel $k$. Consequently, $S^\varrho(t)$ is given by the kernel

$$k^\varrho(t, x, y) = k(t, x, y) e^{\varrho(\psi(y) - \psi(x))}.$$ 

Since by the Dunford–Pettis criterion again one has

$$k^\varrho(t, x, y) \leq c t^{-N/2} e^{\omega(1+e^\varrho) t},$$

it follows that

$$k(t, x, y) \leq c t^{-N/2} e^{\omega t} e^{\omega^\varrho t \pm \varrho(\psi(y) - \psi(x))}$$

for all $\varrho \in \mathbb{R}$. Now, $d(x, y) = \sup \{ \psi(x) - \psi(y) : \psi \in \mathcal{W}\}$ defines a metric on $\mathbb{R}^N$ which is equivalent to the given metric, see [17, pp. 200–202]. Hence $d(x, y) \leq \beta |x - y|$ for all $x, y \in \mathbb{R}^N$ and some $\beta > 0$. Thus

$$k(t, x, y) \leq c t^{-N/2} e^{\omega t} e^{\omega^\varrho t - \varrho \beta |y - x|}$$

a.e. Choosing

$$\varrho = \left( \frac{\beta |x - y|}{t \omega^m} \right)^{m-1}.$$
we obtain
\[
k(t,x,y) \leq c t^{-N/2} e^{\omega t} \exp\left\{-c_2 \frac{|y-x|^m}{t}\right\} \frac{1}{m!}
\]
where \(c_2 = \beta \pi^{-1/2} \left(m^{-1/2} - m^{-3/2}\right)\).
\[\square\]

Now we have to consider a stronger hypothesis than \((H_0)\), namely
\[
div b \leq \beta V, \quad div c \leq \beta V
\]
for some constant \(0 < \beta < 1\). We also need a condition on the growth of the drift terms \(b\) and \(c\) with respect to \(V\) (assumed nonnegative), namely
\[
V \geq 0, \quad |b| \leq k_1 V^\alpha + k_2, \quad |c| \leq k_1 V^\alpha + k_2,
\]
where \(\frac{1}{2} \leq \alpha < 1, k_1, k_2 \geq 0\), as well as some more regularity on the diffusion coefficients:
\[
a_{ij} \in C_b^1(\mathbb{R}^N).
\]
The following result extends [2, Theorem 5.2] from the case \(\alpha = \frac{1}{2}\) (i.e., \(m = 2\)) to \(\frac{1}{2} \leq \alpha < 1\). Note however, that in contrast to the situation when \(\alpha = \frac{1}{2}\), if \(\alpha > \frac{1}{2}\) then the operator \(-A\) is not associated with a form and the semigroup \(T\) may not be holomorphic (see [2, Section 6] and Section 3 below).

**Theorem 2.3.** Let \(A\) be the minimal realization of the elliptic operator whose coefficients satisfy (1.1), \((H_1)\), \((H_2)\) and \((H_3)\). Let \(T\) be the semigroup generated by \(A\). Then \(T\) satisfies a pseudo-Gaussian estimate of order \(m = \frac{1}{1-\alpha}\).

**Proof.** Let \(\varrho \in \mathbb{R}, \psi \in \mathcal{W}\). It is obvious that
\[
T_\varrho^\varrho(t)f = \lim_{r \to \infty} T_\varrho^r(t)f.
\]
Thus the generator \(A_\varrho\) of \(T_\varrho\) is the minimal realization of the elliptic operator \(A_\varrho\) with coefficients
\[
a_{ij}^\varrho = a_{ij},
\]
\[
b_i^\varrho = b_i - \varrho \sum_{j=1}^N a_{ij} \psi_j,
\]
\[
c_i^\varrho = c_i + \varrho \sum_{i,j=1}^N a_{ki} \psi_k,
\]
\[
V^\varrho = V - \varrho^2 \sum_{i,j=1}^N a_{ij} \psi_i \psi_j + \varrho \sum_{i=1}^N b_i \psi_i - \varrho \sum_{i=1}^N c_i \psi_i,
\]
where \(\psi_i = D_i \psi\), cf. [4, Lemma 3.6]. We will find \(\omega \in \mathbb{R}\) such that for
\[
W^\varrho = V^\varrho + (1 + \varrho^m)\omega
\]
one has
\[
div b^\varrho \leq W^\varrho, \quad div c^\varrho \leq W^\varrho,
\]
where \(\omega\) is independent of \(\varrho \in \mathbb{R}\) and \(\psi \in \mathcal{W}\). Then Proposition 1.3 applied to \(A_\varrho - (1 + \varrho^m)\omega\) implies that
\[
\|T(t)\|_{L^1(L_\infty)} \leq c_1 t^{-N/2} e^{\omega t (1 + \varrho^m)} (t > 0).
\]
Then Proposition 2.2 proves the claim. In order to prove (2.6) we proceed in several steps. We first show that
\[
\varrho V^\varrho \leq \varepsilon^{1/\varrho} |\alpha V + (1 - \alpha) e^{-m \varrho^m} (2.8)
\]
for all $\varepsilon > 0$. In fact, let $q = \frac{1}{\alpha}, \frac{1}{p} = 1 - \frac{1}{q}$ and recall that $m = \frac{1}{1 - \alpha} = p$. Then by Hölder’s inequality

