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KATO'S EQUALITY AND SPECTRAL DECOMPOSITION FOR POSITIVE C_o-GROUPS

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Let A be the generator of a C_o-semigroup $\{T(t); t \geq 0\}$ defined on a Banach lattice E. It is shown that T(t) is a lattice homomorphism for all $t > 0$ if and only if A satisfies

$$<|x|, A'x'> = <\text{sign } q(x)Ax, x'> \quad (x \in D(A),\ x' \in D(A'))$$

(where $q: E \to E''$ is the evaluation mapping). This equality is used to obtain a spectral decomposition for generators of positive groups.

0. Introduction

There are many interesting and important examples of one-parameter semigroups of lattice homomorphisms defined on a Banach lattice E. Typical examples are of the following kind: E is a Banach lattice of functions defined on some set X (like C(X), X compact, or $L^p(X,\nu)$) and the semigroup $\{T(t); t \geq 0\}$ is given by $T(t)f = h_t \cdot f \circ \varphi_t$ ($f \in E$, $t \geq 0$), where $\{\varphi_t; t \geq 0\}$ is a semiflow on X and h_t a positive function on X ($t \geq 0$) (see [4] for the case $E = C(X)$).

We present a contribution to the following problem: Given a C_o-semigroup $\{T(t); t \geq 0\}$ on a Banach lattice E, find an intrinsic condition on the generator A of the

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semigroup which is equivalent to the condition that $T(t)$ is a lattice homomorphism for all $t \geq 0$. If E has order continuous norm R. Nagel and H. Uhlig [10] proved that Kato's equality

$$A|x| = \operatorname{sign}(x)Ax \quad (x \in D(A))$$

provides such a condition (here, sign(x) is the difference of the band projections onto $(x^+)^{\perp\perp}$, resp. $(x^-)^{\perp\perp}$, and we refer to R. Nagel and H. Uhlig [10] for the necessary background and its relation to the classical inequality proved by T. Kato for the Laplacian).

The main part of this paper is devoted to the generalization of Nagel and Uhlig's result to arbitrary Banach lattices. Due to the possible absence of non-trivial band projections in the general case, one has to find another version of the above equality. We will show that the following weak form

$$\langle |x|, A'x' \rangle = \langle \operatorname{sign}(q(x))Ax, x' \rangle \quad (x \in D(A),\ x' \in D(A'))$$

(where $q\colon E \to E''$ denotes the evaluation map) is characteristic for generators of C_o-semigroups of lattice homomorphisms.

Our proof is based on a simple but crucial observation: Any convex function on a Banach lattice has monotone difference quotients and so is differentiable in some sense. It seems that our method gives - besides the improved generality - more insight in the known results and simplifies considerably their proofs.

Besides the intrinsic interest of the above characterization Kato's equality can be used to obtain the following spectral decomposition for positive groups: Let A be the generator of a positive C_o-group on a Banach lattice E. If $\mu \in \mathbb{R}$ is not in the spectrum $\sigma(A)$ of A, then E is the direct sum of two orthogonal bands I_μ and J_μ which are invariant under the group such that

$$\sigma(A_{|I_\mu}) = \{\lambda \in \sigma(A);\ \mathrm{Re}\lambda < \mu\},$$

$$\sigma(A_{|J_\mu}) = \{\lambda \in \sigma(A);\ \mathrm{Re}\lambda > \mu\}.$$

Here $A_{|I_\mu}$ (resp. $A_{|J_\mu}$) denotes the generator of the group restricted to I_μ (resp. J_μ). If E is an order complete Banach lattice this has been proved already by G. Greiner [5]. This theorem is of interest in connection with stability theory. In fact, it allows one to decompose the semigroup in two parts with distinct asymptotic behaviour (see Greiner's paper [5] for a discussion of this aspect).

In section 5 we investigate the spectral decomposition on C(X) (X compact). It follows, e. g., that $\sigma(A) \cap \mathbb{R}$ is a closed interval if A is the generator of a positive group and X is connected.

For notations and results concerning Banach lattices and positive operators we use Schaefer's monograph [11] as a general reference.

1. Differentiation

Let E be a vector space and F a topological vector space for some topology $\mathcal{T}$; $\theta: E \to F$ and $g: [t_o, t_o + \delta) \to F$ (where $t_o \in \mathbb{R}$ and $\delta > 0$) are given functions.

DEFINITION 1.1. a) g *is* $\mathcal{T}$-*differentiable in* t_o, *if*

$$g'(t_o) := \lim_{h \downarrow o} (1/h) \cdot (g(t_o + h) - g(t_o))$$

exists in F *for the topology* $\mathcal{T}$.

b) θ *is* $\mathcal{T}$-*Gateaux differentiable if for every* $x, y \in E$

$$D_x\theta(y) := \lim_{h \downarrow o} (1/h) \cdot (\theta(x + hy) - \theta(x))$$

exists in F *for the topology* $\mathcal{T}$.

If F is a Banach space, we will speak of strong (resp. weak) differentiability (in the sense of 1.1 a) or b)) if $\mathfrak{T}$ is the norm (resp. weak) topology on F. Note that all derivatives in 1.1 as throughout the whole paper are right derivatives. We assume from now on that E, F are Banach spaces and that $\mathfrak{T}$ is coarser than the norm topology on F.

DEFINITION 1.2. *θ is locally Lipschitz continuous if for every $x_o \in E$ there exists a neighbourhood U of x_o and $L > 0$ such that*

$$\|\theta(x) - \theta(y)\| \leq L\|x - y\|$$

for all $x, y \in U$.

