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A GENERALIZATION OF DISSIPATIVITY AND POSITIVE SEMIGROUPS

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INTRODUCTION

The starting point of this paper is a "half-norm" Φ on a Banach space X, that is, a positive homogeneous, subadditive real-valued function on X. Such a half-norm defines a positive cone $X_+ = \{x ; \Phi(-x) \leqslant 0\}$, and we assume in addition that this cone is proper. Conversely, every closed, proper cone X_+ in a Banach space is generated by a continuous half-norm (e.g., $\varphi(x) = \text{dist}(-x, X_+)$).

In Part I we develop a theory of Φ-contraction semigroups and the associated class of Φ-dissipative operators. These semigroups are in particular positive, i.e. they leave the cone X_+ invariant.

If $\Phi(x) = \|x\|$, Φ-dissipativity is simply dissipativity, and we recover the Hille-Yosida theorem (X_+ is trivial in this case). If $\Phi(x) = \|x^+\|$ (X being a Banach lattice), Φ-dissipativity is the same as dispersiveness as introduced by Phillips [15]. As has been done in these special cases, Φ-dissipativity can be expressed in terms of the subdifferential $\mathrm{d}\Phi$ of Φ. We also give a notion of strict Φ-dissipativity and there is a remarkable result: Φ-dissipativity implies strict Φ-dissipativity if the operator is densely defined.

In Part II we consider an ordered Banach space X whose positive cone X_+ has non-empty interior. Every $u \in \text{int}(X_+)$ defines in a natural way a half-norm Φ_u which generates the given cone. Applying the results of Part I to these half-norms we show that, if the cone is normal, a densely defined operator A in X is the infinitesimal generator of a C_0-semigroup if and only if its resolvent $\mathrm{R}(\lambda, A)$ exists and is positive for all large real λ. The latter property, in turn, can be expressed by the usual range conditions together with a minimum principle (P) which has been considered by Evans and Hanche-Olsen [6] for bounded generators.

We conclude with an application of our general theory to the case when X is the space of hermitian elements of a C^*-algebra, where we obtain results recently announced by Jørgensen [10].

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Φ-DISSIPATIVE OPERATORS

1. HALF-NORMS

Let X be a real vector space. A real-valued function Φ on X will be called a *half-norm* if the following conditions (C 1—3) are satisfied.

(C1) $\Phi(x+y) \leqslant \Phi(x) + \Phi(y)$ (subadditivity)

(C2) $\Phi(tx) = t\Phi(x)$ for $t \geqslant 0$ (positive homogeneity)

(C3) $\Phi(x) + \Phi(-x) > 0$ for $x \neq 0$.

Note that (C1) and (C2) imply that Φ is convex with $\Phi(0) = 0$ and that $\Phi(x) + \Phi(-x) \geqslant 0$. In view of (C3),

$$\|x\|_{\Phi} = \Phi(x) + \Phi(-x) \tag{1.1}$$

defines a norm on X. This is why we call Φ a half-norm.

In what follows we assume that X is a real Banach space with its own norm $\|\ \|$. We will always assume that Φ is continuous. Because of the positivehomogeneity this means that there exists a constant $c > 0$ such that

$$|\Phi(x)| \leqslant c\|x\|. \tag{1.2}$$

One can introduce an order relation in X by stipulating that

$$x \leqslant y \quad \text{if and only if } \Phi(x-y) \leqslant 0. \tag{1.3}$$

With this definition X becomes an ordered Banach space whose positive cone X_+ is the closed set consisting of all $x \in X$ such that $\Phi(-x) \leqslant 0$.

Conversely, given an ordered Banach space X with a closed positive cone X_+, there always exists a continuous half-norm Φ that induces the order via (1.3); for example,

$$\Phi(x) = \operatorname{dist}(-x, X_+). \tag{1.4}$$

We call (1.4) the *canonical half-norm* for the given order. Canonical half-norms were considered by Calvert [3], though in a different form.

EXAMPLES 1.1. (a) $\Phi(x) = \|x\|$ gives the trivial cone $X_+ = \{0\}$.

