Evolution Equations Governed by Elliptic Operators

by

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Chapter 1

Unbounded Operators

In this chapter we introduce unbounded operators and put together some properties which will be frequently used.

Moreover, we discuss the spectral theorem for self-adjoint operators which will give us very interesting examples of elliptic operators in the sequel.

1.1 Closed operators

Let E be a complex Banach space.

Definition 1.1.1. *An* **operator** *on E is a linear mapping $A : D(A) \to E$, where $D(A)$ is a subspace of E which we call the* **domain** *of A. The operator A is called* **bounded** *if*

$$\|A\| := \sup_{\|x\| \leq 1, x \in D(A)} \|Ax\| < \infty.$$

If $\|A\| = \infty$, then A is called **unbounded**.

The notion of an operator is too general to allow one to do some analysis. The least thing one needs is to be allowed to exchange limits and the operation. That is made precise in the following definition.

Definition 1.1.2. *An operator A is* **closed** *if for any sequence $(x_n)_{n\in\mathbb{N}}$ in $D(A)$ such that $\lim\limits_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Ax_n = y$ exist in E one has $x \in D(A)$ and $Ax = y$.*

Thus an operator A on E is closed if and only if its **graph**

$$G(A) := \{(x, Ax) : x \in D(A)\}$$

is a closed subspace of $E \times E$.

If $D(A)$ is a closed subspace of E, then the closed graph theorem asserts that A is bounded if and only if A is closed. We will be mainly interested in closed operators with dense domain.

In order to give a first typical example we need the following.

Exercise 1.1.3 (graph norm)**.** Let A be an operator on E. Then

$$\|x\|_A = \|x\| + \|Ax\|$$

defines a norm on $D(A)$ which we call the **graph norm**. The operator A is closed if and only if $(D(A), \|\cdot\|_A)$ is a Banach space.

Example 1.1.4. *Let $E = C[0,1]$ with supremum norm*

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|.$$

Denote by $C^1[0,1]$ the once-continuosly differentiable functions on $[0,1]$.

a) Define the operator A on E by $D(A) = C^1[0,1]$, $Af = f'$. Then the graph norm on $D(A)$ is given by $\|f\|_A = \|f\|_\infty + \|f'\|_\infty$ for which $C^1[0,1]$ is complete. Thus A is closed.

b) Define the operator A_0 on E by $D(A_0) = C^\infty[0,1]$ (the infinitely differentiable functions on $[0,1]$), $A_0 f = f'$. Then A_0 is not closed.

An operator may be not closed for two different reasons. The first reason is that the domain had been chosen too small, but the operator has a closed extension. Example 1.1.4 b) is of this type, the operator of part a) being a closed extension. The second possible reason is that things go basically wrong. Such operators are called unclosable; they do not have any closed extension. The problem is that an operator A is asked to be a function; i.e. to each $x \in D(A)$ only one value Ax is associated. Proposition 1.1.7 now makes clear why an operator may not have a closed extension (see Example 1.1.9 for a concrete case). Here are the precise definitions and statements.

Definition 1.1.5 (extension of operators)**.** *Let A, B be two operators on E.*

a) We say that B is an **extension** *of A and write $A \subset B$ if*

$$\begin{array}{rcll} D(A) & \subset & D(B) & \text{and} \\ Ax & = & Bx & \text{for all } x \in D(A). \end{array}$$

b) Two operators A and B are said to be **equal** *if $A \subset B$ and $B \subset A$, i.e. if $D(A) = D(B)$ and $Ax = Bx$ for all $x \in D(A)$.*

An operator A may have a closed extension or it may not . If it has one there is always a smallest one.

Proposition 1.1.6 (closable operators)**.** *An operator A on E is called* **closable** *if there exists a closed operator B such that $A \subset B$. In that case, there exists a smallest closed extension $\bar{A}$ of A which is called the* **closure** *of A , (i.e $\bar{A}$ is closed, $A \subset \bar{A}$ and $\bar{A} \subset B$ for all closed extensions of A).*

The following criterion for closability is very useful. We leave its verification as well as the proof of Proposition 1.1.6 as exercise.

Proposition 1.1.7 (criterion for closability)**.** *Let A be an operator on E.*

a) The operator A is closable if and only if for $x_n \in D(A)$, $y \in E$ such that $\lim_{n\to\infty} x_n = 0$, $\lim_{n\to\infty} Ax_n = y$ one has $y = 0$.

b) An operator A is closable if and only if there exists a closed operator B on E such that $\overline{G(A)} = G(B)$. In that case $B = \bar{A}$.

Here are two examples which illustrate well the situation.

Example 1.1.8. *The operator A_0 in Example 1.1.4 b) is closable and $\bar{A}_0 = A$, where A is the operator introduced in Example 1.1.4 a).*

Exercise 1.1.9 (a non closable operator)**.** *Let $E = C[0,1]$, $D(A) = C^1[0,1]$, $Af = f'(0) \cdot 1$ (where 1 denotes the constant 1 function).* Then A is not closable.

Exercise 1.1.10. *Prove Proposition 1.1.6 and 1.1.7.*

Exercise 1.1.11. *Let A be an operator on X.*
a) Show that A is continuous if and only if A is bounded.
b) Assume that A is bounded. Show that A is closable and $D(\bar{A}) = \overline{D(A)}$. Conclude that $\bar{A}$ is the continuous extension of A to $\overline{D(A)}$.

1.2 The spectrum

Let E be a complex Banach space. Let A be an operator on E. Frequently, even if A is unbounded, it might have a bounded inverse. In that case, we may use properties and theorems on bounded operators to study A.

For this, it does not matter if A is replaced by $\lambda I - A$ where $\lambda \in \mathbb{C}$ and I is the identity operator on E. The set

$$\rho(A) = \{\lambda \in \mathbb{C} : \lambda I - A : D(A) \to E \text{ is bijective and } (\lambda I - A)^{-1} \in \mathcal{L}(E)\}$$

is called the **resolvent set** of A. Here $\mathcal{L}(E)$ is the space of all bounded operators from E into E. If $\lambda I - A : D(A) \to E$ is bijective, then $(\lambda I - A)^{-1} : E \to D(A)$ is linear. But in the definition we ask in addition that $(\lambda I - A)^{-1}$ is a bounded operator from E into E. This is automatic if A is closed (see Exercise 1.2.1).

For $\lambda \in \rho(A)$, the operator

$$R(\lambda, A) = (\lambda I - A)^{-1} \in \mathcal{L}(E)$$

is called the **resolvent** of A in λ.

Frequently we write $(\lambda - A)$ as short hand for $(\lambda I - A)$. The set $\sigma(A) = \mathbb{C} \backslash \rho(A)$ is called the **spectrum** of A.

Exercise 1.2.1 (closed operators and resolvent)**.** *a)* Let $\lambda \in \mathbb{C}$. Then A is closed if and only if $(\lambda - A)$ is closed.

b) If $\rho(A) \neq \emptyset$, then A is closed.

c) Assume that A is closed and $(\lambda - A) : D(A) \to E$ is bijective. Then $\lambda \in \rho(A)$ (Use the closed graph theorem).

If $B \in \mathcal{L}(E)$, then $\rho(B) \neq \emptyset$. In fact, assume that $\|B\| < 1$. Then

$$(I - B)^{-1} = \sum_{k=0}^{\infty} B^k \qquad (\textbf{ Neumann series }). \tag{1.1}$$

Replacing B by $\frac{1}{\lambda}B$ one sees that $\lambda \in \rho(B)$ whenever $|\lambda| > \|B\|$.

Unbounded closed operators may have empty resolvent set. Also, it may happen that an unbounded operators has empty spectrum (which is not true for operators in $\mathcal{L}(E)$), see Exercise 1.2.7.

Proposition 1.2.2 (analyticity of the resolvent)**.** *Let A be an operator and $\lambda_0 \in \rho(A)$. If $\lambda \in \mathbb{C}$ such that $|\lambda - \lambda_0| < \|R(\lambda_0, A)\|^{-1}$, then $\lambda \in \rho(A)$ and*

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, A)^{n+1}$$

which converges in $\mathcal{L}(E)$.

Proof. One has $(\lambda - A) = (\lambda - \lambda_0) + (\lambda_0 - A) = (I - (\lambda_0 - \lambda)R(\lambda_0, A))(\lambda_0 - A)$. Since $\|(\lambda_0 - \lambda)R(\lambda_0, A)\| < 1$, the operator $((I - (\lambda_0 - \lambda)R(\lambda_0, A))$ is invertible and its inverse is given by

$$\sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, A)^n.$$

Hence $\lambda \in \rho(A)$ and

$$R(\lambda, A) = R(\lambda_0, A)(I - (\lambda_0 - \lambda)R(\lambda_0, A))^{-1}.$$

□

Proposition 1.2.2 shows in particular that

$$\operatorname{dist}(\lambda, \sigma(A)) \geq \|R(\lambda, A)\|^{-1} \tag{1.2}$$

for all $\lambda \in \rho(A)$. Here $\operatorname{dist}(\lambda, M) = \inf\{|\lambda - \mu| : \mu \in M\}$ is the distance of $\lambda \in \mathbb{C}$ to a subset M of $\mathbb{C}$. This has the following useful consequence.

Corollary 1.2.3. *Let $\lambda_n \in \rho(A),\ \lambda = \lim_{n\to\infty} \lambda_n$. If*

$$\sup_{n \in \mathbb{N}} \|R(\lambda_n, A)\| < \infty,$$

then $\lambda \in \rho(A)$.

Remark 1.2.4. *The property expressed in Corollary 1.2.3 is quite remarkable since it is not true for holomorphic functions in general. In fact, if* $\Omega \subset \mathbb{C}$ *is open ,* $z_o \in \Omega$ *and* $f : \Omega \ \{z_0\} \to \mathbb{C}$ *is holomorphic and bounded on some neighborhood* $B(z_o, r) \setminus \{z_o\} := \{z \in \mathbb{C} : 0 < |z - z_0| < r\}$*, then* f *has a holomorphic extension to* Ω*. But it does not suffice, in general, that* f *is bounded on some sequence* $(z_n) \subset \Omega \setminus \{z_0\}$ *converging to* z_0*. For example, let* $\Omega = \mathbb{C}$*,* $z_0 = 0$*,* $f(z) = \exp(1/z), z_n = i/n$*.* □

By
$$\sigma_p(A) = \{\lambda \in \mathbb{K} : \ \exists x \in D(A), x \neq 0, (\lambda - A)x = 0\}$$
we denote the **point spectrum**, or the set of all **eigenvalues** of A. If λ is an eigenvalue, each $x \in D(A) \setminus \{0\}$ such that $(\lambda - A)x = 0$ is called an **eigenvector** of A. There is a natural relation between the spectrum of A and its resolvents.

Proposition 1.2.5 (spectral mapping theorem for resolvents)**.** *Let* $\lambda_0 \in \rho(A)$*. Then*

a) $\sigma(R(\lambda_0, A)) \setminus \{0\} = \{(\lambda_0 - \lambda)^{-1} : \lambda \in \sigma(A)\}$,

b) $\sigma_p(R(\lambda_0, A)) \setminus \{0\} = \{(\lambda_0 - \lambda)^{-1} : \lambda \in \sigma_p(A)\}$.

Proof. a) 1. If $\mu \in \rho(A), \mu \neq \lambda_0$, then
$$\left(\frac{1}{\lambda_0 - \mu} - R(\lambda_0, A)\right)^{-1} = (\lambda_0 - \mu)(\lambda_0 - A)R(\mu, A).$$

2." $\subset$ " Let $\nu \in \sigma(R(\lambda_0, A)), \nu \neq 0$. Assume that $\nu \notin \{(\lambda_0 - \lambda)^{-1} : \lambda \in \sigma(A)\}$. Then $\lambda_0 - 1/\nu \in \rho(A)$. This implies $\nu \in \rho(R(\lambda_0, A))$ by 1.

3. " $\supset$ " Let $\mu = (\lambda_0 - \lambda)^{-1}$ where $\lambda \neq \lambda_0$. Suppose that $\mu \in \rho(R(\lambda_0, A))$. Then one easily sees that $\lambda \in \rho(A)$ and $R(\lambda, A) = \mu R(\lambda_0, A)(\mu - R(\lambda_0, A))^{-1}$.

b) is left to the reader.

□

It follows in particular, that $\sigma(A) = \emptyset$ if and only if there exists $\mu \in \rho(A)$ such that $\sigma(R(\mu, A)) = \{0\}$, and in that case $\sigma(R(\mu, A)) = \{0\}$ for all $\mu \in \rho(A)$. We denote by
$$r(B) = \sup\{|\lambda| : \ \lambda \in \sigma(B)\}$$
the **spectral radius** of an operator $B \in \mathcal{L}(E)$. Then Proposition 1.2.5 gives an improvement of (1.2).
$$r(R(\lambda, A)) = \mathrm{dist}(\lambda, \sigma(A))^{-1} \qquad \text{for all } \lambda \in \rho(A) \tag{1.3}$$
We conclude this section by the following easy identity.

Proposition 1.2.6 (resolvent identity)**.**
$$(R(\lambda, A) - R(\mu, A))/(\mu - \lambda) = R(\lambda, A)R(\mu, A)$$
for all $\lambda, \mu \in \rho(A), \lambda \neq \mu$*.*

Proof. One has,

$$\begin{aligned} R(\lambda, A) - R(\mu, A) &= R(\lambda, A)[I - (\lambda - A)R(\mu, A)] \\ &= R(\lambda, A)[(\mu - A) - (\lambda - A)]R(\mu, A) \\ &= (\mu - \lambda)R(\lambda, A)R(\mu, A) \end{aligned}$$

□

The resolvent identity shows in particular, that resolvents commute.

Exercise 1.2.7 (empty spectrum)**.** *Let* $E = C[0,1]$, $D(A_1) = \{f \in C^1[0,1], f(1) = 0\}$, $A_1 f = f'$. *Then* $\sigma(A_1) = \emptyset$.

Indication: $(R(\lambda, A_1)f)(x) = \int_x^1 e^{\lambda(x-y)} f(y)\, dy \qquad (\lambda \in \mathbb{C})$.

1.3 Operators with compact resolvent

Let E be a Banach space over $\mathbb{C}$. By $\mathcal{K}(E)$ we denote the space of all compact operators on E. The following facts are well-known. $\mathcal{K}(E)$ is a closed subspace of $\mathcal{L}(E)$. It is even an ideal, i.e. $K \in \mathcal{K}(E)$ implies $SK,\ KS \in \mathcal{K}(E)$ for all $S \in \mathcal{L}(E)$.

Compact operators have very particular spectral properties. Let $K \in \mathcal{K}(E)$. Then the spectrum consists only of eigenvalues with 0 as possible exception, i.e.

$$\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\} \ . \tag{1.4}$$

Moreover, $\sigma(K)$ is countable with 0 as only possible accumulation point, i.e., either $\sigma(K)$ is finite or there exists a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ such that $\lim_{n\to\infty} \lambda_n = 0$ and

$$\sigma(K) = \{\lambda_n : n \in \mathbb{N}\} \cup \{0\}.$$

Finally, for each $\lambda \in \sigma_p(K) \setminus \{0\}$, the eigenspace $\ker(\lambda - K)$ is finite dimensional.

The purpose of this section is to find out what all these properties mean for an unbounded operator if its resolvent is compact.

Definition 1.3.1. *An operator* A *on* E *has* **compact resolvent** *if* $\rho(A) \neq \emptyset$ *and* $R(\lambda, A)$ *is compact for all* $\lambda \in \rho(A)$.

From the resolvent identity and the ideal property, it follows that A has compact resolvent whenever $R(\lambda, A) \in \mathcal{K}(E)$ for some $\lambda \in \rho(A)$.

If $\dim E = \infty$ then operators with compact resolvent are necessarily unbounded (otherwise, for $\lambda \in \rho(A)$ we have $R(\lambda, A) \in \mathcal{K}(E)$ and $(\lambda - A) \in \mathcal{L}(E)$. Thus $I = (\lambda - A)R(\lambda, A)$ is compact by the ideal property).

The following criterion is most useful

Exercise 1.3.2 (criterion for compact resolvent)**.** *Let* A *be an operator on* E *with non-empty resolvent set. Then* A *has compact resolvent if and only if the canonical injection* $(D(A), \|\cdot\|_A) \to E$ *is compact.*

The following spectral properties follow easily from those of compact operators with help of the spectral mapping theorem for resolvents (Propositon 1.2.5).

Proposition 1.3.3 (spectral properties of operators with compact resolvent)**.** *Let A be an operator with compact resolvent. Then the following holds.*

a) $\sigma(A) = \sigma_p(A)$*;*

b) either $\sigma(A)$ is finite or there exists a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ such that $\lim_{n\to\infty} |\lambda_n| = \infty$ and $\sigma(A) = \{\lambda_n : n \in \mathbb{N}\}$;

c) $\dim\ker(\lambda - A) < \infty$ *for all* $\lambda \in \mathbb{C}$ *where* $\ker(\lambda - A) := \{x \in D(A) : Ax = \lambda x\}$.

The most simple examples of unbounded operators are diagonal operators. In the next section we will see that selfadjoint operators with compact resolvent are equivalent to such simple operators. This will be exploited to solve the heat equation. We let

$$l^2 := \{(x_n)_{n\in\mathbb{N}} \subset \mathbb{K} : \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$$

where as usual $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then l^2 is a separable Hilbert space over $\mathbb{K}$ with respect to the scalar product

$$(x \mid y) = \sum_{n=1}^{\infty} x_n \bar{y}_n$$

(where $\bar{y}_n$ is the complex conjugate of y_n).

Definition 1.3.4 (diagonal operator)**.** *Let $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$. The operator M_α on l^2 given by*

$$\begin{aligned} D(M_\alpha) &= \{x \in l^2 : (\alpha_n x_n)_{n\in\mathbb{N}} \in l^2\} \\ M_\alpha x &= (\alpha_n x_n)_{n\in\mathbb{N}} \end{aligned}$$

is called the **diagonal operator** *associated with α and is denoted by M_α.*

We define the sequence spaces

$$\begin{aligned} l^\infty &= \{\alpha = (\alpha_n)_{n\in\mathbb{N}} \subset \mathbb{C} : \sup_{n\in\mathbb{N}} |\alpha_n| < \infty\} \\ c_0 &= \{\alpha = (\alpha_n)_{n\in\mathbb{N}} \subset \mathbb{C} : \lim_{n\to\infty} |\alpha_n| = 0\}. \end{aligned}$$

Note that l^∞ is a Banach space for the norm

$$\|\alpha\|_\infty = \sup_{n\in\mathbb{N}} |\alpha_n|$$

and c_0 is a closed subspace.

Exercise 1.3.5. *Let* $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in l^\infty$.

a) The operator M_α is bounded and $\|M_\alpha\| = \|\alpha\|_\infty$.

b) If $\alpha \in c_0$, then $M_\alpha \in \mathcal{K}(l^2)$.

Hint for b): M_α can be approximated by finite rank operators in $\mathcal{L}(l^2)$.

Here an operator $R \in \mathcal{L}(E)$ is said to have **finite rank** if $\dim RE < \infty$. Such an operator is always compact.

Example 1.3.6 (diagonal operators with compact resolvent)**.** *Let* $\alpha = (\alpha_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ *be a sequence such that* $\lim_{n\to\infty} |\alpha_n| = \infty$. *Then* M_α *has compact resolvent.*

This is easy to see with help of Exercise 1.3.5.

Given an operator A, it is easy to define a new operator by similarity which has the same properties as A.

Proposition 1.3.7 (Similarity)**.** *Let* A *be an operator on* E *and let* $V : E \to F$ *be an isomorphism where* F *is a Banach space . Define the operator* B *on* F *by*

$$\begin{aligned} D(B) &= \{y \in F : V^{-1}y \in D(A)\} \\ By &= VAV^{-1}y. \end{aligned}$$

Then the following holds:

a) A *is closed if and only if* B *is closed;*

b) $\rho(B) = \rho(A)$ *and* $R(\lambda, B) = VR(\lambda, A)V^{-1}$ *for all* $\lambda \in \rho(B)$;

c) B *has compact resolvent if and only if* A *has compact resolvent.*

Notation: $VAV^{-1} := B$. The easy proof is left as exercise.

Exercise 1.3.8 (further properties of diagonal operators)**.** Let $\alpha = (\alpha_n)_{n\in\mathbb{N}} \subset \mathbb{C}$.

a) If M_α is bounded, then $\alpha \in l^\infty$ and $\|M_\alpha\| = \|\alpha\|_\infty$.

b) $\sigma(M_\alpha) = \overline{\{\alpha_n : n \in \mathbb{N}\}}$.

c) $\sigma_p(M_\alpha) = \{\alpha_n : n \in \mathbb{N}\}$.

d) M_α is compact if and only if $\alpha \in c_0$.

e) M_α has compact resolvent if and only if $\lim_{n\to\infty} |\alpha_n| = \infty$.

1.4 Selfadjoint operators with compact resolvent.

Here we consider unbounded operators on a Hilbert space. The main result is the spectral theorem which shows that every selfadjoint operator with compact resolvent can be represented as a diagonal operator.

Throughout this section H is a separable complex Hilbert space.

Definition 1.4.1. *An operator A on H is called* **dissipative** *if*

$$\mathrm{Re}(Ax|x) \le 0 \text{ for all } x \in D(A)\ .$$

The following proposition shows a remarkable spectral property of dissipative operators. We denote by

$$\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\}$$

the right half-plane.

Proposition 1.4.2. *Let A be a dissipative operator on H. Assume that there exists $\lambda \in \mathbb{C}_+$ such that $(\lambda - A)$ is surjective. Then $\mu \in \rho(A)$ and $\|R(\mu, A)\| \le 1/\mathrm{Re}\mu$ for all $\mu \in \mathbb{C}_+$.*

Proof. Let $\mu \in \mathbb{C}_+$. Let $x \in D(A)$, $\mu x - Ax = y$. Then

$$\begin{array}{rcccl} \mathrm{Re}\mu\|x\|^2 & = & \mathrm{Re}(\mu x|x) & = & \mathrm{Re}(y + Ax|x) \\ & = & \mathrm{Re}(x|y) & + & \mathrm{Re}(Ax|x) \\ & \le & \mathrm{Re}(x|y) & \le & \|x\|\ \|y\| \end{array}$$

by dissipativity and the Cauchy Schwartz inequality. Thus $(\mathrm{Re}\mu)\|x\| \le \|y\|$. It follows that

$$\|R(\mu, A)\| \le \frac{1}{\mathrm{Re}\mu} \tag{1.5}$$

whenever $\mu \in \rho(A) \cap \mathbb{C}_+ =: M$. Since $\rho(A)$ is open, also M is open. Now (1.5) and Corollary 1.2.3 imply that M is closed in $\mathbb{C}_+$. Since the right half-plane is connected and $M \ne \emptyset$, it follows that $M = \mathbb{C}_+$. □

Definition 1.4.3. *An operator A on H is called* **m-dissipative** *if A is dissipative and $(I - A)$ is surjective.*

From Proposition 1.4.2 we know that the spectrum of an m-dissipative operator A is contained in the left half-plane, and $\|R(\lambda, A)\| \le 1/\mathrm{Re}\lambda$ $(\mathrm{Re}\lambda > 0)$.

Now we consider symmetric operators.

Definition 1.4.4. *An operator A on H is called* **symmetric** *if*

$$(Ax|y) = (x|Ay) \text{ for all } x, y \in D(A)\ .$$

If A is symmetric, then $(Ax|x) = (x|Ax) = \overline{(Ax|x)}$. Hence $(Ax|x) \in \mathbb{R}$ for all $x \in D(A)$. Also the converse is true. Recall the **polarization identity**.

$$(x|y) = \tfrac{1}{4}\{(x+y, x+y) - (x-y|x-y) + i(x+iy|x+iy) - i(x-iy|x-iy)\} \tag{1.6}$$

$(x, y \in H)$, which is an immediate consequence of the properties of the scalar product. Considering $y = Ax$ one sees the following.

Proposition 1.4.5. *Let A be an operator on a complex Hilbert space H. The following assertion are equivalent.*

(i) A is symmetric;

(ii) $(Ax|x) \in \mathbb{R}$ for all $x \in D(A)$;

(iii) $\pm iA$ is dissipative.

Note that (iii) is just a reformulation of (ii). But now Proposition 1.4.2 shows us the following.

Proposition 1.4.6. *Let A be a symmetric operator. Assume that $(\lambda - A)$ is surjective for some $\lambda \in \mathbb{C}$ such that $\mathrm{Im}\lambda > 0$. Then $\lambda \in \rho(A)$ for all λ with $\mathrm{Im}\lambda > 0$. Similarly, if $(\lambda - A)$ is surjective for some $\lambda \in \mathbb{C}$ such that $\mathrm{Im}\lambda < 0$, then $\{\lambda \in \mathbb{C} : \mathrm{Im}\lambda < 0\} \subset \rho(A)$.*

Thus for a symmetric operator A, there are four possibilities:

a) $\sigma(A) = \{\lambda \in \mathbb{C} : \mathrm{Im}\lambda \geq 0\}$

b) $\sigma(A) = \{\lambda \in \mathbb{C} : \mathrm{Im}\lambda \leq 0\}$

c) $\sigma(A) = \mathbb{C}$

d) $\sigma(A) \subset \mathbb{R}$.

The cases (a) - (c) are not of interest for our purposes and we refer to the literature for further investigation (e.g. [RS80]). We are rather interested in the last case (d) which leads to the following definition.

Definition 1.4.7. *An operator A is called* **selfadjoint** *if A is symmetric and if $(i - A)$ and $(-i - A)$ are surjective.*

By our discussion, a selfadjoint operator has real spectrum. Whereas every bounded symmetric operator is selfadjoint, for unbounded operators, this is not true, and the range condition (that $(\pm i - A)$ be surjective) is a severe restriction.

We are particularly interested in the case where A is symmetric and dissipative. Then A is selfadjoint if and only if $(\mathbb{C} \setminus (-\infty, 0]) \cap \rho(A) \neq \emptyset$. In particular, an operator A is **dissipative and selfadjoint** if and only if

$$(Ax|y) = (y|Ax) \quad (x, y \in D(A)); \tag{1.7}$$

$$(Ax|x) \leq 0 \quad (x \in D(A); \tag{1.8}$$

$$\text{for all } y \in H \text{ there exists} \quad x \in D(A) \quad \text{such that } x - Ax = y\ . \tag{1.9}$$

These three conditions will be convenient for many examples. Condition (1.9) is called the **range condition**. The class of dissipative selfadjoint operators is particularly important for applications. Diagonal operators are the most simple examples of selfadjoint operators.

Example 1.4.8 (Selfadjoint diagonal operators)**.** *Let $\lambda = (\lambda_n)_{n\in\mathbb{N}}$ be a real sequence. Then the diagonal operator M_λ on ℓ^2 given by*

$$\begin{aligned} D(M_\lambda) &= \{x \in \ell^2 : (\lambda_n x_n)_{n\in\mathbb{N}} \in \ell^2\} \\ M_\lambda x &= (\lambda_n x_n)_{n\in\mathbb{N}} \end{aligned}$$

is selfadjoint. The operator M_λ is dissipative if and only if $\lambda_n \leq 0$ for $n \in \mathbb{N}$. Moreover, M_λ has compact resolvent if and only if $\lim_{n\to\infty} |\lambda_n| = -\infty$.

This is easy to check. The last assertion had been considered in Exercise 1.3.8.

We recall the spectral theorem for compact symmetric operators.

Proposition 1.4.9. *Let B be a compact, symmetric operator. Then H has an orthonormal basis which consists of eigenvectors of B.*

This result is contained in all standard texts on Functional Analysis (see e.g. [RS80, p. 203]).

For our purpose the following version for unbounded operators is important.