$$\varrho V^\alpha = \frac{1}{\varepsilon} \varrho V^\alpha \varepsilon \leq \frac{1}{p} \varrho \varrho^p + \frac{1}{q} V^{\alpha q} \varepsilon^q = (1 - \alpha)\varepsilon^{-m} \varrho^m + \alpha V^{1/\alpha}.$$ 

Next we show that there exists $\omega_1 \in \mathbb{R}$ such that

$$\beta V \leq V^\varrho + \omega_1 (1 + \varrho^m)$$

(2.9)

for all $\varrho \in \mathbb{R}, \psi \in W$, where $\beta \in (0, 1)$ is the constant in $(H_1)$. In fact, by $(H_2)$ and (2.8),

$$V^\varrho \geq V - k_3 \varrho^2 - k_3 \varrho^\alpha - k_4 \varrho$$

$$\geq V - k_3 \varrho^2 - k_3 \varepsilon^{1/\alpha} \alpha V - k_3 (1 - \alpha)\varepsilon^{-m} \varrho^m - k_4 \varrho$$

$$\geq \beta V - \omega_1 (1 + \varrho^m)$$

for suitable constants $k_3, k_4\omega_1$ where $\varepsilon > 0$ is chosen such that $\beta = 1 - k_3 \varepsilon^{1/\alpha}$. Now we show (2.6). One has by (2.9),

$$\operatorname{div} b^0 = \operatorname{div} b - \varrho \sum_{i,j=1}^N D_i (a_{ij} \psi_j)$$

$$\leq \beta V + k_4 \varrho$$

$$\leq V^\varrho + \omega_1 (1 + \varrho^m) + k_5 \varrho$$

$$\leq V^\varrho + \omega (1 + \varrho^m)$$

for all $\varrho \in \mathbb{R}, \psi \in W$ where $k_5, \omega$ are suitable constants. The estimate for $\operatorname{div} c^0$ is the same. \(\square\)

**Remark 2.4.** It is obvious from the definition that a semigroup $S$ satisfies (pseudo-) Gaussian estimates if and only if $(e^{\omega t} S(t))_{t \geq 0}$ does so for some $\omega \in \mathbb{R}$. Thus in Theorem 2.3 we may replace condition $(H_1)$ by the weaker condition

$$\operatorname{div} b \leq \beta V + \beta'$$

$$\operatorname{div} c \leq \beta V + \beta'$$

(\(H_1'\))

where $0 < \beta < 1, \beta' \in \mathbb{R}$ and the result remains valid.

As application we obtain a result on $p$-independence of the spectrum. Assume that assumptions (1.1) and $(H_1)$ are satisfied. Let $A$ be the minimal realization of the elliptic operator $\mathcal{A}$. Then $A$ generates a $C_0$-semigroup $T$ on $L^2(\mathbb{R}^N)$ and $T$ as well as $T^*$ are submarkovian. As a consequence there exists a consistent family $T_p = (T_p(t))_{t \geq 0}$ of semigroups on $L^p(\mathbb{R}^N)$ such that $T_2 = T$. Here $T_p$ is a $C_0$-semigroup if $1 \leq p < \infty$ and $T_\infty$ is a dual $C_0$-semigroup. We denote by $A_p$ the generator of $T_p$, $1 \leq p \leq \infty$.

**Corollary 2.5.** Assume that (1.1), $(H_1)$, $(H_2)$ and $(H_3)$ are satisfied. Assume that $\alpha < \frac{N+2}{2N}$ Then $\sigma (A_p) = \sigma (A)$ for all $p \in [1, \infty]$. Here $\frac{1}{2} \leq \alpha < 1$ is the constant occurring in hypothesis $(H_2)$.