(b) If X is a Banach lattice, $\Phi(x) = \|x^+\|$ is the canonical half-norm that induces the given order.

(c) Similarly, if X is the space of hermitian elements in a C^*-algebra, then $\Phi(x) = \|x^+\|$ is the canonical half-norm.

2. Φ-DISSIPATIVE OPERATORS

Let X be a real Banach space and Φ a continuous half-norm on X. Let A be a linear operator with domain $\mathrm{D}(A)$ and range $\mathrm{R}(A)$ in X.

We say A is *Φ-dissipative* if

$$\Phi(u - tAu) \geqslant \Phi(u) \quad \text{for } u \in \mathrm{D}(A) \quad \text{and} \quad t \geqslant 0. \tag{2.1}$$

Since $\Phi(u - tAu)$ is a convex function of $t \in \mathbf{R}$, (2.1) is equivalent to

$$[(\mathrm{d}/\mathrm{d}t)^+\Phi(u - tAu)]_{t=0} \geqslant 0 \quad \text{for } u \in D(A), \tag{2.2}$$

where $(\mathrm{d}/\mathrm{d}t)^+$ denotes the right derivative.

We say A is *strictly Φ-dissipative* if

$$[(\mathrm{d}/\mathrm{d}t)^-\Phi(u - tAu)]_{t=0} \geqslant 0 \quad \text{for } u \in \mathrm{D}(A), \tag{2.3}$$

where $(\mathrm{d}/\mathrm{d}t)^-$ is the left derivative. Due to the convexity of $\Phi(u - tAu)$ in t, strict Φ-dissipativity implies Φ-dissipativity.

EXAMPLE 2.1. (cf. Example 1.1) (a) If $\Phi(x) = \|x\|$, (strict) Φ-dissipativity reduces to (strict) *dissipativity* (cf. Browder [2], Kato [11]).

(b) If $\Phi(x) = \|x^+\|$ (assuming that X is a Banach lattice), Φ-dissipativity coincides with the property of *dispersiveness* due to Phillips [15] (see also Hasegawa [8], Sato [16]), although this is not obvious. (The proof will be given below in Example 3.2(b), after another characterization of Φ-dissipativity has been introduced.)

PROPOSITION 2.2. *If A is Φ-dissipative, $1 - tA$ is injective for $t \geqslant 0$.*

Proof. Suppose $u - tAu = 0$, where $u \in \mathrm{D}(A)$ and $t \geqslant 0$. Then (2.1) implies $\Phi(u) \leqslant 0$. Since $-u$ satisfies the same condition as u, we have also $\Phi(-u) \leqslant 0$. Thus we conclude $u = 0$ from (C3).

THEOREM 2.3. *If a Φ-dissipative operator A is closable, the closure $\tilde{A}$ is also Φ-dissipative.*

Proof. This follows directly from (2.1).

REMARK. It is not known whether Φ-dissipativity in Theorem 2.3 can be replaced with *strict* Φ-dissipativity. However, Theorem 2.5 establishes this for *densely defined* operators.

THEOREM 2.4. *If A is densely defined and Φ-dissipative, then A is closable (hence $\tilde{A}$ is Φ-dissipative by Theorem 2.3).*

Proof. Let $u_n \in \mathrm{D}(A)$, $u_n \to 0$, $Au_n \to v \in X$, $n \to \infty$. We have to show that $v = 0$. To this end, let $w \in \mathrm{D}(A)$. Then (2.1) gives

$$\Phi(u_n + tw) \leqslant \Phi(u_n + tw - tA(u_n + tw)), \quad t > 0.$$

Because Φ is continuous we can let $n \to \infty$, getting

$$\Phi(tw) \leqslant \Phi(t(w - v) - t^2 Aw).$$

Hence $\Phi(w) \leqslant \Phi(w - v - tAw)$ by positive-homogeneity. Letting $t \downarrow 0$ finally gives $\Phi(w) \leqslant \Phi(w - v)$. Since $\mathrm{D}(A)$ is dense by hypothesis, we can let $w \to v$, obtaining $\Phi(v) \leqslant \Phi(0) = 0$. Since $A(-u_n) \to -v$ as $n \to \infty$, we have $\Phi(-v) \leqslant 0$ similarly. Hence $v = 0$ by (C3).