Theorem 1.4.10 (Spectral Theorem)**.** *Let A be a selfadjoint operator with compact resolvent. Then there exist an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of $H, \lambda_n \in \mathbb{R}$ such that $e_n \in D(A)$ and $Ae_n = \lambda_n e_n$. Moreover,* $\lim_{n\to\infty} |\lambda_n| = \infty$ *if* $\dim H = \infty$. *Finally, A is given by*

$$\begin{aligned} D(A) &= \{x \in H : (\lambda_n \,|\, x(e_n))_{n\in\mathbb{N}} \in \ell^2\} \\ Ax &= \sum_{n=1}^{\infty} \lambda_n (x|e_n) e_n\ . \end{aligned}$$

Proof. Since A has compact resolvent, by Proposition 1.3.3 there exists $\mu \in (0,\infty)\cap \rho(A)$. Then $R(\mu, A)$ is a compact and symmetric operator (as is easy to see). By Proposition 1.4.9 there exists an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of H and $\alpha_n \in \mathbb{R}$ such that $R(\mu, A)e_n = \alpha_n e_n$. Since $R(\mu, A)$ is injective one has $\alpha_n \neq 0$ $(n \in \mathbb{N})$. Hence $e_n \in D(A)$ and $e_n = \alpha_n(\mu - A)e_n$. It follows that $Ae_n = \lambda_n e_n$ where $\lambda_n = (\mu - \frac{1}{\alpha_n})$. Since $\lim_{n\to\infty} |\alpha_n| = 0$, one has $\lim_{n\to\infty} |\lambda_n| = \infty$. Let $x \in D(A)$. Then $(\lambda_n(x|e_n))_{n\in\mathbb{N}} = ((x|Ae_n))_{n\in\mathbb{N}} \in \ell^2$ and $Ax = \sum_{n=1}^{\infty}(Ax|e_n)e_n = \sum_{n=1}^{\infty} \lambda_n(x|e_n)e_n$. Conversely, assume that $x \in H$ such that $(\lambda_n(x|e_n))_{n\in\mathbb{N}} \in \ell^2$. Let $x_m = \sum_{n=1}^{m}(x|e_n)e_n$, $y_m = \sum_{n=1}^{m} \lambda_n(x|e_n)e_n$. Then $\lim_{m\to\infty} x_m = x$ and y_m converges as $m \to \infty$. Observe that $x_m \in D(A)$ and $Ax_m = y_m$. Since A is closed, it follows that $x \in D(A)$.

□

There is another way to present the Spectral Theorem. Denote by $U : H \to \ell^2$ the unitary operator given by $Ux = ((x|e_n))_{n\in\mathbb{N}}$. Then it follows directly from Theorem 1.4.10 that

$$UAU^{-1} = M_\lambda \tag{1.10}$$

(see Proposition 1.3.7 for the notation). We have obtained the following result.

Corollary 1.4.11 (Diagonalization)**.** *Let A be an operator on H. Suppose that* $\dim H = \infty$*. The following assertions are equivalent.*

(i) A is selfadjoint and has compact resolvent;

(ii) there exists a unitary operator $U : H \to \ell^2$ and a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $\lim_{n\to\infty} |\lambda_n| = \infty$ and

$$UAU^{-1} = M_\lambda \ .$$

We express (ii) by saying that A and M_λ are **unitarily equivalent**. Note that A is dissipative if and only if $\lambda_n \leq 0$ for all $n \in \mathbb{N}$.

The Spectral Theorem establishes a surprising metamorphoses. Frequently the operator A will be given as a differential operator. But identifying H with ℓ^2 via the unitary operator U, the operator A is transformed into the diagonal operator M_λ. This will be most convenient to prove properties of A.

Exercise 1.4.12. *Let A be an m-dissipative operator. Show that $D(A)$ is dense,* **Hint:** *By Hilbert space theory it suffices to show that $D(A)^\perp := \{y \in H : (x|y) = 0$ for all $x \in D(A)\} = 0$.*

Exercise 1.4.13. *Let A be a closed, dissipative operator. Assume that there exists $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda > 0$ and $(\lambda - A)$ has dense range. Show that A is m-dissipative.*

The **range** $R(A)$ of an operator A is by definition the space

$$R(A) := \{Ax : x \in D(A)\} \ . \tag{1.11}$$

Exercise 1.4.14. *Let A be an operator on H. Show that A is dissipative if and only if*

$$\|x - tAx\| \geq \|x\| \tag{1.12}$$

for all $x \in D(A), t > 0$.

Exercise 1.4.15. *Let A be a densely defined dissipative operator on a Hilbert space H. Show that A is closable.*
Hint: *This is not quite obvious. But there is a trick. Let $x_n \in D(A), x_n \to 0$, $Ax_n \to y$ $(n \to \infty)$. One has to show that $y = 0$. Consider first $z \in D(A)$ and apply (1.12) to $x_n + tz$ (where $t > 0$).*

1.5 The Spectral Theorem for general selfadjoint operators.

In this section we give a representation of selfadjoint operators which do not necessarily have a compact resolvent. Another simple example of a selfadjoint operator is obtained if we consider multiplication by a function in L^2 instead of a sequence in ℓ^2. We make this more precise.

Proposition 1.5.1 (Multiplication operators)**.** *Let (Y, Σ, μ) be a σ-finite measure space and let $m : Y \to \mathbb{R}$ be a measurable function. Define the operator A_m on $L^2(Y, \Sigma, \mu)$ by*

$$\begin{aligned} D(A_m) &= \{f \in L^2(Y, \Sigma, \mu) : mf \in L^2(Y, \Sigma, \mu)\} \\ A_m f &= mf \ . \end{aligned}$$

Then A_m is selfadjoint. This is not difficult to see.

Of course multiplication operators contain diagonal operators as special case: It sufficies to take $Y = \mathbb{N}$ and μ the counting measure. But they are more general. In fact, each diagonal operator has eigenvalues whereas a multiplication operator does not, in general (see Exercise 1.5.3). And indeed, multiplication operators are the most general selfadjoint operators as the following theorem shows.

Theorem 1.5.2 (Spectral Theorem: general form)**.** *Let A be a selfadjoint operator on a separable complex Hilbert space. Then there exist a finite measure space (Y, Σ, μ), a measurable function $m : Y \to \mathbb{R}$ and a unitary operator $U : H \to L^2(Y, \Sigma, \mu)$ such that*

$$UAU^{-1} = A_m \ . \tag{1.13}$$

We refer to Exercise 1.3.7 concerning the notion used in (1.13). For the proof of Theorem 2.3.4 we refer to [RS80, Theorem VIII.4, p. 260]. But in a series of exercises we will demonstrate the power of the Spectral Theorem.

Exercise 1.5.3 (No eigenvalues)**.** *Let* $Y = \mathbb{R}$ *with Lebesgue measure,* $m(y) = y \quad (y \in \mathbb{R}),\ H = L^2(\mathbb{R})$*. Then* $\sigma_p(A_m) = \emptyset$*. Deduce from this that* A_m *is not unitarily equivalent to a diagonal operator.*

Exercise 1.5.4 (Spectrum and essential image)**.** *Let* A_m *be a multiplication operator on* $L^2(Y, \Sigma, \mu)$ *(cf. Proposition 1.5.1.). Show that*

$$\sigma(A_m) = \text{ ess image } (m)$$

where the **essential image** *of* m *is defined by*

$$\text{ess image } (m) := \{\lambda \in \mathbb{C} : \forall\ \varepsilon > 0\ \mu(\{x : |m(x) - \lambda| \leq \varepsilon\}) > 0\}\ .$$

Deduce the following assertion from Exercise 1.5.4 and the Spectral Theorem.

Exercise 1.5.5 (Bounded selfadjoint operators)**.** *Let* A *be a selfadjoint operator on a separable Hilbert space. Then* A *is bounded if and only if* $\sigma(A)$ *is bounded. Moreover,* A *is a projection if and only if* $\sigma(A) \subset \{0, 1\}$*.*

In general, it is a most delicate matter to obtain information out of the spectrum of an operator. But selfadjoint operators are of special nature.

Chapter 2

Semigroups

In this chapter we give a short introduction to semigroups. We start with a preliminary technical section.

2.1 The vector valued Riemann integral

Let X be a Banach space, $-\infty < a < b < \infty$. By $C([a,b],X)$ we denote the space of all continuous functions on $[a,b]$ with values in E. Let $u \in C([a,b],X)$. Let π be a **partition** $a = t_0 < t_1 < \ldots < t_n = b$ of $[a,b]$ with **intermediate** points $s_i \in [t_{i-1}, t_i]$. By $|\pi| = \max_{i=1,\ldots n}(t_i - t_{i-1})$ we denote the **norm** of π and by

$$S(\pi,u) = \sum_{i=1}^{n} u(s_i)(t_i - t_{i-1})$$

the **Riemann sum** of u with respect to π. One shows as in the scalar case that

$$\int_a^b u(s)\mathrm{ds} := \lim_{|\pi|\to 0} S(\pi,u) \tag{2.1}$$

exists. If Y is another Banach space and $B \in \mathcal{L}(X,Y)$, then $BS(\pi,u) = S(\pi,Bu)$ where $Bu = B \circ u \in C([a,b],Y)$. It follows that

$$B\int_a^b u(s)\mathrm{ds} = \int_a^b Bu(s)\mathrm{ds}\ . \tag{2.2}$$

In particular,

$$\langle x', \int_a^b u(s)\mathrm{ds}\rangle = \int_a^b \langle x', u(s)\rangle \mathrm{ds}\ . \tag{2.3}$$

Now the Hahn-Banach theorem allows us to carry over the usual properties of scalar Riemann integral to the vector-valued case. For example, the mapping $u \mapsto$

$\int_a^b u(t)dt$ from $C([a,b],X)$ into X is linear. We also note that

$$\| \int_a^b u(s)\mathrm{ds}\| \leq \int_a^b \|u(s)\|\mathrm{ds} \tag{2.4}$$

as is easy to see.

Let A be a closed operator on X. Let $u \in C([a,b],D(A))$, where $D(A)$ is considered as a Banach space with the graph norm; i.e. $u \in C([a,b],X)$ such that $u(t) \in D(A)$ for all $t \in [a,b]$ and $Au \in C([a,b].X)$. Since $A \in \mathcal{L}(D(A),E)$, (2.2) implies that

$$A\int_a^b u(s)\mathrm{ds} = \int_a^b Au(s)\mathrm{ds}\ . \tag{2.5}$$

Exercise 2.1.1. *Let $f : [a,b] \to X$ be continuous (we write $f \in C([a,b];X)$). Let $F(t) = \int_a^t f(s)\mathrm{ds}$. Show that F is continuously differentiable (we write $F \in C^1([a,b];X)$) and $F'(t) = f(t)$ $(t \in [a,b])$.*

Exercise 2.1.2. *Let $u \in C^1([a,b];X)$. Show that $\int_a^t u'(s)\mathrm{ds} = u(t) - u(a)$.*

2.2 Semigroups

In this section we introduce semigroups and their generators. Let X be a Banach space.

Definition 2.2.1. *A C_0-**semigroup** is a mapping $T : \mathbb{R}_+ \to \mathcal{L}(X)$ such that*

a) $T(\cdot)x : \mathbb{R}_+ \to X$ continuous for all $x \in X$;

b) $T(t+s) = T(t)T(s) \quad (s,t \in \mathbb{R}_+)$;

c) $T(0) = I$.

It follows immediately from the definition that

$$T(t)T(s) = T(s)T(t) \text{ for all } t,s \geq 0\ . \tag{2.6}$$

Let $T : \mathbb{R}_+ \to \mathcal{L}(X)$ be a C_0-semigroup. We now define the **generator** of T.

Definition 2.2.2. *The generator A of T is the operator A on X given by*

$$\begin{aligned} D(A) &= \{x : \lim_{h\downarrow 0} \frac{1}{h}(T(h)x - x) \text{ exists in } X\} \\ Ax &= \lim_{h\downarrow 0} \frac{1}{h}(T(h)x - x)\ . \end{aligned}$$

We now investigate relations between the semigroup T and its generator A. One has

$$T(t)x \in D(A) \text{ and } AT(t)x = T(t)Ax \tag{2.7}$$

for all $x \in D(A), t \geq 0$. In fact, $\frac{1}{h}(T(h)T(t)x - T(t)x) = T(t)[\frac{1}{h}(T(h)x - x)] \to T(t)Ax$ $(h \downarrow 0)$. This shows in particular, that the right derivate of $T(t)x$ is $T(t)Ax$ if $x \in D(A)$. More is true.

Proposition 2.2.3. *Let $x \in D(A)$. Then $u(t) = T(t)x$ is the unique solution of the initial value problem*

$$\begin{cases} u \in C^1(\mathbb{R}_+, X) \ , \ u(t) \in D(A) \quad (t \geq 0) \ ; \\ \dot{u}(t) \quad = \quad Au(t) \quad (t \geq 0) \\ u(0) \quad = \quad x \ ; \end{cases} \tag{2.8}$$

Proof. Let $t > 0$. It follows from the uniform boundedness principle that T is bounded on $[0, t]$. Then

$$\begin{aligned} \frac{1}{-h}(T(t-h)x - T(t)x) &= T(t-h)[\frac{T(h)x - x}{h}] = T(t-h)[\frac{T(h)x - x}{h} - Ax] \\ &+ T(t-h)Ax \to T(t)Ax \quad (h \downarrow 0) \ . \end{aligned}$$

This shows that u is also leftdifferentiable and indeed a solution of the problem (2.8). Conversely, let v be another solution. Let $t > 0, w(s) = T(t-s)v(s)$. Then

$$\begin{aligned} \frac{d}{ds}w(s) &= -A(T(t-s)v(s)) + T(t-s)\dot{v}(s) \\ &= -T(t-s)Av(s) + T(t-s)Av(s) = 0 \ . \end{aligned}$$

It follows that w is constant. Hence $T(t)x = w(0) = w(t) = v(t)$. □

Proposition 2.2.3 shows why generators of C_0-semigroups are interesting. The initial value problem (2.2.7) has a unique solution for initial values x in the domain of the generator. Moreover, the orbit $T(\cdot)x$ is the solution. There is another way to describe the generator A.

Proposition 2.2.4. *Let $x, y \in X$. Then $x \in D(A)$ and $Ax = y$ if and only if*

$$\int_0^t T(s)y\mathrm{ds} = T(t)x - x \quad (t \geq 0) \ . \tag{2.9}$$

Proof. Assume (2.9). Then $\lim\limits_{t\downarrow 0} \frac{T(t)x - x}{t} = \lim\limits_{t\downarrow 0} \frac{1}{t}\int_0^t T(s)y\mathrm{ds} = y$. Conversely, let $x \in D(A)$, then $T(\cdot)x$ is the solution of (2.8). By the fundamental theorem of calculus, $T(t)x - x = \int_0^t \frac{d}{dt}T(s)x\mathrm{ds} = \int_0^t T(s)Ax\mathrm{ds}$. □

Corollary 2.2.5. *The operator A is closed.*

Proof. Let $x_n \in D(A), x_n \to x, y_n := Ax_n \to y$ $(n \to \infty)$. Then by (2.9),

$$\int_0^t T(s)y_n \mathrm{ds} = T(t)x_n - x_n \ .$$

Letting $n \to \infty$ shows that (2.9) holds. □

Let $x \in D(A)$. Since A is closed it follows from (2.9) and (2.5) that $\int_0^t T(s)x\mathrm{ds} \in D(A)$ and

$$\begin{aligned} A\int_0^t T(s)x\mathrm{ds} &= \int_0^t AT(s)x\mathrm{ds} = \int_0^t T(s)Ax\mathrm{ds} \\ &= T(t)x - x \end{aligned}$$

for all $t \geq 0$. This identity remains valid for all $x \in X$.

Proposition 2.2.6. *Let $x \in X, t \geq 0$. Then $\int_0^t T(s)x\mathrm{ds} \in D(A)$ and*

$$A\int_0^t T(s)x\mathrm{ds} = T(t)x - x \ . \tag{2.10}$$

Proof. In fact,

$$\begin{aligned} &\frac{1}{h}\{T(h)\int_0^t T(s)x\mathrm{ds} - \int_0^t T(s)x\mathrm{ds}\} \\ &= \frac{1}{h}(\int_0^t T(s+h)x\mathrm{ds} - \int_0^t T(s)x\mathrm{ds}) \\ &= \frac{1}{h}(\int_h^{t+h} T(s)x\mathrm{ds} - \int_0^t T(s)x\mathrm{ds}) \\ &= \frac{1}{h}(\int_t^{t+h} T(s)x\mathrm{ds} - \int_0^h T(s)x\mathrm{ds}) \to T(t)x - x \end{aligned}$$

as $h \downarrow 0$.

□

Corollary 2.2.7. *The domain of A is dense in X.*

Proof. Let $x \in X$. Then $\frac{1}{t}\int_0^t T(s)x\mathrm{ds} \in D(A)$ and $\lim_{t\downarrow 0} \frac{1}{t}\int_0^t T(s)x\mathrm{ds} = x$. □

Since A is closed, the space $D(A)$ is a Banach space with the graph norm. The properties shown above imply that

$$T(\cdot)x \in C(\mathbb{R}_+, D(A)) \cap C^1(\mathbb{R}_+, X)$$

for all $x \in D(A)$. If $x \in X$, then there exist $x_n \in D(A)$ such that $x_n \to x$ $(n \to \infty)$. Then $T(t)x_n$ converges to $T(t)x$ as $n \to \infty$. Since $T(\cdot)x_n$ is a solution of the initial value problem (2.8) with initial value x_n we may consider $u(t) = T(t)x$ as a mild solution of the abstract Cauchy problem

$$(ACP)\begin{cases} \dot{u}(t) &= Au(t) \quad (t \geq 0) \\ u(0) &= x_0 \ . \end{cases}$$

Exercise 2.2.8. *Let A and B be operators such that $A \subset B$ (see Definition 1.1.5). If $\rho(A) \cap \rho(B) \neq \emptyset$, then $A = B$.*

Exercise 2.2.9. *Let S and T be C_0-semigroups with generators A and B, respectively. Assume that $A \subset B$. Show that $A = B$.*

Exercise 2.2.10. *Let $T : \mathbb{R}_+ \to \mathcal{L}(X)$ be a mapping such that*

a) $\lim_{t \downarrow 0} T(t)x = x$ *for all* $x \in X$;

b) $T(t+s) = T(t)T(s)$ $(t, s \geq 0)$.

Show that T is a C_0-semigroup.

Exercise 2.2.11 (Rescaling)**.** *Let A be the generator of a C_0-semigroup T.*

a) Let $\omega \in \mathbb{C}$, $S(t) = e^{-\omega t}T(t)$. Show that S is a C_0-semigroup and $A - \omega I$ its generator.

b) Let $\alpha > 0$, $S(t) = T(\alpha t)$. Show that S is a C_0-semigroup and αA its generator.

2.3 Differentiable semigroups

In this section we consider semigroups which are regular, in the sense that orbits are of class C^∞ on $(0, \infty)$. Particular cases are selfadjoint semigroups.

Let E be a Banach space. If A is a closed operator, we define the operator A^m inductively: $A^1 := A$; and

$$\begin{aligned} D(A^{k+1}) &= \{x \in D(A) : Ax \in D(A^k)\} \\ A^{k+1}x &= A^k(Ax) \ . \end{aligned}$$

Then it is easy to see that $D(A^k)$ is a Banach space for the norm

$$\|x\|_{D(A^k)} := \|x\| + \|Ax\| + \ldots + \|A^k x\| \ . \tag{2.11}$$

The following properties are easy to show by induction and will be used frequently.

Exercise 2.3.1. *Let A be a closed operator. Then*

a) $D(A^{k+1}) \subset D(A^k)$ $(k \in \mathbb{N})$;

b) $A^\ell D(A^k) \subset D(A^{k-\ell})$ $(k > \ell \geq 1)$*;*

c) for $x \in D(A^{k+\ell})$, $A^k x \in D(A^\ell)$ *and* $A^{\ell+k}x = A^\ell(A^k x)$ $(\ell, k \in \mathbb{N})$.

Definition 2.3.2. *A* C_0*-semigroup* T *on a Banach space* E *is called* **differentiable** *if*

$$T(t)x \in D(A) \text{ for all } t > 0, x \in E$$

where A *is the generator of* T.

The orbits of differentiable semigroups are regular in time, i.e. $T(\cdot)x \in C^\infty((0,\infty);E)$. But we will also see that $T(t)x \in D(A^k)$ for all $k \in \mathbb{N}$, $t > 0$, $x \in E$. Frequently $D(A^k)$ is a better space than E. For example, if A is a differentiable operator on $L^2(\Omega)$ ($\Omega \subset \mathbb{R}^n$ open), then $D(A^k)$ may consist of differentiable functions if k is sufficiently large. So the following proposition says that differentiable semigroups are regular in time and improve regularity in space.

Proposition 2.3.3. *Let* T *be a differentiable* C_0*-semigroup. Then*

$$T(\cdot)x \in C^\infty((0,\infty); D(A^k))$$

for all $x \in X$ *and all* $k \in \mathbb{N}$*. Here* $D(A^k)$ *is considered as a Banach space with the graph norm (2.11).*

Proof. 1. We show inductively that $T(t)x \in D(A^k)$ for all $t > 0$, $x \in X$. This is true for $k = 1$. Assume that it holds for $k \in \mathbb{N}$. Then $A^k T(t)x = A^k T(t/2)T(t/2)x = T(t/2)A^k T(t/2)x \in D(A)$ by hypothesis. Thus $T(t)x \in D(A^{k+1})$.
2. Let $k \in \mathbb{N}_0$. We show that $T(\cdot)x \in C^m((0,\infty), D(A^k))$ for all $m \in \mathbb{N}_0$ and all $x \in X$.
a) Let $m = 0$. We have to show that $A^\ell T(\cdot)x \in C((0,\infty);E)$ for $0 \leq \ell \leq k$. But for $t > 0$, $A^\ell T(t+h)x - A^\ell T(t)x = T(t/2+h)A^\ell T(t/2)x - T(t/2)A^\ell T(t/2)x \to 0$ $(h \to 0)$ in X.
b) Assume that the assertion is true for $m \in \mathbb{N}_0$. Let $t_0 > 0$. Then

$$\begin{aligned} \frac{d}{dt}T(t)x &= \frac{d}{dt}T(t-t_0)T(t_0)x \\ &= T(t-t_0)AT(t_0)x \ , \end{aligned}$$

and $T(\cdot - t_0)AT(t_0)x \in C^m((t_0,\infty), D(A^k))$ by the inductive hypothesis. Hence $T(\cdot)x \in C^{m+1}((0,\infty); D(A^k))$. □

It is interesting to reformulate the preceding result in terms of well-posedness of the Cauchy problem. Whereas for general semigroups we merely have mild solutions if the initial value is not in the domain, differentiable semigroups always give us classical solutions.

Theorem 2.3.4. *Let A be the generator of a differentiable C_0-semigroup. Let $x \in E$. Then the problem*

$$\left\{ \begin{array}{l} u \in C^1((0,\infty);X) \cap C([0,\infty);X) \\ u(t) \in D(A) \text{ for all } t > 0 \\ u'(t) = Au(t) \quad (t > 0) \\ u(0) = x \end{array} \right\} \tag{2.12}$$

has a unique solution u. Moreover, $u \in C^\infty((0,\infty), D(A^k))$ for all $k \in \mathbb{N}_0$.

Proof. In view of Proposition 2.3.3 only uniqueness remains to be shown. Substracting two solutions we obtain a solution u of (2.12) with $x = 0$. We have to show that $u(t) = 0$ for all $t > 0$. For $0 \leq \varepsilon \leq t$ let $v_\varepsilon(t) = \int_\varepsilon^t u(s)ds$. Then by (2.5), for $0 < \varepsilon < t$, $v_\varepsilon(t) \in D(A)$ and $Av_\varepsilon(t) = \int_\varepsilon^t Au(s)ds = \int_\varepsilon^t \dot{u}(s)ds = u(t) - u(\varepsilon)$. Since A is closed, it follows that $v_0(t) \in D(A)$ and $Av_0(t) = u(t) = \dot{v}_0(t)$. Now it follows from Proposition 2.2.3 that $v_0 \equiv 0$. Consequently, $u(t) = \dot{v}_0(t) = 0$ $(t > 0)$. ☐

Exercise 2.3.5. *Let A be a closed operator and let $k \in \mathbb{N}$.*

a) Show that $D(A^k)$ is a Banach space for the norm (2.11).

b) Let $\lambda \in \rho(A)$. Show that

$$\|x\|_k := \|(\lambda - A)^k x\|$$

defines a norm which is equivalent to (2.11).

The following is an improvement of Proposition 2.3.2

Exercise 2.3.6. *Let T be a differentiable C_0-semigroup.*
a) Show that $T \in C((0,\infty); \mathcal{L}(X))$.
b) Show that $T \in C^\infty((0,\infty) : \mathcal{L}(X, D(A^k))$ for all $k \in \mathbb{N}$.

Exercise 2.3.7. *Let A be the generator of a differentiable C_0-semigroup.*
a) Show that A has compact resolvent if and only if $T(t)$ is compact for all $t > 0$.
b) In that case $T(t)$ is even compact as operator from X into $D(A^k)$ for all $k \in \mathbb{N}$, $t > 0$ where $D(A^k)$ carries the norm (2.11)).

Hint: Rescaling one can assume that A is invertible. Consider the operator $S(t) = \int_0^t T(s)ds$. Observe that $S(t) = A^{-1}(T(t) - I)$ and use Exercise 2.3.5 a).

2.4 Selfadjoint semigroups

Selfadjoint operators can be transformed into diagonal operators or multiplication operators by the spectral theorem. After this transformation one can write down explicitly the corresponding semigroup. We obtain the most simple C_0-semigroups with unbounded generator. Still we will see in the next two chapters that many concrete examples are of this form. We first consider the case where A has compact resolvent. In fact, in that case, the spectral theorem is particularly easy to prove and the operator is transformed into a diagonal operator. In addition, our prototype example is of this type, namely the Laplacian with Dirichlet boundary conditions on a bounded open set. So the functional analytic tools needed for this important example are particularly simple.

Let H be a complex, separable Hilbert space and let A be a selfadjoint, dissipative operator on H. Assume first that A has compact resolvent. Then, up to unitary equivalence, we can assume that

$$H = \ell^2 \ , \ Ax = -(\lambda_n x_n)_{n\in\mathbb{N}}$$

where $\lambda_n \in \mathbb{R}_+$, $\lim_{n\to\infty} \lambda_n = \infty$ and

$$D(A) = \{x \in \ell^2 : (\lambda_n x_n)_{n\in\mathbb{N}} \in \ell^2\} \ .$$

Define $T(t) \in \mathcal{L}(\ell^2)$ by

$$T(t)x = (e^{-\lambda_n t} x_n)_{n\in\mathbb{N}} \ . \tag{2.13}$$

Then $T(t)$ is a compact, selfadjoint operator and $\|T(t)\| \leq 1$. It is easy to see that $T = (T(t))_{t\geq 0}$ is a differentiable C_0-semigroup and A its generator.

In the general case, if the resolvent is not necessarily compact, then after a unitary transformation we can assume that

$$\begin{aligned} H &= L^2(X, \Sigma, \mu) \\ Af &= -mf \\ D(A) &= \{f \in H : m \cdot f \in H\} \end{aligned}$$

where (X, Σ, μ) is a finite measure space and $m : X \to \mathbb{R}_+$ a measurable function. Now it is easy to see that

$$T(t)f = e^{-tm} f \tag{2.14}$$

defines a differentiable C_0-semigroup of selfadjoint operators. Moreover, $\|T(t)\| \leq 1$. We have proved the following result.

Theorem 2.4.1. *Let A be a selfadjoint, dissipative operator. Then A generates a differentiable C_0-semigroup T of contractive, selfadjoint operators. If A has compact resolvent, then $T(t)$ is compact for all $t > 0$.*

Applying Theorem 2.4.1 it is frequently useful to have the concrete representation (2.13), (2.14) in mind which is valid after a unitary transformation in virtue of the spectral theorem. It shows for example the following, simple result on asymptotics.

Exercise 2.4.2. *Let A be a selfadjoint, dissipative operator with compact resolvent. Assume that* $\ker A = \{0\}$. *Then there exists $\varepsilon > 0$ such that*

$$\|T(t)\| \le e^{-\varepsilon t} \quad (t \ge 0) \ . \tag{2.15}$$

Exercise 2.4.3. *Give a detailed proof of Theorem 2.4.1 in the case where A has compact resolvent.*

Exercise 2.4.4. *Give a detailed proof of Theorem 2.4.1 in the general case.*

Exercise 2.4.5. *Let T be a C_0-semigroup of selfadjoint operators. Show that the generator of A is selfadjoint and*

$$(Ax|x) \le \omega \|x\|^2 \text{ for all } x \in D(A) \tag{2.16}$$

and some $\omega \in \mathbb{R}$. Deduce that

$$\|T(t)\| \le e^{\omega t} \quad (t \ge 0) \ . \tag{2.17}$$

Hint: Show that A is selfadjoint. Use the spectral theorem in order to prove (2.16) and (2.17).

