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On the o-Spektrum of Regular Operators and the Spectrum of Measures

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The Banach algebra $M(G)$ of all bounded regular Borel measures on a locally compact group G is often represented on $L^2(G)$ by the mapping $\mu \to T_{\mu,2}$, $T_{\mu,2}f = \mu * f$ $(f \in L^2(G))$. This representation has the advantage of bringing the Hilbert space structure into play, thus the involution is preserved and the spectrum of $T_{\mu,2}$ is easy to compute (if G is abelian or compact). There is nevertheless an inconvenience: The spectrum of μ in the Banach algebra $M(G)$ does not coincide with the spectrum of the operator $T_{\mu,2}$ in general. This could be overcome by considering $L^1(G)$ instead of $L^2(G)$. Another possibility, which conserves the Hilbert space structure, makes use of the fact that the operators $T_{\mu,2}$ are regular operators on the Banach lattice $L^2(G)$ (i.e. linear combinations of positive operators). And indeed, the representation of $M(G)$ in the smaller Banach algebra $\mathscr{L}^r(L^2(G))$ (of all regular operators on $L^2(G)$) behaves nicely, if G is amenable:

It is an algebraic and lattice isomorphism of $M(G)$ onto the full subalgebra and sublattice of those regular operators which commute with the (right-) translations. Consequently the spectrum of μ in $M(G)$ is the same as the spectrum of $T_{\mu,2}$ in the Banach algebra $\mathscr{L}^r(L^2(G))$.

This spectrum (of a regular operator T on a complex Banach lattice E in the Banach algebra $\mathscr{L}^r(E)$) is called *o-spectrum* and was first investigated by Schaefer [14]. A natural question arrising in the context of this new definition is whether the spectrum and o-spectrum are equal for a given operator.

The main theorem of the first part of this paper gives a positive answer to this question for r-compact operators (i.e. operators which can be approximated in $\mathscr{L}^r(E)$ by operators of finite rank).

In the second part applications to convolution operators are given.

First of all the rich supply of examples known from Harmonic Analysis allows the illustration in this context of the behavior of the o-spectrum. Thus it can be shown that the theorem mentioned above cannot be significantly improved. In fact there exist compact positive operators with uncountable o-spectrum. Conversely, results concerning the o-spectrum can be used to prove properties of the spectrum of measures. For instance, we get as a corollary that

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the group algebra of a compact group is symmetric (which is well known) and are able to calculate the spectrum in the group algebra by means of the Fourier transformation.

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1. r-Compact Operators

In this section we establish some elementary facts about regular operators.

A basic notion is that of a complex Banach lattice, as it is defined in the monograph [13], which we shall use as a general reference.

A *complex Banach lattice* E is the complexification $E_{\mathbb{R}}+iE_{\mathbb{R}}$ of a real Banach lattice $E_{\mathbb{R}}$. The modulus $|z|$ of an element $z=x+iy$ in E is given by

$$|z| = \sup_{\theta\in[0,\,2\pi[} \operatorname{Re}(e^{i\theta}z) = \sup_{\theta\in[0,\,2\pi[} ((\cos\theta)x+(\sin\theta)\,y).$$

Obviously, $|z|\in E_+$ (the positive cone of $E_{\mathbb{R}}$). The norm on E is defined by

$$\|z\| := \|\,|z|\,\|,$$

where $\|\,|z|\,\|$ denotes the given norm of $|z|$ in $E_{\mathbb{R}}$. The spaces $L^p(X,\Sigma,\mu)$ $(1\le p\le\infty)$ and $C_{\mathbb{C}}(K)$ (the continuous complex functions on a compact K) are typical examples of complex Banach lattices.

Let E,F be complex Banach lattices. By $\mathscr{L}(E,F)$ ($\mathscr{L}(E)$ if $E=F$) we denote the space of all bounded operators of E into F. For $T\in\mathscr{L}(E,F)$ there exist canonical operators $\operatorname{Re}T$, $\operatorname{Im}T\in\mathscr{L}(E_{\mathbb{R}},F_{\mathbb{R}})$, such that $Tx=(\operatorname{Re}T)x+i(\operatorname{Im}T)x$ for all $x\in E_{\mathbb{R}}$. An operator $T\in\mathscr{L}(E,F)$ is said to be *positive* ($T\geqq 0$), if T is equal to $\operatorname{Re}T$ and $Tx\geqq 0$ for $x\in E_+$; T is called regular, if T can be written as a linear combination of positive operators. The space of all regular operators of E into F is denoted by $\mathscr{L}^r(E,F)$ ($\mathscr{L}^r(E)$ if $E=F$). Obviously $\mathscr{L}^r(E,F)$ is a subspace of $\mathscr{L}(E,F)$. In the following way a norm (the "r-norm") can be defined on $\mathscr{L}^r(E,F)$ such that it becomes a Banach space (a Banach algebra if $E=F$):

$$\|T\|_r=\inf\{\|S\|\,|\,S\in\mathscr{L}(E,F),\ S\geqq 0,\ |Tz|\leqq S|z| \text{ for all } z\in E\}\quad (T\in\mathscr{L}^r(E,F)).$$

In fact, it is easy to see that $\|\ \|_r$ is a norm, which satisfies in addition

$$\|T\|\leqq\|T\|_r \quad \text{for all } T\in\mathscr{L}^r(E,F),$$
$$\|T\|=\|T\|_r \quad \text{for } T\geqq 0,$$
$$\|T_1T_2\|_r\leqq\|T_1\|_r\,\|T_2\|_r,\quad T_1,T_2\in\mathscr{L}^r(E)\ (\text{if } E=F).$$

1.1. Lemma. *Let E, F be complex Banach lattices. For $S,T\in\mathscr{L}^r(E,F)$ the following are equivalent:*

(i) $|Tz|\leqq S|z|$ *for all* $z\in E$,

(ii) $\operatorname{Re}(e^{-i\theta}T)\leqq S$ *for all* $\theta\in[0,2\pi[$.

Proof. The canonical mapping $j\colon E \to E''$ is a lattice homomorphism (i.e. preserves the modulus). Consequently, $|Tz| \leqq S|z|$ for all $z \in E$ is valid, iff $|jTz| \leqq jS|z|$ for all $z \in E$; and this is satisfied, iff $\operatorname{Re}(e^{i\theta} jT) \leqq jS$ for all $\theta \in [0, 2\pi[$ by [13], IV 1.8, which is equivalent to (ii).

It follows from the lemma that the norm of a real operator $T \in \mathscr{L}^r(E, F)$ (i.e. $T = \operatorname{Re} T$) is given by

$$\|T\|_r = \inf\{\|S\| \mid S \in \mathscr{L}(E_{\mathbb{R}}, F_{\mathbb{R}}),\ \pm T \leqq S\}.$$

From this the completeness of $\mathscr{L}^r(E_{\mathbb{R}}, F_{\mathbb{R}})$ can be easily seen ([13], IV exercise 3) and the completeness of $\mathscr{L}^r(E, F)$ follows from the inequality

$$\max\{\|\operatorname{Re} T\|_r, \|\operatorname{Im} T\|_r\} \leqq \|T\|_r \leqq \|\operatorname{Re} T\|_r + \|\operatorname{Im} T\|_r.$$

1.1 shows in particular that the existence of $\sup\{\operatorname{Re}(e^{i\theta} T) \mid \theta \in [0, 2\pi[\}$ is equivalent to the existence of $\inf\{S \in \mathscr{L}(E)_+ \mid |Tz| \leqq S|z| \text{ for all } z \in E\}$ (in the ordered space $\mathscr{L}^r(E_{\mathbb{R}})$). The common value, if it exists, is called the *modulus* of T and denoted by $|T|$.

