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On Lattice Isomorphisms with Positive Real Spectrum and Groups of Positive Operators

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Introduction

One basic problem in the spectral theory of bounded operators is the following: Which additional properties of an operator T imply it to be the identity when the spectrum $\sigma(T)$ of T is known to consist of the element 1 only? For example, if T is a normal operator on Hilbert space then $T=I$ iff $\sigma(T)=\{1\}$. More recently, Akeman and Ostrand [1] proved the same result for automorphisms of C^*-algebras, and Kamowitz, Scheinberg [4], and Johnson [3] proved it for commutative semisimple Banach algebras. It is one of our aims to prove an analogous result for lattice isomorphisms of a Banach lattice E. In fact, we show the following more general result to hold: A lattice isomorphism T of E has positive real spectrum iff it belongs to the center $Z(E)$ of E. For concrete function lattices (e.g. $C(X)$ or $L^p(X, \mu)$) this means that T is multiplication by a positive function. This general result enables us to prove a generalization of a result of Lotz [5] on the discreteness of groups of positive operators. In addition, we show that a C_0-semi-group of lattice homomorphisms on E which possesses a bounded generator, is contained in the center $Z(E)$.

In Section 1 we explain some notations and give some auxiliar results. The second section contains our main theorem, the third being reserved for the applications mentioned above.

§ 1. Preliminaries

In the following we use [8] as a general reference for the theory of Banach lattices.

1.1. Let E denote a complex Banach lattice with positive cone E_+. A linear operator T on E is called *positive* if $T(E_+)\subset E_+$. T is called a *lattice homomorphism* (isomorphism) iff $|Tx|=T|x|$ holds for all x in E (and T is bijective). Here $|x|$ denotes the absolute value of x (for $x\in E_+-E_+$, x coincides with $\sup(x, -x)$).

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The space $\mathscr{L}(E)$ of all continous linear operators on E is ordered under the relation $\leqq$, defined by $S \leqq T$ iff $T-S$ is positive.

1.2. The *center* $Z(E)$ of E is defined to be the subspace of $\mathscr{L}(E)$ of all operators satisfying $|Tx| \leqq n(T)|x|$ for all $x \in E$ and some $n(T) \in \mathbb{N}$. We recall some basic facts about $Z(E)$. As above, let E denote a complex Banach lattice.

(i) A (linear) subspace $J \subset E$ is called an *ideal* if $y \in J$ and $|x| \leqq |y|$, $x \in E$, implies $x \in J$.

(ii) For $x \in J$ denote by E_x the set $\bigcup_{n \in \mathbb{N}} \{y: |y| \leqq n|x|\}$. E_x is an ideal, called the *principal ideal* generated by x. As a consequence of the theorem of Kakutani, E_x is vector lattice isomorphic to $C(X)$, the space of all continuous complex valued functions on X, for a suitable compact space X ([8], II 7.2 and II 7.4).

The following lemma was proved in a more general form by Martignon [6]; see also Meyer [7]. We include a proof for the sake of completeness.

1.3. **Lemma.** *Let* $0 \leqq T \in \mathscr{L}(E)$. *Then* T *is in* $Z(E)$ *iff* $T(J) \subset J$ *for every ideal* J.

Proof. If $T \in Z(E)$ the conclusion is clear. Conversely, suppose that $T(J) \subset J$ for all ideals J.

(a) We consider the special case where $E = C(X)$ for a compact space X. Define $h = T(1_X)$. We show that $Tf = h \cdot f$ for every $f \in C(X)$. For $x \in X$ we define $J = \{g \in C(X): g(x) = 0\}$. J is an ideal in $C(X)$. The function $f - f(x)\, 1_X$ is in J, so $T(f - f(x)\, 1_X) = Tf - f(x)\, h \in J$ by hypothesis whence $Tf(x) = f(x)\, h(x)$.

