SPECTRAL MAPPING THEOREMS FOR COMPACT AND LOCALLY COMPACT ABELIAN GROUPS OF OPERATORS

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Let G be a locally compact group, E a Banach space and $U\colon G \to \mathfrak{L}(E)$ a representation of G on E. If U is uniformly bounded and suitably continuous (e.g. U may be strongly continuous—see [3] for more general assumptions) it is possible to lift U to $M(G)$, the Banach algebra of all bounded regular Borel measures on G, by means of

$$U(\mu) = \int U_t \, d\mu(t) \,.$$

1. *Example.* Let $E = L^p(G)$, $U_t f(s) = f(t^{-1}s)$ for $f \in E$, $s, t \in G$. Then $U(\mu)f = \mu * f$ for $f \in E$, $\mu \in M(G)$.

In the following we suppose that G is abelian or compact. The purpose of this talk is to show in what form and to what extent the spectrum of $U(\mu)$ can be calculated by means of the Fourier-Stieltjes transform of μ.

2. Definition. *Let $\hat{G}$ be the dual group of G, if G is abelian, or the dual object fo G, if G is compact non-abelian.*

Let $I = \{f \in L^1(G) | U(f) = 0\}$ and $\operatorname{sp}(U) = \{\gamma \in \hat{G} | \hat{f}(\gamma) = 0$ for all $f \in I\}$, where $\hat{f}$ is the Fourier transform of f, i.e. $\hat{f}(\gamma) = \int \gamma(t^{-1}) f(t)\, dt$ for $\gamma \in \hat{G}$.

If G is abelian $\operatorname{sp}(U)$ is the Arveson-spectrum as introduced in [3].

Let us furthermore agree on the following notation. We say: $\mu \in M(G)$ has the spectral mapping property (s.m.p.) if

$$\sigma(U(\mu)) = \hat{\mu}(\operatorname{sp}(U))^{-} \qquad \text{if } G \text{ is abelian}$$

$$\sigma(U(\mu)) = \Big(\bigcup_{\gamma \in \operatorname{sp}(U)} \sigma(\hat{\mu}(\gamma)) \Big)^{-} \qquad \text{if } G \text{ is compact}\,.$$

Here we denote by $\sigma(T)$ the spectrum of a bounded operator T and by $\hat{\mu}(\gamma) = \int \gamma(t^{-1})\, d\mu(t)$ the Fourier-Stieltjes transform of μ at γ $(\gamma \in \hat{G})$. Note that in the compact case $\gamma \in \hat{G}$ can be identified with a unitary finite dimensional representation of G and $\hat{\mu}(\gamma)$ with a matrix.

With these notations we can formulate the following results.

3. *Unitary representations.* If E is a Hilbert space and U a unitary representation, every normal $\mu \in M(G)$ (hence every $\mu \in M(G)$, if G is abelian) has the s.m.p.

If U is an arbitrary representation on a Banach space we get:

4. *Abelian groups* [4]. Every $\mu \in M(G)$ whose continuous part belongs to $L^1(G)$ has the s.m.p.A singular measure doesn't have the s.m.p. in general, as can be seen by taking a self-adjoint measure μ on $\mathbb{R}$ with non-real spectrum ([5], 5.3) and U on $L^1(\mathbb{R})$ as in 1.

5. *Compact groups* [2]. Every absolutely continuous measure has the s.m.p. Moreover, point-measures have the s.m.p. But in contrast to the abelian case, there exists a completely discontinuous measure on $G = SO(3, \mathbb{R})$ which doesn't have the s.m.p. if we chose U on $L^1(G)$ as in 1.

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