If $|T|$ exists its r-norm is $\|T\|_r = \|\,|T|\,\|$.

Of course, if T is real and $|T|$ exists it is $\sup\{T, -T\}$.

If for every $x \in E_+$ the $\sup\{|Tz| \mid |z| \leqq x\}$ exists in F, then the modulus of T exists and is given by

$$|T|x = \sup\{|Tz| \mid |z| \leqq x\} \qquad \text{(s)}.$$

As a consequence $\mathscr{L}^r(E_{\mathbb{R}}, F_{\mathbb{R}})$ is a real Banach space if F is order complete, and $\mathscr{L}^r(E, F)$ is its complexification ([13], IV, 1.8).

Whereas the modulus does not exist for every $T \in \mathscr{L}^r(E, F)$ in general, it always exists for elements of the closed subspace of $\mathscr{L}^r(E, F)$ generated by the operators of finite rank. This is due to Schlotterbeck [15] for the real case (in a slightly different version).

1.2. Definition. Let E, F be (real or complex) Banach lattices. An operator $T \in \mathscr{L}^r(E, F)$ is called *r-compact* if it can be approximated in the r-norm by operators of finite rank. The space of r-compact operators is denoted by $E' \tilde{\otimes}_e F$.

1.3. Theorem. *Let E, F be (real or complex) Banach lattices. Every r-compact operator T possesses a modulus $|T|$ given by* (s) *and $|T|$ is r-compact.*

Thus $E' \tilde{\otimes}_e F$ is a Banach lattice (the complexification of $E'_{\mathbb{R}} \tilde{\otimes}_e F_{\mathbb{R}}$, if E, F are complex). Moreover the mapping $E' \tilde{\otimes}_e F \to \mathscr{L}^r(E, F'')$ $(T \to j \circ T)$ (where $j\colon F \to F''$ is canonical) is an isometric lattice homomorphism (i.e. preserves modulus and norm).

For the proof we need two lemmas. Recall that a subspace F of a complex Banach lattice E is called a sublattice of E if $z \in F$ implies $\operatorname{Re} z \in F$ and $|z| \in F$.

1.4. Lemma. *Let E be a complex Banach lattice, F a subspace of E. Denote by $F_{\mathbb{R}} = \{\operatorname{Re} z \mid z \in F\}$ the real part of F. Suppose that*

(i) $F_{\mathbb{R}} \subset F$

(ii) $F_{\mathbb{R}}$ *is a closed sublattice of* $E_{\mathbb{R}}$.

Then F is a closed sublattice of E.

Proof. Let $x \in F$. Set $A = \{\mathrm{Re}(e^{i\theta}x) \mid \theta \in [0, 2\pi[\}$ and denote by $\tilde{A}$ the set of suprema and infima of finite subsets of A. The space $E_{|x|}$ is isomorphic to some $C_{\mathbb{C}}(K)$ ([13], II, § 11). $\tilde{A}$ is a $\leqq$-directed set which converges to $|x|$ by Dini's theorem in the uniform topology of $C_{\mathbb{C}}(K)$, hence a fortiori in the topology of E. It follows that $|x| \in F$, since F is closed.

1.5. Lemma. *Let E, F be (real or complex) Banach lattices, $T \in \mathscr{L}^r(E, F)$. Assume there exists a sequence $(T_n)_{n \in \mathbb{N}}$ in $\mathscr{L}^r(E, F)$ such that*

(i) $\lim_{n \to \infty} \|T - T_n\|_r = 0$

(ii) $|T_n|$ *exists for every $n \in \mathbb{N}$ and is given by* (s).

Then $|T|$ exists and is given by (s). *Moreover $|T| = \lim_{n \to \infty} |T_n|$ in the r-norm.*

Proof. 1. We show that $\| |T_n| - |T_m| \|_r \leqq \|T_n - T_m\|_r$ for $n, m \in \mathbb{N}$.
Let $S \in \mathscr{L}(E)_+$ such that $|(T_n - T_m)z| \leqq S|z|$ for all $z \in E$.
Then

$$\begin{aligned} |T_n z| &\leqq |(T_n - T_m) z| + |T_m z| \\ &\leqq S|z| + |T_m|\,|z| \quad \text{for all } z \in E. \end{aligned}$$

Hence

$$\begin{aligned} &|T_n| \leqq S + |T_m|, \quad \text{and} \\ &|T_m| \leqq S + |T_n| \quad \text{(by symmetry).} \end{aligned}$$

Consequently $\pm(|T_n| - |T_m|) \leqq S$, which implies

$$\| |T_n| - |T_m| \|_r \leqq \|S\|.$$

2. By 1. $(|T_n|)_{n \in \mathbb{N}}$ is a Cauchy-sequence in $\mathscr{L}^r(E, F)$. Set $S = \lim_{n \to \infty} |T_n|$. Let $x \in E_+$.

We have to show that $Sx = \sup\{|Tz| \mid |z| \leqq x\}$. Since

$$|Tz| = \lim_{n \to \infty} |T_n z| \leqq \lim_{n \to \infty} |T_n|\,|z| = S|z|$$

we have $|Tz| \leqq Sx$ if $|z| \leqq x$.

To show the inverse inequality let $w \geqq |Tz|$ for all $z \in E$ satisfying $|z| \leqq x$.

There exists a sequence $(R_n)_{n \in \mathbb{N}}$ of positive operators on E such that $|(T_n - T) z| \leqq R_n |z|$ for all $z \in E$ and

$$\lim_{n \to \infty} \|R_n\| = 0.$$

For $|z| \leqq x$ we have

$$|T_n z| \leqq |(T_n - T)z| + |Tz| \leqq R_n |z| + w \leqq R_n x + w.$$

This implies $|T_n| x \leqq R_n x + w$, hence $Sx = \lim_{n \to \infty} |T_n| x \leqq w$.

Proof of 1.3. 1. The assertion is true for real Banach lattices E, F

a) If $T \in E' \otimes F$, $|T|$ exists and $|T| \in E' \tilde{\otimes}_e F$. This follows from IV, 4.6 in [13] because $E' \otimes F \subset E' \tilde{\otimes}_m F \subset E' \tilde{\otimes}_e F$. Moreover from the proof of IV, 4.6 in [13] it can be seen that $|T|$ is given by (s) and $|j \circ T| = j \circ |T|$.

b) Let $T \in E' \tilde{\otimes}_e F$. There exists a sequence $(T_n) \subset E' \otimes F$ such that $T = \lim_{n \to \infty} T_n$ in the r-norm.