LEMMA 1.3. *(Chain rule) Assume that θ is $\mathfrak{T}$-Gateaux differentiable and locally Lipschitz continuous. If g is strongly differentiable in t_o, then $\theta \circ g$ is $\mathfrak{T}$-differentiable in t_o and*

$$(\theta \circ g)'(t_o) = D_{g(t_o)}\theta(g'(t_o)).$$

Proof. By hypothesis, there exist functions $O_g\colon (0,\delta) \to E$ and $O\colon (0,\delta) \to F$ such that

$$g(t_o + h) = g(t_o) + g'(t_o)h + O_g(h) \quad (0 < h < \delta) \tag{1.1}$$

$$\lim_{h\downarrow o} O_g(h)/h = 0 \text{ strongly} \tag{1.2}$$

$$\theta(g(t_o) + g'(t_o)h) = \theta(g(t_o)) + hD_{g(t_o)}\theta(g'(t_o)) + O(h) \quad (0 < h < \delta) \tag{1.3}$$

$$\mathfrak{T} - \lim_{h\downarrow o} O(h)/h = 0. \tag{1.4}$$

Hence

$$\mathfrak{T} - \lim_{h\downarrow o} (1/h)\cdot(\theta(g(t_o + h)) - \theta(g(t_o)) - h\cdot D_{g(t_o)}\theta(g'(t_o))) =$$

$$\mathfrak{T} - \lim_{h\downarrow o} (1/h)\cdot(\theta(g(t_o) + g'(t_o)h + O_g(h)) - \theta(g(t_o) + g'(t_o)h))$$
$$+ \mathfrak{T} - \lim_{h\downarrow o} (1/h)(\theta(g(t_o) + g'(t_o)h) - \theta(g(t_o)) - hD_{g(t_o)}\theta(g'(t_o)))=$$

$$\mathcal{T}\text{-}\lim_{h\downarrow o}\,(1/h)\cdot(\theta(g(t_o)+g'(t_o)h+O_g(h))-\theta(g(t_o)+g'(t_o)h))$$

$$+\ \mathcal{T}\text{-}\lim_{h\downarrow o}\,O(h)/h =$$

$= 0$ by (1.2) and (1.4), since the norm of the first expression above is dominated by $L\|O_g(h)\|/h$ for small h.

2. Functions which operate on the domain of a generator

Throughout this section $\{T(t);\ t \geq 0\}$ is a C_o-semigroup on a Banach space E with infinitesimal generator A. Let $\theta\colon E \to E$ be a locally Lipschitz continuous function. By $q\colon E \to E''$ we denote the evaluation map. If no confusion is possible, we identify $x \in E$ with $q(x) \in E''$.

THEOREM 2.1. *If $q \circ \theta$ is $\sigma(E'',E')$-Gateaux differentiable, then the following assertions are equivalent:*

(i) $\theta(T(t)x) = T(t)\theta(x) \quad (x \in E,\ t \geq 0)$

(ii) $\langle\theta(x), A'x'\rangle = \langle D_x(q\circ\theta)(Ax), x'\rangle$
for all $x \in D(A),\ x' \in D(A')$

Proof. Assume (i) and let $x \in D(A)$. By $g(t) := T(t)x$ $(0 \leq t \leq 1)$ we define a strongly differentiable function g with values in E. Applying the chain rule 1.3 one obtains that for $x' \in D(A')$

$$\begin{aligned}\langle\theta(x), A'x'\rangle &= d/dt_{\,|t=o}\langle T(t)\theta(x), x'\rangle\\ &= d/dt_{\,|t=o}\langle q(\theta(g(t))), x'\rangle\\ &= \langle D_x(q\circ\theta)(Ax), x'\rangle.\end{aligned}$$

To prove the other implication assume that (ii) holds. Since θ is continuous and $D(A)$ dense in E, it is enough to show that

(2.1) $\theta(T(s)x) = T(s)\theta(x)$ for $s \geq 0,\ x \in D(A)$.

Let $x \in D(A)$, $x' \in D(A')$, and define $\eta: [0,s] \to E$ by $\eta(t) = T(s-t)\theta(T(t)x)$. We will show that

$$(2.2)\quad d/dt_{|t=t_o}\langle\eta(t),x'\rangle = -\langle\theta(T(t_o)x),A'T(s-t_o)'x'\rangle + \langle D_{T(t_o)x}(q\circ\theta)(AT(t_o)x),T(s-t_o)'x'\rangle$$

for $t_o \in (0,s)$. Then (ii) implies that

$$d/dt_{|t=t_o}\langle\eta(t),x'\rangle = 0 \quad (t_o \in (0,s)).$$

Hence $\langle\eta(0),x'\rangle = \langle\eta(s),x'\rangle$. Since $x' \in D(A')$ is arbitrary, it follows that $\eta(0) = \eta(s)$, that is (2.1) holds.

We prove now (2.2). Let $t \in (0,s)$. Let $g(h) = (1/h)\cdot(\theta(T(t+h)x) - \theta(T(t)x))$ $(h > 0)$. Then by 1.3 $\sigma(E'',E') - \lim_{h\downarrow o} g(h) = D_{T(t)x}(q\circ\theta)(AT(t)x) =: z'' \in E''$. In particular, the set $\{g(h);\ 0 < h \leq 1\}$ is weakly bounded in E; hence strongly bounded. Moreover, since $x' \in D(A')$, $\lim_{h\downarrow o}\|(T(s-t-h)' - T(s-t)')x'\| = 0$. Hence

$$(2.3)\quad \lim_{h\downarrow o}|\langle T(s-t-h)g(h),x'\rangle - \langle z'',T(s-t)'x'\rangle| = 0.$$

In fact, $\overline{\lim_{h\downarrow o}}\,|\langle T(s-t-h)g(h),x'\rangle - \langle z'',T(s-t)'x'\rangle|$

$$\leq \overline{\lim_{h\downarrow o}}\,|\langle g(h),(T(s-t-h)' - T(s-t)')x'\rangle| + \overline{\lim_{h\downarrow o}}\,|\langle g(h) - z'', T(s-t)'x'\rangle| = 0.$$