(b) Let be E arbitrary. By hypothesis, for every $x > 0$ the number $\lambda(x) = \inf\{\lambda > 0: Tx < \lambda x\}$ is finite. Suppose the set $\{\lambda(x): x \in E_+\}$ is unbounded in $\mathbb{R}$. Then for each $n \in \mathbb{N}$ there exists $x_n \in E_+$ such that $\|x_n\| = 2^{-n}$ and $\lambda(x_n) > n$. Let $x = \sum_{n=1}^{\infty} x_n$. Then E_x is invariant under T by hypothesis and $x_n \in E_x$ for all n, while $T_{|E_x}$ is not in the center of E_x. This contradicts (a), since each ideal of E_x is an ideal of E.

1.4. **Proposition.** $Z(E)$ *is a commutative subalgebra of* $\mathscr{L}(E)$, *algebraically and order isomorphic to the Banach algebra* $C(X)$ *for a suitable compact space* X; *moreover,* $Z(E)$ *is a full subalgebra of* $\mathscr{L}(E)$ *(i.e., whenever* $T \in Z(E)$ *is invertible in* $\mathscr{L}(E)$ *then* $T^{-1} \in Z(E)$*). Thus the spectrum* $\sigma(T)$ *of* $T \in Z(E)$ *coincides with the spectrum of the corresponding function in the Banach algebra* $C(X)$.

The proof of the corresponding proposition for a real Banach lattice can be found in [2]. For a complex Banach lattice E, it is easy to see that $Z(E)$ is the complexification of the center of the underlying real Banach lattice of E. (For the complexification of a Banach lattice, see [8], Chap. II, §11.)

1.5. *Examples.* (i) If E is the space $C(X)$ of all continuous complex valued functions on a compact space X, $Z(E)$ is isomorphic to $C(X)$, a function $g \in C(X)$ defining the operator $T \in Z(E)$ given by the equation $Tf = g \cdot f$ for all $f \in E$ (cp. part (a) of the proof to 1.3).

(ii) If E is the space $L^p(X, \Sigma, \mu)$ (where (X, Σ, μ) denotes an arbitrary measure space possessing a lifting), then to every operator T in $Z(E)$ there corresponds a function $g \in L^\infty(X, \Sigma, \mu)$ such that $Tf = g \cdot f$ for all $f \in E$.

§ 2. Lattice Isomorphisms with Positive Real Spectrum

By $\sigma(T)$ we denote the spectrum of an operator $T \in \mathscr{L}(E)$, E a complex Banach space. Our main result is this.

2.1. **Theorem.** *Let T be a lattice isomorphism of a Banach lattice E. The following two statements are equivalent:*

(i) *T is in the center $Z(E)$.*

(ii) *$\sigma(T)$ consists of positive real numbers only.*

The proof is based on several lemmas, some of which appear to be of independent interest. However, we first observe that Theorem 2.1 answers our original problem (see Introduction):

2.2. **Corollary.** *Let T be a lattice homomorphism satisfying $\sigma(T)=\{1\}$. Then T is the identity on E.*

Proof. By 2.1, T is in $Z(E)$. The corollary now follows from 1.4.

2.3. *Remark.* The statement of the corollary is wrong for more general positive operators, as the (2×2)-matrix (a_{ik}) shows, where $a_{11}=a_{12}=a_{22}=1$, $a_{21}=0$.

We proceed to establish the announced lemmas.

2.4. **Lemma.** *Let T be a lattice isomorphism of the Banach lattice E. If $T^{n!}$ is not in $Z(E)$ then there exists an $x>0$ in E such that* $\inf\{T^i x, T^j x\}=0$ *holds for all* $0 \leqq i, j \leqq n$, $i \neq j$.