THEOREM 2.5. *If A is densely defined , A is Φ-dissipative if and only if A is strictly Φ-dissipative. (More generally, it suffices that* $\mathrm{D}(A)$ *is dense with respect to* $\mathrm{R}(A)$.*)*

Proof. Suppose that A is Φ-dissipative. Let $u, w \in \mathrm{D}(A)$, $t > 0$. Then

$$\Phi(u + tAu) \leqslant \Phi(u + tw) + ct\|w - Au\| \leqslant \qquad \text{(by (C1) and (1.2))}$$
$$\leqslant \Phi((1 - tA)(u + tw)) + ct\|w - Au\| = \qquad \text{(by (2.1))}$$
$$= \Phi(u + t(w - Au) - t^2 Aw) + ct\|w - Au\| \leqslant$$
$$\leqslant \Phi(u) + 2ct\|w - Au\| + ct^2\|Aw\|. \qquad \text{(by (1.2))}$$

It follows that $[(\mathrm{d}/\mathrm{d}t)^- \Phi(u - tAu)]_{t=0} \geqslant -2c\|w - Au\|$. If $\mathrm{D}(A)$ is dense with respect to $\mathrm{R}(A)$, we may let $w \to Au$, obtaining (2.3). Thus A is strictly Φ-dissipative.

REMARK. In the special case of dissipativity ($\Phi(x) = \|x\|$), Theorem 2.5 has been mentioned by several authors independently (Chernoff [4], Batty [1]). Theorem 2.4 was proved by Lumer and Phillips [13] in the dissipative case, and by Sato [16] in the dispersive case. Curiously, it appears that Theorem 2.3 has not previously been proved in the dispersive case, probably because condition (2.1) was not known in that case.

3. Φ-DISSIPATIVITY AND THE SUBDIFFERENTIAL OF Φ.

Sometimes it is convenient to express Φ-dissipativity in terms of the subdifferential $\mathrm{d}\Phi$ of Φ; in fact ordinary dissipativity and dispersiveness are usually defined in this manner.

Recall that Φ is a convex, continuous map of X into $\mathbf{R}$. The subdifferential $\mathrm{d}\Phi$ is a map from X to 2^{X^*}. For each $x \in X$, $\mathrm{d}\Phi(x)$ is by definition the set of all $f \in X^*$ such that

$$\langle x, f\rangle = \Phi(x) \text{ and } \langle y, f\rangle \leqslant \Phi(y) \quad \text{for all } y \in X \tag{3.1}$$

(because Φ is positive homogeneous in our case).

The one-sided directional derivatives of Φ have well known representations by means of the subdifferential (see e.g.: Moreau [14, (10.15)]).

$$[(\mathrm{d}/\mathrm{d}t)^+ \Phi(x + ty)]_{t=0} = \max\{\langle y, f\rangle;\ f \in \mathrm{d}\Phi(x)\} \tag{3.2}$$

$$[(\mathrm{d}/\mathrm{d}t)^- \Phi(x + ty)]_{t=0} = \min\{\langle y, f\rangle;\ f \in \mathrm{d}\Phi(x)\}. \tag{3.3}$$

Accordingly, from the definition (2.2—3) of (strict) Φ-dissipativity, we have the following criteria.

THEOREM 3.1. *A is Φ-dissipative if and only if for each $u \in D(A)$, $\langle Au, f\rangle \leqslant 0$ for some $f \in d\Phi(u)$.*

A is strictly Φ-dissipative if and only if for each $u \in D(A)$,

$$\langle Au, f\rangle \leqslant 0 \quad \textit{for every } f \in d\Phi(u).$$

Finally, let us recall that X carries an ordering associated with Φ by (1.3). Defining the order on X^* as usual we obtain from (3.1)

$$f \geqslant 0 \quad \text{if } f \in d\Phi(x). \tag{3.4}$$

EXAMPLE 3.2. (a) $\Phi(x) = \|x\|$. In this case $d\Phi$ is the *duality map:* $d\Phi(x)$ contains each $f \in B^*$ with $\langle x, f\rangle = \|x\|$. This implies that $\|f\| = 1$ if $x \neq 0$, while $d\Phi(0) = B^*$. (Here B^* denotes the unit-ball of X^*.)