Now we obtain (2.2) by the following calculation:

$$\lim_{h\downarrow o}(1/h)\langle\eta(t+h) - \eta(t),x'\rangle$$

$$= \lim_{h\downarrow o}((1/h)\langle T(s-(t+h))\theta(T(t+h)x),x'\rangle - \langle T(s-t)\theta T(t)x,x'\rangle)$$

$$= \lim_{h\downarrow o}(\langle T(s-t-h)g(h),x'\rangle + (1/h)\langle(T(s-t-h) - T(s-t))\theta(T(t)x),x'\rangle)$$

$$= \langle z'',T(s-t)'x'\rangle + d/dr_{|r=t}\langle T(s-r)\theta(T(t)x),x'\rangle \quad \text{(by (2.2))}$$

$$= \langle D_{T(t)x}(q \circ \theta)(AT(t)x), T(s-t)'x'\rangle - \langle \theta(T(t)x), A'T(s-t)'x'\rangle.$$

COROLLARY 2.2. *If θ is $\sigma(E,E')$-Gateaux differentiable then the following assertions are equivalent:*

(i) $\theta(T(t)x) = T(t)\theta(x) \quad (x \in E,\ t \geq 0)$

(ii) *If* $x \in D(A)$, *then* $\theta(x) \in D(A)$ *and* $A\theta(x) = D_x\theta(Ax)$.

EXAMPLE 2.3. Let X be a locally compact space and $\varphi\colon [0,\infty) \times X \to X$ a continuous semiflow on X. Then

$$T(t)f(p) = f(\varphi(t,p)) \quad (f \in C_o(X),\ p \in X,\ t \geq 0)$$

defines a C_o-semigroup $\{T(t);\ t \geq 0\}$ on $C_o(X)$ (the space of all continuous functions on X which vanish at infinity). Let A be the generator of this semigroup. If k is a differentiable function on $\mathbb{R}$ with continuous derivative, then $\theta(f) = k \circ f \quad (f \in C_o(X))$ defines a function from $C_o(X)$ into itself such that $T(t)(\theta(f)) = \theta(T(t)f)$. Moreover, θ is locally Lipschitz continuous and $\sigma(C_o(X), M(X))$-Gateaux differentiable. More precisely,

$$(2.4) \quad D_f\theta(g) = (k' \circ f) \cdot g \quad (f,g \in C_o(X)).$$

In fact, let μ be a bounded Borel measure on X. Then $\lim_{t \downarrow 0} (1/t)[k(f(p) + tg(p)) - k(f(p))] = k'(f(p)) \cdot g(p)$.
It follows from the dominated convergence theorem that $\lim_{t \downarrow 0} \langle (1/t)(k \circ (f + tg) - k \circ f), \mu\rangle = \langle (k' \circ f) \cdot g, \mu\rangle$.
So we obtain from 2.2:
Every differentiable function k *on* $\mathbb{R}$ *with continuous derivative operates on* $D(A)$ (i. e. $f \in D(A)$ *implies* $k \circ f \in D(A)$) *and* $A(k \circ f) = (k' \circ f) \cdot Af$ *for* $f \in D(A)$.

EXAMPLE 2.4. Let (X,Σ,μ) be a σ-finite measure space and $\varphi\colon [0,\infty) \times X \to X$ a mapping such that

(1) $\varphi(0,x) = x \quad \mu$ - a.e.

(2) $\varphi(s+t,x) = \varphi(s,\varphi(t,x)) \quad \mu$ - a.e. for all $s,t \geq 0$

(3) φ_t: $X \to X$ (defined by $\varphi_t(x) = \varphi(t,x)$) is measure preserving for all $t > 0$

(4) $\lim\limits_{t \downarrow 0} \mu(\varphi_t^{-1}(S) \Delta S) = 0$ for all $S \in \Sigma$.

Then $T(t)f = f \circ \varphi_t$ $(t \geq 0,\ f \in L^p(X,\Sigma,\mu))$ defines a C_o-semigroup on $E = L^p(X,\Sigma,\mu)$ $(1 \leq p < \infty)$. Denote by A the generator of this semigroup. Let $k\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function such that $k(0) = 0$. Define $\theta\colon E \to E$ by $\theta(f) = k \circ f$. Then θ commutes with the semigroup, and it is easy to see that θ is strongly Gateaux differentiable with derivative

$$D_f\theta(g) = (k' \circ f) \cdot g \quad (f,g \in E).$$

(Note that k is absolutely continuous, so the derivative k' of k exists almost everywhere. Moreover $|k'(x)| \leq L$ (a. e.) if L denotes the Lipschitz constant of k.) We conclude from 2.2:

<u>Every Lipschitz continuous function</u> $k\colon \mathbb{R} \to \mathbb{R}$ <u>satisfying</u> $k(0) = 0$ <u>operates on</u> $D(A)$ <u>and</u> $A(k \circ f) = k' \cdot Af$ $(f \in D(A))$.

As an application one obtains that every function of that type operates on the first Sobolev space

$$W^{1,p}(\mathbb{R}^n) = \bigcap_{i=1}^{n} D(\partial/\partial x_i)$$

($\partial/\partial x_i$ is the generator of the translation semigroup $\{T_i(t);\ t \geq 0\}$ on $L^p(\mathbb{R}^n)$ given by $(T_i(t)f)(x_1,\ldots,x_n) = f(x_1,\ldots,x_{i-1},x_i + t,x_{i+1},\ldots,x_n)$ $(i = 1,\ldots,n)$).

An example of a function k of the type considered here is the modulus function $k(x) = |x|$. This function will be considered in more detail in the next section.

3. Semigroups of lattice homomorphisms

During this section E and F denote real Banach lattices.

DEFINITION 3.1. <u>A function</u> $\theta: E \to F$ <u>is called convex if</u> $\theta(\lambda x + (1-\lambda)y) \leq \theta(x) + (1-\lambda)\theta(y)$ <u>for all</u> $x, y \in E$, $0 \leq \lambda \leq 1$.