Proof. (a) Let E equal $C(X)$, where X is compact. By [10], § 1 there exists a positive function h and homeomorphism φ of X such that $(Tf)(x)=h(x)f(\varphi(x))$ holds for all $f \in E$ and $x \in X$. For $x \in X$ let $k(x)=\inf\{r \geqq 1 : \varphi^r(x)=x\}$ and suppose that $\sup\{k(x): x \in X\} \leqq n$. Then $X=\bigcup_{p=1}^{n}\{x: k(x)=p\}$. $k(x)=p$ implies $\varphi^p(x)=x$ and hence $\varphi^{n!}(x)=x$. Thus we get $\varphi^{n!}(x)=x$ for all $x \in X$, which yields

$$T^{n!}f(x)=\prod_{j=0}^{n!-1} h(\varphi^j(x))\,f(x),$$

i.e., $T^{n!} \in Z(E)$. So there must be an $x \in X$ satisfying $k(x)>n$; this implies $\varphi^i(x) \neq \varphi^k(x)$ for all $0 \leqq i, k \leqq n$, $i \neq k$. Hence there exists an open neighborhood U of x such that $\varphi^i(U) \cap \varphi^k(U)=\varnothing$ whenever $i \neq k$, $0 \leqq i, k \leqq n$. Choose $f \in C(X)$ satisfying $f>0$ and $f(y)=0$ for $y \notin U$. Then $\inf\{T^i f, T^j f\}=0$ holds for all $0 \leqq i, j \leqq n$, $i \neq j$.

(b) Now let E be an arbitrary Banach lattice and put $S=T^{n!}$. Let $x>0$ be arbitrary and let λ be any real number greater than the spectral radius $r(T)$ of T. If $R(\lambda, T)$ denotes the resolvent of T at λ, then for $z=R(T,\lambda)x$ E_z is invariant under T and contains x ([8], Chap. V, § 4).

Hence there must exist $z>0$ in E satisfying $T(E_z) \subset E_z$ and such that $S_{|E_z}$ is not in $Z(E_z)$. Otherwise $S(J) \subset J$ would be true for every ideal J, implying $S \in Z(E)$ by 1.3. Applying (a) to $S_{|E_z}$ we obtain the proposition.

2.5. **Proposition.** *Let F denote a Banach space, and let C be a closed wedge of the algebra $\mathscr{L}(F)$ of all continuous linear operators on F satisfying*

(i) $C \cdot C \subset C$

(ii) *Whenever $S \in C$ and S^{-1} exists in $\mathscr{L}(F)$, then $S^{-1} \in C$. Let $T \in \mathscr{L}(F)$ satisfy $\sigma(T) \subset \{\lambda : \lambda > 0\}$. If $T^n \in C$ for some $n \in \mathbb{N}$ then $T \in C$.*

Proof. W.l.o.g. we assume the spectral radius $r(T)$ to be <1, i.e., $\sigma(T) \subset [a, b]$ for some a, b, $0 < a < b < 1$.

Let $U = T^n$ and assume U to be in C ($n \geqq 1$). Then $(I - U) \in C$, because $(I - U)^{-1} = \sum_{k=0}^{\infty} U^k$ is in C. Since $r(U - I) < 1$, the series $\sum_{k=0}^{\infty} \binom{-1/n}{k} (U - I)^k$ converges to an element K such that $K^n = U^{-1}$. Now K is in C. In fact, if k is even then $(U - I)^k = (I - U)^k$ is in C. If k is odd, then

$$\binom{-1/n}{k} \leqq 0 \quad \text{and} \quad -(U - I)^k = (I - U)(I - U)^{k-1}$$

is in C. This proves $K \in C$. Now obviously $KT = TK$ holds. Therefore $S = KT$ satisfies $S^n = I$; now the spectral mapping theorem and $\sigma(T) \cup \sigma(K) \subset \{\lambda : \lambda > 0\}$ imply that $\sigma(S) = \{1\}$. A standard spectral theoretical argument shows S to be equal to I; hence $T = K^{-1}$, and therefore $T \in C$.