From 1.5 it follows that $|T|$ exists and is given by (s). Moreover $|T| = \lim_{n \to \infty} |T_n|$; thus $|T| \in E' \tilde{\otimes}_e F$ and

$$|j \circ T| = \lim_{n \to \infty} |j \circ T_n| = \lim_{n \to \infty} j \circ |T_n| = j \circ |T|$$

by a). In addition

$$\|j \circ T\|_r = \| |j \circ T| \| = \|j \circ |T| \| = \| |T| \| = \|T\|_r .$$

2. Let E, F be complex Banach lattices.

a) $j(E' \tilde{\otimes}_e F)$ is a closed sublattice of $\mathscr{L}^r(E, F'')$. In fact, it easy to see that $(E \tilde{\otimes}_e F)_{\mathbb{R}} = E_{\mathbb{R}} \tilde{\otimes}_e F_{\mathbb{R}})$, hence by 1. the hypotheses of 1.4 is satisfied.

b) For $T \in E' \tilde{\otimes}_e F$ and $x \in E_+$,

$$|j \circ T| x = \sup_{F''} \{ |(j \circ T) z| \mid |z| \geqq x \} = \sup_{F''} \{ j |Tz| \mid |z| \leqq x \}$$

is an element of jE by a). This implies that $\sup_F \{ |Tz| \mid |z| \leqq x \}$ exists and $j(\sup_F \{ |Tz| \mid |z| \leqq x \}) = |j \circ T| x$. Hence $|T|$ is given by (s) and $j \circ |T| = |j \circ T|$.

1.6. From the definition it follows that $E' \tilde{\otimes}_e E$ is a closed algebraic ideal in $\mathscr{L}^r(E)$. Whereas $E' \otimes E$ is an ideal in $\mathscr{L}(E)$, it can happen that for $T \in E' \tilde{\otimes}_e E$ there exists an $R \in \mathscr{L}(E)$ such that RT and TR are not regular. This can be seen in the following example:

On the space $E_n = l^2(2^n)$ there exists an operator A_n such that $\|A_n\| = 1$ and $\|A_n\|_r = \sqrt{2^n}$ $(n \in \mathbb{N})$ (see [13], IV § 1, Example 1). Take E the l^2-sum of the spaces E_n, let R be the diagonal operator on E given by A_n on E_n and T be the diagonal operator given on E_n by c_n-times the identity, (c_n) being a sequence such that $\lim_{n \to \infty} c_n = 0$ and $\lim_{n \to \infty} |c_n| \sqrt{2^n} = \infty$. Then $T \in E' \tilde{\otimes}_e E$, $R \in \mathscr{L}(E)$, but RT and TR are not regular.

2. The o-Spectrum of r-Compact Operators

2.1. Definition. Let E be a complex Banach lattice. The spectrum of a regular operator T in the Banach algebra $\mathscr{L}^r(E)$ is called the o-*spectrum* and denoted by $\sigma_o(T)$.

Obviously the spectrum $\sigma(T)$ of T in $\mathscr{L}(E)$ is a subset of $\sigma_o(T)$ and the "pure o-spectrum" $\sigma_o(T) \setminus \sigma(T)$ consists of the complex numbers $\lambda \in \rho(T)$ (the resolvent set of T) such that $R(\lambda, T) := (\lambda - T)^{-1}$ is not regular.

From the formula for the spectral radius it can be seen that $r(T)=r_0(T)$ for $T\geqq 0$, where $r(T)$ denotes the spectral radius for T in $\mathscr{L}(E)$ and $r_0(T)$ the one in $\mathscr{L}^r(E)$. The topological connection between $\sigma(T)$ and $\sigma_o(T)$ was discovered by Schaefer [14] (for E order complete). We want to formulate the result more generally.

Let A be a Banach algebra with unit e and B a subalgebra of A such that $e\in B$. Denote the norm of A by $\| \|_A$ and suppose that there is a finer norm $\| \|_B$ on B such that B is a Banach algebra for that norm.

For $x\in B$ denote by $\sigma_A(x)$ the spectrum of x in A and by $\sigma_B(x)$ the one in B. Obviously $\sigma_A(x)$ is a subset of $\sigma_B(x)$.

2.2. Proposition. *Let $x\in B$. For every clopen subset $D\neq\varnothing$ of $\sigma_B(x)$, $\sigma_A(x)\cap D\neq\varnothing$.*

Proof. D is a compact subset of $\mathbb{C}$ and by hypothesis there exists an open subset O_1 of $\mathbb{C}$ such that $D=\sigma_B(x)\cap O_1$. Suppose $D\cap\sigma_A(x)=\varnothing$. Then we have $D=\sigma_B(x)\cap O$ for the open set $O=O_1\setminus\sigma_A(x)$. There exists a Cauchy domain G in $\mathbb{C}$ such that $D\subset G\subset\bar{G}\subset O$. Therefore $\partial G\subset\complement\sigma_B(x)$, and the mapping $\lambda\to R(\lambda,x)$ from ∂G into B is continuous. Let $\lambda_0\in D$. The integral (in the topology of B) $\frac{1}{2\pi i}\int_{\partial G} R(\lambda,x)/(\lambda-\lambda_0)\,d\lambda$ exists and is an element of B. Because $G\subset O\subset\complement\sigma_A(x)$ one has $R(\lambda_0,x)=\frac{1}{2\pi i}\int_{\partial G} R(\lambda,x)/(\lambda-\lambda_0)\,d\lambda$ (whereby the integral is defined in the topology of A and coincides with the integral above, since the norm of B is finer than the one of A). It follows that $R(\lambda_0,x)\in B$, contradicting $\lambda_0\in\sigma_B(x)$.

2.3. Corollary. *$\sigma_B(x)\setminus\sigma_A(x)$ has no isolated points.*

2.4. Corollary. *If $\sigma_B(x)\neq\sigma_A(x)$, the set $\sigma_A(x)\setminus\sigma_A(x)$ is not countable.*

Proof. Let $\lambda_0\in\sigma_B(x)\setminus\sigma_A(x)$. There exists a real number $r_0>0$ such that $\{\lambda\,|\,|\lambda-\lambda_0|\leqq r_0\}\subset\complement\sigma_A(x)$. Suppose $\sigma_B(x)\setminus\sigma_A(x)$ is countable. Then there exists an r, $0<r\leqq r_0$, such that $\{\lambda\in\complement\,|\,|\lambda-\lambda_0|=r\}\subset\complement\sigma_B(x)$. Set

$$D:=\{\lambda\in\sigma_B(x)\,|\,|\lambda-\lambda_0|<r\}=\{\lambda\in\sigma_B(x)\,|\,|\lambda-\lambda_0|\leqq r\},$$

then $D\cap\sigma_A(x)=\varnothing$, contradicting 2.2.

For the o-spectrum the preceding results obtain the form:

2.5. Corollary. *Let E be a complex Banach lattice, $T\in\mathscr{L}^r(E)$.*

a) *If D is a non-void clopen subset of $\sigma_o(T)$, then $D\cap\sigma(T)\neq\varnothing$.*

b) *The pure* o-*spectrum $\sigma_o(T)\setminus\sigma(T)$ has no isolated points.*

c) *If $\sigma_o(T)\neq\sigma(T)$, the pure* o-*spectrum is uncountable.*

We can now formulate the main theorem of this section. It shows that all the resolvents of a r-compact operator are regular.

2.6. Theorem. *Let E be a complex Banach lattice. For every $T\in E'\tilde{\otimes}_e E$,*

$$\sigma_o(T)=\sigma(T).$$

For the proof of this theorem we need two lemmas.

2.7. Lemma. *Let* $T_n = \sum_{i=1}^{r} x_i' \otimes y_i^n \in E' \otimes E$ $(n \in \mathbb{N})$, $(y_i^n)_{n \in \mathbb{N}}$ *being a Cauchy sequence in* E $(i=1,\ldots,r)$. *Then* $(T_n)_{n \in \mathbb{N}}$ *is a Cauchy sequence in* $\mathscr{L}^r(E)$.