Exercise 2.4.6 (Euler's formula)**.** *Let A be a dissipative, selfadjoint operator and T the C_0-semigroup generated by A. Show that*

$$\lim_{n\to\infty} (I - \frac{t}{n} A)^{-n} x = T(t)x \tag{2.18}$$

for all $x \in H$, $t > 0$.

Formula (2.18) is frequently useful to deduce properties of T from the resolvent. It is **Euler's formula** *in the scalar case and we still use this name here. Euler's formula is true for all C_0-semigroups and can be used to prove the generation theorem of Hille-Yosida (see Chapter 5).*

Chapter 3

The Laplacian in Dimension 1

In this chapter we consider the Laplacian on an interval with diverse boundary conditions. Only few analytical tools are needed and Sobolev spaces are easy in 1 dimension. Still the concepts and ideas are typical and they will be applied again in Chapter 4 where higher dimensions will be considered.

Let us make more precise what we intend to do. For this we consider a prototype example, the **heat equation** on an interval

$$(HE)\begin{cases} u_t &= u_{xx} & (t>0, x\in(0,1)) \\ u(t,1) &= u(t,0)=0 & (t>0) \\ u(0,x) &= u_0(x) & x\in[0,1]\ . \end{cases}$$

Given is a continuous function $u_0:[0,1]\to\mathbb{R}$ such that $u_0(0)=u_0(1)=0$. This is the **initial value**. Physically it may be interpreted as a heat distribution on a wire. The given initial temperature at the point $x\in[0,1]$ is $u_0(x)$. We look for a function $u:\mathbb{R}_+\times[0,1]\to\mathbb{R}$, differentiable on $(0,\infty)\times(0,1)$ and satisfying (HE). The first line of (HE) is the law of propagation. Here $u_t=\frac{\partial u}{\partial t}$ and $u_{xx}=\frac{\partial^2 u}{\partial^2 x}$ are the partial derivatives. The second line is the **boundary condition**; Dirichlet boundary condition in this case. It says that the wire is kept at temperature 0 at the end points. The solution $u(t,x)$ is interpreted as the temperature at the time $t>0$ at the point $x\in[0,1]$. How can we solve sucht an initial-value/boundary-value problem?

The basic idea is to transform the partial differential equation (HE) into an ordinary differential equation in the following way. Let $H=L^2(0,1)$ with Lebesgue measure. Assume that $u:\mathbb{R}_+\to H$ is continuous and differentiable on $(0,\infty)$. Consider the operator A on H given by

$$\begin{aligned} D(A) &:= \{f\in C^2[0,1]: f(0)=f(1)=0\}\ , \\ Af &= f''\ . \end{aligned}$$

Assume that $u(t) \in D(A)$ for all $t > 0$ and

$$(CP) \begin{cases} \dot{u}(t) &= Au(t) \quad (t > 0) \\ u(0) &= u_0 \ . \end{cases}$$

This is a Cauchy problem as considered before. Then, letting $u(t,x) = u(t)(x)$ $(t > 0, x \in [0,1])$ we obtain a solution of (HE). Thus, we transformed (HE) into the ordinary differential equation (CP). The boundary condition is incorporated into the definition of the opertor A (or more precisely its domain). The prize we pay for this approach is that instead of the scalar-valued functions considered in (HE) we now have to treat functions with values in an infinite dimensional vector space. It is natural that this space is a function space over $[0,1]$. In order to apply the spectral theorem and our results from Chapter 1 and 2, a Hilbert space is needed. But now the difficulty is that the operator A as we defined it above is not closed. The domain, consisting of twice differentiable functions in the classical sense, is too small. This leads us to the concept of weak derivatives and Sobolev spaces. Indeed, Sobolev spaces will be the right domain for differential operators on L^2-spaces. Things are quite easy on intervals and we consider this case systematically in this chapter.

Exercise: Show that the operator A on $L^2(0,1)$ defined above is not closed.

3.1 Sobolev spaces on an interval

Let $-\infty < a < b < \infty$. We consider the Hilbert space $L^2(a,b)$ defined with respect to Lebesgue measure. By $C[a,b]$, $C^1[a,b]$ we denote the spaces of all continuous functions on $[a,b]$ and of all continuously differentiable functions on $[a,b]$, respectively. By $C_c^1(a,b)$ we denote the space of all functions $f \in C^1[a,b]$ with compact support in (a,b); i.e. such that $f(x) = 0$ for all $x \in [a, a+\varepsilon] \cup [b-\varepsilon, b]$ and some $\varepsilon > 0$. We recall from integration theory that

$$C_c^1(a,b) \text{ is dense in } L^2(a,b) \ . \tag{3.1}$$

If $f \in C^1[a,b]$, then integration by parts gives

$$-\int_a^b f\varphi' dx = \int_a^b f'\varphi dx \quad (\varphi \in C_c^1(a,b)) \ . \tag{3.2}$$

This leads us to the following definition. Let $f \in L^2(a,b)$. A function $f' \in L^2(a,b)$ is called a **weak derivative** of f if (3.2) is satisfied. Since $C_c^1(a,b)$ is dense in $L^2(a,b)$ there exists at most one weak derivative of f. Now we define the **first Sobolev space** by

$$H^1(a,b) = \{f \in L^2(a,b) : f \text{ has a weak derivative } f' \in L^2(a,b)\} \ .$$

Exercise 3.1.1. *Let* $f(x) = |x|$. *Show that* $f \in H^1(-1,1)$ *and* $f'(x) = \operatorname{sgn} x$, *where*

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \ . \end{cases}$$

Proposition 3.1.2. *The space* $H^1(a,b)$ *is a separable Hilbert space for the scalar product*

$$(f|g)_{H^1} := (f|g)_{L^2} + (f'|g')_{L^2}$$

where $(f|g)_{L^2} = \int_a^b f\bar{g}dx$ *denotes the scalar product in* $L^2(a,b)$.

Proof. The space $H = L^2(a,b) \oplus L^2(a,b)$ is a separable Hilbert space for the norm $\|(f,g)\|_H^2 = \int_a^b |f|^2dx + \int_a^b |g|^2dx$. The mapping $j : H^1(a,b) \to H$ given by $j(f) = (f,f')$ is linear and isometric. It suffices to show that the image H of j is closed. Let $(f,g) \in H$ such that $f = \lim_{n\to\infty} f_n$ und $g = \lim_{n\to\infty} f_n'$ in $L^2(a,b)$ where $f_n \in H^1(a,b)$. Let $\varphi \in C_c^1(a,b)$. Then

$$\begin{aligned} -\int_a^b f\varphi' dx &= \lim_{n\to\infty} -\int_a^b f_n\varphi' dx \\ &= \lim_{n\to\infty} \int_a^b f_n'\varphi dx \\ &= \int_a^b g\varphi dx \end{aligned}$$

by the Dominated Convergence Theorem. Hence g is the weak derivative of f; i.e. $f \in H^1(a,b)$ and $j(f) = (f,g)$. □

Lemma 3.1.3. *Let* $f \in H^1(a,b)$ *such that* $f' = 0$. *Then* f *is constant.*

Proof. Let $\psi \in C_c(a,b)$ such that $\int_a^b \psi dx = 1$. Let $w \in C_c(a,b)$. Then there exists $\varphi \in C_c^1(a,b)$ such that

$$\varphi' = w - (\int_a^b wdx)\psi \ .$$

In fact, one has $v := w - (\int_a^b wdx)\psi \in C_c(a,b)$ and $\int_a^b vdx = 0$. Define $\varphi(x) = \int_a^x v(y)dy$.

It follows from the assumption that

$$\begin{aligned} 0 &= \int_a^b f\varphi' \\ &= \int_a^b fw dx - \int_a^b w dx \int_a^b f\psi dx \\ &= \int_a^b (f(x) - \int_a^b f\psi dy)\ w(x) dx\ . \end{aligned}$$

Since this holds for all $w \in C_c(a,b)$ it follows from (3.1) that

$$\int_a^b (f(x) - \int_a^b f\psi dy) w(x) \quad dx = 0 \text{ for all } w \in L^2(a,b)\ .$$

Hence $f(x) - \int_a^b f\psi dy = 0$ $x-$a.e. □

Theorem 3.1.4. *a) Let $g \in L^2(a,b)$, $c \in \mathbb{C}$. Let $f(x) = c + \int_a^x g(y)dy$ $(x \in (a,b))$. Then $f \in H^1(a,b)$ and $f' = g$.*
b) Conversely, let $f \in H^1(a,b)$. Then there exists $c \in \mathbb{C}$ such that

$$f(x) = c + \int_a^x f'(y)dy \qquad\qquad \text{a.e.}$$

Proof. a) Let $\varphi \in C_c^1(a,b)$. Then by Fubini's theorem

$$\begin{aligned} -\int_a^b f(x)\varphi'(x)dx &= -\int_a^b \int_a^x g(y)dy\varphi'(x)dx - c\int_a^b \varphi'(x)dx \\ &= -\int_a^b \int_y^b \varphi'(x)dx g(y)dy \\ &= \int_a^b \varphi(y)g(y)dy\ . \end{aligned}$$

Thus g is the weak derivative of f.
b) Let $f \in H^1(a,b)$, $w(x) = f(x) - \int_a^x f'(y)dy$. Then by a), $w \in H^1(a,b)$ and $w' = 0$. Hence w is constant by Lemma 3.1.3. □

Corollary 3.1.5. *One has*

$$H^1(a,b) \subset C[a,b]\ .$$

Proof. Let $f \in H^1(a,b)$. Then by Theorem 3.1.4, $f(x) = c + \int_a^x f'(y)dy$ for some constant and almost all $x \in (a,b)$. It follows from the dominated convergence theorem that the right hand side is continuous. □

By definition, two functions in $L^2(a,b)$ are identified if they coincide almost everywhere. Moreover, if two continuous functions coincide almost everywhere, then they coincide everywhere. So Corollary 3.1.5 says more precisely, that for each f there exists exactly one continuous function g which coincides with f almost everywhere. By Theorem 3.1.4 we may choose the representive such that

$$f(x) = c + \int_a^x f'(y)dy \quad (x \in [a, b])$$

which we will always do. Thus $c = f(0)$. Recall that the space $C[a, b]$ is a Banach space for the uniform norm

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)| \ .$$

It follows immediately from the closed graph theorem that the injection of $H^1(a, b)$ into $C[a, b]$ is continuous (where each space carries its proper norm). More is true:

Theorem 3.1.6. *The injection of*

$$\begin{array}{rcl} H^1(a,b) & \text{into} & C[a,b] \ \textit{and} \\ \textit{of}\ H^1(a,b) & \text{into} & L^2(a,b) \end{array}$$

are compact.

Proof. a) Let $B = \{f \in H^1(a, b) : \|f\|_{H^1} \leq 1\}$ be the unit ball of $H^1(a, b)$. We have to show that B is relatively compact in $C[a, b]$. For $f \in B$ we have

$$\begin{aligned} |f(x) - f(y) \ | &= | \int_x^y f'(y)dy| \\ &\leq \|f'\|_{L^2(a,b)}|x - y|^{1/2} \\ &\leq |x - y|^{1/2} \end{aligned}$$

by Hölder's inequality. This shows that B is equicontinuous. Since the injection is continuous, B is bounded in $C[a, b]$. Now it follows from the Arzela-Ascoli theorem that B is relatively compact in $C[a, b]$.
b) Since the injection of $C[a, b]$ into $L^2(a, b)$ is continuous, and the composition of a bounded operator and a compact operator is compact, the second assertion follows from the first. □

Next we establish the usual rule for derivatives of products also in the weak sense.

Proposition 3.1.7. *Let $f, g \in H^1(a, b)$. Then*

a) $fg \in H^1(a, b)$ and $(fg)' = f' \cdot g + f \cdot g'$;

b) $\int_a^b fg'dx = f(b)g(b) - f(a)g(a) - \int_a^b f'gdx$.

Proof. 1. By Fubini's theorem we have

$$\begin{aligned}
\int_a^b f\cdot g' dx &= \int_a^b (f(a)+\int_a^x f'(y)dy)g'(x)dx \\
&= f(a)(g(b)-g(a)) + \int_a^b \int_y^b g'(x)dx f'(y)dy \\
&= f(a)(g(b)-g(a)) + \int_a^b (g(b)-g(y))f'(y)dy \\
&= f(a)(g(b)-g(a)) + g(b)(f(b)-f(a)) \\
&\quad - \int_a^b g(y)f'(y)dy \\
&= f\cdot g|_a^b - \int_a^b g(y)f'(y)dy \ .
\end{aligned}$$

This proves b).
2. Replacing b by x in 1. we obtain

$$\begin{aligned}
\int_a^x f(y)g'(y)dy &= f(x)g(x) - f(a)g(a) \\
&\quad - \int_a^x f'(y)g(y)dy \ .
\end{aligned}$$

$$\text{Hence } f(x)\cdot g(x) = f(a)g(a) + \int_a^x \{f(y)g'(y)+f'(y)g(y)\}dy \ .$$

Now Theorem 3.1.4 implies assertion a). □

We define the higher order Sobolev spaces inductively as follows

$$\begin{aligned}
H^{k+1}(a,b) &= \{f\in H^1(a,b) : f'\in H^k(a,b)\} \\
f^{(k+1)} &= f'^{(k)} \quad (f\in H^{k+1}(a,b)) \ .
\end{aligned}$$

$(k=1,2,\ldots)$. Then $H^k(a,b)$ is a Hilbert space for the norm

$$\|f\|^2_{H^k} = \|f\|^2_{L^2} + \sum_{m=1}^k \|f^{(m)}\|^2_{L^2} \ .$$

Exercise 3.1.8. *Give a proof of the above statement.*

Exercise 3.1.9. *Let X, Y be two Banach spaces such that $X\subset Y\subset L^2(0,1)$ and such that $f_n\to f$ $(n\to\infty)$ in X or in Y implies that $f_{n_k}(x)\to f(x)$ a.e. for some subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$. Show that the injection of X into Y is continuous. Use the Closed Graph Theorem.*

Exercise 3.1.10. *Consider the Banach space $C^k[a,b]$ of all k-times continuously differentiable functions with the norm*

$$\|f\|_{c^k} = \max_{m=1,\dots k} \|f^{(m)}\|_\infty \ .$$

a) Show that $H^{k+1}(a,b) \subset C^k[a,b]$ $k = 1, 2 \dots$).

b) Show that the injection in a) is continuous (use Exercise 3.1.9).

c) Show that the injection in a) is compact.

Exercise 3.1.11. *Show that $H^1(a,b)$ is a Banach algebra (i.e. $f_n \to f$, $g_n \to g$ in $H^1(a,b)$ implies $f_n \cdot g_n \to f \cdot g$ in $H^1(a,b)$).*

Exercise 3.1.12. *Let $a < c < b$. Let $f_1 \in H^1(a,c)$, $f_2 \in H^1(c,b)$ such that $f_1(c) = f_2(c)$. Define $f : (a,b) \to \mathbb{C}$ by*

$$f(x) = \begin{cases} f_1(x) & \text{if} \quad x \in (a,c] \ , \\ f_2(x) & \text{if} \quad x \in (c,b) \ . \end{cases}$$

Show that $f \in H^1(a,b)$.

Exercise 3.1.13. *a) Let $f \in H^k(a,b)$ where $k \in \mathbb{N}$. Assume that*

$$\sum_{m=0}^{k} a_m f^{(m)} = 0$$

where $a_m \in \mathbb{C}$, $m = 1, \dots, k$; $a_k \neq 0$. Show that $f \in C^k[a,b]$.
b) Let $f \in H^2(a,b)$ such that $f'' = 0$. Show that $f(x) = \alpha x + \beta$ $(x \in (a,b))$ for some $\alpha, \beta \in \mathbb{C}$.
c) Let $f \in H^2(a,b)$ such that $f'' = -\lambda f$ where $\lambda > 0$. Show that $f(x) = \alpha \cos(\sqrt{\lambda} \cdot x) + \beta \sin(\sqrt{\lambda} \cdot x)$.

3.2 The Laplacian with Dirichlet and Neumann boundary conditions

Now we define the Laplacian on $L^2(a,b)$ where $-\infty < a < b < \infty$. At first we consider Dirichlet boundary conditions. Recall that $H^1(a,b) \subset C[a,b]$. We define

$$H_0^1(a,b) = \{f \in H^1(a,b) : f(a) = f(b) = 0\} \ .$$

Since the embedding of $H^1(a,b)$ into $C[a,b]$ is continuous, it follows that $H_0^1(a,b)$ is a Hilbert space for the scalar product defined in Proposition 3.1.2. In the following example the range conditions is a simple consequence of the Riesz-Fréchet lemma (saying that each continuous linear form φ on a Hilbert space H is of the form $\varphi(x) = (x|y)$ for a unique $y \in H$).

Theorem 3.2.1 (The Dirichlet Laplacian). *Define the operator A on $L^2(a,b)$ by* $D(A) = H_0^1(a,b) \cap H^2(a,b)$

$$Af = f'' \ .$$

Then A is a dissipative selfadjoint operator with compact resolvent.

Proof. a) Let $f, g \in D(A)$. Then

$$\begin{aligned} (Af|g)_{L^2} &= \int_a^b f''\bar{g}dx = f'\bar{g}|_a^b - \int_a^b f'\bar{g}' \\ &= -\int_a^b f'\bar{g}' \\ &= (f|Ag)_{L^2} \end{aligned}$$

since $g(b) = g(a) = f(a) = f(b) = 0$. Thus A is symmetric and $(Af|f) = -\int_a^b |f'|^2 \leq 0$. So A is dissipative.
b) We prove the range condition. Let $g \in L^2(a,b)$. Then $\Phi(\varphi) = \int_a^b \varphi\bar{g}dx$ defines a continuous linear form on $H_0^1(a,b)$. By the Riesz-Fréchet lemma there exists a unique $f \in H_0^1(a,b)$ such that

$$\int_a^b \varphi\bar{f}dx + \int_a^b \varphi'\bar{f}'dx = (\varphi|f)_{H^1} = \int_a^b \varphi\bar{g}dx \tag{3.3}$$

for all $\varphi \in H_0^1(0,1)$. Replacing φ by $\bar{\varphi}$ and taking the complex conjugate of (3.3) we deduce that

$$-\int_a^b \varphi' f' = \int_a^b \varphi(f-g)dx$$

for all $\varphi \in C_c^1(a,b)$. This means that $(f-g)$ is the weak derivative of f'. Hence $f \in H^2(a,b)$ and $f'' = f - g$. Thus $f \in D(A)$ and $f - Af = g$.
c) It follows from Exercise 3.1.9 that the injection of $D(A)$ into $H^1(a,b)$ is continuous. By Theorem 3.1.6 the injection of $H^1(a,b)$ into $L^2(a,b)$ is compact. Consequently, the injection of $D(A)$ into $L^2(a,b)$ is compact, as composition of a compact and a bounded operator. This means that A has compact resolvent by Exercise 1.3.2. □

We call the operator A defined in Theorem 2.1 the **Dirichlet Laplacian** on $L^2(a,b)$, or more precisely, the **Laplacian with Dirichlet boundary conditions**, and denote A by $\Delta^D_{(a,b)}$ or Δ^D if the interval is fixed. By Theorem 2.4.1, Δ^D generates a contraction semigroup T^D on $L^2(a,b)$. Moreover, each $T^D(t)$ is compact and selfadjoint ($t > 0$). Later we will study the Cauchy problem governed by Δ^D in more detail (see Section 3.4).

Next we consider Neumann boundary conditions. Recall that $H^2(a,b) \subset C^1[a,b]$. So the following definition makes sense.

Theorem 3.2.2 (the Neumann Laplacian). *Define the operator* Δ^N *on* $L^2(a,b)$ *by*

$$\begin{aligned} D(\Delta^N) &= \{f \in H^2(a,b) : f'(a) = f'(b) = 0\} \ , \\ \Delta^N f &= f'' \ . \end{aligned}$$

Then Δ^N *is selfadjoint, dissipative and has compact resolvent. We call* Δ^N *the* **Neumann Laplacian**.

Proof. a) Let $f, g \in D(\Delta^N)$. Then by Proposition 3.1.7

$$\begin{aligned} (Af|g) &= \int_a^b f''\bar{g}dx \\ &= f'(b)\overline{g(b)} - f'(a)\overline{g(a)} - \int_a^b f'\bar{g}'dx \\ &= -\int_a^b f'\bar{g}'dx = (\int_a^b g'\bar{f}'dx)^- \\ &= \overline{(Ag|f)} = (f|Ag) \ . \end{aligned}$$

Thus A is symmetric and $(Af|f) = -\int_a^b |f'|^2dx \leq 0$. Hence A is dissipative.
b) We show that $I - A$ is surjective. Let $g \in L^2(a,b)$. Then $G(\varphi) = \int_a^b \varphi g dx$ defines a continuous linear form on $H^1(a,b)$. By the lemma of Riesz-Fréchet there exists $f \in H^1(a,b)$ such that

$$\int_a^b \varphi f dx + \int_a^b \varphi' f' dx = (\varphi|\bar{f})_{H^1} = G(\varphi) = \int_a^b \varphi g dx \tag{3.4}$$

for all $\varphi \in H^1(a,b)$. In particular,

$$-\int_a^b \varphi' f' dx = \int_a^b \varphi(f-g)dx \quad (\varphi \in C_c^1(a,b)) \ .$$

It follows that $f' \in H^1(a,b)$ and $f'' = f - g$. Hence $f \in H^2(a,b)$ and $f - f'' = g$. It remains to show that $f'(a) = f'(b) = 0$. Let $\varphi \in H^1(a,b)$ such that $\varphi(a) = 1$ and $\varphi(b) = 0$. Since $g = f - f''$, it follows from (3.4) that

$$\begin{aligned} \int_a^b \varphi f dx + \int_a^b \varphi' f' dx &= \int_a^b \varphi(f - f'')dx \\ &= \int_a^b \varphi f dx + \int_a^b \varphi' f' dx - \varphi(x)f'(x)|_a^b \\ &= \int_a^b \varphi \cdot f dx + \int_a^b \varphi' f' dx + f'(a) \ . \end{aligned}$$

Hence $f'(a) = 0$. Choosing $\varphi \in H^1(a,b)$ such that $\varphi(a) = 0$ and $\varphi(b) = 1$ we obtain that $f'(b) = 0$. We have shown that $f \in D(\Delta^N)$ and $f - \Delta^N f = g$. Thus

Δ^N is selfadjoint and dissipative. Compactness of the resolvent is obtained by the same reasons given for the Dirichlet Laplacian. □

As a consequence, the Neumann Laplacian generates a contraction semigroup T^N of selfadjoint compact operators on $L^2(a,b)$.

In the following exercises we determine the asymptotic behaviour of the semigroups T^D and T^N.

Exercise 3.2.3. *a) Let $a = 0 < b$. Determine the spectrum of the Dirichlet Laplacian Δ^D on $L^2(0,b)$.*
b) Denote by T^D the semigroup generated by Δ^D. Show that

$$\|T^D(t)\| \le e^{-\pi^2/b^2 t} \quad (t \ge 0) \ . \tag{3.5}$$

Hint: Use (2.13).

Exercise 3.2.4 (Convergence to an equilibrium)**.** *Let A be a selfadjoint dissipative operator on a separable Hilbert space H. Assume that A has compact resolvent. Denote by T the semigroup generated by A. We adopt the notions of the Spectral Theorem 1.4.10. Assume that $\lambda_1 = 0 > \lambda_2$. Let $Px = (x|e_1)e_1$ (i.e. P is the orthogonal projection onto $\mathbb{C} \cdot e_1$). Show that*

$$\|T(t) - P\| \le e^{\lambda_2 t} \quad (t \ge 0) \ .$$

In particular, $\lim_{t\to\infty} T(t) = P$ in $\mathcal{L}(H)$.

Exercise 3.2.5. *Let T^N be the semigroup generated by the Neumann Laplacian on $L^2(a,b)$. Consider the operator P on $L^2(a,b)$ given by*

$$Pf = \frac{1}{b-a}\int_a^b f(x)dx \cdot 1_{(a,b)} \ .$$

Show that there exists $\varepsilon > 0$ such that

$$\|T^N(t) - P\| \le e^{-\varepsilon t} \quad (t \ge 0) \ . \tag{3.6}$$

In fact, one may choose $-\varepsilon = \lambda_2$, the second eigenvalue of Δ_N. Compute λ_2 (choosing $a = 0 < b$).

Hint: Use Exercise 2.4.

We will study these examples further and give interpretations of the physical models they describe. But first we want to generalize our method in order to treat further boundary conditions.

3.3 Classical solutions

In this section we want to find classical solutions of the heat equation

$$u_t(t,x) = u_{xx}(t,x) \quad (t \in (0,\tau], x \in [a,b]) \ . \tag{3.7}$$

Here $-\infty < a < b < \infty$ and $\tau > 0$ are given. By a **classical solution** of (3.7) we mean a continuous function u defined on $[0,\tau] \times [a,b]$ such that $u_t(t,x) := \frac{\partial u}{\partial t}(t,x)$, $u_x(t,x) := \frac{\partial u}{\partial x}(t,x)$ and $u_{xx}(t,x) := \frac{\partial^2 u}{\partial x^2}(t,x)$ exist and are continuous on $(0,\tau) \times (a,b)$. First we prove the parabolic maximum principle, which will give us uniqueness. For that we consider the rectangle $Q_\tau = [0,\tau] \times [a,b]$ and its **parabolic boundary** $\Gamma_\tau = \{0\} \times [a,b] \cup (0,\tau] \times \{a\} \cup (0,\tau] \times \{b\}$.

Theorem 3.3.1. *Let u be a classical solution of (3.7). Then*

$$\max_{Q_\tau} u(t,x) = \max_{\Gamma_\tau} u(t,x) \ . \tag{3.8}$$

Proof. a) Let $\varepsilon > 0$, $v(t,x) = u(t,x) - \varepsilon t$. Then

$$v_t(t,x) = u_t(t,x) - \varepsilon = u_{xx}(t,x) - \varepsilon = v_{xx}(t,x) - \varepsilon < v_{xx}(t,x) \ . \tag{3.9}$$

Let $0 < \varrho < \tau$. We show that $\sup\limits_{Q_\varrho} v = \sup\limits_{\Gamma_\varrho} v$. If not, there exist $t_0 \in (0,\varrho]$ and $x_0 \in (a,b)$ such that $v(t_0,x_0) = \max\limits_{Q_\varrho} v(t,x)$. Hence $v_t(t_0,x_0) \geq 0$ and $v_{xx}(t_0,x_0) \leq 0$. This contradicts (3.8).
Since $\varrho < \tau$ was arbitrary, it follows that $\sup\limits_{Q_\tau} v = \sup\limits_{\Gamma_\tau} v$.
b) We have shown that for all $\varepsilon > 0$

$$\max_{Q_\tau}\{u(t,x) - \varepsilon t\} = \max_{\Gamma_\tau}\{u(t,x) - \varepsilon t\} \ .$$

This implies (3.8). □

If we suppose Dirichlet boundary conditions, the parabolic maximum principle implies uniqueness of the initial value problem. The results on semigroups and selfadjoint operators of the previous sections give us existence. Recall that $\mathbb{R}_+ = [0,\infty)$.