2.6. **Corollary.** *Let T be in $\mathscr{L}(F)$ and denote by $A(T)$ the closed subalgebra generated by T and I. If $\sigma(T) \subset \{\lambda : \lambda > 0\}$, then $A(T)$ equals $A(T^n)$ for all $n \in \mathbb{N}$.*

Proof. Let $C = A(T^n)$. We have only to show that condition (ii) is satisfied. To this end let $S \in C$ be invertible in $\mathscr{L}(F)$. Then there exists a sequence of polynomials P_k with $S = \lim_{k \to \infty} P_k(T^n)$. For large k we get $0 \notin \sigma(P_k(T^n)) = P_k(\sigma(T^n))$. Since $\mathbb{C} \setminus \sigma(T^n)$ is connected, $\sigma(T^n)$ is the same as the spectrum $\sigma_C(T^n)$ of T^n considered as an element of the abstract algebra C (cf. [12], 17.5). This implies $0 \notin P_k(\sigma_C(T^n)) = \sigma_C(P_k(T^n))$. Hence $(P_k(T^n))^{-1}$ lies already in C. This implies S to be in C.

2.7. *Remark.* 2.5 together with 2.6 have an obvious analogue in the setting of abstract Banach algebras with unit (through use of the regular representation). Let us mention one more easy consequence: If T is a bounded operator on a Hilbert space H, if in addition $\sigma(T)$ is contained in the open half line $(0, \infty)$, and if T^n is normal then T is a positive hermitian operator.

2.8. **Corollary.** *Let E be a Banach lattice, and let T be a continuous linear operator on E satisying $\sigma(T) \subset (0, \infty)$.* If $T^n \in Z(E)$ for some $n \in \mathbb{N}$, then $T \in Z(E)$.

Proof. By 1.4, $C = Z(E)$ satisfies the assumptions of 2.5.

2.9. *Proof of 2.1.* (i) implies (ii) by 1.4.

(ii)$\Rightarrow$(i). If there is an $n \in \mathbb{N}$ with $T^n \in Z(E)$ then T is in $Z(E)$ by 2.8. Now suppose $T^n \notin Z(E)$ for all $n \geqq 1$. In addition we assume w.l.o.g. that $r(T) = 1$, i.e., $\sigma(T) \subset [b, 1]$ for a suitable b satisfying $0 < b < 1$. This implies $r(I - T) < 1$. On the other hand by 2.4 for every $k \in \mathbb{N}$ there exists an element $x>0$ such that

$$\inf\{T^i x, T^j x\}=0 \quad \text{for } 0 \leqq i, j \leqq k,\ i \neq j.$$

This implies

$$|(I-T)^k x| = \left|\sum_{j=0}^{k} \binom{k}{j} (-1)^j T^j x\right| = \sum_{j=0}^{k} \binom{k}{j} T^j x \geqq x$$

(see [8], p. 52, Cor. 1). This in turn yields $\|(I-T)^k\| \geqq 1$. Since k was arbitrary, we obtain $r(I-T) \geqq 1$. Hence our assumption that no power of T is in $Z(E)$ leads to a contradiction. This completes the proof of Theorem 2.1.

§ 3. Applications

The first application of the main theorem shows how rather general properties of the spectrum of a lattice isomorphism give a fair amount of information about the operator. For a Markovian lattice isomorphism on $C(X)$, the following result is an easy consequence of ([10], 3.2).

3.1. **Proposition.** *Let T be a lattice isomorphism of the Banach lattice E. Assume that there exists a real number c, $0<c<2\pi$, such that the ray $\{r \cdot \exp(ci): r>0\}$ is not contained in the spectrum $\sigma(T)$ of T.*

Then $T^n \in Z(E)$ for a suitable integer $n \geqq 1$.

Proof. Since T is an isomorphism (hence invertible), 0 is not in $\sigma(T)$. Moreover if $a \in \sigma(T)$ and $b=|a|^{-1} a$, then $|a| \cdot b^n \in \sigma(T)$ for all $n \in \mathbb{Z}$ ([9], 2.3)[1]. Thus the assumption and the compactness of $\sigma(T)$ imply that the set $\{|a|^{-1} a: a \in \sigma(T)\}$ contains only roots of unity of a universally bounded order. Therefore, there exists an integer $n \neq 0$ such that $a^n>0$ for all $a \in \sigma(T)$. The spectral mapping theorem yields $\sigma(T^n) \subset (0, \infty)$ and hence the assertion follows from 2.1.