Proof. For $n, m \in \mathbb{N}$, $T_n - T_m = \sum_{i=1}^{r} x_i' \otimes (y_i^n - y_i^m)$, and therefore

$$|T_n - T_m| \leqq \sum_{i=1}^{r} |x_i'| \otimes |y_i^n - y_i^m|.$$

This implies that

$$\|T_n - T_m\|_r \leqq \sum_{i=1}^{r} \|x_i'\| \, \|y_i^n - y_i^m\|,$$

and the assertion follows.

2.8. Lemma. *Let* $T \in E' \tilde{\otimes}_e E$ *and* $(S_n)_{n \in \mathbb{N}}$ *be a sequence of regular operators such that* $\|S_n\|_r \leqq M$ $(n \in \mathbb{N})$ *for a positive real number* M. *If* $(S_n)_{n \in \mathbb{N}}$ *is a Cauchy sequence in* $\mathscr{L}(E)$, $(S_n T)_{n \in \mathbb{N}}$ *is a Cauchy sequence in* $\mathscr{L}^r(E)$.

Proof. Let $\varepsilon > 0$. There exists an operator of finite rank $T_0 = \sum_{i=1}^{r} x_i' \otimes x_i$ such that $\|T - T_0\|_r < \frac{\varepsilon}{4M}$. By 2.7 there exists a natural number n_0 such that for $n, m \geqq n_0$,

$$\|S_n T_0 - S_m T_0\|_r = \left\| \sum_{i=1}^{r} x_i' \otimes S_n x_i - \sum_{i=1}^{r} x_i' \otimes S_m x_i \right\|_r \leqq \frac{\varepsilon}{2}.$$

Thus for $n, m \geqq n_0$,

$$\begin{aligned} \|S_n T - S_m T\|_r &\leqq \|S_n T - S_n T_0\|_r + \|S_n T_0 - S_m T_0\|_r + \|S_m T_0 - S_m T\|_r \\ &\leqq 2M \, \|T - T_0\|_r + \|S_n T_0 - S_m T_0\|_r \leqq \varepsilon. \end{aligned}$$

Proof. of 2.6. Set $A = \{S \in \mathscr{L}^r(E) \mid ST = TS\}$. A is a closed subalgebra of $\mathscr{L}^r(E)$ with $I_E \in A$.

Because $(\lambda - T^2)^{-1} T = T(\lambda - T^2)^{-1}$ for $\lambda \in \sigma_o(T^2)$, $\sigma_o(T^2) = \sigma_A(T^2)$. Therefore $\sigma_0(T^2) = \sigma(R)$, where $R \in \mathscr{L}(A)$ is defined by $R(S) = ST^2$ for $S \in A$ (see [2], § 5, Prop. 4 (ii)).

The proof is finished when it can be shown that R is a compact operator. Then $\sigma(R) = \sigma_o(T^2)$ is countable and $\sigma_o(T)$ is so, too, because $\sigma_o(T^2) = \sigma_o(T)^2$ (spectral mapping theorem), and the proposition follows from 2.5c).

We now prove that R is compact.[1]

Let $(S_n)_{n \in \mathbb{N}}$ be a sequence in A, $\|S_n\|_r \leqq 1$ for $n \in \mathbb{N}$. We have to show that $(R(S_n))_{n \in \mathbb{N}}$ has a convergent subsequence. Denote by U the unit ball of E. The set $K := \overline{TU}$ (norm closure in E) is compact, T being a compact operator. The set $\{S_{n|K} \mid n \in \mathbb{N}\}$ is a relatively compact subset of $C_E(K)$, the space of all continuous E-valued functions on K with the supremum-norm. This follows from the theorem of Arzela-Ascoli, because

1 For the following argument cf. 9.1 of [1]

1. $\{S_{n|K} | n \in \mathbb{N}\}$ is equicontinuous. (For $x_0, x \in K$,

$$\|S_n x - S_n x_0\| \leqq \|S_n\| \, \|x - x_0\| \leqq \|S_n\|_r \, \|x - x_0\| \leqq \|x - x_0\|$$

for every $n \in \mathbb{N}$).

2. For every $x \in K$ the set $\{S_n x | n \in \mathbb{N}\}$ is relatively compact in E. (Since $STU = TSU \subset TU \subset K$, it follows that $SK \subset K$ for all $S \in A$, $\|S\| \leq 1$).

Thus the sequence $(S_{n|K})_{n \in \mathbb{N}}$ has a convergent subsequence $(S_{n_l|K})_{l \in \mathbb{N}}$ in $C_E(K)$. Consequently $(S_{n_l} T)_{l \in \mathbb{N}}$ is a Cauchy-sequence in $\mathscr{L}(E)$. By 2.8, $(S_{n_l} T^2)_{l \in \mathbb{N}}$ is a Cauchy-sequence in $\mathscr{L}^r(E)$, hence in A, which had to be proved.

2.9. Corollary. *Let E be a complex Banach lattice, $T \in \mathscr{L}^r(E)$. Suppose that there exists a non-constant holomorphic function f defined in a neighborhood of $\sigma_0(T)$ such that $f(T) \in E' \tilde{\otimes}_e E$. Then $\sigma(T) = \sigma_0(T)$.*

Proof. By 2.6 and the spectral mapping theorem, $\sigma(f(T)) = \sigma_o(f(T)) = f(\sigma_o(T))$. Hence $f(\sigma_o(T))$ is at most countable. Since f is not constant and $\sigma_o(T)$ is compact, the set $\{\lambda \in \sigma_o(T) | f(\lambda) = \mu\}$ is finite for every $\mu \in f(\sigma_o(T))$. Consequently, $\sigma_o(T)$ being a subset of $f^{-1}(f(\sigma_o(T))$, is at most countable too, and the assertion follows from 2.5c).

The next result gives spectral properties of a special class of positive operators.

Let E be a complex Banach lattice, T a positive operator on E. The *peripheral* o-*spectrum* $r\sigma_o(T)$ is the set $r\sigma_o(T) = \{\lambda \in \sigma_o(T) | \, |\lambda| = r(T)\}$. A positive operator is said to have *disjoint powers* if

$$\inf\{T^n, T^m\} = 0 \qquad \text{for } n, m \in \mathbb{N} \cup \{0\}, \; n \neq m$$

(this means that if for $n \neq m$ S is a real operator on E such that $S \leq T^n, T^m$, then $S \leq 0$).

2.10. Proposition. *Let E be a complex Banach lattice. If T is a positive operator on E with disjoint powers, then the peripheral order spectrum of T is given by*

$$r\sigma_o(T) = \{\lambda \in \mathbb{C} \,|\, |\lambda| = r(T)\}.$$

Proof. For $|\lambda| > r(T)$, $R(\lambda, T) = \sum_{n=0}^{\infty} T^n / \lambda^{n+1}$ (Neumann's series, converging in $\mathscr{L}^r(E)$). Routine arguments involving the disjointness of the powers of T show that

$$|R(\lambda, T)| = \sup\{\operatorname{Re}(e^{-i\theta} R(\lambda, T) | \theta \in [0, 2\pi[\}$$

exists and is given by

$$|R(\lambda, T)| = \sum_{n=0}^{\infty} |\lambda|^{-(n+1)} T^n. \tag{$*$}$$

There is a sequence $(r_n)_{n \in \mathbb{N}}$ such that $r_n > r(T)$ for all $n \in \mathbb{N}$, $\lim_{n \to \infty} r_n = r(T)$ and $\lim_{n \to \infty} \|R(r_n, T)\| = \infty$ (see V, 4.1 of [13]). Suppose $r(T) > 0$ (otherwise the proposition is trivial) and let $\mu \in \mathbb{C}$, $|\mu| = r(T)$. Set $\alpha = r(T)$. Then $\mu = \lim_{n\to\infty} r_n \alpha$ and by (*)

$$\lim_{n\to\infty} \|R(r_n\alpha, T)\|_r = \lim_{n\to\infty} \|R(r_n, T)\| = \infty.$$

Since the mapping $v \to R(v, T)$ of $\complement\, \sigma_0(T)$ into $\mathscr{L}^r(E)$ is continuous, this implies $\mu \in \sigma_0(T)$.