(b) $\Phi(x) = \|x^+\|$ (X is a Banach lattice). In this case it is easy to see that $f \in d\Phi(x)$ is characterized by

$$f \in B_+^* \ (= B^* \cap X_+^*) \quad \text{with } \langle x, f\rangle = \|x^+\|. \tag{3.5}$$

(3.5) implies that $\langle x^-, f\rangle = 0$, and that $\|f\| = 1$ if $x^+ \neq 0$ while $0 \in d\Phi(x)$ if $x^+ = 0$. Thus Theorem 3.1 shows that in this case Φ-dissipativity coincides with dispersiveness as defined by Phillips [15]. Indeed, Phillips calls an operator A dispersive if $[Ax, x^+] \leqslant 0$. Here $[x, y] = \langle x, g\rangle$ is a *semi-inner product:* $g = g(y)$ is some vector in X^* such that $\|g\|^2 = \|y\|^2 = \langle y, g\rangle$ and $g \in X_+^*$ if $y \in X_+$. Thus $[Ax, x^+] \leqslant 0$ is equivalent to $\langle Ax, f\rangle \leqslant 0$ for some $f \in X_+^*$ such that $\|f\|^2 = \|x^+\|^2 = \langle x^+, f\rangle$. In view of the remarks made above, this condition is equivalent to (3.5) by a simple change of normalization of f.

4. Φ-CONTRACTION SEMIGROUPS

In this section we consider the infinitesimal generator A of a strongly continuous semigroup $\{U(t); t > 0\}$ on X, and we write $U(t) = e^{tA}$. We assume that $\{e^{tA}; t > 0\}$ is of a class (0, A) and type ω, as defined by Hille-Phillips [9].

For the convenience of the reader, we recall that the type of $U(t)$ is the infimum of the set of real numbers ω_0 such that $\|U(t)\| \leqslant Me^{\omega_0 t}$ for some M and all large t. If $\lambda > \omega$ then the resolvent $(\lambda - A)^{-1} = R(\lambda, A)$ exists. The semigroup $U(t) = e^{tA}$ is said to be of class (A) provided that $\lambda R(\lambda, A) \to I$ in the strong operator topology as $\lambda \to \infty$. The semigroup is of class (0, A) if in addition the integral $\int_0^1 \|U(t)x\|\, dt$ is finite for each $x \in X$ (see [9, § 10.6]).

THEOREM 4.1. *Let* $\{e^{tA}; t > 0\}$ *be of class* $(0, A)$ *and type* ω. *Let* Φ *be a half-norm on* X. *Then*

$$\Phi(e^{tA}x) \leqslant \Phi(x) \quad (t > 0,\ x \in X) \tag{4.1}$$

if and only if A *is* Φ*-dissipative. In this case* e^{tA} *is positive for* $t > 0$.

Proof. It is known that $(1 - tA)^{-1}$ exists and is bounded on X for $t^{-1} > \omega$. If $\Phi(u - tAu) \geqslant \Phi(u)$, it follows that

$$\Phi((1 - tA)^{-1}x) \leqslant \Phi(x) \quad \text{for all } x \in X. \tag{4.2}$$

Then $\Phi((1 - tA)^{-n}x) \leqslant \Phi(x)$ by iteration, and (4.1) follows from the formula $e^{tA}x = \lim_{n\to\infty}(1 - (t/n)A)^{-n}x$ (see [9, (11.6.6)]).