Theorem 3.3.2. *Let $f \in C[a,b]$ such that $f(a) = f(b) = 0$. Then there exists a unique function*

$$u \in C(\mathbb{R}_+ \times [a,b]) \cap C^\infty((0,\infty) \times [a,b])$$

such that

$$u_t(t,x) = u_{xx}(t,x) \quad (t > 0, x \in [a,b]) \ ; \tag{3.10}$$

$$u(t,a) = u(t,b) = 0 \quad (t > 0) \ ; \tag{3.11}$$

$$u(0,x) = f(x) \quad (x \in [a,b]) \ . \tag{3.12}$$

Proof. Uniqueness follows directly from Theorem 3.3.1. In fact, if u_1, u_2 are two solutions, then for $\tau > 0$, $u = u_1 - u_2$ satisfies (3.7) and vanishes on Γ_τ. Hence $u(t,x) \leq 0$ by Theorem 3.3.1. For the same reason $u_2 - u_1 \leq 0$. Hence $u = 0$. Existence. Let A be the Dirichlet Laplacian and T the semigroup generated by A. Let $f \in C[a,b]$. Since T is differentiable, $v = T(\cdot)f \in C^\infty((0,\infty); D(A^k))$ for all $k \in \mathbb{N}$. It follows from the definition of A that $D(A^k) \subset H^{2k}(a,b)$. By Exercise 3.1.10, $H^{2k}(a,b) \subset C^{2k-1}[a,b]$ with continuous injection. Hence $v \in C^\infty((0,\infty); C^m[a,b])$ for all $m \in \mathbb{N}$. Now let $u(t,x) = v(t)(x)$ $(t > 0, x \in [a,b])$. It follows that $u \in C^\infty((0,\infty) \times [a,b])$ and u satisfies (3.10) and (3.11). In order to prove that u is continuous at $t = 0$ we first assume that $f \in D(A)$. Then $AT(t)f = T(t)Af$. Hence $T(\cdot)f \in C(\mathbb{R}_+, D(A))$. Since $D(A) \subset C[a,b]$ with continuous injection it follows that $T(\cdot)f \in C(\mathbb{R}_+, C[a,b])$. Hence $u \in C(\mathbb{R}_+ \times [a,b])$ satisfies (3.12). If $f \in C[a,b]$ we choose $f_n \in D(A)$ such that $f = \lim_{n\to\infty} f_n$ in $C[a,b]$. Let $u_n(t,x) = (T(t)f_n)(x)$. Let $\tau > 0$. It follows from the maximum principle that $\|T(t)f_n - T(t)f_m\|_\infty \leq \|f_n - f_m\|_\infty$ for $0 < t \leq \tau$. Hence u_n is a Cauchy sequence in $C([0,\tau] \times [a,b])$. Let u be its limit. Then $u(t,x) = (T(t)f)(x)$ for all $t > 0$, $x \in [a,b]$. Since $\lim_{t\downarrow 0} T(t)f = f$ in $L^2(a,b)$ it follows that $u(0,x) = f(x)$ a.e. Hence $u(0,x) = f(x)$ for all $x \in [a,b]$, since both functions are continuous. □

Next we discuss physical models which are described by equations (3.10), (3.11) and (3.12). The most initiative is heat conduction. In that case we imagine a thin homogeneous metal rod with end points a and b. Given a point $x \in [a,b]$ on the rod and a time $t \geq 0$, the value $u(t,x)$ is the temperature at the point x at time t. The initial temperature $u(0,x) = f(x)$ is given for each $x \in [a,b]$. Dirichlet boundary conditions (3.11) express that the endpoints a and b of the rod are kept at temperature 0.

Equation (3.10) may be derived as follows. The temperature in a small part of the rod is proportional to the amount of heat divided by the volume. Imagine the rod divided into small sections of lenght $\varepsilon > 0$ and let x be the midpoint of one section. We only consider this and the two adjacent sections and assume that the temperature is approximately constant in each section. Now the heat flow in the section centered at x is proportional to the temperature difference $u(x+\varepsilon,t) - u(x,t)$ to the right neighbour section and inversely proportional to the distance ε. Considering also the left neighbour section we obtain the following approximate expression of change of heat with respect to time:

$$\varepsilon u_t(t,x) = c\varepsilon^{-1}[u(x+\varepsilon,t) - u(x) + u(x-\varepsilon,t) - u(x)] \ . \tag{3.13}$$

Note that the heat in the section is the volume multiplied by the temperature; i.e., proportional to $\varepsilon \cdot u(t,x)$. Hence

$$u_t(t,x) = c\varepsilon^{-2}[u(t,x+\varepsilon) + u(t,x-\varepsilon) - 2u(t,x)] \ . \tag{3.14}$$

Letting $\varepsilon \downarrow 0$, by the rule of de l'Hôspital, we obtain

$$u_t(t,x) = u_{xx}(t,x) \ . \tag{3.15}$$

Here the constant $c > 0$ is the heat conductivity of the metal. For simplicity, we will assume $c = 1$, in general.

As another example one may consider diffusion of dye in a tube of water. In that case $u(t,x)$ is the density of the dye at the point x at time t. This is why equation (3.7) is also called the **diffusion equation**. Experienced colleagues say that the diffusion equation also models the population density during the late stages of a large party in a narrow room.

One may also consider Neumann boundary conditions

$$u_x(t,a) = u_x(t,b) = 0 \quad (t > 0) \tag{3.16}$$

instead of Dirichlet boundary conditions. In the case where we describe heat conduction, they signify that the rod is insulated. If we consider party population, Neumann boundary conditions signify that the two doors at the end of the corridor are closed; whereas Dirichlet boundary conditions signify that the doors allow exit but no entrance.

Exercise 3.3.3. *a) Let* $v \in C([0,\tau]; C[a,b])$*,* $u(t,x) = v(t)(x)$*. Show that* $u \in C([0,\tau] \times [a,b])$*.*
b) Let $u \in C([0,\tau] \times [a,b])$*. For* $t \in [0,\tau]$ *let* $v(t) = u(t,\cdot)$*. Show that* $v \in C([0,\tau]; C[a,b])$*.*

Exercise 3.3.4. *Denote by* T *the semigroup generated by the Dirichlet Laplacian on* $L^2(a,b)$*. Show that* T *is positive (i.e., if* $f \in L^2(a,b)$*,* $f \geq 0$*, then* $T(t)f \geq 0$*).*

Hint: Consider first $f \in C[a,b]$. Use Theorem 3.3.2 and Theorem 3.3.1.

Exercise 3.3.5 (arbitrary selfadjoint realisations of the Laplacian)**.** *Let* A *be a selfadjoint dissipative operator on* $L^2(a,b)$ *such that* $D(A) \subset H^2(a,b)$ *and* $Af = f''$ *for all* $f \in D(A)$*. Denote by* T *the* C_0*-semigroup on* $L^2(a,b)$ *generated by* A*. Let* $f \in L^2(a,b)$ *and* $u(t,x) = (T(t)f)(x)$ $(t > 0, x \in [a,b])$*. Show that* u *is the unique solution of the initial-boundary value problem*

$$\begin{cases} u \in C^\infty((0,\infty) \times [a,b]) & \\ u_t(t,x) = u_{xx}(t,x) & (t > 0, x \in [a,b]) \\ u(t) \in D(A) & (t > 0) \\ \lim_{t \downarrow o} u(t,\cdot) = f \quad \text{in} & L^2(a,b) \end{cases} \tag{3.17}$$

Hint: Use the proof of Theorem 3.3.2 and Theorem 2.3.4.

Exercise 3.3.6 (Neumann boundary conditions)**.** *Let $f \in L^2(a,b)$. Show that there exists a unique function $u \in C^\infty((0,\infty) \times [a,b])$ satisfying (3.10), (3.16) such that*

$$\lim_{t \downarrow 0} u(t,\cdot) = f \ \text{ in } L^2(a,b) \ .$$

Use Exercise 3.3.5.

Exercise 3.3.7 (The Dirichlet Laplacian on $C_0(a,b)$)**.** *Let $C_0(a,b) := \{f \in C[a,b] : f(a) = f(b) = 0\}$. Let T be the semigroup generated by the Dirichlet Laplacian on $L^2(a,b)$. Show that $T(t)L^2(a,b) \subset C_0(a,b)$ $(t > 0)$ and that $T_0(t) := T(t)_{|_{C_0(a,b)}}$ defines a differentiable C_0-semigroup on $C_0(a,b)$. Determine the generator of T_0.*

3.4 Variational Methods: The symmetric case

It looks like hazard that the range condition for the Dirichlet and Neumann Laplacian can be proved with help of the Riesz-Fréchet lemma. At least, it shows that the scalar product in the first Sobolev space and the Laplacian fit well together. However, it turns out that this simple method can be applied in much more general situations.
For this reason we interrupt our investigation of the Laplacian in dimension 1 to introduce a more general framework. It will then be tested by considering more general boundary conditions for the Laplacian on an interval.

Let V be a complex vector space. A **sesquilinear form** on V is a mapping $a : V \times V \to \mathbb{C}$ satisfying

$$\begin{aligned} a(x_1 + x_2, y) &= a(x_1, y) + a(x_2, y) \ ; \\ a(x, y_1 + y_2) &= a(x, y_1) + a(x, y_2) \ ; \\ a(\lambda x, y) &= \lambda a(x, y) \\ a(x, \lambda y) &= \bar{\lambda} a(x, y) \end{aligned}$$

for all $x, y, x_1, x_2, y_1, y_2 \in V, \lambda \in \mathbb{C}$. In other words, $a(\cdot, y) : V \to \mathbb{C}$ is linear and $a(y, \cdot) : V \to \mathbb{C}$ is antilinear for all $y \in V$. Frequently, we simply use the word **form** instead of sesquilinear form. A form a is called **symmetric** if

$$a(x,y) = \overline{a(y,x)} \quad (x, y \in V) \ . \tag{3.18}$$

In that case, $a(x,x) \in \mathbb{R}$ for all $x \in V$. A **positive** form is a symmetric sesquilinear form a satisfying

$$a(x,x) \geq 0 \quad (x \in V) \ . \tag{3.19}$$

Frequently, the domain V of the form a is a Hilbert space in its own right with scalar products $(\ |\ )_V$ and norm $\|\ \|_V$. Then it is easy to see that a sesquilinear form $a : V \times V \to \mathbb{C}$ is continuous if and only if

$$|a(x,y)| \leq M \|x\|_V \|y\|_V \text{ for all } x, y \in V \tag{3.20}$$

and some $M \geq 0$. Moreover, the sesquilinear form a is called **coercive** if for some $\alpha > 0$

$$\text{Re } a(x,x) \geq \alpha \|x\|_V^2 \quad (x \in V) \ . \tag{3.21}$$

Thus, given a Hilbert space V, a symmetric, continuous, coercive form a is the same as an equivalent scalar product on V; i.e., a scalar product defining an equivalent norm.

So far we considered a continuous sesquilinear form a on a Hilbert space V. A concrete example we have in mind is

$$V = H^1(a,b) \ , \quad a(f,g) = \int_a^b f' \bar{g}' dx \ .$$

However this form is not coercive (since $a(1,1) = 0$). Here $H^1(a,b)$ is continuously injected into the Hilbert space $L^2(a,b)$ and the form $a_1(f,g) = \int_a^b f'\bar{g}'dx + \int_a^b f\bar{g}dx$ is coercive. We introduce the notion of "ellipticity" to describe this situation: Assume that V and H are Hilbert spaces with scalar products $(\ |\ )_V$, $(\ |\ )_H$, and norms $\|\ \|_V$ and $\|\ \|_H$. We write $V \hookrightarrow H$ if V is continuously embedded into H; i.e. $V \subset H$ and

$$\|x\|_H \leq c\|x\|_V \quad (x \in V)$$

for some constant $c \geq 0$. We write $V \underset{d}{\hookrightarrow} H$ if in addition V is dense in H. Now assume that $V \hookrightarrow H$. Let $a : V \times V \to \mathbb{C}$ be a continuous sesquilinear form. We say that a is **H-elliptic** or **elliptic with respect to** H if for some $w \in \mathbb{R}$ and some $\alpha > 0$,

$$\text{Re } a(x,x) + w(x \,|\, x)_H \geq \alpha \|x\|_V^2 \tag{3.22}$$

for all $x \in V$. Note that

$$a_w(x,y) := a(x,y) + w(x \,|\, y)_H \tag{3.23}$$

defines a new continuous form on V. Thus, the form a is H-elliptic if and only if a_w is coercive for some $w \in \mathbb{R}$. If a is a symmetric form, then a_w is also symmetric. Thus a is H-elliptic if and only if a_w defines an equivalent scalar product on V for some $w \in \mathbb{R}$. We note the following simple fact.

Exercise 3.4.1. *Let $V \hookrightarrow H$ and let $a : V \times V \to \mathbb{C}$ be a positive H-elliptic form. Then a_w is an equivalent scalar product on V for all $w > 0$.*

Theorem 3.4.2. *Let H, V be a Hilbert space such that $V \underset{d}{\hookrightarrow} H$. Let $a : V \times V \to \mathbb{C}$ be a positive, continuous form which is H-elliptic. Define the operator A on H by*

$$\begin{aligned} D(A) &= \{x \in V : \exists\, y \in H \text{ such that } a(x,\varphi) = (y|\varphi)_H \quad \forall\, \varphi \in V\} \\ Ax &= -y \ . \end{aligned}$$

Then A is selfadjoint and dissipative. Thus A generates a C_0-semigroup T on H. We call A **the operator** *and T* **the semigroup associated with** *a.*

Note that the operator A is well-defined. In fact, if $x \in V$ and $y_1, y_2 \in H$ such that $a(x,\varphi) = (y_1 \,|\, \varphi)_H = (y_2 \,|\, \varphi)_H$ for all $\varphi \in V$, then $(y_1 - y_2 \,|\, \varphi)_H = 0$ for all $\varphi \in V$. Since V is dense in H, this implies that $y_1 - y_2 = 0$.

Proof of Theorem 3.4.1. Let $x_1, x_2 \in D(A)$. Then $(Ax_1 \,|\, x_2)_H = -a(x_1, x_2) = \overline{-a(x_2, x_1)} = \overline{(Ax_2 \,|\, x_1)_H} = (x_1 \,|\, Ax_2)_H$. Thus A is symmetric. Moreover, $(Ax_1 \,|\, x_1) = -a(x_1, x_1) \leq 0$. Hence A is dissipative. It remains to show the range condition. Recall that

$$a_1(x,y) = a(x,y) + (x|y)_H$$

defines an equivalent scalar product on V (see Exercise 3.4.1). Let $y \in H$. Then $F(\varphi) = (\varphi \,|\, y)_H$ defines a continuous linear form on V (since $V \hookrightarrow H$). By the Riesz-Fréchet lemma there exists a unique $x \in V$ such that $a_1(\varphi, x) = (\varphi \,|\, y)_H$ for all $\varphi \in V$. Hence $a(x,\varphi) = a_1(x,\varphi) - (x \,|\, \varphi)_H = (y - x \,|\, \varphi)_H$ for all $\varphi \in V$. Thus $x \in D(A)$ and $Ax = x - y$. We have shown that $I - A$ is surjective. □

Exercise 3.4.3. *Let $H = L^2(a,b)$. Let $V = H_0^1(a,b)$ or $V = H^1(a,b)$ and let $a(f,g) = \int_a^b f'\bar{g}'dx$. Then $V \underset{d}{\hookrightarrow} H$ and a is a continuous, positive, H-elliptic form. The operator associated with a is the Dirichlet Laplacian in the case $V = H_0^1(a,b)$ and the Neumann Laplacian if $V = H^1(a,b)$.*

Using the spectral theorem we now show that every dissipative operator is associated with a positive form. It will be convenient to introduce the following terminology. The analogy to the notion of closed operators will be made clear later.

Definition 3.4.4. *Let H be a Hilbert space. A* **closed positive form** *on H is a couple (V,a) where V is a Hilbert space such that $V \underset{d}{\hookrightarrow} H$ and $a : V \times V \to \mathbb{C}$ is a continuous, positive sesquilinear form which is H-elliptic.*

Theorem 3.4.5. *Let H be a separable Hilbert space and A a dissipative selfadjoint operator on H. Then there exists a unique positive closed form (V,a) on H such that A is associated with V, a.*

Proof. We first show uniqueness. Let (V,a) be a closed positive form such that A is associated with a. Then $a_1(x,y) = a(x,y) + (x \,|\, y)_H$ defines an equivalent scalar product on V. We show that $D(A)$ is dense in V. Let $y \in V$ such that $a_1(x,y) = 0$ for all $x \in D(A)$. We have to show that $y = 0$. By the definition of A we have $(-Ax \,|\, y)_H + (x \,|\, y)_H = 0$ for all $x \in D(A)$. Since $I - A$ is surjective, it follows that $y = 0$.
Since $a_1(x,x)^{1/2} = (\|Ax\|_H^2 + \|x\|_H^2)^{1/2}$ and $a(x,y) = -(Ax \,|\, y)_H$ for all $x, y \in D(A)$, uniqueness of (V,a) follows from the density $D(A)$ in V. In order to show existence we use the spectral theorem. It allows us to assume that $H = L^2(Y, \Sigma, \mu)$, $Af = m \cdot f$, $D(A) = \{f \in H : m \cdot f \in H\}$ where (Y, Σ, μ) is a finite measure space and $m : Y \to [0, \infty)$ is measurable. Let

$$V = \{f \in L^2(Y, \Sigma, \mu) : \int_Y m|f|^2 d\mu < \infty\}$$

with scalar product

$$(f \mid g)_V = \int_Y f\bar{g}(1 + m)d\mu \ .$$

Then V is a Hilbert space such $V \underset{d}{\hookrightarrow} H$. Moreover,

$$a(f,g) = \int_Y f\bar{g}m d\mu$$

defines a closed positive form (V, a) on H. It is easy to see that A is associated with (V, a). □

Remark 3.4.6. *In the situation of Theorem 3.4.5 we use the following terminology: Let A be a selfadjoint dissipative operator. Then we call (V, a)* **the form associated with** *A.*

Of special interest is the case where A has compact resolvent. We assume throughout that H is a separable Hilbert space.

Theorem 3.4.7. *Let (V, a) be a closed, positive form on H. The operator A associated with a has compact resolvent if and only if the embedding $V \hookrightarrow H$ is compact.*

Proof. 1. Considering $D(A)$ with the graph norm the embedding $D(A) \hookrightarrow V$ is continuous. This is clear since $a_1(x,x)^{1/2}$ is an equivalent norm on V, but also follows from Exercise 3.1.9. If the injection $V \hookrightarrow H$ is compact, then $D(A) \hookrightarrow H$ is compact as composition of a compact and a continuous mapping.
2. The converse follows from the spectral theorem. The details are worked out in Exercise 3.4.10. □

As a concrete example we now consider the Laplacian with Robin's boundary conditions, which are also called **boundary conditions of the third kind**.

Theorem 3.4.8 (the Laplacian with Robin's boundary conditions)**.** *Let $\alpha, \beta \geq 0$. We define the operator A on $L^2(0,1)$ by*

$$\begin{aligned} D(A) &= \{f \in H^2(0,1) : f'(1) = -\beta f(1) \ , \ f'(0) = \alpha f(0)\} \\ Af &= f'' \ . \end{aligned}$$

Then A is a selfadjoint dissipative operator with compact resolvent.

Proof. 1. In order to find the form associated with A note that for $f, g \in D(A)$,

$$\begin{aligned} a(f,g) &= -(Af \mid g) = \int_0^1 f''\bar{g}dx \\ &= -f'(1)\bar{g}(1) + f'(0)\bar{g}(0) + \int_0^1 f'\bar{g}'dx \\ &= \beta f(1)\bar{g}(1) + \alpha f(0)\bar{g}(0) + \int_0^1 f'\bar{g}'dx \ . \end{aligned}$$

This leads us to defining the form (V, a) by $V = H^1(0,1)$, $a(f,g) = \beta f(1)\bar{g}(1) + \alpha f(0)\bar{g}(0) + \int_0^1 f'\bar{g}'dx$. Since $H^1(0,1) \hookrightarrow C[0,1]$ it follows that $a : V \times V \to \mathbb{C}$ is continuous. It is clear that a is symmetric and positive. Since $a(f,f) + (f \mid f)_{L^2} \geq \|f\|_{H^1}^2$, the form is elliptic with respect to $L^2(0,1)$.
2. Let B be the operator associated with a. We show that $A = B$. Let $f \in D(A)$, $Af = g$. Then for $\varphi \in V = H^1(0,1)$, $-(g \mid \varphi)_{L^2} = -\int_0^1 f''\bar{\varphi}dx = -f'(1)\bar{\varphi}(1) + f'(0)\bar{\varphi}(0) + \int_0^1 f'\bar{\varphi}'dx = a(f,\varphi)$. Thus $f \in D(B)$ and $Bf = g$. We have shown that $A \subset B$.
Conversely, let $f \in D(B)$, $Bf = g$. Then

$$-\int_0^1 g\bar{\varphi}dx = \beta f(1)\bar{\varphi}(1) + \alpha f(0)\bar{\varphi}(0) + \int_0^1 f'\bar{\varphi}'dx \tag{3.24}$$

for all $\varphi \in H^1(0,1)$. In particular, $-\int_0^1 g\varphi dx = \int_0^1 f'\varphi' dx$ for all $\varphi \in C_c^1(0,1)$. This implies that $f' \in H^1(0,1)$ and $f'' = g$. Now introducing this into (3.24) gives

$$\begin{aligned} & \int_0^1 f'\bar{\varphi}'dx - f'(1)\bar{\varphi}(1) + f'(0)\bar{\varphi}(0) = -\int_0^1 f''\bar{\varphi}dx \\ = & \ -\int_0^1 g\bar{\varphi}dx = \beta f(1)\bar{\varphi}(1) + \alpha f(0)\bar{\varphi}(0) + \int_0^1 f'\bar{\varphi}'dx \ . \end{aligned}$$

Hence $-f'(1)\bar{\varphi}(1) + f'(0)\bar{\varphi}(0) = \beta f(1)\bar{\varphi}(1) + \alpha f(0)\bar{\varphi}(0)$ for all $\varphi \in H^1(0,1)$. This implies that $f'(1) = -\beta f(1)$ and $f'(0) = \alpha f(0)$. Thus $f \in D(A)$ and $Af = Bf$. □

Note that Robin's boundary conditions contain Neumann's boundary conditions (choosing $\alpha = \beta = 0$) but not Dirichlet boundary conditions.

Exercise 3.4.9 (mixed boundary conditions)**.** *Let* $\alpha \geq 0$. *Define the operator* A *on* $L^2(0,1)$ *by*

$$\begin{aligned} D(A) &= \{f \in H^2(0,1) : f(1) = 0 \ , \ f'(0) = \alpha f(0)\} \\ Af &= f'' \ . \end{aligned}$$

Show that A *is selfadjoint and dissipative.*

Exercise 3.4.10. *Let* A *be a selfadjoint, dissipative operator with compact resolvent. By the spectral theorem we can assume that* $H = \ell^2$, $Ax = (-\lambda_n x_n)_{n\in\mathbb{N}}$, $D(A) = \{x \in \ell^2 : (\lambda_n x_n)_{n\in\mathbb{N}} \in \ell^2\}$ *where* $\lambda_n \geq 0$, $\lim_{n\to\infty} \lambda_n = \infty$.

a) *Show that* A *is associated with the positive form* (V, a) *given by*

$$\begin{aligned} V &= \{x \in \ell^2 : \sum_{n=1}^{\infty} \lambda_n |x_n|^2 < \infty\} \\ (x \mid y)_V &= \sum_{n\in\mathbb{N}} (\lambda_n + 1) x_n \overline{y_n} \ ; \\ a(x,y) &= \sum \lambda_n x_n \overline{y_n} \ . \end{aligned}$$

b) Show that the embedding $V \hookrightarrow H$ is compact.

Exercise 3.4.11 (periodic boundary conditions)**.** *Define the operator A on $L^2(0,1)$ by*

$$\begin{aligned} D(A) &= \{f \in H^2(0,1) : f(0) = f(1)\ ,\ f'(0) = f'(1)\} \\ Af &= f''\ . \end{aligned}$$

Show that A is selfadjoint, dissipative and has compact resolvent.

Exercise 3.4.12 (antiperiodic boundary condtions)**.** *Define A on $L^2(0,1)$ by*

$$\begin{aligned} D(A) &= \{f \in H^2(0,1) : f(0) = -f(1)\ ,\ f'(0) = -f'(1)\} \\ Af &= f''\ . \end{aligned}$$

Show that A is selfadjoint, dissipative and has compact resolvent.

Chapter 4

The Laplacian on open sets in $\mathbb{R}^{\mathbf{n}}$

In this chapter we study the Laplacian on $L^2(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is an open set. Motivated by the 1-dimensional examples we consider Dirichlet and Neumann boundary conditions. They define selfadjoint realisations of the Laplacian on $L^2(\Omega)$. The semigroups generated by these operators give the solution of the heat equation with these two different boundary conditions. We use Sobolev imbedding to prove that the solutions are regular in the interiour of Ω. For the Dirichlet Laplacian we also obtain semigroups on $L^p(\Omega)$ $(1 \leq p \leq \infty)$. We study monotonicity properties of the corresponding semigroup with respect to the domain. As a consequence, we establish the existence of a Green's function, which one frequently calls the heat kernel defined by the Dirichlet Laplacian. All these properties are quite easy to obtain directly for Dirichlet boundary conditions. However, in the case of Neumann boundary conditions more elaborate techniques are necessary. Those will be presented in the subsequent chapters.

4.1 The Dirichlet and Neumann Laplacian on open sets in $\mathbb{R}^{\mathbf{n}}$

We start introducing the first Sobolev space on an open set of $\mathbb{R}^n$. Not much more than the definition is needed to show that the Laplacian with Dirichlet or Neumann boundary conditions generates a C_0-semigroup.

First we introduce some notation. Let $\Omega \subset \mathbb{R}^n$ be an open set. The space

$L^p(\Omega)$, $1 \le p \le \infty$, is understood with respect to Lebesgue measure. We define

$$\begin{aligned} L^1_{\mathrm{loc}}(\Omega) &= \{f : \Omega \to \mathbb{C} \text{ measurable } : \int_K |f(x)|\,dx < \infty \text{ for all compact } K \subset \Omega\} . \\ C(\Omega) &:= \{f : \Omega \to \mathbb{C} \text{ continuous}\} , \\ C(\bar{\Omega}) &:= \{f : \bar{\Omega} \to \mathbb{C} \text{ continuous}\} , \\ C^k(\Omega) &:= \{f : \Omega \to \mathbb{C} : k\text{-times continuously differentiable}\} , \end{aligned}$$

where $k \in \mathbb{N}$. For $f \in C^1(\Omega)$ we let $D_j f = \frac{\partial f}{\partial x_j}$ $(j = 1, \dots, n)$. By $C_c(\Omega)$ we denote the space of all continuous functions $f : \Omega \to \mathbb{C}$ such that the **support** $\mathrm{supp} f = \{x \in \Omega : f(x) \neq 0\}^-$ is a compact subset of Ω. We let

$$\begin{aligned} C^k_c(\Omega) &:= C^k(\Omega) \cap C_c(\Omega) , \\ C^\infty(\Omega) &:= \bigcap_{k \in \mathbb{N}} C^k(\Omega) \text{ ; and by} \\ \mathcal{D}(\Omega) &:= C^\infty(\Omega) \cap C_c(\Omega) \end{aligned}$$

we denote the space of all **test functions**. Let $f \in C^1(\Omega)$, $\varphi \in C^1_c(\Omega)$. Then

$$-\int_\Omega f D_j \varphi dx = \int_\Omega D_j f \varphi dx . \tag{4.1}$$

We use this relation (4.1) to define weak derivatives.

Definition 4.1.1. *Let $f \in L^1_{\mathrm{loc}}(\Omega)$. Let $j \in \{1, \dots, n\}$. A function $g \in L^1_{\mathrm{loc}}(\Omega)$ is called the* **weak j-th partial derivative** *of f (in Ω) if*

$$-\int_\Omega f D_j \varphi dx = \int_\Omega g \varphi dx$$

for all $\varphi \in \mathcal{D}(\Omega)$. Then we set $D_j f := g$.