2.11. Corollary. *A positive operator $T \in E' \tilde{\otimes}_e E$ with disjoint powers is quasi-nilpotent.*

It will be shown in the next section that the corollary is no longer true for positive compact operators.

3. Convolution Operators

Throughout this section we assume that G is a locally compact group. By a measure on G we understand a complex regular Borel measure. The space of the continuous complex-valued functions on G having compact support is denoted by $C_c(G)$, and by $C_0(G)$ we denote the space of the continuous functions vanishing at infinity.

The space $M(G)$ of all bounded measures on G may be identified with the dual space of $C_0(G)$.

$M(G)$ is a Banach algebra with unit when multiplication is defined by convolution: For $\mu, v \in M(G)$ $\mu * v \in M(G)$ is given by

$$\langle f, \mu * v \rangle = \iint f(st)\, d\mu(s)\, dv(t)$$

for $f \in C_0(G)$. The unit of $M(G)$ is the Dirac measure δ_e in the unit e of G. There is an *involution* $\sim$ on $M(G)$ which is defined by

$$\langle f, \mu^{\sim} \rangle = \overline{\langle \bar{f}, \check{\mu} \rangle}, \quad (f \in C_0(G))$$

where the bar denotes complex conjugation and $\check{\mu}$ is defined by $\langle f, \check{\mu} \rangle = \langle \check{f}, \mu \rangle$ for $f \in C_0(G)$, $\check{f}(s) = f(s^{-1})$ $(s \in G)$. The spectrum of an element μ of $M(G)$ related to the Banach algebra $M(G)$ is denoted by $\sigma(\mu)$ and its spectral radius by $r(\mu)$.

On the positive cone $M(G)_+$ of $M(G)$ the norm is multiplicative, i.e.

$$\|\mu * v\| = \|\mu\|\, \|v\| \qquad \text{for } \mu, v \in M(G)_+.$$

In particular, $\|\mu^n\| = \|\mu\|^n$ (for all $n \in \mathbb{N}$), hence $r(\mu) = \|\mu\|$ for $\mu \in M(G)_+$, as can be seen from the formula for the spectral radius.

The positive cone defines the structure of a Banach lattice on $M(G)$.

In the following, the spaces $L^p(G)$ $(1 \leqq p \leqq \infty)$ are always understood relative to the left Haar measure on G, as is the expression "almost everywhere" (a.e.). Invoking the Radon-Nykodim theorem, $L^1(G)$ can be identified with a band in $M(G)$, in particular $L^1(G)$ is a closed sublattice of $M(G)$. Moreover, $L^1(G)$ is an algebraic ideal in $M(G)$. Consequently the spectrum of an element f of $L^1(G)$

related to the Banach algebra $L^1(G)$ is the same as the one related to the Banach algebra $M(G)$, we denote it by $\sigma(f)$.

For a measure μ on G and $f \in C_c(G)$ a continuous function $\mu * f$ can be defined on G by the formula

$$\mu * f(s) = \int f(t^{-1}s)\, d\mu(t) \qquad (s \in G).$$

If μ is bounded $\mu * f$ is in $L^p(G)$ and

$$\|\mu * f\|_p \leqq \|\mu\|\, \|f\|_p \qquad (1 \leqq p \leqq \infty).$$

We call the continuous (resp., weakly continuous if $p = \infty$) extension to $L^p(G)$ of the mapping $(f \to \mu * f)$ of $C_c(G)$ into $L^p(G)$ the *convolution operator* $T_{\mu, p}$ on $L^p(G)$.

In this way we get a Banach algebra homorphism τ_p of $M(G)$ into $\mathscr{L}(L^p(G))$ $(\tau_p(\mu) = T_{\mu, p})$.

τ_2 preserves the involution

$$(T_{\mu, 2})^* = T_{\mu^\sim, 2}$$

(S^* denotes the Hilbert space conjugate of $S \in \mathscr{L}(L^2(G))$). For $p = 1$

$$(T_{\mu, 1})' = T_{\check{\mu}, \infty}$$

($S' \in \mathscr{L}(L^\infty(G))$ is the Banach space adjoint of $S \in \mathscr{L}(L^1(G))$).

Up to now all these facts are standard and can be found in the monograph [7].

For our purposes the following observation is important: The operator $T_{\mu, p}$ is positive if μ is positive; since every measure in $M(G)$ is a linear combination of positive bounded measures, every convolution operator is regular. Thus the mapping τ_p in fact gives a representation of $M(G)$ into the Banach algebra $\mathscr{L}^r(L^p(G))$ $(1 \leqq p \leqq \infty)$; we are going to investigate this representation in what follows.

Every $a \in G$ defines a translation operator R_a on $L^p(G)$ by

$$(R_a f)(s) = f(sa) \qquad (s \in G).$$

It is easy to see that the convolution operators commute with the translations. Moreover, this property characterizes the (positive) convolution operators among the positive operators on $L^p(G)$ $(1 \leqq p < \infty)$ if the group is amenable. This can be seen from the following two theorems.

3.1. Theorem (Brainerd-Edwards [3]). *Let T be a positive operator on $L^p(G)$ $(1 \leqq p \leqq \infty)$ satisfying $R_a T = T R_a$ for every $a \in G$. Then there is a positive measure μ on G such that*

$$Tf = \mu * f \qquad \textit{for every } f \in C_c(G).$$

It is not difficult to see that for $p = 1, \infty$, the following is valid:

a) If μ is a positive measure on G such that

$$\mu * f \in L^p(G) \quad \text{and} \quad \|\mu * f\|_p \leqq c\, \|f\|_p$$

for a positive constant c and all $f \in C_c(G)$, then μ is bounded.

b) $\|T_{\mu,p}\| = \|\mu\|$ for $\mu \in M(G)_+$.

For $1 < p < \infty$ the following theorem is valid (see [5] and [10]).

3.2. Theorem. *For every p, $1 < p < \infty$, the following assertions are equivalent:*

(i) *G is amenable*

(ii) $\|T_{\mu,p}\| = \|\mu\|$ *for every positive* $\mu \in M(G)$

(iii) *If* μ *is a positive measure on G such that* $\mu * f \in L^p(G)$ *and* $\|\mu * f\|_p \leqq c\, \|f\|_p$ *for a positive constant c and every* $f \in C_c(G)$, *then* μ *is bounded.*

Every compact and every Abelian locally compact group is amenable. For this and the usual definition of "amenable" we refer to [6] and [10]. In the following we will use the notion "amenable" only in the sense (ii) and (iii) of 3.2.