Suppose, conversely, that (4.1) is true. The relation

$$\lambda(\lambda - A)^{-1}x = \int_0^\infty \lambda e^{-\lambda t}e^{tA}x \, dt \quad (\lambda > \omega)$$

is true for a semigroup of class $(0, A)$ (see [9, 11.5.2]), whence the convexity of Φ gives, for $\lambda > \omega$,

$$\Phi(\lambda(\lambda - A)^{-1}x) \leqslant \int_0^\infty \lambda e^{-\lambda t}\Phi(e^{tA}x) \, dt \leqslant \int_0^\infty \lambda e^{-\lambda t}\Phi(x) \, dt = \Phi(x).$$

This gives (2.1) on writing $\lambda = t^{-1}$ and $u = (1 - tA)^{-1}x$.

(4.1) implies that $\Phi(e^{tA}(-x)) \leqslant \Phi(-x) \leqslant 0$ for $x \in X_+$. Hence $x \in X_+$ implies $e^{tA}x \in X_+$, showing that e^{tA} is positive.

REMARK 4.2. We have assumed that A is the infinitesimal generator of a semigroup. Suppose we do not make this assumption and try to generate a semigroup $\{e^{tA}; t > 0\}$. Then we have to assume in addition to A being Φ-dissipative, that $(1 - tA)^{-1}$ exists as a bounded linear operator on X for sufficiently small t. But this does not seem to suffice.

If we assume in addition that Φ has the property:

$$\Phi(x) + \Phi(-x) \geqslant \delta\|x\| \quad (x \in X) \tag{C4}$$

for some constant $\delta > 0$, then it is easy to show that A generates a bounded C_0-semigroup $\{e^{tA}; t > 0\}$. Indeed, we then obtain (4.2) as above. Now (C4) implies that $\|x\|_\Phi =: \Phi(x) + \Phi(-x)$ is an equivalent norm on X. Moreover, (4.2) implies $\|(1 - tA)^{-1}x\|_\Phi \leqslant \|x\|_\Phi$. Hence A generates a *contraction* C_0-semigroup $\{e^{tA}; t > 0\}$ on $(X, \|\ \|_\Phi)$, which is also a bounded C_0-semigroup in the original norm of X.

Note that if X is an ordered Banach space with closed positive cone X_+ and Φ is the canonical half-norm, then property (C4) simply says that X_+ is a *normal* cone.

SEMIGROUPS WHICH LEAVE INVARIANT A POSITIVE CONE WITH NONEMPTY INTERIOR

5. The result

Let X be a real Banach space and X_+ a proper closed cone in X. Let A be a linear operator in X with domain $D(A)$. We consider the following property:

(P) If $x \in D(A) \cap X_+$ and $f \in X_+^*$ such that $\langle x, f \rangle = 0$, then $\langle Ax, f \rangle \geqslant 0$.

Property (P) has been considered by Evans and Hanche-Olsen [6]. They prove that (P) is equivalent to e^{tA} being positive for all $t \geqslant 0$ if A is a bounded operator and X_+ has the "nearest point property".

We denote by $\rho(A)$ the resolvent set of A, that is the set of all complex numbers λ such that $\lambda - A_C$ has a bounded inverse $R(\lambda, A) := (\lambda - A_C)^{-1}$. Here A_C denotes the $\mathbf{C}$-linear extension of A in X_C, the complexification of X. That is, A_C has the domain $D(A_C) = D(A) + iD(A)$ and $A_C(x + iy) = Ax + iAy$ for $x + iy \in$ $\in D(A_C)$.

For $\lambda \in \rho(A)$ we say $R(\lambda, A)$ is *positive* ($R(\lambda, A) \geqslant 0$) if $R(\lambda, A)_x \in X_+$ whenever $x \in X_+$.

By $\sigma(A)$ we denote the spectrum of A, that is the complement of $\rho(A)$ in $\mathbf{C}$. The *spectral bound* $s(A)$ of A is the number

$$s(A) = \sup\{\mathrm{Re}\lambda\,;\ \lambda \in \sigma(A)\}.$$

If A is the generator of a C_0-semigroup of type ω the following inequality holds:

$$-\infty \leqslant s(A) \leqslant \omega < \infty. \tag{5.1}$$

For the rest of this paper we assume that X_+ has non-empty interior. Then we have the following characterization of operators A with positive resolvent.