An easy but interesting corollary is the following (for notation, see [8]).

3.2. **Corollary.** *Let T be a lattice isomorphism of the Banach lattice E and suppose the spectrum $\sigma(T)$ to be finite. Then E is the direct sum of finitely many pairwise disjoint bands E_j ($j=1, \ldots, n$, say) such that E_j is invariant under T, and the restriction T_j of T to E_j is of the form $a_j S_j$, where $a_j>0$ for all j and S_j is a periodic lattice isomorphism of E_j (i.e. $S_j^{r_j}=I_{E_j}$ for a suitable $r_j \in \mathbb{N}$).*

Proof. By 3.1 there is an $r \in \mathbb{N}$ such that $T^r \in Z(E)$. T^r has finite spectrum $\sigma(T^r) =\{c_1, \ldots, c_n\}$, where $c_i \neq c_j$ for $i \neq j$ and $c_j>0$. Now $Z(E)$ is isomorphic to $C(X)$ for some compact space X, and if h is the image of T^r under this isomorphism, we obtain $h=\sum_{j=1}^{n} c_j 1_{A_j}$ with $A_j=h^{-1}(c_j)$. Hence $T^r=\sum_{j=1}^{n} c_j P_j$, where $0<P_j \leqq I$ and $P_i P_j=P_j P_i=\delta_{ij} P_i$. Each P_j is a band projection ([8], II 2.9) and the bands $E_j =P_j(E)$ are mutually orthogonal.

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[1] A subset of $\mathbb{C}$ with this property is called *cyclic*

Next we prove E_j $(j=1, \ldots, n)$ to be invariant under T. If not, T being a lattice isomorphism, there exists $x \in E_j$, $x>0$, such that $Tx \in E_k$ where $j \neq k$. Then $c_j x = T^r x = T^{-1} T^r T x = T^{-1} c_k T x = c_k x$ which implies $x=0$, a contradiction. Letting $a_j = c_j^{1/r}$ (>0) and $S_j = a_j^{-1} T_{|E_j}$ the proposition follows.

We now turn to the application of §2 to groups of operators. The following lemma is the key to the rest of this paper.

3.3. **Lemma.** *Let T be a lattice isomorphism of a Banach lattice E such that $r(I-T) < \sqrt{b^2+b+1}$ holds for $b = r(T^{-1})^{-1}$. Then T is in the center $Z(E)$ of E.*

Proof. We show that $\sigma(T) \subset (0, \infty)$, and apply 2.1.

Suppose $\sigma(T)$ not to be in $(0, \infty)$. $\sigma(T)$ is cyclic (cf. the proof of 3.1), hence there must be an element $a \in \sigma(T)$ such that $\frac{2}{3}\pi \leqq \arg(a) \leqq \frac{4}{3}\pi$ holds; this implies $-1 \leqq \operatorname{Re}(|a|^{-1} a) \leqq -1/2$. From this inequality we obtain

$$
\begin{aligned}
r(I-T)^2 &\geqq |a-1|^2 = |a|^2 - 2|a| \cdot |\operatorname{Re}(|a|^{-1} a)| + 1 \\
&\geqq |a|^2 + |a| + 1 \geqq b^2 + b + 1
\end{aligned}
$$

because of $\sigma(T) = \sigma(T^{-1})^{-1}$ (spectral mapping theorem).

Remark. The inequality is sharp as shown by the (3×3)-matrix (a_{ik}), where $a_{12} = a_{23} = a_{31} = 1$, $a_{ij} = 0$ otherwise.