In order to characterize the convolution operators in $\mathscr{L}^r(L^p(G))$ we define the following subspace of $\mathscr{L}^r(L^p(G))$:

$$\mathscr{F}^p = \{T \in \mathscr{L}^r(L^p(G)) \mid R_a T = T R_a \text{ for all } a \in G\} \quad \text{for } 1 \leqq p < \infty.$$

3.3. Proposition. *The space* $\mathscr{F}^p$ $(1 \leqq p < \infty)$ *is a closed sublattice and a full subalgebra of* $\mathscr{L}^r(L^p(G))$. *Consider the mapping*

$$\tau_p \colon M(G) \to \mathscr{F}^p, \quad \mu \to T_{\mu,p}.$$

If $p = 1$ *or G is amenable, then* τ_p *is an isometric algebra and lattice isomorphism onto* $\mathscr{F}^p$.

Remarks. 1. Let B be a closed subalgebra of a Banach algebra A with unit e, such that $e \in B$. B is said to be *full* in A when the following condition is satisfied: If $x \in B$ is invertible in A the inverse x^{-1} is an element of B.
This condition implies $\sigma_A(x) = \sigma_B(x)$ for $x \in B$.

2. Let E_1, E_2 be Banach lattices. An injective linear mapping T of E_1 onto a sublattice of E_2 is called a *lattice isomorphism*, if

$$Tz \geqq 0 \quad \text{iff } z \geqq 0 \text{ for all } z \in E_1.$$

This is equivalent to

$$|Tz| = T|z| \quad \text{for all } z \in E_1.$$

Proof. of 3.3. a) For $S, T \in \mathscr{F}^p$, $a \in G$, $R_a(ST) = SR_a T = (ST) R_a$. Thus $ST \in \mathscr{F}^p$. If $T \in \mathscr{F}^p$ is invertible in $\mathscr{L}^r(L^p(G))$, $R_a T^{-1} = T^{-1} T R_a T^{-1} = T^{-1} R_a T T^{-1} = T^{-1} R_a$ for every $a \in G$, hence $T^{-1} \in \mathscr{F}^p$.
We have proved that $\mathscr{F}^p$ is a full subalgebra of $\mathscr{L}^r(L^p(G))$.

b) $\mathscr{F}^p$ is a sublattice of $\mathscr{L}^r(L^p(G))$.

In fact, take $T \in \mathscr{F}^p$. Since R_a $(a \in G)$ is a real operator, one has

$$R_a(\operatorname{Re} T) = \operatorname{Re}(R_a T) = \operatorname{Re}(T R_a) = (\operatorname{Re} T) R_a.$$

Hence $\operatorname{Re} T \in \mathscr{F}^p$. Since R_a is a lattice isomorphism, the mappings $(S \to R_a S)$ and $(S \to SR_a)$ of $\mathscr{L}^r(L^p(G))$ onto $\mathscr{L}^r(L^p(G))$ are lattice isomorphisms, too (they are positive with positive inverse). Hence $R_a|T| = |R_a T| = |TR_a| = |T|R_a$ for all $a \in G$, which implies $|T| \in \mathscr{F}^p$.

c) For $f \in C_c(G)$, $\langle f, \mu \rangle = (\mu * \check{f})(e)$, and $\mu * \check{f}$ is a continuous function. Hence τ_p is injective and $T_{\mu,p} \geqq 0$ iff $\mu \geqq 0$.

d) The assertion concerning τ_p follows now for $p=1$ from 3.1 and the remark after it, and from 3.1 and 3.2 for $1<p<\infty$ (note that by c) $|T_{\mu,p}| = T_{|\mu|,p}$).

The following theorem is a consequence of 3.3.

3.4. Theorem. 1. *For* $p = 1, \infty$,

$$\sigma(\mu) = \sigma_0(T_{\mu,p}) = \sigma(T_{\mu,p}) \quad \text{for every } \mu \in M(G).$$

2. *If G is amenable,* $1 \leqq p \leqq \infty$, *then*

$$\sigma(\mu) = \sigma_0(T_{\mu,p}) \quad \text{for every } \mu \in M(G).$$

And conversely,

3. *If there exists p,* $1<p<\infty$, *such that*

$$\sigma(\mu) = \sigma_0(T_{\mu,p}) \quad \text{for every } \mu \in M(G),$$

then G is amenable.

Proof. a) Since every operator on $L^p(G)$ is regular if $p=1,\infty$ ([13], IV 1.5), $\sigma_0(T_{\mu,p}) = \sigma(T_{\mu,p})$ for every $\mu \in M(G)$ in this case.

b) Suppose G is amenable or $p=1$. Since $\mathscr{F}^p$ is a full subalgebra of $\mathscr{L}^r(L^p(G))$ (3.3), $\sigma_0(T_{\mu,p})$ coincides with the spectrum of $T_{\mu,p}$ in the Banach algebra $\mathscr{F}^p$ and consequently with $\sigma(\mu)$, $\tau_p\,(\mu \to T_{\mu,p})$ being an Banach algebra isomorphism of $M(G)$ onto $\mathscr{F}^p$ (3.3).

c) Let $p = \infty$.

By a) and b),

$$\begin{aligned} \sigma_0(T_{\mu,\infty}) &= \sigma(T_{\mu,\infty}) = \sigma((T_{\check{\mu},1})') = \sigma(T_{\check{\mu},1}) \\ &= \sigma(\check{\mu}) = \sigma(\mu). \end{aligned}$$

We have now proved the assertion 1. and 2. of the theorem. To prove 3., let $p \in (1,\infty)$ and suppose G is not amenable. By 3.2(ii) there exists $\mu \in M(G)_+$ such that $\|T_{\mu,p}\| < \|\mu\|$. It follows that

$$r_0(T_{\mu,p}) = r(T_{\mu,p}) \leqq \|T_{\mu,p}\| < \|\mu\| = r(\mu),$$

consequently $\sigma_0(T_{\mu,p}) \neq \sigma(\mu)$.

If G is Abelian with dual group G we denote by $\hat{\mu}$ the Fourier-Stieltjes transform of $\mu \in M(G)$, i.e.

$$\hat{\mu}(\gamma) = \int \overline{(s,\gamma)}\, d\mu(s) \quad \text{for } \gamma \in \hat{G}.$$

Since the Fourier transformation coincides with the Gelfand transformation on $L^1(G)$ the spectrum of $f \in L^1(G)$ is

$$\sigma(f) = \{\hat{f}(\gamma) \mid \gamma \in \hat{G}\}^-,$$

whereas

$$\sigma(\mu) \supsetneqq \{\hat{\mu}(\gamma) \mid \gamma \in \hat{G}\}^-$$

for $\mu \in M(G)$ in general (see [12], Chap. 5).

3.5. Corollary. *Let G be a locally compact Abelian group with dual group $\hat{G}$, $\mu \in M(G)$. If D is a non-void clopen subset of $\sigma(\mu)$ there exists $\gamma \in \hat{G}$ such that $\hat{\mu}(\gamma) \in D$.*

Proof. Denote by $\mathscr{F}$: $L^2(G) \to L^2(\hat{G})$ the Plancherel-transformation. Let $\hat{T} := \mathscr{F} T_{\mu,2} \mathscr{F}^{-1}$. $\hat{T}$ is the multiplication operator on $L^2(\hat{G})$ given by $\hat{T}\hat{f} = \mu \cdot \hat{f}$ for all $\hat{f} \in L^2(\hat{G})$. Therefore

$$\sigma(T_{\mu,2}) = \sigma(\hat{T}) = \{\hat{\mu}(\gamma) \mid \gamma \in \hat{G}\}^-.$$

From 2.5 it follows that $\sigma(T_{\mu,2}) \cap D \neq \varnothing$. Since D is open in $\sigma(\mu)$, there exists an open subset O of $\mathbb{C}$ such that $D = \sigma(\mu) \cap O$.