Theorem 5.1. *If A is densely defined the following assertions are equivalent.*

(1) *A satisfies* (P) *and there exist arbitrarily large real λ such that* $(\lambda - A)D(A) = X$.

(2) *There exist arbitrarily large real $\lambda \in \rho(A)$ such that* $R(\lambda, A) \geqslant 0$.

Remark. Suppose that it is known that for sufficiently large real $\lambda \in \rho(A)$ there is an estimate of the form $\|R(\lambda, A)\| \leqslant M/\lambda$. Then (2) is equivalent to:

(3) $\rho(A)$ contains an interval of the form (λ_0, ∞) and, for all $\lambda > \lambda_0$, $R(\lambda, A) \geqslant 0$.

That (3) implies (2) is trivial. To see that (2) implies (3), let $\lambda_0 > 0$ be such that $\|R(\lambda, A)\| \leqslant M/\lambda$ for all $\lambda \in \rho(A)$ with $\lambda > \lambda_0$. Now take any $\mu \in \rho(A)$ with $\mu > \lambda_0$ and $R(\mu, A) \geqslant 0$. Then we see by the usual geometric series expansion that $R(\lambda, A)$ exists if $(1 - M^{-1})\mu < \lambda \leqslant \mu$, and

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\mu - \lambda)^n R(\mu, A)^{n+1}$$

is obviously a positive operator. By iterating this argument we get the existence and positivity of $R(\lambda, A)$ for $\lambda_0 < \lambda \leqslant \mu$. Since μ can be arbitrarily large, (3) is established.

COROLLARY 5.2. *If A is the infinitesimal generator of a strongly continuous semigroup $\{e^{tA}; t > 0\}$ of class $(0, A)$ the following are equivalent.*

(i) $e^{tA} \geqslant 0$ *for all* $t > 0$.

(ii) *A satisfies* (P).

(iii) *For infinitely many (equivalently, for all) sufficiently large real* λ, $R(\lambda, A) \geqslant 0$.

REMARK. Concerning the proof of 5.2, note that A certainly satisfies the estimate $\|R(\lambda, A)\| \leqslant M/\lambda$ for some M and all large λ, so that the preceding remark is applicable. The equivalence of (i), (ii), and (iii) then follows from 5.1 together with the formulas connecting e^{tA} and $R(\lambda, A)$ discussed in Section 4.

From the assumption (1) (equivalently (2)) of Theorem 5.1 alone we can conclude that A is a generator if the cone X_+ has a stronger property. We need that for an interior point u of X_+ the order interval $\{x \in X; -u \leqslant x \leqslant u\}$ is norm-bounded. This is equivalent to X_+ being a *normal* cone (see [12, 2.2]).

THEOREM 5.3. *Suppose that X_+ is a normal cone and A is densely defined. Then A is the generator of a positive C_0-semigroup if and only if A satisfies condition* (1) *(equivalently* (2)*) of Theorem* 5.1.

In that case, the following formula holds for the type ω and the spectral bound $s(A)$ of the semigroup generated by A.

$$s(A) = \omega = \inf\{\lambda \in \mathbf{R};\ Au \leqslant \lambda u \text{ for some } u \in D(A) \cap \operatorname{int}(X_+)\}. \tag{5.2}$$

REMARK. 5.4. If A is densely defined $D(A) \cap \operatorname{int}(X_+)$ is non-empty, so that the set which appears in (5.2) is non-empty.

(5.2) is well known for positive matrices; in fact, (5.2) is the upper estimate for $s(A)$ of Collatz's inclusion theorem [5]. Moreover, in our situation an analogous lower estimate for $s(A)$ is valid (and easier to prove): If $x \in D(A)$, $x > 0$ and $\lambda \in \mathbf{R}$ such that $Ax \geqslant \lambda x$, then $\lambda \leqslant s(A)$. In fact, if $\lambda > s(A)$, $R(\lambda, A)$ exists and is positive. Since $Ax - \lambda x \geqslant 0$ it follows that $-x = R(\lambda, A)(Ax - \lambda x) \geqslant 0$, a contradiction.