Note that the weak j-th partial derivative is unique. Here we identify functions in $L^1_{\mathrm{loc}}(\Omega)$ which coincide almost everywhere. We let

$$W(\Omega) = \{f \in L^1_{\mathrm{loc}}(\Omega) : \text{ the weak derivative } D_j f \in L^1_{\mathrm{loc}}(\Omega) \text{ exists for all } j = 1, \dots, n\} .$$

Note that $L^p(\Omega) \subset L^1_{\mathrm{loc}}(\Omega)$ for all $1 \le p \le \infty$. Now we define the first Sobolev space $H^1(\Omega)$ by

$$H^1(\Omega) := \{f \in L^2(\Omega) \cap W(\Omega) : D_j f \in L^2(\Omega) \quad j = 1, \dots, n\} .$$

Proposition 4.1.2. *The space $H^1(\Omega)$ is a separable Hilbert space for the scalar pro-duct*

$$(f \,|\, g)_{H^1(\Omega)} = \int_\Omega f \bar{g} dx + \sum_{j=1}^n \int D_j f D_j \bar{g} dx .$$

Proof. Consider the separable Hilbert space $H = L^2(\Omega)^{n+1}$ with norm

$$\|(u_0, u_1, \ldots, u_n)\|_H^2 = \sum_{j=0}^{n} \int |u_j|^2 dx \ .$$

Then $\Phi : H^1(\Omega) \to H,\ f \mapsto (f, D_1 f, \ldots, D_n f)$ is isometric and linear. Thus it suffices to show that the image of Φ is closed. Let $f_k \in H^1(\Omega)$ such that $\lim_{k\to\infty} \Phi(f_k) = (f, g_1, \ldots, g_n)$ in H. Then $\lim_{k\to\infty} f_k = f$ and $\lim_{k\to\infty} D_j f_k = g_j$ in $L^2(\Omega)$ $(j = 1, \ldots, n)$. Let $\varphi \in C_c^1(\Omega)$. Then by the dominated convergence theorem

$$\begin{aligned} -\int_\Omega D_j\varphi f dx &= \lim_{k\to\infty} (-\int_\Omega D_j \varphi f_k dx) \\ &= \lim_{k\to\infty} \int_\Omega \varphi D_j f_k dx \\ &= \int_\Omega \varphi g_j dx \ . \end{aligned}$$

Thus g_j is the weak j-th partial derivative of f and $\Phi(f) = (f, g_1, \ldots, g_n)$. □

Next we talk about Dirichlet boundary conditions. If $n \geq 2$, then $H^1(\Omega)$ is no longer a subspace $C(\Omega)$ (see Exercise 4.1.15). Thus the elements of $H^1(\Omega)$ are merely equivalence classes; we identify functions which coincide almost everywhere. So we cannot define $H_0^1(\Omega)$ as we did when Ω is a bounded interval. In fact, in general $\partial\Omega$ will be of measure 0, so it does not make sense to talk about the restriction to $\partial\Omega$ for functions in $H^1(\Omega)$. This leads us to the following definition:

$$H_0^1(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)} \ ;$$

i.e., $H_0^1(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$.

Later we will investigate further properties of $H_0^1(\Omega)$ and show that this definition coincides with the one given in Chapter 3 if Ω is an interval.

Now we want to introduce the Dirichlet Laplacian. For $f \in C^2(\Omega)$ we define the **Laplacian** Δf by

$$\Delta f = \sum_{j=1}^{n} D_j^2 f \ .$$

Similarly as for the first order derivatives we define the **weak Laplacian** as follows. Let $f \in L^1_{\text{loc}}(\Omega), g \in L^1_{\text{loc}}(\Omega)$. We say that $\Delta f = g$ **weakly**, if

$$\int_\Omega \Delta\varphi f dx = \int_\Omega \varphi g dx \tag{4.2}$$

for all $\varphi \in \mathcal{D}(\Omega)$. In that case we write

$$\Delta f = g \quad \text{weakly (on } \Omega) \ .$$

Remark 4.1.3. *a) Again g is unique up to a set of measure 0.*
b) In the language of distributions, (4.2) just means that the distribution Δf equals g.

For $f \in W(\Omega)$ we denote by $\mathrm{grad} f(x) = \nabla f(x) = (D_1 f(x), \dots, D_n f(x))$ the **gradient** of f. For $x, y \in \mathbb{R}^n$ we dentoe by $x \cdot y = \sum\limits_{j=1}^{n} x_j y_j$ the scalar product in $\mathbb{R}^n$. Similarly, for $f, g \in W(\Omega)$ we let $\nabla f \cdot \nabla g = \sum\limits_{j=1}^{n} D_j f \cdot D_j g$.

Theorem 4.1.4 (the Dirichlet Laplacian)**.** *Define the operator A on $L^2(\Omega)$ by $D(A) = \{f \in H_0^1(\Omega) :$ there exists $g \in L^2(\Omega)$ such that $\Delta f = g$ weakly$\}$.*

$$Af = \Delta f \ .$$

Then A is a selfadjoint, dissipative operator. We denote A by Δ_Ω^D and call A **the Laplacian with Dirichlet boundary conditions** *or simply the* **Dirichlet Laplacian**.

Proof. Let $V = H_0^1(\Omega)$. Then $V \underset{d}{\hookrightarrow} L^2(\Omega)$. Define $a : V \times V \to \mathbb{C}$ by

$$a(f,g) = \int_\Omega \nabla f \cdot \nabla \bar{g} dx \ .$$

Then

$$\begin{aligned} |a(f,g)| &\le (\int_\Omega (|\nabla f|^2 dx)^{1/2} (\int |\nabla g|^2 dx)^{1/2} \\ &\le \|f\|_{H_0^1} \cdot \|g\|_{H_0^1} \ . \end{aligned}$$

Moreover, $a(f,f) + \|f\|_{L^2}^2 = \|f\|_{H_0^1}^2$. Thus a is a closed positive form on $L^2(\Omega)$. Denote by B the operator associated with a. Let $f \in D(B)$, $Bf = g$. Then

$$\begin{aligned} (g \,|\, \varphi)_{L^2} = -a(f,\varphi) &= -\sum_{j=1}^{n} \int_\Omega D_j f D_j \bar{\varphi} dx \\ &= \int_\Omega f \Delta \bar{\varphi} dx \end{aligned}$$

for all $\varphi \in \mathcal{D}(\Omega)$. Thus $\Delta f = g$ weakly. Conversely, let $f \in H_0^1(\Omega)$ such that there exists $g \in L^2(\Omega)$ such that $\Delta f = g$ weakly. Then for all $\varphi \in \mathcal{D}(\Omega)$

$$\begin{aligned} -a(f,\varphi) &= -\sum_{j=1}^{n} \int_\Omega D_j f D_j \bar{\varphi} dx \\ &= \int_\Omega f \Delta \bar{\varphi} dx \\ &= \int_\Omega g \bar{\varphi} dx \ . \end{aligned}$$

Since $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, this identity remains true for all $\varphi \in H_0^1(\Omega)$. Thus $f \in D(\Delta_\Omega^D)$ and $\Delta_\Omega^D f = g$. We have shown that $B = \Delta_\Omega^D$. □

Thus the operator Δ_Ω^D generates a contractive C_0-semigroup of selfadjoint operators on $L^2(\Omega)$. We sometime use the symbolic notation

$$e^{t\Delta_\Omega} := T(t) \quad (t \geq 0) \ .$$

This semigroup governs the heat equation with Dirichlet boundary conditions. Indeed, if $f \in L^2(\Omega)$, then $u(t) = e^{t\Delta_\Omega^D} f$ is the unique solution of

$$\begin{cases} u \in C^\infty((0,\infty); L^2(\Omega)) \cap C([0,\infty); L^2(\Omega)) \\ u(t) \in H_0^1(\Omega) \quad (t > 0) \\ \dot{u}(t) = \Delta u(t) \quad \text{weakly} \\ u(0) = f \ . \end{cases}$$

This follows from Theorem 2.3.4. In fact, we will see later that u is a classical solution.

Next we consider Neumann boundary conditions. It is remarkable that they can be defined for arbitrary open sets.

Theorem 4.1.5 (the Neumann Laplacian)**.** *Let $\Omega \subset \mathbb{R}^n$ be open. Define the operator A on $L^2(\Omega)$ by*

$$\begin{aligned} D(A) &= \{f \in H^1(\Omega) : \text{ there exists } g \in L^2(\Omega) \text{ such that} \\ & \quad -\int_\Omega \nabla f \nabla \varphi dx = \int_\Omega g\varphi dx \text{ for all } \varphi \in H^1(\Omega)\} \\ Af &= g \ . \end{aligned}$$

Then A is selfadjoint and dissipative. We call A the **Laplacian with Neumann boundary conditions** *or simply the* **Neumann Laplacian***. We denote the operator by Δ_Ω^N.*

Proof. The operator A is associated with the positive closed form (V, a) where $V = H^1(\Omega)$ and $a(f,g) = \int_\Omega \nabla f \nabla \bar{g}$. □

Remark that

$$\Delta_\Omega^N f = \Delta f \quad \text{weakly}$$

for all $f \in D(\Delta_\Omega^N)$. This follows clearly from the definition.

We saw in Theorem 3.2.2 and Exercise 3.4.3 that in the case where $n = 1$ and $\Omega = (a,b)$ then

$$D(\Delta_\Omega^N) = \{f \in H^1(a,b) : f'(a) = f'(b) = 0\} \ .$$

This justifies the name of "Neumann Laplacian". In higher dimension the boundary conditions are satisfied in a weak form. We make this more precise in the following remark.

Remark 4.1.6 (comparison of classical and weak Neumann boundary conditions)**.** *Assume that $\Omega \subset \mathbb{R}^n$ is open, bounded with boundary of class C^1. Recall* **Green's formula**

$$\int_\Omega \Delta f g dx = \int_{\partial\Omega} \frac{\partial f}{\partial \nu} g \, d\sigma - \int_\Omega \nabla f \nabla g dx \tag{4.3}$$

($f \in C^2(\bar{\Omega})$, $g \in C^1(\bar{\Omega})$), where σ denotes the surface measure on $\partial\Omega$. By $\nu(x)$ we denote the **exteriour normal** *in each $x \in \partial\Omega$; and for $f \in C^1(\bar{\Omega})$, $\frac{\partial f}{\partial \nu}(x) = \nabla f(x) \cdot \nu(x)$ is the* **normal derivative** *of f in $x \in \partial\Omega$. Now define the operator B on $L^2(\Omega)$ by*

$$\begin{aligned} D(B) &= \{f \in C^2(\bar{\Omega}) : \frac{\partial f}{\partial \nu}_{|\partial\Omega} = 0\} , \\ Bf &= \Delta f . \end{aligned}$$

Then $\quad (a) \quad B \subset \Delta_\Omega^N$ *and*

$\qquad (b) \quad C^2(\bar{\Omega}) \cap D(\Delta_\Omega^N) \subset D(B)$.

Proof. a) Let $f \in D(B)$. Then by (4.3), $-(\Delta f \mid \varphi) = \int_\Omega \nabla f \nabla \varphi dx$ for all $\varphi \in C^1(\bar{\Omega})$. Since Ω is of class C^1, the space $C^1(\bar{\Omega})$ is dense in $H^1(\Omega)$ (see [Bre87, Corollaire IX.8, p. 162]). Hence, going to the limit yields

$$-(\Delta f \mid \varphi) = \int_\Omega \nabla f \nabla \varphi dx = a(f, \varphi)$$

for all $\varphi \in H^1(\Omega)$. Hence $f \in D(\Delta_\Omega^N)$ and $\Delta_\Omega^N f = \Delta f$.
b) Let $f \in C^2(\bar{\Omega}) \cap D(\Delta_\Omega^N)$. Then

$$\int_\Omega \Delta f \varphi = - \int_\Omega \nabla f \nabla \varphi d\sigma$$

for all $\varphi \in C^1(\bar{\Omega})$. Comparison with (3) shows that $\int_{\partial\Omega} \frac{\partial f}{\partial \nu} \varphi d\sigma = 0$ for all $\varphi \in C^1(\bar{\Omega})$. This implies $\frac{\partial f}{\partial \nu} = 0$ on $\partial\Omega$. □

The operator Δ_Ω^N generates a C_0-semigroup T on $L^2(\Omega)$. We frequently use the notation

$$e^{t\Delta_\Omega^N} := T(t) \quad (t > 0) .$$

This semigroup governs the heat equation with Neumann boundary conditions. Later we will see that also in this case solutions are of class C^∞ in space and time.

A natural question occurs: If $\Omega = \mathbb{R}^n$, then there is no boundary. So one expects that the Dirichlet and Neumann Laplacian coincide in this case. We will give a proof of this and related results introducing the technique of regularisation which will also be useful later.

First we recall Young's inequalities for convolution.

Proposition 4.1.7. *Let $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$ where $1 \leq p \leq \infty$. Then*

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

exists for almost all $x \in \mathbb{R}^n$ as Lebesgue integral and defines a function $f * g \in L^p(\mathbb{R}^n)$. Moreover,

$$\|f * g\|_p \leq \|f\|_1 \; \|g\|_p \; .$$

We deduce from this that also $H^1(\mathbb{R}^n)$ is invariant by convolution.

Proposition 4.1.8. *Let $h \in L^1(\mathbb{R}^n)$, $f \in H^1(\mathbb{R}^n)$, then $h * f \in H^1(\mathbb{R}^n)$ and*

$$D_j(h * f) = h * D_j f \quad (j = 1, \ldots, n) \; .$$

Proof. By Proposition 4.3.1 we have $h * f$, $h * D_j f \in L^2(\mathbb{R}^n)$. We show that $D_j(h * f) = h * D_j f$ weakly. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Then by Fubini's theorem

$$\begin{aligned}
&-\int_{\mathbb{R}^n} D_j\varphi(h * f)dx = \\
&-\int_{\mathbb{R}^n} (D_j\varphi)(x) \int_{\mathbb{R}^n} f(x-y)h(y)dydx = \\
&-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (D_j\varphi)(x)f(x-y)dxh(y)dy = \\
&-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\partial}{\partial x_j}\varphi(x+y)f(x)dxh(y)dy = \\
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi(x+y)D_jf(x)dxh(y)dy = \\
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi(x)D_jf(x-y)dxh(y)dy = \\
&\int_{\mathbb{R}^n} (h * D_jf)(x)\varphi(x)dx \; .
\end{aligned}$$

This proves the claim. ☐

Now let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then for $\rho \in \mathcal{D}(\mathbb{R}^n)$,

$$\begin{aligned} \rho * f(x) &= \int_{\mathbb{R}^n} f(y)\rho(x-y)dy \\ &= \int_{\text{supp}\rho} f(x-y)\rho(y)dy \end{aligned}$$

defines a function $\rho * f \in C^\infty(\mathbb{R}^n)$ and

$$D_j(\rho * f) = D_j\rho * f \quad (j = 1, \ldots, n)$$

by the usual rules allowing the commutation of integral and differentiation.

Definition 4.1.9. *Let* $0 \leq \rho \in \mathcal{D}(\mathbb{R}^n)$ *such that* $\text{supp}\rho \subset B(0,1) = \{x \in \mathbb{R}^n : |x| < 1\}$ *and* $\int_{\mathbb{R}^n} \rho(x)dx = 1$. *Let* $\rho_k(x) = k^n \rho(k \cdot x)$ $(x \in \mathbb{R}^n)$. *Then* $\rho_k \in \mathcal{D}(\mathbb{R}^n)$ $\int_{\mathbb{R}^n} \rho_k dx = 1$ *and* $\text{supp}\rho_k \subset B(0, 1/k)$. *We call* $(\rho_k)_{k\in\mathbb{N}}$ *a* **regularizing sequence** *or* **mollifier**.

The following property is well-known.

Proposition 4.1.10. *Let* $f \in L^p(\mathbb{R}^n)$. *Then* $\rho_k * f \in L^p(\mathbb{R}^n)$ *and* $\lim_{k\to\infty} \rho_k * f = f$ *in* $L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$.

Lemma 4.1.11. *Let* $f \in H^1(\Omega)$ *such that* $f(x) = 0$ *for* $x \in \Omega \setminus K$ *where* $K \subset \Omega$ *is compact. Then* $f \in H^1_0(\Omega)$.

Proof. Let $f_k = \rho_k * f$. Then $f_k \in \mathcal{D}(\Omega)$ if $k > \text{dist}(K, \partial\Omega)^{-1}$. Moreover, $f_k \to f$ $(k \to \infty)$ in $H^1(\Omega)$ by Proposition 4.1.10. □

Proposition 4.1.12. *One has* $H^1_0(\mathbb{R}^n) = H^1(\mathbb{R}^n)$. *Consequently,* $\Delta^D_{\mathbb{R}^n} = \Delta^N_{\mathbb{R}^n}$.

Proof. Let $f \in H^1(\mathbb{R}^n)$. Let $\xi \in \mathcal{D}(\mathbb{R}^n)$ such that $\text{supp}\xi \subset B(0,2)$, $0 \leq \xi(x) \leq 1$ $(x \in \mathbb{R}^n)$ and $\xi(x) = 1$ for $x \in B(0,1)$. Let $\xi_k(x) = \xi(x/k)$. Then $f_k = \xi_k f \in H^1(\mathbb{R}^n)$ and $D_j f_k = (D_j\xi_k)f + \xi_k D_j f$ (see Exercise 4.1.15 a)). Since $D_j\xi_k(x) = \frac{1}{k}D_j\xi(x/k)$ it follows from the dominated convergence theorem that $f_k \to f$ in $H^1(\mathbb{R}^n)$. Since $f_k \in H^1_0(\mathbb{R}^n)$ by Lemma 4.1.11, it follows that $f \in H^1_0(\mathbb{R}^n)$. □

The next two exercises concern the Dirichlet or Neumann Laplacian. The others give more information on Sobolev spaces and weak derivatives.

Exercise 4.1.13. *Give a direct proof of Theorem 4.1.4 without using forms as for Theorem 3.2.2.*

Exercise 4.1.14. *Let Ω_1, Ω_2 be two disjoint open subsets of $\mathbb{R}^n, \Omega = \Omega_1 \cup \Omega_2$. Then $L^2(\Omega) = L^2(\Omega_1) \oplus L^2(\Omega_2)$. Denote by T, T_1, T_2 the semigroups on $L^2(\Omega), L^2(\Omega_1)$ and $L^2(\Omega_2)$ generated by the Dirichlet or Neumann Laplacian. Show that*

$$T(t)(f_1, f_2) = (T_1(t)f_1 \ , \ T_2(t)f_2)$$

for all $(f_1, f_2) \in L^2(\Omega_1) \oplus L^2(\Omega_2)$.

Exercise 4.1.15. *a) Let $f \in W(\Omega)$ and $\xi \in C^1(\Omega)$. Show that $f\xi \in W(\Omega)$ and $D_j(\xi f) = (D_j\xi) \cdot f + \xi D_j f$.*
b) Let $f \in H^1(\Omega)$, $\psi \in H_0^1(\Omega)$. Show that

$$-\int_\Omega D_j\psi \cdot f dx = \int_\Omega \psi D_j f dx \ , \ j = 1, \dots, n \ .$$

c) Let $f \in H^1(\Omega)$, $\xi \in W^{1,\infty}$ (i.e., $\xi \in W(\Omega) \cap L^\infty(\Omega)$ such that $D_j\xi \in L^\infty(\Omega)$, $j = 1, \dots n$). Show that $\xi f \in H^1(\Omega)$ and

$$D_j(\xi f) = (D_j\xi) \cdot f + \xi D_j f \quad (j = 1 \dots n) \ .$$

In the next exercise we show that in dimension $n \geq 2$ for each open set Ω in $\mathbb{R}^n$ and each $a \in \Omega$ there exists a function $f \in H^1(\Omega) \cap \mathcal{D}(\Omega \setminus \{a\})$ such that $\lim_{x \to a} f(x) = \infty$. We recall integration of radial functions.

Remark 4.1.16 (radial functions)**.** *Let $\Omega = \{x \in \mathbb{R}^n : r_1 < |x| < r_2\}$ be a ring where $0 \leq r_1 < r_2 \leq \infty$. Let $f : \Omega \to \mathbb{R}_+$ be a* **radial function** *i.e.*

$$f(x) = g(|x|)$$

for some measurable function $g : (r_1, r_2) \to \mathbb{R}_+$. Then

$$\int_\Omega f dx = \sigma_{n-1} \int_{r_1}^{r_2} g(r) r^{n-1} dr$$

where σ_{n-1} is the surface of the sphere $\{x \in \mathbb{R}^n : |x| = 1\}$.

Exercise 4.1.17 (singularities of functions in $H^1(\Omega)$)**.** *a) Let $n \geq 3$, $\Omega = B(0,1) := \{x \in \mathbb{R}^n : |x| < 1\}$. Show that there exists $\alpha > 0$ such that $f_\alpha \in H^1(\Omega)$ where*

$$f_\alpha(x) = |x|^{-\alpha} \ .$$

b) Let $n = 2$, $f(x) = (\log \frac{1}{|x|})^\alpha$ where $0 < \alpha < \frac{1}{2}$. Show that $f \in H^1(\Omega)$ where $\Omega = B(0, 1/2) = \{x \in \mathbb{R}^2 : |x| < 1/2\}$.

Exercise 4.1.18. *Show that $\mathcal{D}(\mathbb{R}^n)$ is dense in $H^2(\mathbb{R}^n)$.*

4.2 The Gaussian Semigroup

In this section we consider the Laplacian on $L^2(\mathbb{R}^n)$. We have seen that the spectral theorem allows us to transform each selfadjoint operator into a multiplication operator by unitary equivalence. In general, the unitary transform is abstract and cannot be described explicitely. This is different for the Laplacian on $\mathbb{R}^n$; here it is the Fourier transform which is doing the work. Another thing is remarkable. Whereas the semigroup generated by the Dirichlet or Neumann Laplacian on an open set cannot be given by an explicit formula, the semigroup generated by the Laplacian on $\mathbb{R}^n$ is given by the Gauss kernel in very explicit form.

We start recalling some facts about the Fourier transform. If $x, y \in \mathbb{R}^n$ we denote by $x \cdot y = \sum\limits_{j=1}^{n} x_j \cdot y_j$ the scalar product and let $x^2 = x \cdot x$. For $f \in L^1(\mathbb{R}^n)$

$$\mathcal{F}f(x) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-ixy} f(y)dy$$

is the **Fourier transform** of f. One has $\mathcal{F}f \in C_0(\mathbb{R}^n) := \{g : \mathbb{R}^n \to \mathbb{C}$ continuous, $\lim\limits_{|x|\to\infty} g(x) = 0\}$. Moreover $\|\mathcal{F}f\|_2 = \|f\|_2$ for all $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and thus $\mathcal{F}$ extends to a unitary mapping from $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$, the **Plancherel transform**, which we still denote by $\mathcal{F}$. For $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $\mathcal{F}^{-1} = \mathcal{F}^*$ is given by

$$(\mathcal{F}^{-1}f)(x) = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} e^{ixy} f(y)dy \ .$$

We now determine the image of the Sobolev space $H^1(\mathbb{R}^n)$ under $\mathcal{F}$. For this, consider first $f \in C_c^1(\mathbb{R}^n)$. Then integration by parts yields

$$\begin{aligned}
\mathcal{F}D_j f(x) &= (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} e^{-ixy} D_j f(y)dy \\
&= (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} (ix_j) e^{-ixy} f(y)dy \\
&= ix_j(\mathcal{F}f)(x) \quad (x \in \mathbb{R}^n) \ .
\end{aligned}$$

Proposition 4.2.1. *The Plancherel transform induces an isomorphism from $H^1(\mathbb{R}^n)$ onto*

$$L^2(\mathbb{R}^n; (1 + x^2)dx) := \{f \in L^2(\mathbb{R}^n) : \int |f(x)|^2(1 + x^2)dx < \infty\}$$

and $(\mathcal{F}D_j f)(x) = ix_j\mathcal{F}f(x)$ a.e. for all $f \in H^1(\mathbb{R}^n)$.

Proof. For $f \in C_c^1(\mathbb{R}^n)$ one has

$$\begin{aligned}
\|f\|^2_{H^1(\mathbb{R}^n)} &= \|f\|^2_{L^2(\mathbb{R}^n)} + \sum_{j=1}^n \|D_j f\|^2_{L^2(\mathbb{R}^n)} \\
&= \|\mathcal{F}f\|^2_{L^2(\mathbb{R}^n)} + \sum_{j=1}^n \|\mathcal{F}D_j f\|^2_{L^2(\mathbb{R}^n)} \\
&= \int_{\mathbb{R}^n} |\mathcal{F}f(x)|^2 dx + \sum_{j=1}^n \int_{\mathbb{R}^n} x_j^2 |\mathcal{F}f(x)|^2 dx \\
&= \int_{\mathbb{R}^n} (1+x^2)|\mathcal{F}f(x)|^2 dx\ .
\end{aligned}$$

Since $C_c^1(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$, it follows that $\mathcal{F}$ is an isometric isomorphism of $H^1(\mathbb{R}^n)$ onto a closed subspace of $L^2(\mathbb{R}^n;\,(1+x^2)dx)$. Conversely, let $g \in L^2(\mathbb{R}^n;$ $(1+x^2)dx)$. Let $f = \mathcal{F}^{-1}g \in L^2(\mathbb{R}^n)$. Let $j \in \{1,\dots,n\}$. Since $x \mapsto ix_j g(x) \in L^2(\mathbb{R}^n)$, there exists $f_j \in L^2(\mathbb{R}^n)$ such that $\mathcal{F}f_j(x) = ix_j g(x)$ $(x \in \mathbb{R}^n)$. Let $\varphi \in C_c^1(\mathbb{R}^n)$. Then $-\int_{\mathbb{R}^n} f\overline{D_j\varphi}dx = -\int_{\mathbb{R}^n} (\mathcal{F}f)(x)\overline{\mathcal{F}(D_j\varphi)}dx = \int_{\mathbb{R}^n} ix_j\mathcal{F}f(x)\overline{\mathcal{F}\varphi(x)}dx = \int_{\mathbb{R}^n} \mathcal{F}f_j(x)\overline{\mathcal{F}\varphi(x)}dx = \int_{\mathbb{R}^n} f_j\overline{\varphi}dx$. It follows that $f_j = D_j f$ weakly. This shows that $g \in \mathcal{F}H^1(\mathbb{R}^n)$. □

We define $H^k(\mathbb{R}^n)$ inductively by

$$H^{k+1}(\mathbb{R}^n) := \{f \in H^1(\mathbb{R}^n) : D_j f \in H^k(\mathbb{R}^n) \text{ for } j = 1,\dots,n\}\ .$$

Then we deduce from Proposition 4.2.1 by an inductive argument.

Corollary 4.2.2. *The Plancherel transformation induces an isomorphism from $H^k(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n, (1+(x^2)^k)dx)$ for all $k \in \mathbb{N}$.*

Now we define the Laplacian A on $L^2(\mathbb{R}^n)$ by

$$D(A) = H^2(\mathbb{R}^n)\ ;\ Af = \Delta f\ .$$

Consider the operator $B = \mathcal{F}A\mathcal{F}^{-1}$ on $L^2(\mathbb{R}^n)$ similar to A via Plancherel transform; i.e.,

$$\begin{aligned}
D(B) &= \{g \in L^2(\mathbb{R}^n) : \mathcal{F}^{-1}f \in D(A) = H^2(\mathbb{R}^n)\} \\
Bg &= \mathcal{F}A\mathcal{F}^{-1}g\ .
\end{aligned}$$

Then B is the multiplication operator defined by $-x^2$. More precisely,

Proposition 4.2.3. *One has*

$$\begin{aligned}
D(B) &= L^2(\mathbb{R}^n; (1+(x^2)^2)dx) \\
Bg(x) &= -x^2 g(x)\ .
\end{aligned}$$

Proof. Let $g \in L^2(\mathbb{R}^n)$. Then $g \in D(B)$ if and only if $\mathcal{F}^{-1}g \in D(A) = H^2(\mathbb{R}^n)$; i.e., $g \in L^2(\mathbb{R}^n;\ (1+(x^2)^2)dx)$ by Corollary 4.2.2. Let $f \in D(A)$. Then $B\mathcal{F}f = \mathcal{F}Af = \mathcal{F}\Delta f = \Sigma \mathcal{F}D_j^2 f$. Hence $(B\mathcal{F}f)(x) = \sum\limits_{j=1}^{n}(ix_j)^2\mathcal{F}f(x) = -x^2(\mathcal{F}f)(x)$. □

It follows that A is selfadjoint and dissipative. Hence A generates a C_0-semigroup G on $L^2(\mathbb{R}^n)$. Moreover,

$$\mathcal{F}(G(t)f)(x) = e^{-tx^2}(\mathcal{F}f)(x) \quad (t > 0\ ,\ x \in \mathbb{R}^n)$$

for all $f \in L^2(\mathbb{R}^n)$.

In order to compute $G(t)$ we need to compute the inverse Fourier transform of e^{-tx^2}.