Hence $\{\hat{\mu}(\gamma) \mid \gamma \in \hat{G}\}^- \cap O = \{\hat{\mu}(\gamma) \mid \gamma \in \hat{G}\} \cap D \neq \varnothing$, which implies

$$\{\hat{\mu}(\gamma) \mid \gamma \in \hat{G}\} \cap D = \{\hat{\mu}(\gamma) \mid \gamma \in \hat{G}\} \cap O \neq \varnothing,$$

as O is open.

3.6. Corollary. *If λ is an isolated point of $\sigma(\mu)$, then there exists $\gamma \in \hat{G}$ such that $\hat{\mu}(\gamma) = \lambda$. If $\sigma(\mu)$ is contable it is equal to $\{\hat{\mu}(\gamma) \mid \gamma \in \hat{G}\}^-$.*

3.7. *A Counterexample.* We give an example of a compact (positive) operator T such that $\sigma(T) \neq \sigma_o(T)$. Thus Theorem 2.6 cannot be essentially improved.

Chose $E = L^2(G)$, G the one dimensional torus. It follows from [17] (see also [11]) that a positive measure μ exists on G such that

1. $\mu = \mu^{\sim}$, $\|\mu\| = 1$
2. $\mu^n \wedge \mu^m = 0$ for $n \neq m$, $n, m \in \mathbb{N} \cup \{0\}$
   (μ^n denotes the n-th convolution power of μ).
3. $\hat{\mu} \in c_0(\mathbb{Z})$.

Let $T = T_{\mu,2}$.

Then T has the properties:

a) *T is positive (in the sense of Banach lattices), selfadjoint, $\|T\| = 1$*

b) *T has disjoint powers.*

c) *T is compact.*

d) *$\sigma(T)$ is countable and a subset of $\mathbb{R}$.*

e) $\sigma_o(T) \supset \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$

Proof. a) $(T_{\mu,2})^* = T_{\mu^{\sim},2} = T_{\mu,2}$; $\|T_{\mu,2}\| = \|\mu\| = 1$ by 3.2

b) follows from 3.3

c) Denote by $\mathscr{F}: L^2(G) \to l^2(\mathbb{Z})$ the Plaucherel transformation. $\hat{T} := \mathscr{F} T \mathscr{F}^{-1}$ is given by $\hat{T}(a_n)_{n\in\mathbb{Z}} = (\hat{\mu}(n)\, a_n)_{n\in\mathbb{Z}}$; so $\hat{T}$ is compact, because $\mu \in c_0(\mathbb{Z})$.

d) follows from c) and a)

e) follows from b) and 2.10.

4. Compact Groups

In this section we consider convolution operators related to compact groups. Therefore we will assume that G is a compact group in the following, otherwise keeping the notations of the last section.

Moreover, the Haar measure on G will be denoted by m, the dual object of G by $\hat{G}$. For every $\alpha \in \hat{G}$ we denote by $u_{\alpha ij}$ $(1 \leqq i, j \leqq n_\alpha)$ the coordinate functions corresponding to a representation in α (see [7], § 27 or [4], Chap. 8). In particular

$$\begin{aligned}
&u_{\alpha ij} \quad \text{is a continuous function on } G \\
&u_\alpha(s) = (u_{\alpha ij}(s))_{ij} \quad \text{is a unitary } (n_\alpha \times n_\alpha)\text{-matrix} \\
&u_\alpha(s \cdot t) = u_\alpha(s) \cdot u_\alpha(t) \\
&u_\alpha(s^{-1}) = u_\alpha(s)^{-1} = u_\alpha(s)^* \quad \text{for } s, t \in G.
\end{aligned}$$

To establish a connection to the main theorem of section 2 we note first of all:

4.1. Proposition. *With* p, $1 \leqq p \leqq \infty$, *set* $E = L^p(G)$. *For every* $f \in L^1(G)$, $T_{fm,p} \in E' \tilde{\otimes}_e E$.

Proof. 1. Assume f is a coordinate function, say $f = u_{\alpha ij}$. Then for $g \in L^p(G)$,

$$\begin{aligned}
(T_{mf,p}(g))(s) &= (m \cdot f) * g(s) = \int g(t^{-1}s) f(t)\, dt \\
&= \int g(t) f(st^{-1})\, dt = \int g(t)\, u_{\alpha ij}(st^{-1})\, dt \\
&= \sum_{k=1}^{n_\alpha} \int g(t)\, u_{\alpha ik}(s) \cdot u_{\alpha kj}(t^{-1})\, dt \\
&= \sum_{k=1}^{n_\alpha} \int g(t)\, u^*_{\alpha kj}(t)\, dt\, u_{ik}(s) \qquad \text{(a.e.)}
\end{aligned}$$

Hence $T_{fm,p} = \sum_{k=1}^{n_\alpha} u^*_{\alpha kj} \otimes u_{\alpha ik}$ is an operator of finite rank.

2. It follows from 1. that $T_{f \cdot m} \in E' \otimes E$ for f a trigonometric polynomial (i.e. a linear combination of coordinate functions).

3. Let f be arbitrary. There exists a sequence $(f_n)_{n\in\mathbb{N}}$ of trigonometric polynomials such that $\lim_{n\to\infty} \|f - f_n\|_1 = 0$ (see 27.39 in [7]). By 3.2 (resp. the remark preceding 3.2 for $p = 1, \infty$),

$$\|T_{f_n m,p} - T_{fm,p}\|_r = \|T_{|f_n - f|m,p}\| = \|f_n - f\|_1, \qquad (n \in \mathbb{N}).$$

Hence $\lim_{n\to\infty} T_{f_n m,p} = T_{fm,p}$ in $\mathscr{L}^r(E)$, consequently $T_{fm,p} \in E' \tilde{\otimes}_e E$.

A Banach algebra with involution is called *symmetric* if every self-adjoint element has real spectrum. In this notation we get the well known fact (see [9, 16] for example):

4.2. Corollary. *Let G be a compact group. $L^1(G)$ is a symmetric Banach algebra.*

Proof. If f is self-adjoint, the operator $T_{fm,2}$ is self-adjoint, hence $\sigma(T_{f\cdot m,2})$ is real. Since $\sigma(f)=\sigma_o(T_{f\cdot m,2})=\sigma(T_{f\cdot m,2})$ by 4.1 and 2.6 f has real spectrum.

More can be said.

We denote by $\hat{f}$ the Fourier transform of $f\in L^1(G)$. It is a family $(\hat{f}_\alpha)_{\alpha\in\hat{G}}$ of matrices defined by

$$\hat{f}_{\alpha ij}=\int f(s)\,u_{\alpha ij}(s^{-1})\,ds\,(\alpha\in\hat{G},\ 1\leqq i,j\leqq n_\alpha).$$

The mapping $\mathscr{F}$ defined by $\mathscr{F}f=\hat{f}$ for $f\in L^2(G)$ going from $L^2(G)$ onto the Hilbert space.

$$\mathscr{L}^2(\hat{G})=\{(\phi_\alpha)_{\alpha\in\hat{G}}\,|\,\phi_\alpha\ a\ n_\alpha\times n_\alpha\text{-matrix},\ \sum_{\alpha\in\hat{G}} n_\alpha\|\phi_\alpha\|_2^2<\infty\}$$

$(\|\phi_\alpha\|_2^2=\operatorname{trace}(\phi_\alpha^*\circ\phi_\alpha))$, with the scalar product

$$(\phi,\Psi)=\sum_{\alpha\in\hat{G}} n_\alpha \operatorname{trace}\ (\Psi_\alpha^*\circ\phi_\alpha),$$

is a unitary operator, called the Plancherel transformation (see [4], Chap. 8 for all that).