This lower estimate also shows that $\sigma(A)$ is not empty; in fact, if $u \in D(A) \cap \operatorname{int}(X_+)$, then $Au \geqslant \lambda u$ for some λ. But note that this argument fails if

X_+ has no interior, and indeed it can happen that $\sigma(A)$ is empty even for generators of positive C_0-semigroups if $\operatorname{int}(X_+) = \emptyset$ (see [7]). In this case it can also happen that $\omega > -\infty$, so that $\omega \neq s(A)$ (see [7]).

6. The proofs

For $u \in \operatorname{int}(X_+)$ we define a function $\Phi_u : X \to \mathbf{R}_+$ by

$$\Phi_u(x) = \inf\{\lambda > 0 \; ; \; x \leqslant \lambda u\}. \tag{6.1}$$

Proposition 6.1. *Φ_u is a continuous half-norm which induces the given order of X by* (1.3). *Moreover,*

$$\Phi_u(x)u - x \in X_+ \quad \textit{for every } x \in X. \tag{6.2}$$

Finally, if X_+ is normal Φ_u satisfies (C4) *(see Section* 4*).*

Proof. Since $u \in \operatorname{int}(X_+)$ there exists $\varepsilon > 0$ such that $\|u - y\| \leqslant \varepsilon$ implies $y \in X_+$. Let $x \in X$, $\|x\| \leqslant \varepsilon$. Then $\|u - (u - x)\| \leqslant \varepsilon$. Hence $u - x \geqslant 0$, and $\Phi_u(x) \leqslant 1$. This implies

$$\Phi_u(z) \leqslant (1/\varepsilon)\|z\| \quad \text{for all } z \in X. \tag{6.3}$$

Thus Φ_u is well defined. It is easy to see that Φ_u satisfies (C1) and (C2). So (6.3) implies that Φ_u is continuous.

We prove (6.2). Let $x \in X$. By the definition of Φ_u, $(\Phi_u(x) + (1/n))u \geqslant x$ for all positive integers n. Hence $n\langle x - \Phi_u(x)u, f\rangle \leqslant \langle u, f\rangle$ for all $n \in \mathbf{N}$ and all $f \in X_+^*$. This implies $\langle \Phi_u(x)u - x, f\rangle \geqslant 0$ for all $f \in X_+^*$. Hence $\Phi_u(x)u - x \geqslant 0$ by [11, Corollary 1.3].

Φ_u induces the given order. In fact, let $x \in X$. Clearly, $\Phi_u(-x) = 0$ if $x \in X_+$. Conversely, if $\Phi_u(-x) = 0$, then $-x \leqslant \Phi_u(-x)u = 0$ by (6.2). Hence $x \geqslant 0$.

(C3) follows now because X_+ is a proper cone. We have thus proved that Φ_u is a continuous half-norm.

Suppose that X_+ is normal. Then there exists $M > 0$ such that $\|x\| \leqslant M$ whenever $-u \leqslant x \leqslant u$. Using (6.2) we obtain:

$$\max\{\Phi_u(x), \Phi_u(-x)\} \leqslant 1 \text{ implies } \|x\| \leqslant M.$$

By the positive homogeneity of $\| \; \|$ and (C3),

$$\|x\|_{\Phi_u} = \Phi_u(x) + \Phi_u(-x) \geqslant \max\{\Phi_u(x), \Phi_u(-x)\} \geqslant M^{-1}\|x\|.$$

That is, (C4) is valid.

Note: the norm $\| \; \|_{\Phi_u}$ is equivalent to the order-unit norm $\| \; \|_u$ defined by u, namely

$$\|x\|_u = \inf\{\lambda \; ; \; -\lambda u \leqslant x \leqslant \lambda u\}.$$

In fact, $\|x\|_u = \max\{\Phi_u(x), \Phi_u(-x)\}$.