Lemma 4.2.4. *Let $h(x) = e^{-x^2/2}$. Then*

$$\mathcal{F}h = h\ .$$

Proof. a) Let $n = 1$, $g(x) = (2\pi)^{-1/2}\int\limits_{\mathbb{R}} e^{-ixy}e^{-y^2/2}dy$. It is well-known that $\int\limits_{\mathbb{R}} e^{-z^2}dz = \sqrt{\pi}$. Substitung $z = y/\sqrt{2}$ we conclude that

$$g(0) = (2\pi)^{-1/2}\int\limits_{\mathbb{R}} e^{-y^2/2}dy = (2\pi)^{-1/2}\int\limits_{\mathbb{R}} e^{-z^2}\sqrt{2}dz = 1\ .$$

Differentiation under the integral gives

$$\begin{aligned} g'(x) &= (2\pi)^{-1/2}\int\limits_{\mathbb{R}} e^{-ixy}(-iy)e^{-y^2/2}dy \\ &= (2\pi)^{-1/2}\int\limits_{\mathbb{R}} ie^{-ixy}\frac{d}{dy}e^{-y^2/2}dy \\ &= -i(2\pi)^{-1/2}\int\limits_{\mathbb{R}} \frac{d}{dy}e^{-ixy}e^{-y^2/2}dy \\ &= -xg(x)\ . \end{aligned}$$

Hence $g(x) = ce^{-x^2/2}$. Since $c = g(0) = 1$, the claim is proved.
b) Since $h(x) = e^{-x_1^2/2}\cdot\ldots\cdot e^{-x_n^2/2}$, the result follows from a) for arbitrary dimension. □

We recall that for $f \in L^1(\mathbb{R}^n)$, $g \in L^2(\mathbb{R}^n)$ one has

$$\mathcal{F}(f * g) = (2\pi)^{n/2}(\mathcal{F}f)(\mathcal{F}g)\ .$$

Finally, if $(\mathcal{F}f)(x) = 0$ for all $x \in \mathbb{R}^n$, then $f = 0$ in $L^1(\mathbb{R}^n)$. Using these standard facts, we now obtain the following

Theorem 4.2.5. *The Gaussian semigroup G on $L^2(\mathbb{R}^n)$ is given by $G(t)f = k_t * f$ where $k_t(x) = (4\pi t)^{-n/2} e^{-x^2/4t}$; i.e.,*

$$(G(t)f)(x) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-(x-y)^2/4t} f(y) dy \qquad x - a.e.$$

for all $f \in L^2(\mathbb{R}^n)$.

Proof. We only have to show that $\mathcal{F}k_t(x) = (2\pi)^{-n/2} e^{-tx^2}$. Using Lemma 4.2.4, substituting $z = y/\sqrt{2t}$, we have

$$\begin{aligned}
\mathcal{F}k_t(x) &= (2\pi)^{-n/2} (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-ixy} e^{-y^2/4t} dy \\
&= (2\pi)^{-n/2} (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-iz\sqrt{2t}x} e^{-z^2/2} (2t)^{n/2} dz \\
&= (2\pi)^{-n/2} \mathcal{F}h(\sqrt{2t}x) \\
&= (2\pi)^{-n/2} h(\sqrt{2t}x) \\
&= (2\pi)^{-n/2} e^{-tx^2} .
\end{aligned}$$

□

Corollary 4.2.6. *The operator A is associated with the positive closed form (V, a) where $V = H^1(\mathbb{R}^n)$ and $a(f, g) = \int_{\mathbb{R}^n} \nabla f \overline{\nabla g} dx$.*

Proof. We know from the proof of Theorem 3.4.5 that the operator B is associated with the form (W, b) where $W = L^2(\mathbb{R}^n, (1 + x^2)dx)$, $b(k, h) = \int_{\mathbb{R}^n} x^2 k(x) \overline{h(x)} dx$. Since

$$\begin{aligned}
\mathcal{F}^{-1}W &= V \text{ and} \\
(\mathcal{F}f, \mathcal{F}g) &= \sum_{j=1}^{n} \int ix_j \mathcal{F}f(x) \overline{ix_j \mathcal{F}g(x)} dx \\
&= \sum_{j=1}^{n} \int \mathcal{F}(D_j f) \overline{\mathcal{F}(D_j g)} dx \\
&= \sum_{j=1}^{n} \int D_j f \overline{D_j g} dx \\
&= a(f, g)
\end{aligned}$$

for all $f, g \in V$ the claim follows (cf. Exercise 4.2.7). □

The operator A on $L^2(\mathbb{R}^n)$ given by $D(A) = H^2(\mathbb{R}^n)$, $Af = \Delta f$ is the Dirichlet Laplacian on $L^2(\mathbb{R}^n)$ (which coincides with the Neumann Laplacian by Proposition 4.1.12).

Exercise 4.2.7. *Let (V_j, a_j) be a closed positive form on the Hilbert space H_j such that $V_j \underset{d}{\hookrightarrow} H_j$. Let A_j be the operator associated with a_j on H_j and T_j the semigroup generated by A_j. Let $U : H_1 \to H_2$ be a unitary operator. The following are equivalent:*

(i) $UV_1 = V_2$ *and* $a_2(Ux, Uy) = a_1(x, y)$ *for all* $x, y \in V_1$;

(ii) $UA_1U^{-1} = A_2$;

(iii) $UR(\lambda, A_1) = R(\lambda, A_2)U \ (\lambda > 0)$;

(iv) $UT_1(t) = T_2(t)U \ (t \geq 0)$.

Hint: Use Euler's formula (2.18) to prove that *(iii)* implies *(iv)*.

Exercise 4.2.8. *a) Let A be a selfadjoint dissipative operator on a separable Hilbert space H. Show that $-A^2$ is selfadjoint and dissipative.*
b) Show that the operator B on $L^2(\mathbb{R}^n)$ given by $D(B) = H^4(\mathbb{R}^n)$, $Bf = -\Delta^2 f$ is dissipative and selfadjoint.
c) Show that B is unitarily equivalent to the multiplication operator on $L^2(\mathbb{R}^n)$ defined by $m(x) = -(x^2)^2 = -(\sum_{j=1}^{n} x_j^2)^2$.

Exercise 4.2.9. *Give a detailed proof of Corollary 4.2.2.*

4.3 Order Properties of $\mathbf{H^1(\Omega)}$

In this section we establish some order properties of weak derivatives. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f : \Omega \to \mathbb{R}$ be measurable. We define $f^+, f^-, |f| : \Omega \to \mathbb{R}$ by $f^+(x) = \max\{f(x), 0\}$, $f^- = (-f)^+$, $|f|(x) = \max\{f(x), -f(x)\}$. Observe that $f = f^+ - f^-$ and $|f| = f^+ + f^-$. Moreover, we define $\mathrm{sign} f : \Omega \to \mathbb{R}$ by

$$\mathrm{sign} f(x) = \begin{cases} 1 & \text{if} \quad f(x) > 0 \\ 0 & \text{if} \quad f(x) = 0 \\ -1 & \text{if} \quad f(x) < 0\ . \end{cases}$$

Thus $|f| = (\mathrm{sign} f) \cdot f$. Moreover, we define $\{f > 0\} := \{x \in \Omega : f(x) > 0\}$ and similarly $\{f < 0\}$. Thus $\mathrm{sign} f = 1_{\{f>0\}} - 1_{\{f<0\}}$.

Proposition 4.3.1. *Let $f \in W^1(\Omega)$. Then $f^+, f^-, |f| \in W^1(\Omega)$ and*

$$D_j f^+ = 1_{\{f>0\}} D_j f \tag{4.4}$$

$$D_j f^- = 1_{\{f<0\}} D_j f \tag{4.5}$$

$$D_j |f| = (\mathrm{sign} f) \cdot D_j f \tag{4.6}$$

$(j = 1, \ldots, n)$.

We refer to p. 152 in D. Gilbarg, N.S. Trudinger: *Elliptic Partial Differential Equations of Second Order,* Springer, Berlin 1998 for the proof (which is not difficult).

Note that the identities in Proposition 4.3.1 have to be understood in $W^1(\Omega)$; i.e., almost everywhere on Ω. The first is equivalent to

$$-\int_{\Omega} f^+(x) D_j\varphi(x)dx = \int_{\{f>0\}} f(x)D_j\varphi(x)dx$$

for all $\varphi \in \mathcal{D}(\Omega)$.

We note the following corollary:

Corollary 4.3.2. *Let $u \in W^1(\Omega)$, $k \in \mathbb{R}$. Then*

$$D_j f(x) = 0 \text{ a.e. for } x \in \{y \in \Omega : f(y) = k\} \ .$$

Proof. Replacing f by $f - k$ we can assume that $k = 0$. Since $f = f^+ - f^-$ we have $D_j f = D_j f^+ - D_j f^- = 1_{\{f>0\}} D_j f - 1_{\{f<0\}} D_j f$. □

Corollary 4.3.3. *Let $f \in H^1(\Omega)$. Then $|f|, f^+, f^- \in H^1(\Omega)$. Moreover, if $k > 0$, then $(f \wedge k)(x) := \min\{f(x), k\}$ defines a function $f \wedge k \in H^1(\Omega)$ and*

$$D_j(f \wedge k) = 1_{\{f<k\}} D_j f \ .$$

Proof. It follows from Proposition 4.3.1 that $D_j|f| = \text{sign} f D_j f$, $D_j f^+ = 1_{\{f>0\}} D_j f$ and $D_j f^- = 1_{\{f<0\}} D_j f \in L^2(\Omega)$ $(j = 1, \ldots n)$. This implies that $|f|, f^+, f^- \in H^1(\Omega)$. Moreover, since $f \vee k = f + (k - f)^+$, one has

$$\begin{aligned} D_j(f \vee k) &= D_j f + 1_{\{k-f>0\}} D_j(k - f) \\ &= D_j f + 1_{\{f<k\}}(-D_j f) \\ &= 1_{\{f \geq k\}} D_j f \\ &= 1_{\{f>k\}} D_j f \end{aligned}$$

by Corollary 4.3.2. Hence $D_j(f \vee k) \in L^2(\Omega)$ $(j = 1, \ldots, n)$ and so $f \vee k \in H^1(\Omega)$. □

It follows from Proposition 4.3.1 that

$$\|f\|_{H^1(\Omega)} = \| \ |f| \ \|_{H^1(\Omega)} \tag{4.7}$$

for all $f \in H^1(\Omega)$.

Remark 4.3.4. *However, $H^1(\Omega)$ is not a Banach lattice, since $0 \le f \le g$ does* **not** *imply $\|f\|_{H^1} \le \|g\|_{H^1}$ (see Exercise 4.3.8a)).*

Proposition 4.3.5. *The mappings $f \mapsto |f|$, $f \mapsto f^+$ and $f \mapsto f^-$ are continuous from $H^1(\Omega)$ into $H^1(\Omega)$.*

For the proof we use the following well-known result of measure theory which shows that the Theorem of Dominated Convergence describes the most general case of convergence if we are ready to pass to subsequences. It is obtained as Corollary of the usual proof of the completeness of $L^2(\Omega)$.

Lemma 4.3.6. *Let $f_m \to f$ in $L^2(\Omega)$ $(m \to \infty)$. Then there exists a subsequence $(f_{m_k})_{k\in\mathbb{N}}$ and $h \in L^2(\Omega)$ such that*

(a) $f_{m_k} \to f(x)$ $(k \to \infty)$ a.e.;

(b) $|f_{m_k}(x)| \le h(x)$ $(x \in \Omega)$.

Proof of Proposition 4.3.5 a) Let $f_m \to f$ in $H^1(\Omega)$ as $m \to \infty$. We want to show that $f_m^+ \to f^+$ in $H^1(\Omega)$. Let $j \in \{1, \dots n\}$. We have to prove that $D_j f_m^+ \to D_j f^+$ in $L^2(\Omega)$. For that, it suffices to show that each subsequence of $(f_m)_{m\in\mathbb{N}}$ has a subsequence $(g_k)_{k\in\mathbb{N}}$ such that $D_j g_k^+ \to D_j f^+$ $(k \to \infty)$ in $L^2(\Omega)$. Thus passing to a subsequence by Lemma 4.3.6 we can assume that there exists a null set $N \subset \Omega$ such that $g_k(x) \to f(x)$, $D_j g_k(x) \to D_j f(x)$ $(k \to \infty)$, and $|D_j g(x)| \le h(x)$ for all $x \in \Omega \setminus N$, $k \in \mathbb{N}$ where $h \in L^2(\Omega)$. Let $x \in \Omega \setminus N$ such that $f(x) > 0$. Then $g_k(x) > 0$ for large $k \in \mathbb{N}$. Hence by (4.4), $D_j g_k^+(x) = D_j g_k(x) \to D_j f(x) = D_j f^+(x)$ $(k \to \infty)$. Let $x \in \Omega \setminus N$ such that $f(x) < 0$. Then $g_k(x) < 0$ for large $k \in \mathbb{N}$ and $D_j g_k^+(x) = 0 = D_j f^+(x)$. Since by Corollary 4.3.2 $D_j f(x) = 0$ a.e. on $\{y \in \Omega : f(y) = 0\}$ we conclude that $D_j g_k(x) \to D_j f(x)$ $(k \to \infty)$ a.e. Now it follows from the Dominated Convergence Theorem that $D_j g_k \to D_j f$ $(k \to \infty)$ in $L^2(\Omega)$.
b) We have shown that the mapping $f \mapsto f^+$ is continuous. Since $f^- = (-f)^+$ and $|f| = f^+ + f^-$, continuity of the mapping $f \mapsto f^-$ and $f \mapsto |f|$ are immediate consequences.

Corollary 4.3.7. *Let $f \in H_0^1(\Omega)$. Then $f^+, f^-, |f| \in H_0^1(\Omega)$.*

Proof. a) Let $\varphi \in \mathcal{D}(\Omega)$. Then $\varphi^+ \in H^1(\Omega)$ by Proposition 4.3.1. Since φ^+ has compact support, it follows from Lemma 4.1.11 that $\varphi^+ \in H_0^1(\Omega)$.
b) Let $f \in H_0^1(\Omega)$. Let $f_m \in \mathcal{D}(\Omega)$ such that $f_m \to f$ in $H^1(\Omega)$ as $m \to \infty$. Then $f_m^+ \in H_0^1(\Omega)$ by a) and $\lim\limits_{m\to\infty} f_m^+ \to f^+$ in $H^1(\Omega)$ by Proposition 4.3.5. Thus $f^+ \in H_0^1(\Omega)$. Hence also $f^- = (-f)^+$ and $|f| = f^+ + f^- \in H_0^1(\Omega)$. □

Further order properties of $H^1(\Omega)$ are established in the following exercises.

Exercise 4.3.8. *Let $f, g \in H^1(\Omega)$ such that $f \leq g$.*
a) Show that the **order interval**

$$[f, g] = \{h \in H^1(\Omega) : f \leq h \leq g\} \ .$$

is closed in $H^1(\Omega)$.
b) Show by an example that order intervals are not bounded in general.
c) Show that $[f, g] \subset H_0^1(\Omega)$ if $f, g \in H_0^1(\Omega)$.
d) Let $f \in H_0^1(\Omega)$, $k > 0$. Show that $f \wedge k \in H_0^1(\Omega)$.

It follows from Corollary 4.3.7 and Exercise 4.3.8 that $H_0^1(\Omega)$ is a closed **ideal** of $H^1(\Omega)$. Here a subspace $\mathcal{J}$ of $H^1(\Omega)$ is called in **ideal** if

a) $f \in \mathcal{J} \Rightarrow |f| \in \mathcal{J}$ and

b) $0 \leq g \leq f$, $f \in \mathcal{J}$, $g \in H^1(\Omega) \Rightarrow g \in \mathcal{J}$.

Exercise 4.3.9. *For $f : \Omega \to \mathbb{R}$ we denote by $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ the extension of f by 0; i.e.,*

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if} \quad x \in \Omega \\ 0 & \text{if} \quad x \in \mathbb{R}^n \setminus \Omega \end{cases}$$

a) Let $f \in H_0^1(\Omega)$. Show that $\tilde{f} \in H^1(\mathbb{R}^n)$ and

$$D_j \tilde{f} = \widetilde{D_j f} \quad (j = 1, \dots n) \ .$$

b) Let

$$\tilde{H}_0^1(\Omega) = \{f : \Omega \to \mathbb{R} : \tilde{f} \in H^1(\mathbb{R}^n)\} \ .$$

Show that $\tilde{H}_0^1(\Omega)$ is a closed ideal in $H^1(\Omega)$ such that $H_0^1(\Omega) \subset \tilde{H}_0^1(\Omega)$.
c) Let $n = 1$, $\Omega = (-1, 0) \cup (0, 1)$. Show that $\tilde{H}_0^1(\Omega)$ can be identified with $H_0^1(-1, 1)$. Deduce that $\tilde{H}_0^1(\Omega) \neq H_0^1(\Omega)$.

Exercise 4.3.10. *Let*

$$\mathcal{D}(\Omega)_+ = \{\varphi \in \mathcal{D}(\Omega) : \varphi \geq 0\} \ , \ H_0^1(\Omega) = \{f \in H_0^1(\Omega) : f \geq 0\} \ .$$

Show that $\overline{\mathcal{D}(\Omega)_+}^{H^1(\Omega)} = H_0^1(\Omega)_+$.

4.4 Positivity and Monotonicity

The aim of this section is to establish order properties of the semigroups generated by the Dirichlet and Neumann Laplacian. First of all we show that these semigroups are positive.
In the case of Dirichlet boundary conditions, the semigroups are monotonic with

respect to the domain. In particular, the semigroup is always dominated by the Gaussian semigroup. This shows us that it is always given by a kernel, the so-called **heat kernel**.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leq p \leq \infty$. By $L^p(\Omega)_+ = \{f \in L^p(\Omega) : f \geq 0\}$ we denote the **positive cone** in $L^p(\Omega)$ (where $f \geq 0$ means that $f(x) \in \mathbb{R}_+$ a.e.). A bounded operator B on $L^p(\Omega)$ is called **positive** (we write $B \geq 0$) if $BL^p(\Omega)_+ \subset L^p(\Omega)_+$. Finally, a C_0-semigroup T on $L^p(\Omega)$ is called **positive** if $T(t) \geq 0$ for all $t \geq 0$.

Now we consider the C_0-semigroups $(e^{t\Delta_\Omega^D})_{t\geq 0}$ and $(e^{t\Delta_\Omega^N})_{t\geq 0}$ on $L^2(\Omega)$ generated by the Dirichlet Laplacian Δ_Ω^D and the Neumann Laplacian Δ_Ω^N, respectively.

Proposition 4.4.1. *The semigroups* $(e^{t\Delta_\Omega^D})_{t\geq 0}$ *and* $(e^{t\Delta_\Omega^N})_{t\geq 0}$ *on* $L^2(\Omega)$ *are positive.*

Proof. Let $A = \Delta_\Omega^D$ or Δ_Ω^N, $T(t) = e^{tA}$. Since by Euler's formula (Exercise(2.4.6)),

$$e^{tA} = \lim_{n\to\infty} (I - \frac{t}{n}A)^{-n}$$

strongly, it suffices to show that $R(\lambda, A) \geq 0$ for $\lambda > 0$. Let $a(f, g) = \int_\Omega \nabla f \cdot \overline{\nabla g} dx$ $(f, g \in H^1(\Omega))$. Let $V = H_0^1(\Omega)$ if $A = \Delta_\Omega^D$ and $V = H^1(\Omega)$ in the case $A = \Delta_\Omega^N$. Let $0 \geq f \in L^2(\Omega)$, $u = R(\lambda, A)f \in V$. It is clear that u is real-valued. We have to show that $u \leq 0$. One has

$$\begin{aligned}
\lambda\|u^+\|_{L^2}^2 &= (\lambda u \mid u^+) \\
&= (\lambda u - Au \mid u^+) + (Au \mid u^+) \\
&= (f \mid u^+) - a(u, u^+) && (\text{since } \ u^+ \in V) \\
&\leq -a(u, u^+) && (\text{since } \ f \leq 0) \\
&= -\sum_{j=1}^n \int_\Omega D_j u D_j u^+ dx \\
&= -\sum_{j=1}^n \int_\Omega (D_j u^+)^2 dx && (\text{by } (4.4)) \\
&\leq 0\ .
\end{aligned}$$

Hence $u^+ = 0$; i.e., $u \leq 0$. □

Next we want to compare the semigroups on different open sets. For this, the following convention is useful. If f is a scalar-valued function we identify f with its extension by 0 to $\mathbb{R}^n$. In this way $L^2(\Omega)$ is a closed subspace of $L^2(\mathbb{R}^n)$. Moreover, f is positive in $L^2(\Omega)$ if and only if it is positive as element of $L^2(\mathbb{R}^n)$. Note also that, we may write $L^2(\mathbb{R}^n)$ as direct sum of $L^2(\Omega)$ and $L^2(\Omega^c)$, the decomposition of $f \in L^2(\mathbb{R}^n)$ being given by

$$f = f \cdot 1_\Omega + f \cdot 1_{\Omega^c}$$

where $\Omega^c = \mathbb{R}^n \setminus \Omega$. It is an **order direct sum**; i.e. $f \geq 0$ if and only if $f \cdot 1_\Omega \geq 0$ and $f \cdot 1_{\Omega^c} \geq 0$.

If B is a bounded operator on $L^p(\Omega)$ we may extend B to $L^p(\mathbb{R}^n)$ by defining

$$Bf := B(1_\Omega f) \quad (f \in L^p(\mathbb{R}^n)) \ .$$

In that way $\mathcal{L}(L^p(\Omega))$ becomes a subspace of $\mathcal{L}(L^p(\mathbb{R}^n))$ such that $B \geq 0$ in $\mathcal{L}(L^p(\Omega))$ if and only if $B \geq 0$ in $\mathcal{L}(L^p(\mathbb{R}^n))$.

If T is a semigroup on $L^p(\Omega)$, considering $T(t)$ as operator on $L^p(\mathbb{R}^n)$ the semigroup property

$$T(t+s) = T(t)T(s) \quad (t, s \geq 0)$$

still holds. But $T(0)$ is the projection onto $L^2(\Omega)$ along $L^2(\Omega^c)$. Moreover, the mapping $t \mapsto T(t) : \mathbb{R}_+ \to \mathcal{L}(\mathbb{R}^n)$ is strongly continuous.

If B_1, B_2 are bounded operators on $L^2(\mathbb{R}^n)$ we write $B_1 \leq B_2$ if $B_2 - B_1 \geq 0$. Our aim is to prove the following comparison result.

Theorem 4.4.2. *a) One has always*

$$0 \leq e^{t\Delta_\Omega^D} \leq e^{t\Delta_\Omega^N} \ . \tag{4.8}$$

b) If $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ are open such that $\Omega_1 \subset \Omega_2$, then

$$0 \leq e^{t\Delta_{\Omega_1}^D} \leq e^{t\Delta_{\Omega_2}^D} \ . \tag{4.9}$$

For the proof we use the notion of positive distributions. By $\mathcal{D}(\Omega)'$ we denote the space of all distributions. For $u \in \mathcal{D}(\Omega)'$ we write

$$u \geq 0 \text{ if } u(\varphi) \geq 0 \text{ for all } \varphi \in \mathcal{D}(\Omega)_+ \ .$$

Here $\mathcal{D}(\Omega)_+ := \{\varphi \in \mathcal{D}(\Omega) : \varphi \geq 0\}$. We identify $L^1_{\text{loc}}(\Omega)$ with a subspace of $\mathcal{D}(\Omega)'$ by defining $u_f \in \mathcal{D}(\Omega)'$ by

$$u_f(\varphi) = \int_\Omega f\varphi dx \quad (\varphi \in \mathcal{D}(\Omega))$$

whenever $f \in L^1_{\text{loc}}(\Omega)$. Then

$$u_f \geq 0 \text{ if and only if } f \geq 0 \ .$$

If $u \in \mathcal{D}(\Omega)'$ the Laplacian $\Delta u \in \mathcal{D}(\Omega)'$ is defined by

$$(\Delta u)(\varphi) = u(\Delta\varphi) \quad (\varphi \in \mathcal{D}(\Omega)) \ .$$

For $u, v \in \mathcal{D}(\Omega)'$ we write

$$u \leq v \text{ if and only if } u(\varphi) \leq v(\varphi) \text{ for all } \varphi \in \mathcal{D}(\Omega)_+ \ .$$

Moreover, we let $H^1(\Omega)_+ := L^2(\Omega)_+ \cap H^1(\Omega)$.

Lemma 4.4.3. *Let $\lambda > 0$, $u \in H_0^1(\Omega)$, $0 \le v \in H^1(\Omega)$ such that*

$$\lambda u - \Delta u \le \lambda v - \Delta v \quad \text{in } \mathcal{D}(\Omega)' .$$

Then $u \le v$.

Proof. Let $0 \le \varphi \in \mathcal{D}(\Omega)$. Then

$$\int_\Omega \lambda u \varphi dx + \int_\Omega \nabla u \nabla \varphi dx \le \lambda \int_\Omega v \varphi dx + \int_\Omega \nabla v \nabla \varphi dx \tag{4.10}$$

for all $0 \le \varphi \in \mathcal{D}(\Omega)$. It follows by density that (4.10) remains true for all $\varphi \in H_0^1(\Omega)_+$ (see Exercise 4.3.10). Note that $(u-v)^+ \in H_0^1(\Omega)$. In fact, let $u_k \in \mathcal{D}(\Omega)$ such that $u_k \to u$ in $H^1(\Omega)$ as $k \to \infty$. Then $(u_k - v)^+$ has compact support, hence $(u_k - v)^+ \in H_0^1(\Omega)$ by Lemma 4.1.11. It follows that $(u-v)^+ = \lim_{k\to\infty} (u_k - v)^+ \in H_0^1(\Omega)$. Now it follows from (4.10) applied to $\varphi := (u-v)^+$ that

$$\begin{aligned} &\int_\Omega \lambda u (u-v)^+ dx + \int_\Omega \nabla u \nabla (u-v)^+ dx \\ \le\ &\lambda \int_\Omega v(u-v)^+ dx + \int_\Omega \nabla v \nabla (u-v)^+ dx . \end{aligned}$$

Hence

$$\begin{aligned} \int_\Omega \lambda (u-v)^{+2} dx &= \int_\Omega \lambda (u-v)(u-v)^+ dx \\ &\le \int_\Omega \nabla (v-u) \nabla (u-v)^+ \\ &= -\int_\Omega |\nabla (u-v)^+|^2 dx \quad \text{(by (4.4))} \\ &\le 0 . \end{aligned}$$

It follows that $(u-v)^+ \le 0$; i.e., $u \le v$. □

Proof of Theorem 4.4.2. a) Since $e^{tA} = \lim_{n\to\infty} (I - \frac{t}{n} A)^{-n}$ strongly, where $A = \Delta_\Omega^D$ or $A = \Delta_\Omega^N$, it suffices to show that

$$R(\lambda, \Delta_\Omega^D) \le R(\lambda, \Delta_\Omega^N) \quad (\lambda > 0) .$$

Let $0 < \lambda$ and $R(\lambda, \Delta_\Omega^D) f = u$, $R(\lambda, \Delta_\Omega^N) f = v$ where $0 \le f \in L^2(\Omega)$. Then $u \in H_0^1(\Omega)_+$, $v \in H^1(\Omega)_+$ and

$$\lambda u - \Delta u = f = \lambda v - \Delta v \text{ in } \mathcal{D}(\Omega)' .$$

It follows from Lemma 4.4.3 that $u \le v$.
b) Let $\lambda > 0$, $0 \le f \in L^2(\Omega_1)$. We have to show that

$$u := R(\lambda, \Delta_{\Omega_1}^D) f \le R(\lambda, \Delta_{\Omega_2}^D) f := v .$$

One has $u \in H_0^1(\Omega_1)_+$, $v_{|\Omega_1} \in H^1(\Omega_1)$ and $\lambda u - \Delta u = f = \lambda v - \Delta v$ in $\mathcal{D}(\Omega_1)'$. It follows from Lemma 4.4.3 that $u \leq v$. □

It follows from Theorem 4.4.2 that

$$0 \leq e^{t\Delta_\Omega^D} \leq G(t) \quad (t \geq 0)$$

where G denotes the Gaussian semigroup on $L^2(\mathbb{R}^n)$. From this we can deduce that $e^{t\Delta_\Omega^D}$ is a kernel operator. We need a simple criterion, the Dunford-Pettis Theorem, which describes operators defined by bounded kernels.