3.4. **Corollary.** *If T is a lattice isomorphism satisfying $\|I-T\| \leqq 1$, then T is in the center $Z(E)$ of E.*

3.5. **Corollary.** *Let $G = (T_t)_{t \geqq 0}$ be a C_0-semigroup of lattice homomorphisms on E. If the generator A of G is bounded then G is contained in the center $Z(E)$.*

Proof. G is continuous in the norm topology, hence there exists $t_0 > 0$ such that $\|I - T_t\| < 1$ for $0 \leqq t \leqq t_0$. This forces T_t to be bijective hence a lattice isomorphism. By 3.4, $T_t \in Z(E)$ for all $t \leqq t_0$. Thus A is in $Z(E)$ and $T_t = e^{tA}$ for all $t \in \mathbb{R}_+$.

Remarks. 1) The converse is not true. A counterexample is given by $E = L^1((0, 1)$ (Lebesgue measure))

$$D(A) = \{f \in E \colon x \mapsto x^{-1/2} f(x) \text{ is in } E\},$$

$(Af)(x) = -x^{-1/2} f(x)$. $(T_t f)(x) = \exp(-t x^{-1/2}) f(x)$. Here $T_t (t>0)$ is in $Z(E)$, but A is unbounded.

2) In applications one is interested in C_0-groups $G \neq \{I\}$ of lattice isomorphisms with $G \cap Z(E) = \{I\}$. 3.5 shows that such groups necessarily have unbounded generators. For example, in physical applications there are C_0-groups of lattice isomorphisms on an L^2-space originating from space symmetries (translation invariance, etc.). Our result shows that one cannot obtain a norm continuous group through the use of any other (possibly more convenient) Banach lattice in place of L^2.

The analogue for semi-simple Banach algebras of the following proposition is established in [4] (cor. of Thm. 3).

3.6. **Corollary.** *Let $T \neq I$ be a lattice isomorphism such that $\sigma(T) \subset \{a \colon |a| = 1\}$. Then the inequality $\sqrt{3} \leqq r(I-T) \leqq 2$ holds. If T is not periodic, then $r(I-T) = 2$.*

Proof. The inequality follows from 3.4, since $b=1$ in the present case. If T is not periodic, then $\sigma(T)=\{a: |a|=1\}$ by 3.1 and 2.2. Hence $2=1-(-1)$ is in $\sigma(I-T)$.

The following, again an easy consequence of 3.3, is a rather complete generalization of a theorem of Lotz [5], which we obtain as a corollary (see 3.10).

3.7. **Theorem.** *Let G be a group of lattice isomorphisms of a Banach lattice E, satisfying $G\cap Z(E)=\{I\}$. Then G is discrete in the norm topology. More precisely,*

$$1<\|S-T\| \min(\|S^{-1}\|, \|T^{-1}\|)$$

for all $S\neq T$; $S, T\in G$.

Proof. $S\neq T$ implies $S^{-1}T\notin Z(E)$, hence by 3.4

$$1<\|I-S^{-1}T\|=\|S^{-1}(S-T)\|\leqq\|S^{-1}\|\,\|S-T\|$$

holds. Similarly we have $1<\|T^{-1}\|\,\|S-T\|$.

If $\sigma(T)\subset\{a: |a|=1\}$ holds for all T in G, then $G\cap Z(E)=\{I\}$ is automatically satisfied by 2.2. From 3.6 we obtain this stronger inequality.

3.8. **Corollary.** *Let G be a group of lattice isomorphisms of E such that $\sigma(T)\subset\{a: |a|=1\}$ holds for all T in G. Then*

$$\sqrt{3}\leqq\|S-T\| \min(\|S^{-1}\|, \|T^{-1}\|)$$

holds for all $S, T\in G$, $S\neq T$. If, in addition, G contains no element of finite order then $\sqrt{3}$ may be substituted by 2.

Remark. Needless to say that other (more cumbersome) assumptions on T suffice to ensure the better inequality; in fact, all we need by 3.6 is to guarantee that $-1\in\sigma(T)$ for all T.

3.8 combined with the reduction method adopted from Lotz' result [5] yields the following theorem.