LEMMA 3.2. <u>Let</u> $\theta: E \to F$ <u>be a convex function. Given</u> $x, y \in E$ <u>let</u> $\Delta: \mathbb{R} \setminus \{0\} \to F$ <u>denote the function</u>

$$\Delta(t) = (1/t) \cdot (\theta(x + ty) - \theta(x)).$$

<u>Then</u> Δ <u>is monotone increasing</u> (i. e. $\Delta(t) \leq \Delta(s)$ <u>whenever</u> $t \leq s$).

The easy proof is omitted.

PROPOSITION 3.3. <u>Let</u> $\theta: E \to F$ <u>be a convex function. Then</u> $q \circ \theta: E \to F''$ <u>is</u> $\sigma(F'', F')$-<u>Gateaux differentiable</u> (<u>where</u> $q: F \to F''$ <u>denotes the evaluation mapping</u>). <u>If</u> F <u>has order continuous norm, then</u> θ <u>is strongly Gateaux differentiable</u>.

<u>Proof</u>. Let $x, y \in E$. It follows from 3.2 that for any $x' \in F'_+$

$$\lim_{t \downarrow 0} \langle \Delta(t), x' \rangle = \inf_{t \downarrow 0} \langle \Delta(t), x' \rangle \tag{3.1}$$

exists. Moreover,

$$\langle \Delta(-1), x' \rangle \leq \lim_{t \downarrow 0} \langle \Delta(t), x' \rangle \leq \langle \Delta(1), x' \rangle$$

Hence $\langle D_x\theta(y), x' \rangle := \lim_{t \downarrow 0} \langle \Delta(t), x' \rangle$ exists for all $x' \in F'$ and defines a continuous linear form $D_x\theta(y)$ on F'. If F has order continuous norm, then $D_x\theta(y) = \lim_{t \downarrow 0} \Delta(t)$ exists strongly.

We say the element $a \in E_+$ is <u>projectable</u> if the band $a^{\perp\perp}$ generated by a is a projection band. In that case we denote by P_a the band projection onto $a^{\perp\perp}$. An arbitrary element a of E is said to be projectable if both a^+ and a^- are so. Note: in a σ-order complete Banach lattice (e. g. $L^p(X, \Sigma, \mu)$) every element is projectable. But if X is locally compact, then $u \in C_0(X)_+$ is projectable if and only if $\mathrm{supp}(u) := \overline{\{p \in X;\ u(p) > 0\}}$ is open. In that case P_u is given by $P_u v = 1_{\mathrm{supp}(u)} \cdot v \quad (v \in C_o(X))$ (where $1_{\mathrm{supp}(u)}$ denotes the characteristic function of supp(u)).

DEFINITION 3.4. *If $x \in E$ is projectable the functions* sign(x) *and* $\widehat{\mathrm{sign}}(x)$ *from* E *into* E *are defined by*

$$\mathrm{sign}(x)y = P_{x^+}y - P_{x^-}y \quad (y \in E)$$
$$\widehat{\mathrm{sign}}(x)y = \mathrm{sign}(x)y + (1 - P_{|x|})|y| \quad (y \in E).$$

Note: sign(x) *is a linear operator, but* $\widehat{\mathrm{sign}}(x)$ *is not unless* $|x|^{\perp\perp} = E$.

LEMMA 3.5. *Let* $x \in E$ *be projectable. Then*

$$\inf_{t>o} \tfrac{1}{t}(|x + ty| - |x|) = \widehat{\mathrm{sign}}(x)y \quad (y \in E). \tag{3.2}$$

Proof. Let $Q = 1 - P_{|x|}$. Then $Qx = 0$. Since Q is a band projection, we obtain

$$Q(\inf_{t>o} \tfrac{1}{t}(|x + ty| - |x|)) = Q|y|. \tag{3.3}$$

Since $Q + P_{x^+} + P_{x^-} = 1$, (3.2) will follow from (3.3) and the following two equalities:

$$\inf_{t>o} \tfrac{1}{t}(|x^+ + tP_{x^+}y| - x^+) = P_{x^+}y \tag{3.4}$$

$$\inf_{t>o} \tfrac{1}{t}(|-x^- + tP_{x^-}y| - x^-) = -P_{x^-}y. \tag{3.5}$$

Since (3.5) can be obtained from (3.4) by replacing x with -x and y with -y, it is enough to show (3.4).

Let $z = x^+ + |y|$. There exists a compact space K such that E_z is isomorphic to C(K). P_{x^+} and $1 - P_{x^+}$ leave E_z invariant. So x^+ is also projectable in C(K). Since E_z is an ideal in E, it is enough to show that (3.4) is valid if the infimum is taken in E_z. We identify x^+ and y with continuous functions on K. Let $B = \{p \in K;\ x^+(p) > 0\}$. Then $\overline{B}$ is open and closed and $P_{x^+}f = 1_{\overline{B}} \cdot f$ for $f \in C(K)$. But for

$p \in B$,

$$\inf_{t>0} \frac{1}{t}(|x^+(p) + t(P_{x^+}y)(p)| - x^+(p)) =$$

$$\inf_{t>0} \frac{1}{t}(|x^+(p) + ty(p)| - x^+(p)) = y(p) = (P_{x^+}y)(p).$$

Moreover, for $p \in K \setminus \overline{B}$,

$$\inf_{t>0} \frac{1}{t}(|x^+(p) + t(P_{x^+}y)(p)| - x^+(p)) = 0 = (P_{x^+}y)(p).$$

So the pointwise infimum in (3.4) coincides with $P_{x^+}y$ outside a rare set. This implies (3.4).

Let $\{T(t);\ t \geq 0\}$ be a C_0-semigroup with generator A. The following is a generalization of [10, 3.3].

PROPOSITION 3.5. *If the semigroup consists of lattice homomorphisms, then the following holds:*
If $x \in D(A)$, $y'' \in E''_+$ *such that* $|x| \wedge y'' = 0$, *then*

(3.6) $\quad <|Ax| \wedge y'',x'> = 0 \quad$ *for all* $x' \in D(A')$.