Let $k \in L^\infty(\Omega \times \Omega)$. Then

$$(B_k f)(x) = \int_\Omega k(x,y)f(y)dy \tag{4.11}$$

defines a bounded operator $B_k \in \mathcal{L}(L^1(\Omega), L^\infty(\Omega))$ and

$$\|B_k\|_{\mathcal{L}(L^1(\Omega),L^\infty(\Omega))} \leq \|k\|_{L^\infty(\Omega\times\Omega)} \ .$$

If $E \subset \mathbb{R}^n$ is a Borel set we denote by $|E|$ the Lebesgue measure of E.

Theorem 4.4.4 (Dunford Pettis)**.** *The mapping $k \mapsto B_k$ is an isometric isomorphism from $L^\infty(\Omega \times \Omega)$ onto $\mathcal{L}(L^1(\Omega), L^\infty(\Omega))$. Moreover*

$$B_k \geq 0 \textit{ if and only if } k \geq 0 \tag{4.12}$$

for all $k \in L^\infty(\Omega \times \Omega)$.

Proof. For $f, g \in L^1(\Omega)$ we define $f \otimes g \in L^1(\Omega \times \Omega)$ by $(f \otimes g)(x,y) = f(x)g(y)$. Then $\|f \otimes g\|_{L^1(\Omega\times\Omega)} = \|f\|_{L^1(\Omega)} \cdot \|g\|_{L^1(\Omega)}$. It follows from the construction of the product measure that the space

$$F := \{\sum_{i=1}^n c_i 1_{E_i} \otimes 1_{F_i} : n \in \mathbb{N} \ , \ c_i \in \mathbb{C} \ , \ E_i, F_i \subset \Omega \text{ measurable of finite measure}\}$$

is dense in $L^1(\Omega \times \Omega)$. Let $B \in \mathcal{L}(L^1, L^\infty)$. Define $\phi : F \to \mathbb{C}$ by

$$\phi(u) = \sum_{i=1}^m c_i \int_\Omega (B1_{E_i})(y) \cdot 1_{F_i}(y)dy$$

where $u = \sum_{i=1}^m c_i 1_{E_i} \otimes 1_{F_i}$. It is easy to see that this definition is independent of the representation of u (see Exercise 4.4.11). Hence $\phi : F \to \mathbb{C}$ is a linear mapping. We show that

$$|\phi(u)| \leq \|B\|_{\mathcal{L}(L^1,L^\infty)} \cdot \|u\|_{L^1(\Omega\times\Omega)} \ .$$

For that we can assume that $E_i \cap E_j = \emptyset$ for $i \neq j$. This implies that

$$\|u\|_{L^1(\Omega\times\Omega)} = \sum_{i=1}^{m} c_i |E_i|\ |F_i|\ .$$

$$\begin{aligned}\text{Hence } |\phi(u)| &\leq \sum_{i=1}^{m} |c_i|\ \|B1_{E_i}\|_{L^\infty(\Omega)}\|1_{F_i}\|_{L^1(\Omega)} \\ &\leq \sum_{i=1}^{m} |c_i|\ \|B\|_{\mathcal{L}(L^1,L^\infty)}\|1_{E_i}\|_{L^1}\|1_{F_i}\|_{L^1} \\ &= \|B\|_{\mathcal{L}(L^1,L^\infty)}\|u\|_{L^1(\Omega\times\Omega)}\ .\end{aligned}$$

Since $(L^1(\Omega\times\Omega))' = L^\infty(\Omega\times\Omega)$, there exists a function $k \in L^\infty(\Omega\times\Omega)$ such that $\|k\|_{L^\infty(\Omega\times\Omega)} \leq \|B\|_{\mathcal{L}(L^1,L^\infty)}$ and

$$\phi(u) = \int_\Omega \int_\Omega u(y,x)k(x,y)dydx$$

for all $u \in F$. In particular, for simple functions $f, g \in L^1(\Omega)$ we have

$$\begin{aligned}\int_\Omega (Bf)gdy = \phi(f\otimes g) &= \int_\Omega \int_\Omega f(y)k(x,y)dyg(x)dx \\ &= \int (B_k f)(x)g(x)dx\ .\end{aligned}$$

It follows that $Bf = B_k f$ for all simple functions f in $L^1(\Omega)$. Hence $B = B_k$. We have shown that the mapping $k \mapsto B_k : L^\infty(\Omega\times\Omega) \to \mathcal{L}(L^1(\Omega), L^\infty(\Omega))$ is surjective and isometric. Finally, since functions of the type

$$u = \sum_{j=1}^{m} f_j \otimes g_j \text{ with } f_j, g_j \in L^1(\Omega)_+$$

are dense in $L^1(\Omega\times\Omega)_+$ it follows that $B_k \geq 0$ if and only if $\int\limits_{\Omega\times\Omega} uk \geq 0$ for all $u \in L^1(\Omega\times\Omega)_+$; i.e., if and only if $k \geq 0$ a.e. □

Let $B \in \mathcal{L}(L^p(\Omega))$ where $1 \leq p < \infty$. We define

$$\|B\|_{\mathcal{L}(L^1,L^\infty)} := \sup\{\|Bf\|_{L^p} : f \in L^1 \cap L^p\ ,\ \|f\|_{L^1} \leq 1\}\ .$$

Corollary 4.4.5. *Let $1 \leq p < \infty$, $B \in \mathcal{L}(L^p(\Omega))$ such that*

$$\|B\|_{\mathcal{L}(L^1,L^\infty)} < \infty\ . \tag{4.13}$$

Then there exists a function $k \in L^\infty(\Omega \times \Omega)$ *such that*

$$(Bf)(x) = \int_\Omega k(x,y)f(y)dy \ a.e. \tag{4.14}$$

for all $f \in L^1(\Omega) \cap L^p(\Omega)$. *In that case* $B \geq 0$ *if and only if* $k \geq 0$. *Let* $1 \leq p < \infty$.

We say that an operator $B \in \mathcal{L}(L^p(\Omega))$ has a **bounded kernel** if $\|B\|_{\mathcal{L}(L^1,L^\infty)} < \infty$. Then there exists a unique $k \in L^\infty(\Omega \times \Omega)$ such that $Bf = B_k f$ for all $f \in L^1(\Omega) \cap L^p(\Omega)$. We call k **the kernel of** B. Note, if Ω has finite measure, then $L^p(\Omega) \subset L^1(\Omega)$, thus B is given by (4.14) for all $f \in L^p(\Omega)$. It is worth it to state explicitely the following (obvious) domination property.

Corollary 4.4.6. *Let* $1 \leq p < \infty$, $B_1, B_2 \in \mathcal{L}(L^p(\Omega))$ *such that* $0 \leq B_1 \leq B_2$. *Assume that* B_2 *has a bounded kernel* k_2. *Then* B_1 *has a bounded kernel* $k_1 \in L^\infty(\Omega \times \Omega)$ *and*

$$0 \leq k_1(x,y) \leq k_2(x,y) \quad a.e. \ .$$

We apply the preceding results to the semigroup generated by the Dirichlet Laplacian.

Theorem 4.4.7. *Let* $\Omega \subset \mathbb{R}^n$ *be open. Then* $e^{t\Delta_\Omega^D}$ *has a bounded kernel* k_t *satisfying*

$$0 \leq k_t(x,y) \leq (4\pi t)^{-n/2} e^{-|x-y|^2/4t} \quad a.\ e.$$

for all $t > 0$.

Proof. It follows from Theorem 4.4.2 that $0 \leq e^{t\Delta_\Omega^D} \leq G(t)$. By the results of Section 4.3 the operator $G(t)$ has the bounded kernel $(4\pi t)^{-n/2}e^{-|x-y|^2/4t}$. So the claim follows from Corollary 4.4.6. □

Corollary 4.4.8. *Let* $\Omega \subset \mathbb{R}^n$ *be open of finite measure. Then the operator* $e^{t\Delta_\Omega^D}$ *is compact for every* $t > 0$. *Consequently,* Δ_Ω^D *has compact resolvent and the embedding*

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega)$$

is compact.

Proof. Since $e^{t\Delta_\Omega^D}$ has a bounded kernel, it is a Hilbert-Schmidt operator. Thus $e^{t\Delta_\Omega^D}$ is compact for $t > 0$. Consequently, $R(\lambda, \Delta_\Omega^D)$ is compact for $\lambda > 0$ (see Exercise (4.4.9c)). It follows from Theorem (3.4.7) that the injection $H_0^1 \hookrightarrow L^2(\Omega)$ is compact. □

Exercise 4.4.9. *Let* T *be a* C_0*-semigroup on a Banach space* X. *Assume that* $T(t)$ *is compact for all* $t > 0$.
a) Show that $T : (0, \infty) \to \mathcal{L}(X)$ *is continuous for the operator norm on* $\mathcal{L}(X)$.
b) Deduce that $Q(\lambda) = \int_0^1 e^{-\lambda t}T(t)dt = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 e^{-\lambda t}T(t)dt$ *is a compact operator.*
c) Show that $R(\lambda, A)$ *is compact for all* $\lambda \in \varrho(A)$.

Hint: Use that $(\lambda - A)Q(\lambda)x = x - e^{-\lambda}T(1)x$.

Exercise 4.4.10. *Let $\Omega \subset \mathbb{R}^n$ be open of finite measure. Show that*

$$\|e^{t\Delta_\Omega^D}\|_{\mathcal{L}(L^2(\Omega))} \leq ct^{-n/2} \quad (t > 0) \ .$$

Hint: Use that $0 \leq e^{t\Delta_\Omega^D} \leq G(t)$.

The next exercise gives a measure theoretic argument needed in the proof of the Dunford-Pettis Theorem. Let $\Omega \subset \mathbb{R}^n$ be open. If A is a Borel set in $\Omega \times \Omega$ then it is well-known that $A_1(y) = \{x \in \Omega : (x, y) \in A\}$ is a Borel set for all $y \in \Omega$.

Exercise 4.4.11. *a) Let $A \subset \Omega \times \Omega$ be a Borel set. Show that the following assertions are equivalent:*

(i) A has a Lebesgue measure 0;

(ii) there exists a Borel null set N in Ω such that for each $y \in \Omega \setminus N$, the set $A_1(y)$ has measure zero.

Hint: *Use Fubini's theorem.*

b) Convince yourself that assertion a) can be reformulated in the following way. Let $P(x, y)$ be an assertion for each $(x, y) \in \Omega \times \Omega$. Then $P(x, y)$ is true for almost all $(x, y) \in \Omega \times \Omega$ if and only if for almost all $y \in \Omega$, $P(x, y)$ holds x -a.e..

c) Let $f_i, g_i \in L^1(\Omega)$ such that $u(x, y) = \sum_{i=1}^{m} f_i(x)g_i(y) = 0 \quad (x, y)$-a.e.. Let B be a bounded operator on $L^1(\Omega)$. Show that

$$\sum_{i=1}^{n} \int_\Omega (Bf_i)(y)g_i(y)dy = 0 \ .$$

4.5 The Neumann Laplacian and the extension property

We have seen in the last section that Δ_Ω^D has compact resolvent whenever $\Omega \subset \mathbb{R}^n$ is a bounded open set. This is not true for the Neumann Laplacian, in general. We give at first a 1-dimensional example which illustrates the situation very well.

Example 4.5.1. *Let $\Omega = (0, 1) \setminus \{\frac{1}{n} : n \in \mathbb{N}, n \geq 2\}$. Then Δ_Ω^N has not a compact resolvent in $L^2(\Omega)$. By Theorem 3.4.7, this assertion is equivalent to saying that the injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is not compact.*

Proof. Let $v_n = c_n 1_{(\frac{1}{n+1}, \frac{1}{n})}$ where $c_n > 0$ is chosen such that $\|v_n\|_{L^2(\Omega)} = 1$. Then $v_n \in H^1(\Omega)$ and $v_n' = 0$. Thus, the sequence $(v_n)_{n \in \mathbb{N}}$ is bounded in $H^1(\Omega)$. However, $\|v_n - v_m\|_{L^2(\Omega)}^2 = 2$ whenever $n \neq m$. Hence the sequence has no convergent subsequence in $L^2(\Omega)$. □

In the preceding example, the set $\Omega \subset \mathbb{R}$ is open and bounded, but not connected. This might look artificial. However, it is easy to modify the example a little bit in order to produce an example of a connected bounded open set Ω in $\mathbb{R}^2$ such that Δ_Ω^N does not have a compact resolvent:

Example 4.5.2. *Let*

$$\Omega = \{(x,y) \in (0,1) \times (0,1) : x \neq \frac{1}{n} \text{ for all } n \in \mathbb{N} \text{ whenever } 0 < y \leq \frac{1}{2}\} .$$

Then Ω is open, bounded, connected, but the injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is not compact.

Proof. Let $\varphi \in C^\infty[0,1]$ such that $\int_0^1 \varphi(x)^2 dx = 1$ and $\operatorname{supp}\varphi \subset [0, \frac{1}{4}]$. Define

$$u_n(x,y) = \begin{cases} \sqrt{(n+1)n}\varphi(y) & \text{if} \quad x \in (\frac{1}{n+1}, \frac{1}{n}) \\ 0 & \text{if} \quad x \notin (\frac{1}{n+1}, \frac{1}{n}) . \end{cases}$$

Then $u_n \in C^\infty(\Omega)$ and

$$\begin{aligned} \|u_n\|^2_{L^2(\Omega)} &= \int_0^1 \int_{\frac{1}{n+1}}^{1/n} u_n(x,y)^2 dxdy \\ &= \int_0^1 (\frac{1}{n} - \frac{1}{n+1})(n+1)n\varphi(y)^2 dy \\ &= 1 . \end{aligned}$$

Moreover, $\dfrac{\partial u_n}{\partial x} \equiv 0$ and

$$\frac{\partial u_n}{\partial y}(x,y) = \begin{cases} \sqrt{(n+1)n}\varphi'(y) & \text{if} \quad x \in (\frac{1}{n+1}, \frac{1}{n}) \\ 0 & \text{if} \quad x \notin (\frac{1}{n+1}, \frac{1}{n}) . \end{cases}$$

Thus $u_n \in H^1(\Omega)$ and

$$\begin{aligned} \|u_n\|^2_{H^1(0,1)} &= 1 + (\frac{1}{n} - \frac{1}{n+1})(n+1)n \int_0^1 \varphi'(y)^2 dy \\ &= 1 + \int_0^1 \varphi'(y)^2 dy . \end{aligned}$$

Thus $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $H^1(\Omega)$. However, since

$$\|u_n - u_m\|^2_{L^2(\Omega)} = 2$$

whenever $n \neq m$, this sequence does not have a convergent subsequence. $\square$

The bounded sets considered in these two examples have a very irregular boundary. And indeed, if the boundary of a bounded open set Ω is sufficiently regular, then the Neumann Laplacian in $L^2(\Omega)$ has a compact resolvent. In order to see this, we introduce the following extension property.

Definition 4.5.3. *An open set $\Omega \subset \mathbb{R}^n$ has the* **extension property** *if there exists a bounded linear operator*

$$Q : H^1(\Omega) \to H^1(\mathbb{R}^n)$$

such that $(Qu)_{|\Omega} = u$ for all $u \in H^1(\Omega)$. Such an operator is called an **extension operator**.

Examples 4.5.4. *a) Assume that Ω is a bounded open set whose boundary is Lipschitz continuous. Then Ω has the extension property.*
b) If Ω is an open bounded set with boundary of class C^1, then Ω has the extension property. This is a special case of a).
c) If Ω is unbounded with finite volume, then Ω has not the extension property.

For these examples we refer to R. Adams: *Sobolev Spaces*, Acad. Press 1975, and also to H. Brézis: *Analyse Fonctionelle*, Masson, Paris 1983 in the case b).

Theorem 4.5.5. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set. If Ω has the extension property, then the injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Consequently, Δ_Ω^N has a compact resolvent.*

Proof. Denote by $Q : H^1(\Omega) \to H^1(\mathbb{R}^n)$ an extension operator. Let B be an open ball containing $\bar{\Omega}$. Let $\xi \in C_c^\infty(B)$ such that $\xi_{|\bar{\Omega}} = 1$. Define $Q_1 : H^1(\Omega) \to H_0^1(B)$ by $Q_1 u = (\xi Q u)_{|B}$ $(u \in H^1(\Omega))$. Note that $\xi Q u$ has a compact support contained in B. So $(\xi Q u)_{|B} \in H_0^1(B)$ for all $u \in H^1(\Omega)$ by Lemma 4.1.11 . Then Q_1 is a bounded operator and $(Q_1 u)_{|\Omega} = u$ for all $u \in H^1(\Omega)$. Since the injection of $H_0^1(B)$ into $L^2(B)$ is compact, for each bounded sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1(\Omega)$, there is a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that $(Q_1 u_{n_k})_{k\in\mathbb{N}}$ converges in $L^2(B)$, and consequently also in $L^2(\Omega)$. Thus $(u_{n_k})_{k\in\mathbb{N}}$ converges in $L^2(\Omega)$. We have shown that the injection of $H^1(\Omega)$ into $L^2(\Omega)$ is compact. $\square$

In the following exercise we show that the extension porperty is not necessary to have a compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$.

Exercise 4.5.6. *Let $\Omega = (-1, 0) \cup (0, 1) \subset \mathbb{R}$. Show that Ω has not the extension property. Still, the injection of $H^1(\Omega)$ into $L^2(\Omega)$ is compact.*

Exercise 4.5.7. *Let $\Omega \subset \mathbb{R}^n$ be open. For $u : \Omega \to \mathbb{R}$ we set*

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if} \quad x \in \Omega \\ 0 & \text{if} \quad x \in \mathbb{R}^n \setminus \Omega \ . \end{cases}$$

Let $\tilde{H}_0^1(\Omega) = \{u \in L^2(\Omega) : \tilde{u} \in H^1(\mathbb{R}^n)\}$.
a) Assume that Ω *has the extension property. Show that* $\tilde{H}_0^1(\Omega)$ *is an ideal in* $H^1(\Omega)$.
b) Show that for $\Omega = (-1,0) \cup (0,1) \subset \mathbb{R}$*, the space* $\tilde{H}_0^1(\Omega)$ *is not an ideal in* $H^1(\Omega)$.

Exercise 4.5.8. *Let* $\Omega = \{(x,y) \in (0,1) \times (0,1) : y > \frac{1}{2} \text{ if } x = \frac{1}{2}\}$*. Show that the injection of* $H^1(\Omega)$ *into* $L^2(\Omega)$ *is compact.*

Hint: Use that the injection $H^1(\Omega_j) \hookrightarrow L^2(\Omega_j)$ are compact, $j = 1, 2$, where

$$\Omega_1 = (0, \frac{1}{2}) \times (0,1) \text{ and } \Omega_2 = (\frac{1}{2}, 1) \times (0,1) \; .$$

Exercise 4.5.9. *Let* $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ *be open such that* $\Omega_1 \subset \Omega_2$ *and such that* $\Omega_2 \setminus \Omega_1$ *is a null set. Assume that the injection of* $H^1(\Omega_1)$ *into* $L^2(\Omega_1)$ *is compact. Show that also the injection of* $H^1(\Omega_2)$ *into* $L^2(\Omega_2)$ *is compact.*

Compactness of the injection of $H^1(\Omega)$ into $L^2(\Omega)$ depends on local regularity of the boundary of Ω. This is made more precise in the following two exercises. The first shows that we may perturb any bounded open set in an arbitrarily small neighborhood of a boundary point in such a way that the new open set does not have the compact embedding property.

Exercise 4.5.10. *Let* $\Omega \subset \mathbb{R}^2$ *an arbitrary bounded open set,* $z \in \partial\Omega$*,* $\varepsilon > 0$*. Then there exists an open set* $\tilde{\Omega} \subset \mathbb{R}^2$ *such that*

$$\Omega \triangle \tilde{\Omega} \subset B(z, \varepsilon)$$

and the injection of $H^1(\tilde{\Omega})$ *into* $L^2(\tilde{\Omega})$ *is not compact. Here* $\Omega \triangle \tilde{\Omega} := (\Omega \setminus \tilde{\Omega}) \cup (\tilde{\Omega} \setminus \Omega)$ *denotes the symmetric difference of the two open sets.*

Hint: Take a tiny version Ω_1 of Example 4.5.2 such that $\Omega_1 \subset B(z, \varepsilon)$. It yields a sequence of functions $u_n \in C^\infty(\Omega_1)$ with compact support in $B(z, \varepsilon)$ bounded in $H^1(\Omega_1)$ but having no converging subsequence in $L^2(\Omega_1)$. Now choose $\tilde{\Omega} = (\Omega \setminus \bar{B}(z, \varepsilon)) \cup \Omega_1$. Of course, if Ω is connected by a slight modification one can arrange things such that also $\tilde{\Omega}$ is connected.

In the next exercise we show how a local version of the compact embedding property leads to the corresponding global property.

Exercise 4.5.11. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set. Assume that for each* $z \in \partial\Omega$ *there exists an* $\varepsilon > 0$ *such that the injection* $H^1(B(z,\varepsilon) \cap \Omega) \hookrightarrow L^2(B(z,\varepsilon) \cap \Omega)$ *is compact . Show that the injection of* $H^1(\Omega)$ *into* $L^2(\Omega)$ *is compact.*

Hint: Cover $\bar{\Omega}$ by a finite number of balls B_k such that the injection $H^1(\Omega \cap B_k)$ into $L^2(\Omega \cap B_k)$ is compact for each k.

Finally we show that the extension property implies density of smooth functions in $H^1(\Omega)$.

Exercise 4.5.12. *a) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with extension property. show that the space*

$$\{\varphi_{|\Omega} : \varphi \in \mathcal{D}(\mathbb{R}^n)\}$$

is dense in $H^1(\Omega)$.
b) Show that the assertion of a) is wrong for $\Omega = (-1,0) \cup (0,1)$.

Hint: Use Proposition 4.1.12.

We should mention that by the density theorem due to Meyers and Serrin, the space $C^\infty(\Omega) \cap H^1(\Omega)$ is always dense in $H^1(\Omega)$, without any restriction to the open subset Ω of $\mathbb{R}^n$.

4.6 Classical Solutions

In this section we prove interior regularity and deduce from this that the semigroups generated by the Dirichlet and Neumann Laplacian yield classical solutions of the heat equation. However, we do not investigate the behaviour of the solutions at the boundary of the open set.

Let $\Omega \subset \mathbb{R}^n$ be an open set. Let

$$L^2_{\text{loc}}(\Omega) := \{f : \Omega \to \mathbb{C} \text{ measurable: } \int_K |f(x)|^2 dx < \infty \text{ for all compact } K \subset \Omega\} \ .$$

Note that $L^2_{\text{loc}}(\Omega) \subset L^1_{\text{loc}}(\Omega)$. We define

$$H^1_{\text{loc}}(\Omega) := \{f \in L^2_{\text{loc}}(\Omega) \cap W(\Omega) : D_j f \in L^2_{\text{loc}}(\Omega) \text{ for } j = 1, \ldots, n\} \ ,$$

where $W(\Omega)$ was defined in Section 4.1. Then we define inductively,

$$H^{k+1}_{\text{loc}}(\Omega) := \{f \in H^1_{\text{loc}}(\Omega) : D_j f \in H^k_{\text{loc}}(\Omega) \ , \ j = 1, \ldots, n\} \ .$$

We need the following characterization of functions in $H^k_{\text{loc}}(\Omega)$. As before, for $f : \Omega \to \mathbb{C}$ we define $\tilde{f} : \mathbb{R}^n \to \mathbb{C}$ by

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if} \quad x \in \Omega \\ 0 & \text{if} \quad x \in \mathbb{R}^n \setminus \Omega \ . \end{cases}$$

But we identify $\mathcal{D}(\Omega)$ with a subset of $\mathcal{D}(\mathbb{R}^n)$; i.e., we omit the $\sim$ sign for test functions.

Lemma 4.6.1. *Let $k \in \mathbb{N}$. Then*

$$H^k_{\text{loc}}(\Omega) = \{f \in L^2_{\text{loc}}(\Omega) : (\psi f)^\sim \in H^k(\mathbb{R}^n) \text{ for all } \psi \in \mathcal{D}(\Omega)\} \ .$$

Proof. "$\subset$". Let $k = 1$. Let $f \in H^1_{\text{loc}}(\Omega)$, $\psi \in \mathcal{D}(\Omega)$. We show that $(\psi f)^\sim \in H^1(\mathbb{R}^n)$ and

$$D_j(\psi f)^\sim = (D_j\psi \cdot f + \psi \cdot D_j f)^\sim \ . \tag{4.15}$$

In fact, let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Then

$$\begin{aligned} -\int_{\mathbb{R}^n} D_j\varphi(\psi f)^\sim dx &= -\int_\Omega D_j(\varphi \cdot \psi) f + \int_\Omega \varphi(D_j\psi)\cdot f dx \\ &= \int_\Omega \varphi\psi D_j f + \int_\Omega \varphi D_j\psi \cdot f dx \\ &= \int_{\mathbb{R}^n} \varphi[\psi \cdot D_j f + D_j\psi \cdot f]^\sim dx \ . \end{aligned}$$

This proves the claim for $k = 1$. Now assume that the inclusion "$\subset$" is proved for $k \in \mathbb{N}$. Let $f \in H^{k+1}_{\text{loc}}(\Omega)$, $\psi \in \mathcal{D}(\Omega)$. Then it follows from (4.15) and the inductive hypothesis that $D_j(\psi f)^\sim \in H^k(\mathbb{R}^n)$, $j = 1, \ldots, n$. Hence $(\psi f)^\sim \in H^{k+1}(\mathbb{R}^n)$.

"$\supset$". Let $k = 1$. Let $f \in L^2_{\text{loc}}(\Omega)$ such that $(\psi f)^\sim \in H^1(\mathbb{R}^n)$ for all $\psi \in \mathcal{D}(\Omega)$. Choose $\omega_k \subset \Omega$ open and bounded such that $\bar{\omega}_k \subset \omega_{k+1}$ and $\bigcup_{k\in\mathbb{N}} \omega_k = \Omega$. (One may take $\omega_k = \{x \in \Omega : |x| < k,\ \text{dist}(x, \partial\Omega) > 1/k\}$. Let $\psi_k \in \mathcal{D}(\Omega)$ such that $\psi_k \equiv 1$ on ω_k. Then

$$D_j(\psi_k f)^\sim = D_j(\psi_{k+1} f)^\sim \text{ on } \omega_k \ . \tag{4.16}$$

In fact, let $\varphi \in \mathcal{D}(\omega_k)$. Then

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi D_j(\psi_k f)^\sim dx &= -\int_{\mathbb{R}^n} D_j\varphi \cdot \psi_k \tilde{f} dx \\ &= -\int_{\mathbb{R}^n} D_j\varphi(\psi_{k+1}\cdot f)^\sim dx \\ &= \int_{\mathbb{R}^n} \varphi D_j(\psi_{k+1} f)^\sim dx \ . \end{aligned}$$

Since φ is arbitrary this implies (4.16). In virtue of (4.16) we may define $g_j \in L^2_{\text{loc}}(\Omega)$ by

$$g_j(x) := (D_j(\psi_k f)^\sim)(x)$$

for $x \in \omega_k$. We claim that

$$D_j f = g_j \text{ weakly on } \Omega \ .$$

In fact, let $\varphi \in \mathcal{D}(\Omega)$. Choose $k \in \mathbb{N}$ such that $\operatorname{supp} \varphi \subset \omega_k$. Then

$$\begin{aligned} -\int_\Omega D_j\varphi \cdot f dx &= -\int_{\mathbb{R}^n} D_j\varphi(\psi_k f)^\sim dx \\ &= \int_{\mathbb{R}^n} \varphi D_j(\psi_k f)^\sim dx \\ &= \int_{\mathbb{R}^n} \varphi g_j dx \ . \end{aligned}$$

This proves that $f \in H^1_{\mathrm{loc}}(\Omega)$ and $D_j f = g_j$.