We denote by $\sigma(\hat{f}_\alpha)$ the spectrum of the matrix $\hat{f}_\alpha (f\in L^1(G),\ \alpha\in\hat{G})$. The following formula for $\sigma(f)$ is analogous to the corresponding expression in the Abelian case.

4.3. Theorem. *For $f\in L^1(G)$ the spectrum $\sigma(f)$ of f in the Banach algebra $L^1(G)$ is the set*

$$\sigma(f)=\bigcup_{\alpha\in\hat{G}}\sigma(\hat{f}_\alpha)\cup\{0\}\qquad \textit{if } G \textit{ is infinite}$$

and

$$\sigma(f)=\bigcup_{\alpha\in\hat{G}}\sigma(\hat{f}_\alpha)\qquad \textit{if } G \textit{ is finite.}$$

Proof. Since $\sigma(f)=\sigma(T_{fm,2})$ by 4.1 and 2.6, we have to show that $\sigma(T_{fm,2})$ is given by the expression of the theorem. Set $\hat{T}=\mathscr{F}T_{fm,2}\mathscr{F}^{-1}$, then $\sigma(\hat{T})=\sigma(T_{fm,2})$. $\hat{T}$ is the operator on $\mathscr{L}^2(\hat{G})$ given by

$$T((\phi_\alpha)_{\alpha\in\hat{G}})=(\phi_\alpha\hat{f}_\alpha)_{\alpha\in\hat{G}}.$$

We first show that $\bigcup_{\alpha\in\hat{G}}\sigma(\hat{f}_\alpha)\subset\sigma(\hat{T})$.

Let $\beta\in\hat{G}$ and $\lambda\in\sigma(\hat{f}_\beta)$. Then there is an $n_\beta\times n_\beta$-matrix $B_\beta\neq 0$ such that $B_\beta(\lambda-\hat{f}_\beta)=0$. (use [2], § 5 Prop. 4(ii)). Thus $\hat{T}\phi=\lambda\phi$ for $\phi=(\phi_\alpha)_{\alpha\in\hat{G}}\in\mathscr{L}^2(\hat{G})$ defined by $\phi_\alpha=0$ for $\alpha\neq\beta$ and $\phi_\beta=B_\beta$. Consequently λ is in the (point) spectrum of $\hat{T}$.

If G is infinite, $L^2(G)$ has infinite dimension, hence $0 \in \sigma(\hat{T})$, $\hat{T}$ being compact. If G is finite, then $\hat{G}$ is finite and it is obvious that $\sigma(\hat{T}) \subset \bigcup_{\alpha\in\hat{G}} \sigma(\hat{f}_\alpha)$. It remains to show that $\sigma(T)\setminus\{0\} \subset \bigcup_{\alpha\in\hat{G}} \sigma(\hat{f}_\alpha)$. Take $\lambda \in \sigma(\hat{T})$, $\lambda \neq 0$. Since $(\|\hat{f}_\alpha\|)_{\alpha\in\hat{G}} \in c_0(\hat{G})$, there exists a finite subset A of $\hat{G}$ such that $\|\hat{f}_\alpha\| < 1/2|\lambda|$ for $\alpha \notin A$. Set $H_1 = \{\phi \in \mathscr{L}^2(\hat{G}) | \phi_\alpha = 0$ for all $\alpha \notin A\}$, $H_2 = \{\phi \in \mathscr{L}^2(\hat{G}) | \phi_\alpha = 0$ for all $\alpha \in A\}$. Then $\mathscr{L}^2(\hat{G})$ is the direct Hilbert sum of H_1 and H_2, and $\hat{T}$ leaves H_1 and H_2 invariant. Set $\hat{T}_i = \hat{T}_{|H_i}$ $(i=1,2)$ then $\sigma(\hat{T}) = \sigma(\hat{T}_1) \cup \sigma(\hat{T}_2)$.

Since $\|\hat{T}_2\| < 1/2|\lambda|$, $\lambda \notin \sigma(\hat{T}_2)$. Thus $\lambda \in \sigma(\hat{T}_1) = \bigcup_{\alpha\in A} \sigma(\hat{f}_\alpha)$.

4.4. Corollary. *For $f \in L^1(G)$,*

$$\begin{aligned}\sigma(T_{fm,p}) &= \bigcup_{\alpha\in\hat{G}} \sigma(\hat{f}_\alpha) \cup \{0\} && \text{if } G \text{ is infinite and} \\ &= \bigcup_{\alpha\in\hat{G}} \sigma(\hat{f}_\alpha) && \text{if } G \text{ is finite } (1 \leqq p \leqq \infty).\end{aligned}$$

In particular the spectrum of $T_{fm,p}$ is independent of p.

Proof. $\sigma(T_{fm,p}) = \sigma_o(T_{fm,p}) = \sigma(f)$ by 4.1, 2.6 and 3.4. The assertion follows now from 4.3.

Of course, 3.5 and 3.6 can be extended to the compact non-abelian case using the same arguments:

4.4. Proposition. *Let $\mu \in M(G)$, $\hat{\mu}_\alpha = \int u_\alpha(s^{-1})\, d\mu(s)$ for $\alpha \in \hat{G}$.*

a) *If $D \subset \sigma(\mu)$ is clopen, there exists $\alpha \in \hat{G}$ such that $D \cap \sigma(\hat{\mu}_\alpha) \neq \varnothing$.*

b) *Every singular point of $\sigma(\mu)$ is element of $\sigma(\hat{\mu}_\alpha)$ for some $\alpha \in \hat{G}$.*

c) *If $\sigma(\mu)$ is countable it is equal to $\overline{\bigcup_{\alpha\in\hat{G}} \sigma(\hat{\mu}_\alpha)}$.*

4.5. Corollary. *If λ is an isolated point in $\sigma_o(T_{\mu,p})$, then λ is an eigenvalue of $T_{\mu,p}$ $(1 \leqq p \leqq \infty)$.*

Proof. Since $\sigma_o(T_{\mu,p}) = \sigma(\mu)$ there exists $\alpha \in \hat{G}$ such that $\lambda \in \sigma(\hat{\mu}_\alpha)$ by 4.4b). Consequently there exists a $n_\alpha \times n_\alpha$-matrice $B_\alpha \neq 0$ such that $\hat{\mu}_\alpha B_\alpha = \lambda B_\alpha$. Let f be the trigonometric polynomial satisfying $\hat{f}_\alpha = B_\alpha$ and $\hat{f}_\beta = 0$ for $\beta \neq \alpha$. Then $\mu * f = \lambda f$ by the uniqueness of the Fourier transform. Since $f \in L^p(G)$ $(1 \leq p \leq \infty)$ and $f \neq 0$ λ is an eigenvalue of $T_{\mu,p}$.

Finally we want to formulate for $L^1(G)$ the analoguous statement of 2.11.

4.5. Proposition. *The only positive function in $L^1(G)$ with disjoint convolution powers is the 0-function.*

Proof. If $f \in L^1(G)_+$ has disjoint powers the same is true for $T_{fm,p}$ $(1 \leqq p \leqq \infty)$, hence $r(T_{f\cdot m}) = r(f) = 0$ by 2.11. This implies $f = 0$ since $r(f) = \|f\|$.

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