3.9. **Theorem.** *Let G be a group of positive operators on E with unit P. In addition denote by $T^{(-1)}$ the inverse of T in G*[2] *and assume that $r(T)=1$ for all T in G.*

Then for all S, T in G, $S\neq T$ implies

$$\sqrt{3}\leqq\|S-T\| \min(\|S^{(-1)}\|, \|T^{(-1)}\|).$$

If, in addition, G contains no elements of finite order then $\sqrt{3}$ may be substituted by 2.

Proof. Obviously it is enough to show that $\sqrt{3}\leqq r(P-T)$ whenever $T\neq P$ ($2\leqq r(P-T)$, respectively) (compare the proof of 3.7)

I) If $P=I$, we obtain the assertion from 3.8.

II) Assume that $P|x|=0$ implies $x=0$.

2 Note that T need not be invertible in $\mathscr{L}(E)$

Then $F := P(E)$ is a closed sublattice ([8], III, 11.5) and $TP = PT = T$ for all T in G implies the invariance of F under G. From I) we have

$$\sqrt{3} \leqq r(P|_F - T|_F) \leqq r(T - P) \quad \text{for } T \neq P.$$

Here $S|_F$ denotes the restriction of S to F. Similarly we prove the last assertion in this case.

III) Let now P be arbitrary. Then $J = \{x \in E : P|x| = 0\}$ is a closed G-invariant ideal in E. Denote by Q the quotient mapping. By $\bar{S}Q = QS$ for $S \in G$ we obtain a positive operator $\bar{S}$ on $E_1 = E/J$, and $\bar{G} = \{\bar{S} : S \in G\}$ is a group of positive operators on E_1 with unit $\bar{P}$.

Now $\bar{P}|x + J| = J$ implies $P|x| \in J$ since P is positive, hence $0 = P^2|x| = P|x|$, i.e. x is in J, thus $\bar{P}$ satisfies the assumption of case II) above.

Assume $\bar{T} = \bar{P}$. This implies $(T - P)(E) \subset J$, hence for $x \in E$

$$0 = P(|Tx - Px|) \geqq |PTx - Px| = |Tx - Px|, \quad \text{i.e. } T = P.$$

Now if J^0 denotes the polar of J and S' the adjoint of S, for $T \neq P$ we have $\bar{T} \neq \bar{P}$ and therefore by II),

$$\sqrt{3} \leqq r(\bar{P} - \bar{T}) = r(P'_{|J^0} - T'_{|J^0}) \leqq r(P - T).$$

The remainder is now obvious.

3.10. **Corollary.** *Let G be a group of positive operators on E with unit P, and let $M = \sup\{\|T\| : T \in G\}$ be finite.*

Then for $S, T \in G$, $S \neq T$ implies $M^{-1}\sqrt{3} \leqq \|S - T\|$.

If, in addition, G contains no element of finite order, then $\sqrt{3}$ may be substituted by 2.

Proof. $S \neq T$ implies $P \neq 0$, hence $r(P) = 1$. For arbitrary $T \in G$ we have $r(T) \leqq 1$. From

$$1 = r(P) = r(TT^{(-1)}) \leqq r(T)\, r(T^{(-1)}) \leqq 1,$$

we obtain $r(T) = 1$. The assertion follows.

3.11. *Remarks.* 1) Lotz proved this last corollary with $\sqrt{3}$ replaced by 1. As pointed out above, the reduction of 3.9 to 3.8 is due to him.

2) The same method of reduction yields a generalization of 3.7 to groups G of positive operators. The condition $G \cap Z(E) = \{I\}$ has to be replaced by the following: If T is in G and $-mP \leqq TP \leqq mP$ for some $m \in \mathbb{N}$, then $T = P$ (P the neutral element of G).

3) As mentioned in the introduction, there are other classes of operators T satisfying $\sigma(T) \neq \{1\}$ whenever $T \neq I$. In fact, automorphisms of commutative semisimple Banach algebras with unit have this property and in addition have cyclic spectrum (see [4], Thm. 3). Thus 3.6 and 3.8 are also true for such groups of operators.

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