The inclusion "$\supset$" is proved for $k = 1$. Assume that it is valid for $k \in \mathbb{N}$. Let $f \in L^2_{\mathrm{loc}}(\Omega)$ such that

$$(\psi f)^\sim \in H^{k+1}(\mathbb{R}^n) \text{ for all } \psi \in \mathcal{D}(\Omega) \ .$$

By the case $k = 1$, we have $f \in H^1_{\mathrm{loc}}(\Omega)$ and for all $\psi \in \mathcal{D}(\Omega)$,

$$D_j(\psi f)^\sim = D_j\psi \cdot \tilde{f} + \psi(D_j f)^\sim$$

by (4.15). Thus $\psi \cdot (D_j f)^\sim = D_j(\psi f)^\sim - (D_j\psi \cdot f)^\sim \in H^k(\mathbb{R}^n)$ by the hypothesis. Now the inductive hypothesis implies that $D_j f \in H^k_{\mathrm{loc}}(\Omega)$, $j = 1, \dots, n$. The definition implies that $f \in H^{k+1}_{\mathrm{loc}}(\Omega)$. □

If $\varphi, \psi \in \mathcal{D}(\Omega)$, then the following product formula holds

$$\Delta(\varphi \cdot \psi) = (\Delta\varphi) \cdot \psi + 2\nabla\varphi \cdot \nabla\psi + \varphi \cdot \Delta\psi \ .$$

We now prove a weak version of this identity.

Lemma 4.6.2. *Let $u \in H^1_{\mathrm{loc}}(\Omega)$, $v \in L^2_{\mathrm{loc}}(\Omega)$. Assume that $\Delta u = v$ weakly. Then*

$$\Delta(\psi u)^\sim = (\Delta\psi \cdot u + 2\nabla\psi \cdot \nabla u + \psi \cdot \Delta u)^\sim \quad \text{in } \mathcal{D}(\mathbb{R}^n)'$$

for all $\psi \in \mathcal{D}(\Omega)$.

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Then

$$\begin{aligned}
\langle \Delta(\psi u)^{\sim}, \varphi\rangle &= \int_{\Omega} \psi u \Delta\varphi dx \\
&= \int_{\Omega} (\Delta(\psi\varphi) - 2\nabla\psi\nabla\varphi - \Delta\psi\cdot\varphi) u dx \\
&= \int_{\Omega} \psi\varphi\Delta u dx - 2\int_{\Omega}\sum_{j=1}^{n} D_j(\varphi D_j\psi) u dx \\
&\quad +2\int_{\Omega}\varphi\Delta\psi\cdot u dx - \int_{\Omega}\Delta\psi\cdot\varphi\cdot u dx \\
&= \int_{\Omega}\varphi\psi\Delta u dx + 2\int_{\Omega}\varphi\sum_{j=1}^{m} D_j\psi\cdot D_j u dx + \int_{\Omega}\varphi\Delta\psi u dx \\
&= \int_{\mathbb{R}^n}\varphi(\psi\cdot\Delta u + 2\nabla\psi\nabla u + \Delta\psi\cdot u)^{\sim} dx\ .
\end{aligned}$$

□

The preceding results allow us to reduce regularity in the interior of Ω to regularity results on $\mathbb{R}^n$. Those are easily obtained with help of the Fourier transform. We first prove a Sobolev embedding result. Recall that

$$C_0(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{C} \text{ continuous } : \lim_{|x|\to\infty} |f(x)| = 0\}\ .$$

Proposition 4.6.3. *One has*

(a) $H^k(\mathbb{R}^n) \subset C_0(\mathbb{R}^n)$ *if* $k > \frac{n}{2}$*;*

(b) $H^{k+m}(\mathbb{R}^n) \subset C^m(\mathbb{R}^n)$ *if* $k > \frac{n}{2}$*;*

(c) $\bigcap_{k\in\mathbb{N}} H^k(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$.

Proof. (a) Let $f \in H^k(\mathbb{R}^n)$. Then $\mathcal{F}f \in L^2(\mathbb{R}^n, (1+x^{2k})dx)$. Hence

$$\begin{aligned}
\int_{\mathbb{R}^n} |(\mathcal{F}f)(x)| dx &= \int_{\mathbb{R}^n} |\mathcal{F}f(x)|\cdot\frac{1}{1+x^{2k}}(1+x^{2k})dx \\
&\le \|\mathcal{F}f\|_{L^2(\mathbb{R}^n,(1+x^{2k})dx)}\ (\int_{\mathbb{R}^n}\frac{1}{(1+x^{2k})^2}(1+x^{2k})dx)^{\frac{1}{2}} \\
&< \infty \qquad \text{if } k > \frac{n}{2}\ .
\end{aligned}$$

In fact

$$\int_{|x|>1} \frac{1}{1+x^{2k}}dx \leq \int_{|x|>1} \frac{1}{x^{2k}}dx = \int_1^\infty r^{-2k}r^{n-1}dr < \infty \quad \text{if } k > \frac{n}{2} \ .$$

Thus $\mathcal{F}f \in L^1(\mathbb{R}^n)$. It follows from the Fourier inversion theorem that $f \in C_0(\mathbb{R}^n)$.

If $f \in H^{k+1}(\mathbb{R}^n)$, $k > \frac{n}{2}$, then $D_j f \in C_0(\mathbb{R}^n)$. Hence $f \in C^1(\mathbb{R}^n)$. We obtain (b) by an inductive argument. Moreover, (c) follows from (b). □

Combining Proposition 4.6.3 and Lemma 4.6.1 we obtain the following local embedding theorem.

Proposition 4.6.4. *Let $k \in \mathbb{N}$, $k > \frac{n}{2}$. Then $H^{k+m}_{\text{loc}}(\Omega) \subset C^m(\Omega)$ for all $m \in \mathbb{N}_0$. Moreover,*

$$\bigcap_{m\in\mathbb{N}} H^m_{\text{loc}}(\Omega) = C^\infty(\Omega) \ .$$

Here we let $C^0(\Omega) = C(\Omega)$. With help of the Fourier transform one easily proves the following global regularity result.

Proposition 4.6.5 (global elliptic regularity)**.** *Let $f \in L^2(\mathbb{R}^n)$, $u \in H^1(\mathbb{R}^n)$ such that $u - \Delta u = f$ weakly. Then*

(a) $u \in H^2(\mathbb{R}^n)$;

(b) $f \in H^k(\mathbb{R}^n)$ implies $u \in H^{k+2}(\mathbb{R}^n)$.

Proof. It follows from the definition of the Dirichlet Laplacian on $L^2(\mathbb{R}^n)$ and the fact that $H^1_0(\mathbb{R}^n) = H^1(\mathbb{R}^n)$ that $u \in D(\Delta^D_{\mathbb{R}^n})$. But $D(\Delta^D_{\mathbb{R}^n}) = H^2(\mathbb{R}^n)$ by Corollary 4.2.6. This proves (a). Moreover taking Fourier transforms we obtain

$$(1 + x^2)\mathcal{F}u = \mathcal{F}(u - \Delta u) = \mathcal{F}f \ .$$

Since, by Corollary 4.2.2, $f \in H^k(\mathbb{R}^n)$ if and only if $\mathcal{F}f \in L^2(\mathbb{R}^n, (1 + x^{2k})dx)$, (b) follows. □

Next we deduce local regularity from Proposition 4.6.5. Let $\Omega \subset \mathbb{R}^n$ be open.

Proposition 4.6.6 (local elliptic regularity)**.** *Let $f \in L^2_{\text{loc}}(\Omega)$, $u \in H^1_{\text{loc}}(\Omega)$ such $u - \Delta u = f$ weakly. Then*

(a) $u \in H^2_{\text{loc}}(\Omega)$;

(b) $f \in H^k_{\text{loc}}(\Omega)$ implies $u \in H^{k+2}_{\text{loc}}(\Omega)$;

(c) $f \in C^\infty(\Omega)$ implies $u \in C^\infty(\Omega)$.

Proof. Let $\psi \in \mathcal{D}(\Omega)$. Then by Lemma 4.6.2

$$(\psi u)^{\sim} - \Delta(\psi u)^{\sim} = (\psi f)^{\sim} - 2(\nabla\psi\nabla u)^{\sim} - ((\Delta\psi)u)^{\sim} \tag{4.17}$$

in $\mathcal{D}(\mathbb{R}^n)'$.
a) Since the right side is in $L^2(\mathbb{R}^n)$ it follows from Proposition 4.6.5 that $(\psi u)^{\sim} \in H^2(\mathbb{R}^n)$. Since $\psi \in \mathcal{D}(\Omega)$ is arbitrary we conclude that $u \in H^2_{\text{loc}}(\Omega)$ by Lemma 4.6.1.
b) We prove b) by induction. Letting $H^0_{\text{loc}}(\Omega) = L^2_{\text{loc}}(\Omega)$, it holds for $k = 0$ by a). Assume that $k \in \mathbb{N}_0$ such that (b) holds. Assume that $f \in H^{k+1}_{\text{loc}}(\Omega)$. Then by the inductive assumption $u \in H^{k+2}_{\text{loc}}(\Omega)$. Hence $(\nabla\psi\nabla u)^{\sim} \in H^{k+1}(\mathbb{R}^n)$. Thus the right side of (4.17) is in $H^{k+1}(\mathbb{R}^n)$. Now it follows from Proposition 4.6.5 that $(\psi u)^{\sim} \in H^{k+3}(\mathbb{R}^n)$. Thus $u \in H^{k+3}_{\text{loc}}(\Omega)$ by Lemma 4.6.1.
c) Since $\bigcap_{k\in\mathbb{N}} H^k_{\text{loc}}(\Omega) = C^\infty(\Omega)$, the last assertion follows from (b). □

Now we are ready to prove the existence of classical solutions of the heat equation with Dirichlet or Neumann boundary conditions. In fact, we formulate the result more generally in order to treat both problems simultaneously. Let $\Omega \subset \mathbb{R}^n$ be an open set. A closed operator A on $L^2(\Omega)$ is called a **realization of the Laplacian** in $L^2(\Omega)$ if

$$Af = \Delta f \text{ weakly} \tag{4.18}$$

for all $f \in D(A)$ and

$$D(A) \subset H^1_{\text{loc}}(\Omega) \ . \tag{4.19}$$

Remark 4.6.7. *a) Recall that (4.18) means that $\langle Af, \varphi\rangle = \langle f, \Delta\varphi\rangle$ for all $\varphi \in \mathcal{D}(\Omega)$, where $\langle f, g\rangle = \int_{\mathbb{R}^n} fg dx$ for $f, g \in L^2(\mathbb{R}^n)$.*
b) It will be shown in Exercise 4.6.16 that (4.18) actually implies (4.19).

Lemma 4.6.8. *Let A be a realization of the Laplacian in $L^2(\Omega)$ such that $\varrho(A) \neq \emptyset$. Let $k, m \in \mathbb{N}_0$, $k > \frac{n}{4}$. Then*

$$D(A^{k+m}) \subset C^m(\Omega)$$

and the injection is continuous.

Proof. In view of Proposition 4.6.4 it suffices to show that

$$D(A^k) \subset H^{2k}_{\text{loc}}(\Omega) \tag{4.20}$$

for all $k \in \mathbb{N}_0$. Let $\lambda \in \varrho(A)$. Then $D(A^k) = R(\lambda, A)^k L^2(\Omega)$. Let $k = 0$, $f \in L^2(\Omega)$, $R(\lambda, A)f = u$. Then $\Delta u = Au = f - \lambda u$ weakly. It follows from Proposition 4.6.4 that $u \in H^2_{\text{loc}}(\Omega)$. Now assume that assertion (4.20) holds for $k \in \mathbb{N}_0$. Let $u = R(\lambda, A)^{k+1} f$, where $f \in L^2(\Omega)$. Then $\Delta u = Au = R(\lambda, A)^k f - \lambda u \in H^{2k}_{\text{loc}}(\Omega)$ by the inductive hypothesis. It follows from Proposition 4.6.6 that $u \in H^{2k+2}_{\text{loc}}(\Omega)$. □

Now we need the following lemma which describes the identification of a function in two variables with a vector-valued function in one variable.

Lemma 4.6.9. *Let $v \in C((0,\infty),\, C(\Omega))$. Define $u(t,x) = v(t)(x)$ $(t > 0, x \in \Omega)$. Then $u \in C((0,\infty) \times \Omega)$.*
If $v \in C^\infty((0,\infty);\, C^m(\Omega))$ for all $m \in \mathbb{N}$, then $u \in C^\infty((0,\infty) \times \Omega)$.

Proof. a) Let $t_n \to t$ in $(0,\infty)$, $x_n \to x$ in Ω. Choose $K \subset \Omega$ compact such that $x_n \in K$ for all $n \in \mathbb{N}$. Then $v(t_n) \to v(t)$ in $C(K)$; i.e., $u(t_n, y) \to u(t,y)$ $(n \to \infty)$ uniformly in $y \in K$. Hence $u(t_n, x_n) - u(t,x) = u(t_n,x_n) - u(t,x_n) + u(t,x_n) - u(t,x) \to 0$ $(n \to \infty)$.
b) If $v \in C^\infty((0,\infty),\, C^m(\Omega))$ for all $m \in \mathbb{N}$, it follows that all partial derivatives of u exist and are continuous on $(0,\infty) \times \Omega$ by a). □

Now we obtain the following regularity result for semigroups generated by a realization of the Laplacian in $L^2(\Omega)$.

Theorem 4.6.10. *Let A be the realization of the Laplacian in $L^2(\Omega)$ which generates a differentiable C_0-semigroup T on $L^2(\Omega)$. Let $f \in L^2(\Omega)$, $u(t,x) = (T(t)f)(x)$ $(t > 0, x \in \Omega)$. Then $u \in C^\infty((0,\infty) \times \Omega)$.*

Proof. By Proposition 2.3.3 we have $T(\cdot)f \in C^\infty((0,\infty);\, D(A^k))$ for all $k \in \mathbb{N}$. Hence $T(\cdot)f \in C^\infty((0,\infty);\, C^m(\Omega))$ for all $m \in \mathbb{N}$ by Lemma 4.6.8. Now the claim follows from Lemma 4.6.9. □

Remark 4.6.11. *Theorem 4.6.10 should be interpreted in the right way. A priori, $T(t)f$ is only defined almost everywhere. The result says, that we may choose a representative $T(t)f$ in $C^\infty(\Omega)$, which we always do.*

Thus the orbits $u(t,x) = (e^{t\Delta_\Omega^D} f)(x)$ and $u(t,x) = (e^{t\Delta_\Omega^N} f)(x)$ are both classical solutions of the heat equation

$$\begin{cases} u \in C^\infty((0,\infty) \times \Omega) \\ u_t(t,x) = \Delta u(t,x) \quad (t > 0, x \in \Omega) \end{cases} \tag{4.21}$$

(where the Laplacian is understood with respect to the space variable x only).

We mention that the boundary conditions are satisfied in a weak sense only. For example, in the case of Dirichlet boundary conditions, we have

$$u(t,\cdot) \in H_0^1(\Omega)$$

for all $t > 0$. Regularity conditions on the boundary of Ω are needed in order to deduce that $u(t,\cdot) \in C_0(\Omega)$. This is a different subject.

We conclude this section by showing that both semigroups $(e^{t\Delta_\Omega^D})_{t\geq 0}$ and $(e^{t\Delta_\Omega^N})_{t\geq 0}$ are strictly positive. This follows from Theorem 4.6.10 with help of the following classical maximum principle.

Proposition 4.6.12 (strict parabolic maximum principle). *Let $\Omega \subset \mathbb{R}^n$ be open and connected, and let $\tau > 0$. Let*

$$u \in C^2((0,\tau)\times\Omega)\cap C([0,\tau]\times\bar{\Omega}) \text{ such that } u_t(t,x) = \Delta u(t,x) \quad (t \in (0,\tau)\ ,\ x \in \Omega)\ .$$

Assume that there exist $x_0 \in \Omega$, $t_0 \in (0,\tau]$ such that

$$u(t_0,x_0) = \max_{\substack{t\in(0,\tau]\\ x\in\Omega}} u(t,x)\ .$$

Then u is constant.

Theorem 4.6.13 (strict positivity). *Let $\Omega \subset \mathbb{R}^n$ be an open connected set. Let A be a realization of the Laplacian in $L^2(\Omega)$ which generates a differentiable, positive C_0-semigroup T. For example, $A = \Delta_\Omega^D$ or $A = \Delta_\Omega^N$. Let $0 \le f \in L^2(\Omega)$, $f \ne 0$. Then $T(t)f \in C^\infty(\Omega)$ and*

$$(T(t)f)(x) > 0$$

for all $t > 0$, $x \in \Omega$.

Proof. We know from Theorem 4.6.10 that $u \in C^\infty((0,\infty)\times\Omega)$ satisfies the heat equation (4.21). Moreover, $u(t,x) \ge 0$ for all $t > 0$, $x \in \Omega$, by hypothesis. Assume that there exists $t_0 > 0$, $x_0 \in \Omega$ such that $u(t_0,x_0) = 0$. Let ω be open, bounded, connected such that $\bar{\omega} \subset \Omega$. The strict parabolic maximum principle applied to $-u$ shows that $u(t,x) = 0$ for all $t \in (0,t_0]$, $x \in \omega$. A simple connectedness argument shows that $u(t,x) = 0$ for all $t \in (0,t_0]$, $x \in \Omega$. Since $f = \lim_{t\downarrow 0} u(t,\cdot)$ in $L^2(\Omega)$, it follows that $f = 0$. □

In the following exercises we give some further results on elliptic regularity.

Exercise 4.6.14. *Let $f, g \in L^2(\mathbb{R}^n)$ such that $\Delta f = g$ weakly. Show that $f \in H^2(\mathbb{R}^n)$.*

Hint: a) Let $k \in L^1(\mathbb{R}^n)$. Show that $\Delta(k * f) = k * g$ weakly.
b) Consider the Gaussian semigroup G on $L^2(\mathbb{R}^n)$. We know that its generator A is given by $D(A) = H^2(\mathbb{R}^n)$, $Au = \Delta u$ ($u \in H^2(\mathbb{R}^n)$). Show that $\langle G(s)f,\ \Delta\varphi\rangle = \langle G(s)g,\ \varphi\rangle$ for all $s \ge 0$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Show that $\int_0^t G(s)g\,ds = G(t)f - f$ $(t \ge 0)$.

Exercise 4.6.15. *Show that the space $F = \{\varphi - \Delta\varphi : \varphi \in \mathcal{D}(\mathbb{R}^n)\}$ is dense in $L^2(\mathbb{R}^n)$.*

Hint: Let $f \in L^2(\mathbb{R}^n)$ be orthogonal to F. Use Exercise 4.6.14 in order to show that $G(t)f = e^t f$ $(t \ge 0)$. Since G is contractive, this implies that $f = 0$.

Exercise 4.6.16. *Let $u, v \in L^2_{\text{loc}}(\Omega)$ such that*

$$\Delta u = v \text{ weakly}\ .$$

Show that $u \in H^2_{\text{loc}}(\Omega)$.

Hint: Show the result by proving the following three steps. Let $\psi \in \mathcal{D}(\Omega)$, $f = (\psi u)^{\sim}$.
a) Show that

$$|\langle f, \varphi - \Delta\varphi \rangle| \leq c\|\varphi\|_{H^1}$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.
b) Use the Riesz-Fréchet lemma to show that there exists $g \in H^1(\mathbb{R}^n)$ such that

$$\langle f, \varphi - \Delta\varphi \rangle = \langle g, \varphi - \Delta\varphi \rangle \ .$$

c) Conclude with help of Exercise 4.6.15 that $f = g$.

Chapter 5

Forms generating holomorphic semigroups

Sequilinear forms give a nearly algebraic tool to define holomorphic semigroups on a Hilbert space. In addition, most interesting examples can be treated by this method.

Let V be a complex Hilbert space. A sesquilinear form $a : V \times V \to \mathbb{C}$ is a mapping satisfying

$$\begin{aligned} a(u+v,w) &= a(u,w)+a(v,w) \\ a(\lambda u,w) &= \lambda a(u,w) \\ a(u,v+w) &= a(u,v)+a(u,w) \\ a(u,\lambda u) &= \bar{\lambda} a(u,v) \end{aligned}$$

for $u, v, w \in V, \lambda \in \mathbb{C}$. In other words, a is linear in the first and antilinear in the second variable. We frequently say simply **form** instead of sesquilinear form. The form a is **continuous** if (and only if) there exists a constant $M \geq 0$ such that

$$|a(u,v)| \leq M \|u\|_V \|v\|_V \tag{5.1}$$

for all $u, v \in V$.

Finally, the form a is called **coercive** if there exists $\alpha > 0$ such that

$$\operatorname{Re} a(u,u) \geq \alpha \|u\|_V^2 \quad (u \in V) \ . \tag{5.2}$$

A mapping $f : V \to \mathbb{C}$ is called **antilinear** if

$$f(u+v) = f(u) + f(v) \quad \text{and} \quad f(\lambda u) = \bar{\lambda} f(u)$$

for all $u, v \in V, \lambda \in \mathbb{C}$. The space V' of all continuous antilinear forms is a Banach space for the norm

$$\|f\| = \sup_{\|u\|_V \leq 1} |f(u)| \ .$$

We call it the **antidual** of V. Frequently we write

$$\langle f, u \rangle = f(u) \quad (u \in V, f \in V') \ .$$

A form $a : V \times V \to \mathbb{C}$ is called **symmetric** if

$$a(u, v) = \overline{a(v, u)} \ .$$

Thus, a continuous, coercive, symmetric form on V is the same as a scalar product on V (we say an **equivalent scalar product**). So the following theorem is a generalization of the Theorem of Riesz-Fréchet to a non-symmetric form.

Theorem 5.0.17 (Lax-Milgram)**.** *There exists an isomorphism $\mathcal{A} : V \to V'$ such that*

$$\langle \mathcal{A}u, v \rangle = a(u, v) \tag{5.3}$$

for all $u, v \in V$. Moreover, $\|A^{-1}\|_{\mathcal{L}(V',V)} \leq \frac{1}{\alpha}$.

The space V' is always isomorphic to V (and thus a Hilbert space). We may apply Theorem 5.017 to the usual scalar product. But for the applications we have in mind another identification of V' will be more useful.

Now we assume that the Hilbert space V is continuously and densely injected into another Hilbert space H, i.e. $V \subset H$ and there exists a constant $c > 0$ such that

$$\|u\| \leq c\|u\|_V \quad (u \in V)$$

and V is dense in H for the norm of H. We define a mapping from H into V' using the scalar product of H in the following way. For $u \in V$ let $j(u) \in V'$ be given by

$$\langle j(u), v \rangle = (u \mid v)_H \quad (v \in V) \ .$$

$$\text{Then} \quad \begin{aligned} \|j(u)\|_{V'} &= \sup_{\|v\|_v \leq 1} |(u \mid v)_H| \\ &\leq \sup_{\|v\|_v \leq 1} \|u\|_H \|v\|_H \\ &\leq c\|u\|_H \ . \end{aligned}$$

Thus j is a continuous, linear mapping. Moreover, j is injective. In fact, if $j(u) = 0$. Then $\|u\|_H^2 = (u \mid u)_H = \langle j(u), u \rangle = 0$. Hence $u = 0$.

In the following we identify V with a subspace of V' omitting the identification mapping j, i.e., we write

$$\langle u, v \rangle = (u \mid v)_H$$

for all $u, v \in V$ where $\langle u, v \rangle = \langle j(u), v \rangle$.

Lemma 5.0.18. *V is a dense subspace of V' and $H = V'$ if and only if $H = V$.*

Proof. a) The first assertion means more precisely that $j(V)$ is dense in V'. Since V is reflexive, this is equivalent to saying that for $u \in V, \langle j(v), u\rangle = 0$ for all $v \in V$ implies that $u = 0$. Taking $v = u$ we have $0 = \langle j(u), u\rangle = (u \mid u)_H$, hence $u = 0$.
b) If $H = V'$, then the norms $\| \ \|_H$ and $\| \ \|_{V'}$ are equivalent on V'. Hence also $\| \ \|_H$ and $\| \ \|_V$ are equivalent on V. □

Example 5.0.19. *Let (Ω, Σ, μ) be a σ-finite measure space and $m : \Omega \to [1, \infty)$ measurable. Let $H = L^2(\Omega, \mu), V = L^2(\Omega, m\mu)$. Then $V \hookrightarrow H$. We have $V' = L^2(\Omega, \frac{1}{m}\mu)$ if we write the duality as*

$$\langle v, u\rangle = \int v\bar{u}d\mu(x)$$

$(u \in V, v \in V)$.

Now we consider a continuous coercive form $a : V \times V \to \mathbb{C}$ and the associated operator $\mathcal{A} : V \to V'$ given by (5.3). We may see $\mathcal{A}$ as an unbounded closed operator on the Banach space V'.

Theorem 5.0.20. *The operator $-\mathcal{A}$ generates a bounded holomorphic semigroup on V'.*

Proof. For $\operatorname{Re}\lambda \geq 0$ we consider the form a_λ defined by $a_\lambda(u, v) = \lambda(u \mid v)_H + a(u, v)$. Then a_λ is continuous and coercive and the associated operator is $\lambda + \mathcal{A}$. Thus $\lambda + \mathcal{A} : V \to V'$ is an isomorphism and $\|(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(V',V)} \leq \frac{1}{\alpha}$ for all $\operatorname{Re}\lambda \geq 0$ by Theorem 5.0.20. Since $\lambda(\lambda + cA)^{-1} + \mathcal{A}(\lambda + \mathcal{A})^{-1} = I$ it follows that

$$\begin{aligned} \|\lambda(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(V')} &\leq 1 + \|\mathcal{A}(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(V')} \\ &\leq 1 + \|\mathcal{A}\|_{\mathcal{L}(V')}\|(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(V',V)} \\ &\leq 1 + \frac{1}{\alpha}\|\mathcal{A}\|_{\mathcal{L}(V,V')} \end{aligned}$$

for $\operatorname{Re}\lambda \geq 0$. This proves the claim. □

Now we consider the part A of $\mathcal{A}$ in H, i.e., the operator A is defined by

$$\begin{aligned} D(A) &= \{u \in V : \mathcal{A}u \in H\} \\ &= \{u \in V : \exists\ f \in H \quad a(u, v) = (f \mid v)_H \quad \text{for all} \quad v \in V\} \\ Au &= v\ . \end{aligned}$$

The operator A is called **the operator associated with** a (on H). This is the operator we are really interested in for most applications. In fact, $-A$ generates a bounded, holomorph semigroup on H.

Theorem 5.0.21. *The operator $-A$ generates a bounded holomorphic semigroup on H.*

Proof. Since $(\lambda + \mathcal{A}) : V \to V'$ is an isomorphism and $V \hookrightarrow H \hookrightarrow V'$ it follows that $(\lambda + A)$ is invertible and $(\lambda + A)^{-1} = (\lambda + \mathcal{A})^{-1}|_H$ for all $\operatorname{Re}\lambda \geq 0$. Let $f \in H$ and $u = (\lambda + A)^{-1} f$, i.e.,

$$\lambda(u \mid v)_H + a(u, v) = (f \mid v)_H \quad (v \in V) \ .$$

In particular

$$\lambda \|u\|_{H^2} + a(u, u) = (f \mid u)_H \ .$$

This implies

$$\begin{aligned} \alpha\|u\|_{V^2} &\leq \operatorname{Re} a(u, u) = \\ \operatorname{Re}(f \mid u)_H - \operatorname{Re}\lambda\|u\|_{H^2} &\leq \|f\|_H \|u\|_H \quad \text{and} \\ |\lambda|\|u\|_{H^2} &\leq M\|u\|_{V^2} + \|f\|_H\|u\|_H \ . \end{aligned}$$

Hence $|\lambda|\|u\|_{H^2} \leq (\frac{M}{\alpha} + 1)\|f\|_H\|u\|_H$ and so $|\lambda|\|u\|_H \leq (\frac{M}{\alpha} + 1)\|f\|_H$. We have proved that $\|\lambda(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(H)} \leq (\frac{M}{\alpha} + 1)$. □

Bibliography

[Adams] R. Adams: *Sobolev Spaces.* Acad. Press 1975.

[Brezis] *Analyse Fonctionelle*, Masson, Paris 1983.

[Bre87] Brezis: Corollaire IX.8, p. 162.

[RS80] M. Reed, B. Simon: *Functional Analysis.* Academic Press, New York, Vol. 1, 1980.