**Proof.** This follows from a result of Karrmann [9, Corollary 6.2] which in turn is a consequence of a result of Kunstmann [10, Theorem 1.1]. \(\square\)

The restriction

$$\alpha < \frac{N+2}{2N}$$

is due to the fact that Karrmann proves spectral $p$-independence in the case of quasi-Gaussian estimates of order $m$ if $m < \frac{2N}{N-2}$. We do not know whether these conditions are optimal.
3. An example

In order to show that Theorem 2.3 is optimal we consider the one-dimensional example

\[ Au = u'' - x^3 u' + |x|^{\gamma} u, \]

where \( \gamma > 2 \). Then condition \((H_1')\) is satisfied (see Remark 2.4). Let \( A \) be the minimal realization of \( A \) in \( L^2(\mathbb{R}) \) and let \( T \) be the semigroup generated by \( A \). If \( \gamma \geq 6 \), then it follows from Theorem 2.3 that \( T \) satisfies Gaussian estimates. If \( 6 > \gamma > 3 \), then Theorem 2.3 says that \( T \) satisfies pseudo-Gaussian estimates of order \( m = \frac{\gamma}{\gamma - 3} \). We show that \( T \) does not satisfy Gaussian estimates in that case.

Proposition 3.1. Let \( 3 < \gamma < 6 \). Then \( T \) does not satisfy Gaussian estimates.

Proof. Assume that \( T(t) \) is given by a kernel \( k_t \) satisfying

\[ 0 \leq k_t(x, y) \leq c_1 e^{\omega t} \frac{1}{\sqrt{t}} e^{-c_2 |x - y|^2 / t}. \]  

Consider the operator \( I_n \in \mathcal{L}(L^2) \) given by

\[ (I_n u)(x) = u \left( \frac{x - n}{\lambda_n} \right), \]

where \( \lambda_n = n^{3-\beta}, \gamma < \beta < 6 \). Then

\[ \| I_n u \|_2 = \sqrt{\lambda_n} \| u \|_2 \quad (u \in L^2(\mathbb{R})) \]

and \( (I_n^{-1} u)(x) = u(\lambda_n x + n) \). Define the semigroup \( T_n \) on \( L^2(\mathbb{R}) \) by

\[ T_n(t) = I_n^{-1} T(n \lambda_n t) I_n, \]

where \( r_n = n^{-\beta} \). It follows from the Trotter–Kato Theorem that

\[ \lim_{n \to \infty} T_n(t) f = S(t) f \]

for all \( f \in L^2(\mathbb{R}) \) where \( S \) is the shift semigroup given by \( (S(t)u)(x) = u(x - t) \) (see [2, Proposition 6.4]) One has for \( f \in L^2(\mathbb{R}) \)

\[ T_n(t) f(x) = \left( T(n \lambda_n t) (I_n f) \right)(n + \lambda_n x) \]

\[ = \int k_{n \lambda_n t}(n + \lambda_n x, y) f\left( \frac{y - n}{\lambda_n} \right) dy \]

\[ = \int k_{n \lambda_n}(n + \lambda_n x, n + \lambda_n y) f(y) dy \]

\[ = \int k_{t}^\lambda(x, y) f(y) dy \]

where \( k_{t}^\lambda(x, y) = \lambda_n k_{n \lambda_n t}(n + \lambda_n x, n + \lambda_n y) \). By (3.1) we obtain

\[ k_{t}^\lambda(x, y) \leq n^{3-\beta} c_1 e^{\omega t n} \frac{1}{\sqrt{r_n t}} e^{-c_2 n^2 |x - y|^2 / n^{-\beta} t} \]

\[ = n^{3-\beta/2} c_1 e^{\omega t n} \frac{1}{\sqrt{t}} e^{-c_2 n^6 |x - y|^2 / t}. \]

Denoting by \( G = (G(t))_{t \geq 0} \) the Gaussian semigroup, this implies that for \( 0 \leq f \in L^2(\mathbb{R}^N) \),

\[ (T_n(t) f)(x) \leq c_{\text{astr}} \left( G\left( t / 4 c_2 n^6 \right) f \right)(x). \]
Thus

\[ S(t)f = \lim_{n \to \infty} T_n(t)f \leq \lim_{n \to \infty} c e^{ot/t} G\left(\frac{t}{4c^2 n^{6-\beta}}\right) f = c_1 f. \]

This is a contradiction. \(\square\)

**Remark 3.2.** It was shown in [2, Proposition 6.4] that for \(2 \leq \gamma < 6\), the semigroup \(T\) is not holomorphic. It seems not to be known whether Gaussian estimates for positive semigroups imply holomorphy. They do not without positivity assumption as Voigt’s example

\[ Au = u'' + ix \]

on \(L^2(\mathbb{R})\) shows (see Liskevich and Manavi [11] for more details).

**References**