In particular, if $y'' \in E$, *then* $|Ax| \wedge y'' = 0$.

Proof. Since $D(A')_+ - D(A')_+ = D(A')$, it is enough to show (3.6) for $x' \in D(A')_+ := E_+ \cap D(A')$.

So let $x \in D(A)$, $y'' \in E''_+$ such that $|x| \wedge y'' = 0$, and let $x' \in D(A')_+$. Since $T(t)''$ is a lattice homomorphism [8, (1.2)], it follows that

(3.7) $\quad |T(t)x| \wedge T(t)''y'' = 0 \quad (t \geq 0)$.

Hence

$$<|Ax| \wedge y'',x'> = \lim_{t\downarrow 0} <\frac{1}{t}|T(t)x - x| \wedge y'',x'>$$

$$\leq \overline{\lim_{t\downarrow 0}} <\frac{1}{t}|T(t)x| \wedge y'',x'> \quad \text{(by [11, II 1.6 Corollary])}$$

$$= \overline{\lim_{t\downarrow 0}} <\frac{1}{t}|T(t)x| \wedge (y'' - T(t)''y'' + T(t)''y''),x'>$$

$$\leq \overline{\lim_{t\downarrow 0}} (<\frac{1}{t}|T(t)x| \wedge (y'' - T(t)''y''),x'> + <\frac{1}{t}|T(t)x| \wedge T(t)''y'',x'>)$$

(by [11, II 1.6 Corollary])

$$= \overline{\lim_{t\downarrow 0}} \langle \tfrac{1}{t}|T(t)x| \wedge (y'' - T(t)''y'')x'\rangle \qquad \text{(by (3.7))}$$

$$\leq \overline{\lim_{t\downarrow 0}} \langle y'' - T(t)''y'', x'\rangle$$

$$= \overline{\lim_{t\downarrow 0}} \langle y'', T(t)'x' - x'\rangle = 0 \quad \text{since } x' \in D(A').$$

Now let $\theta: E \to E$ denote the modulus function, that is, $\theta(x) = |x|$ for all $x \in E$. θ is clearly convex and Lipschitz continuous. So it follows from 3.3 that $q \circ \theta$ is $\sigma(E'',E')$-Gateaux differentiable and by 3.5

(3.8) $D_x(q \circ \theta)(y) = (\hat{\mathrm{sign}}(q(x))y \quad (x,y \in E)$

THEOREM 3.6. *The following assertions are equivalent.*

(i) $|T(t)x| = T(t)|x| \quad (x \in E,\ t \geq 0)$

(ii) *For* $x \in D(A)$, $x' \in D(A')$
$\langle |x|, A'x'\rangle = \langle \mathrm{sign}(q(x))Ax, x'\rangle$ (*Kato's equality*).

Proof. It follows from 3.5 that $\langle(\mathrm{sign}(q(x))Ax,x'\rangle = \langle(\hat{\mathrm{sign}}(q(x))Ax,x'\rangle$ for all $x' \in D(A')$. So the theorem is a consequence of 2.1 and (3.8).

COROLLARY 3.7. *Let* $\{T(t);\ t \geq 0\}$ *consist of lattice homomorphisms. If* $x \in D(A)$ *is projectable, then*

$|x| \in D(A)$ *and* $A|x| = \mathrm{sign}(x)Ax.$

Proof. It follows from 3.5 that $\langle \mathrm{sign}(x)Ax,x'\rangle = \langle \mathrm{sign}(q(x))Ax,x'\rangle$ for all $x' \in D(A')$. So the assertion follows from 3.6 (ii).

COROLLARY 3.8. *If* E *is* σ-*order complete, then the following assertions are equivalent:*

(i) $|T(t)x| = T(t)|x| \quad (x \in E,\ t \geq 0)$

(ii) *If* $x \in D(A)$, *then* $|x| \in D(A)$ *and*
$A|x| = \mathrm{sign}(x)Ax$ (*Kato's equality*).

REMARK 3.9. 3.8 is due to H. Uhlig [12], and for the case that the norm is order continuous a proof can be found in [10]. In [10], [12] strong Gateaux differentiability is used exclusively. Indeed it is proved that the modulus function is strongly Gateaux differentiable in certain directions. However, their proof (see [10, 2.2]) uses complicated estimates coming from the algebra structure of E_z via Kakutani's theorem. So the simple observations 3.2 and 3.3 above seem to clarify considerably the matter.

REMARK 3.10. If $\{T(t);\ t \geq 0\}$ is merely a positive semigroup, a slight modification of our arguments shows that

$$(3.9)\quad \langle |x|, A'x'\rangle \geq \langle (\operatorname{sign}(q(x))Ax, x'\rangle \quad \text{for } x \in D(A),\ x' \in D(A')_+ \qquad \text{(Kato's inequality)}$$

Moreover, it is easy to see that

$$(3.10)\quad D(A)_+ := D(A) \cap E_+ \text{ is dense in } E_+$$

$$(3.11)\quad D(A')_+ \text{ is } \sigma(E',E)\text{-dense in } E'_+.$$

So one can reformulate the conjecture by Nagel and Uhlig [10, 1.3] in the following way:

QUESTION: Does Kato's inequality (3.9) together with condition (3.10) and (3.11) imply the positivity of the semigroup? The answer is positive if $E = C(X)$, X compact. In that case one can show (using [2, 5.2]) that the semigroup is positive if (3.9) and (3.11) hold (cf. proof of 5.1).

4. Spectral decomposition

Now we are able to apply the previous results to obtain the spectral decomposition of positive C_0-groups on arbitrary Banach lattices. Let $\{T(t);\ t \geq 0\}$ be a C_0-semigroup on a Banach lattice E with generator A. If M is a closed subspace of E which is invariant under $\{T(t);$

$t \geq 0\}$, then the restrictions of $T(t)$ to M define a C_0-semigroup $\{T(t)_{|M};\ t \geq 0\}$ whose generator will be denoted by $A_{|M}$. Moreover, every $T(t)$ defines an operator $T(t)_{/M}$ on the quotient E/M by $T(t)_{/M}(x + M) = T(t)x + M$. Then $\{T(t)_{/M};\ t \geq 0\}$ is a C_0-semigroup on E/M whose generator will be denoted by $A_{/M}$. By $\rho(A) := \sigma(A) \setminus \mathbb{C}$ we denote the resolvent set of A.

THEOREM 4.1. *Let A be the generator of a C_0-semigroup of lattice homomorphisms. Let $\mu \in \rho(A) \cap \mathbb{R}$. The space $I_\mu = \{x \in E;\ R(\mu,A)|x| \geq 0\}$ is a closed ideal of E which is invariant under $\{T(t);\ t \geq 0\}$. The spectrum of A is decomposed by I_μ in the following sense:*

(4.1) $\quad \sigma(A_{|I_\mu}) = \{\lambda \in \sigma(A);\ \mathrm{Re}\lambda < \mu\}$

(4.2) $\quad \sigma(A_{/I_\mu}) = \{\lambda \in \sigma(A);\ \mathrm{Re}\lambda > \mu\}$

THEOREM 4.2. *Suppose that $\{T(t);\ t \in \mathbb{R}\}$ is a positive C_0-group with generator A. Let $\mu \in \rho(A) \cap \mathbb{R}$. Then E is the direct sum of the orthogonal projection bands*

$$I_\mu = \{x \in E;\ R(\mu,A)|x| \geq 0\} \quad \textit{and}$$

$$J_\mu = \{x \in E;\ R(\mu,A)|x| \leq 0\}.$$

Moreover, I_μ and J_μ are invariant under $T(t)$ $(t \in \mathbb{R})$ and

$$\sigma(A_{|I_\mu}) = \{\lambda \in \sigma(A);\ \mathrm{Re}\lambda < \mu\}$$

$$\sigma(A_{|J_\mu}) = \{\lambda \in \sigma(A);\ \mathrm{Re}\lambda > \mu\}.$$

Our proofs of 4.1 and 4.2 are modifications of those given by Greiner [5] in the order complete case. We first prove 4.1. It is known that I_μ is a closed invariant ideal [5, Lemma 2]. So it remains to show (4.1) and (4.2). By the arguments given in the proof of [5, Proposition], we are done, as soon as the following lemma is proved.

LEMMA 4.3. Let $\mu \in \rho(A) \cap \mathbb{R}$. Then

(4.3) $(R(\mu,A)y)^+ \in I_\mu$ for all $y \in E_+$ and

hence $(R(\mu,A)_{/I_\mu} \leq 0$.

Proof of 4.3. Let $y \in E_+$. First we show that

(4.4) $|R(\mu,A)y| = R(\mu,A)''\ \mathrm{sign}(q(R(\mu,A)y))y$.

This is a consequence of Kato's equality 3.6 (ii). In fact, let $y' \in E'$. Set $x := R(\mu,A)y$, $x' := R(\mu,A)'y'$. It follows from 3.6 that
$<|x|,A'x'> = <\mathrm{sign}(q(x))Ax,x'>$. Hence
$<|x|,(\mu - A')x'> = <\mathrm{sign}(q(x))(\mu - A)x,x'>$.
(Note that $\mathrm{sign}(q(x))x = |x|$ by definition of $\mathrm{sign}(q(x))$).
We obtain
$<|R(\mu,A)y|,y'> = <R(\mu,A)''\mathrm{sign}(q(R(\mu,A)y))y,y'>$.
Since $y' \in E'$ is arbitrary (4.4) follows.

Let $y_1'' = \frac{1}{2}(\mathrm{sign}(q(R(\mu,A)y))y + y) \in E''$. It follows from (4.4) that

(4.5) $(R(\mu,A)y)^+ = R(\mu,A)''y_1''$.

In order to show that $(R(\mu,A)y)^+ \in I_\mu$, by [5, Lemma 2] it is enough to show that

(4.6) $$\sup_{r>0} \left\| \int_0^r e^{-\mu s}\ T(s)\ (R(\mu,A)y)^+\ ds \right\| < \infty.$$

We show (4.6). For $r > 0$ let $V(r)z = \int_0^r e^{-\mu s}\ T(s)z\ ds$ $(z \in E)$. Then $V(r)$ is a positive linear operator on E. By a well known formula, $V(r) = R(\mu,A)(1 - e^{-\mu r}T(r))$ $(r > 0)$.

$$\begin{aligned} \text{Hence } 0 \leq V(r)''y_1'' &= R(\mu,A)''y_1'' - e^{-\mu r}T(r)R(\mu,A)''y_1'' \\ &= R(\mu,A)''y_1'' - e^{-\mu r}T(r)(R(\mu,A)y)^+ \\ &\leq R(\mu,A)''y_1''. \end{aligned}$$

So $\sup_{r>0} \|V(r)''y_1''\| \leq \|R(\mu,A)''y_1''\| = := c$.

Using (4.5) one obtains

$$\sup_{r>0} \|V(r)(R(\mu,A)y)^+\| = \sup_{r>0} \|R(\mu,A)''V(r)''y_1''\| \leq \|R(\mu,A)\|\cdot c<\infty,$$

so that (4.6) holds, and the proof is finished.

Proof of 4.2. Let $\mu \in \rho(A) \cap \mathbb{R}$. It is enough to show that $E = I_\mu + J_\mu$. We first show

$$(4.7) \qquad D(A) \subset I_\mu + J_\mu.$$

Let $y \in E_+$. By (4.5) $(R(\mu,A)y)^+ \in I_\mu$ and

$(R(\mu,A)y)^- = (-R(\mu,A)y)^+ = (R(-\mu,-A)y)^+ \in J_\mu$

$(= \{z \in E;\ R(-\mu,-A)|z| \geq 0\})$. Thus

$R(\mu,A)y = (R(\mu,A)y)^+ - (R(\mu,A)y)^- \in I_\mu + J_\mu$. It follows that

$D(A) = R(\mu,A)E \subset I_\mu + J_\mu$.

From (4.7) we obtain:

$E = \overline{D(A)} \subset \overline{I_\mu + J_\mu} = I_\mu + J_\mu$ (by [11, III 1.2]).

The spectral decomposition obtained in 4.1 and 4.2 has several interesting consequences. Those given in [5] can now easily be generalized to arbitrary Banach lattices, and we refer to the discussion given there.

5. Spectral decomposition on C(X)

Let X be a compact space and C(X) the Banach lattice of all continuous functions on X. The spectral decomposition which we obtained in section 4 can be improved on C(X). In order to reformulate Kato's equality we identify the dual space of C(X) with the space M(X) of all bounded regular Borel measures on X. By $q\colon C(X) \to M(X)'$ we denote the evaluation mapping. Let $f \in C(X)_+$. The band projection $P_{q(f)}$ on $M(X)'$ is given by

$$(5.1) \qquad \langle P_{q(f)}q(g),\mu\rangle = \int_{\{x;f(x)>0\}} g(x)\,d\mu(x)$$

for all $g \in C(X)_+$. [In fact,

$$\langle P_{q(f)} q(g), \mu\rangle = \sup_{n\in\mathbb{N}} \langle nf \wedge g, \mu\rangle = \int_{\{x;f(x)>o\}} g(x)d\mu(x),$$

since $(nf \wedge g)_{n\in\mathbb{N}}$ converges to $1_{\{x;f(x)>o\}} \cdot g$ everywhere.]

Let A be the generator of a C_o-semigroup $\{T(t); t \geq 0\}$ on $C(X)$. It follows from 3.6 and 5.1 that $T(t)$ is a lattice homomorphism for all $t > 0$ if and only if

$$(5.2) \qquad \langle |f|, A'\mu\rangle = \int_{\{x;f(x)>o\}} Af(x)d\mu(x) - \int_{\{x;f(x)<o\}} Af(x)d\mu(x)$$

for all $f \in D(A)$, $\mu \in D(A')$ (Kato's equality).

PROPOSITION 5.1. *Let A be the generator of a C_o-semigroup $\{T(t); t \geq 0\}$ of lattice homomorphisms on $C(X)$. Then A generates a group if and only if $\sigma(A) \cap \mathbb{R}$ is bounded.*

Note: Of course, by saying that A generates a group we mean that $T(t)$ is invertible for $t > 0$. In that case $\{T(t); t > 0\}$ can be embedded in a positive C_o-group.

Proof. In order to prove the non-trivial implication assume that $\sigma(A) \cap \mathbb{R}$ is bounded. By [2, 5.1] it is enough to show

$$(5.3) \qquad f \in D(A)_+,\ \mu \in M(X)_+,\ \langle f,\mu\rangle = 0 \text{ implies } \langle Af,\mu\rangle \leq 0.$$

From (5.2) we obtain

$$(5.4) \qquad \langle Af,\nu\rangle = \int_{\{x;f(x)>o\}} Af(x)d\nu(x) \quad \text{for all } \nu \in D(A').$$

Since $D(A')$ is $\sigma(M(X),C(X))$-dense in $M(X)$, (5.4) is also valid for $\nu = \mu$. Consequently, $\langle Af,\mu\rangle = 0$.

REMARK 5.2. Proposition 5.1 is of interest in connection with the spectral decomposition theorem 4.1 for $E = C(X)$. Indeed, keeping the notation of 4.1, Proposition 5.1 shows that $A_{/I_\mu}$ generates a *positive group* on $C(X)_{/I_\mu}$ (which again can be identified with a space $C(X_1)$, $X_1 \subset X$ compact).

REMARK 5.3. Proposition 5.1 is no longer true on arbitrary Banach lattices. In fact, there is an example of a C_o-semigroup of lattice homomorphisms whose generator has empty spectrum [6, § 4]. On the other hand, generators of positive groups never have empty spectrum [6, 3.4].

Recall: The type ω_o of a C_o-semigroup $\{T(t);\ t \geq 0\}$ is defined by

$$(5.5)\qquad \omega_o = \inf\{\omega \in \mathbb{R};\ \text{there exists } M \geq 1 \text{ such that } \|T(t)\| \leq Me^{t\omega} \text{ for all } t \geq 0\}.$$

If A is the generator of the semigroup for the sake of convenience we let $\omega(A) := \omega_o$. The following holds:

$$(5.6)\qquad e^{t\omega(A)} = r(T(t)) \quad (t \geq 0).$$

The spectral bound $s(A)$ of A is defined by

$$(5.7)\qquad s(A) = \sup\{\mathrm{Re}\lambda;\ \lambda \in \sigma(A)\}.$$

One has always $-\infty \leq s(A) \leq \omega(A)$, but $\omega(A)$ and $s(A)$ may be different even if A is the generator of a positive group on a Banach lattice [14, sec. 4]. If $E = C(X)$, L^1 of L^2, then spectral bound and type for positive semigroups are equal (see [9] for a detailed discussion of this matter).

PROPOSITION 5.4. *Let E be an AM-space and A the generator of a positive C_o-semigroup $\{T(t);\ t \geq 0\}$. If $T(t)'$ is a lattice homomorphism for all $t \geq 0$, then*

$$\omega(A) = s(A).$$

Proof. Denote by $\{T(t)^{\odot};\ t \geq 0\}$ the adjoint semigroup [7, 14.4] with generator $A^{\odot}$ defined on $E^{\odot}$. Then $E^{\odot} = \{x' \in E';\ \lim_{t\to 0} T(t)'x' = x' \text{ strongly}\}$ (by [7, 14.4.2]). Since $T(t)'$ is a lattice homomorphism for every $t \geq 0$, $E^{\odot}$ is a closed sublattice of E' and hence an AL-space. It follows from [3, (3.3)] that $s(A^{\odot}) = \omega(A^{\odot})$. This gives the desired conclusion since $s(A) = s(A^{\odot})$ and $\omega(A) = \omega(A^{\odot})$ as a consequence of [7, 14.3.3].

We are now in the position to prove a "real spectral mapping theorem" for C_o-semigroups of surjective lattice homomorphisms and to improve the spectral decomposition on $C(X)$.

THEOREM 5.5. *Let A be the generator of a C_o-semigroup $\{T(t);\ t \geq 0\}$ of surjective lattice homomorphisms on $C(X)$.*

a) $\exp(t\sigma(A) \cap \mathbb{R}) = \sigma(T(t)) \cap \mathbb{R}_+$ *holds for all* $t > 0$.

b) *Let* $\mu \in \rho(A) \cap \mathbb{R}$. *Then* $I_\mu = \{f \in C(X);\ R(\mu,A)|f| \geq 0\}$ *and* $J_\mu = \{f \in C(X);\ R(\mu,A)|f| \leq 0\}$ *are orthogonal projection bands such that* $I_\mu + J_\mu = C(X)$. *Moreover*, I_μ *and* J_μ *are invariant under* $T(t)$ $(t \geq 0)$ *and*

$$\sigma(A_{|I_\mu}) = \{\lambda \in \sigma(A);\ \mathrm{Re}\lambda < \mu\}$$

$$\sigma(A_{|J_\mu}) = \{\lambda \in \sigma(A);\ \mathrm{Re}\lambda > \mu\}$$

The band projection P *on* I_μ *is given by*

$$P = \frac{1}{2\pi i}\int_\Gamma R(\lambda,T(t))d\lambda$$

for every $t > 0$, *where* $\Gamma = \{z \in \mathbb{C};\ |z| = \exp(\mu \cdot t)\}$. *In particular*, P *commutes with all* $T(t)$ $(t \geq 0)$.

Proof. a) The inclusion $\exp(t\sigma(A)) \subset \sigma(T(t))$ $(t > 0)$ always holds. So one has to verify that $\exp(t\rho(A) \cap \mathbb{R}) \subset \rho(T(t)) \cap \mathbb{R}_+$. Given $\mu \in \rho(A) \cap \mathbb{R}$ we find the closed ideal I_μ according to 4.1. In particular, $s(A_{|I_\mu}) < \mu$. Since $T(t)$ is a surjective lattice homomorphism, $T(t)'$ is a lattice homomorphism as well (use [8, 1.2]). So by 5.4, $\omega(A_{|I_\mu}) = s(A_{|I_\mu}) < \mu$. Consequently, $r(T(t)_{|I_\mu}) < \exp(t\mu)$ $(t > 0)$. Moreover, by 5.1 the operator $A_{/I_\mu}$ generates a positive group. Since $\sigma(A_{/I_\mu}) \subset \{\lambda \in \mathbb{C};\ \mathrm{Re}\lambda > \mu\}$ and $s(-A_{/I_\mu}) = \omega(-A_{/I_\mu})$, it follows that $r((T(t)_{/I_\mu})^{-1}) < \exp(-\mu t)$. Hence
$\exp(t\mu) \in \rho(T(t)_{|I_\mu}) \cup \rho(T(t)_{/I_\mu}) \subset \rho(T(t))$ $(t > 0)$.

b) Let $\mu \in \rho(A) \cap \mathbb{R}$, $t > 0$. By a) $\exp(t\mu) \in \rho(T(t))$. It follows from [1, 4.2] that the spectral projection

$P = \frac{1}{2\pi i} \int_\Gamma R(\lambda, T(t))d\lambda$ is a band projection (where $\Gamma = \{z \in \mathbb{C};$ $|z| = \exp(t\mu)\}$). Let $I = PC(X)$. We show that $I = I_\mu$. Let $x \in I_\mu$. Since $r(T(t)_{|I_\mu}) < \exp(t\mu)$, $R(\lambda, T(t))x = \sum_{n=o}^{\infty} T(n \cdot t)x/\lambda^{n+1}$ for $\lambda \in \Gamma$. Consequently $Px = x$, i. e. $x \in I$. We have proved that $I_\mu \subset I$. Suppose that $I_\mu \neq I$. Then there exists $x \in I \setminus I_\mu$ such that $x > 0$. Since $T(t)_{/I_\mu}$ is a lattice isomorphism and $r((T(t)_{/I_\mu})^{-1}) < \exp(-t\mu)$ (by a) applied to $A_{/I_\mu}$), it follows that $R(\exp(t\mu), T(t)_{/I_\mu}) \leq 0$ (use Neumann's series). So there exists $y \in I_\mu$ such that $R(\exp(t\mu), T(t))x + y \leq 0$. On the other hand, since $x \in I$, we have $R(\exp(t\mu), T(t))x \geq 0$. Hence $0 \leq R(\exp(t\mu), T(t))x \leq -y \in I_\mu$. Since I_μ is an ideal, we conclude that $R(\exp(t\mu), T(t))x \in I_\mu$ and so $x \in I_\mu$, because I_μ is invariant under $T(t)$. This is a contradiction.

COROLLARY 5.6. *If in addition to the assumptions of Theorem 5.4 X is connected, then one has the following alternative:*

Either A generates a positive group and

$$\sigma(A) \cap \mathbb{R} = [-s(-A), s(A)] \quad \textit{or}$$

$$\sigma(A) \cap \mathbb{R} = (-\infty, s(A)].$$

Concluding, we want to point out that, due to 5.4, one can obtain a "real spectral mapping theorem" for positive groups on AM-spaces in the same way as in 5.5:

PROPOSITION 5.7. *Let A be the generator of a positive C_o-group $\{T(t);\ t \in \mathbb{R}\}$ on an AM-space E. Then*

$$\sigma(T(t)) \cap \mathbb{R}_+ = \exp(t\sigma(A) \cap \mathbb{R})$$

holds for all $t \in \mathbb{R}$.

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