Lemma 6.2. *If $x \in X$ is such that $\Phi_u(x) > 0$, then there exists $f \in \mathrm{d}\Phi_u(x)$ such that $\langle u, f\rangle = 1$.*

Proof. Since $\Phi_u(x + tu) = \Phi_u(x) + t$ $(t > 0)$, $[(\mathrm{d}/\mathrm{d}t)^+\Phi_u(x + tu)]_{t=0} = 1$. By (3.2), there exists $f \in \mathrm{d}\Phi_u(x)$ such that $\langle u, f\rangle = 1$.

LEMMA 6.3. *Let $u \in \mathrm{D}(A) \cap \mathrm{int}(X_+)$ be such that $Au \leqslant 0$. If A satisfies* (P), *then A is Φ_u-dissipative.*

Proof. Let $x \in \mathrm{D}(A)$. We show that the criterion of Theorem 3.1 is satisfied. If $\Phi_u(x) = 0$, then $f := 0 \in \mathrm{d}\Phi_u(x)$ and $\langle Ax, f\rangle = 0$. So suppose that $\Phi_u(x) > 0$. By 6.2 there exists $f \in \mathrm{d}\Phi_u(x)$ such that $\langle u, f\rangle = 1$. So $\langle \Phi_u(x)u - x, f\rangle = 0$. Since $\Phi_u(x)u - x \geqslant 0$ by (6.2) and $f \geqslant 0$ by (3.4) the hypothesis (P) implies that $\langle A(\Phi_u(x)u - x), f\rangle \geqslant 0$. Because $Au \leqslant 0$ this gives $\langle Ax, f\rangle \leqslant \Phi_u(x)\langle Au, f\rangle \leqslant 0$.

LEMMA 6.4. *If $u \in \mathrm{D}(A) \cap \mathrm{int}(X_+)$ and A is strictly Φ_u-dissipative, then A satisfies* (P).

Proof. Let $x \in \mathrm{D}(A)$, $x \geqslant 0$, $f \in X^*_+$, $\langle x, f\rangle = 0$. Assume that $f \neq 0$ (otherwise nothing has to be proved).Then $\langle u, f\rangle > 0$ ([12, Corollary 1.4]). Let $g = (1/\langle u, f\rangle)f$. Then $\langle u, g\rangle = 1$ and $g \geqslant 0$. So $g \in \mathrm{d}\Phi_u(-x)$. In fact, $\langle -x, g\rangle = 0 = \Phi_u(-x)$, and for $z \in X$, $\langle z, g\rangle \leqslant \Phi_u(z)$, because $\Phi_u(z)u - z \geqslant 0$, hence $\langle \Phi_u(z)u - z, g\rangle \geqslant 0$. It follows from the assumption and Theorem 3.1 that $\langle A(-x), g\rangle \leqslant 0$. Hence $\langle Ax, f\rangle \geqslant 0$.

Note that A satisfies (P) if and only if $(A - \lambda)$ satisfies (P), where $\lambda \in \mathbf{R}$. This follows immediately from the definition of (P).

COROLLARY 6.5. *If A is densely defined and satisfies* (P), *then A is closable and its closure $\tilde{A}$ satisfies* (P).

Proof. Since $\mathrm{D}(A)$ is dense in X and $\mathrm{int}(X_+)$ is a non-empty open set, there exists $u \in \mathrm{D}(A) \cap \mathrm{int}(X_+)$. There exists $\lambda > 0$ such that $Au \leqslant \lambda u$. Let $B = A - \lambda$. Then $Bu \leqslant 0$. B also satisfies (P) and is densely defined. So by 6.3 B is Φ_u-dissipative, hence closable by 2.4. Consequently, A is also closable. Moreover, the closure $\tilde{B}$ of B is Φ_u-dissipative by 2.3, hence strictly Φ_u-dissipative by 2.5. From 6.4 it follows that $\tilde{B}$ satisfies (P). Hence, $\tilde{A} = \tilde{B} + \lambda$ also satisfies (P).

PROPOSITION 6.6. *Suppose that A is densely defined. If there exist arbitrarily large real $\lambda \in \rho(A)$ such that* $\mathrm{R}(\lambda, A) \geqslant 0$, *then A satisfies* (P).

Proof. Let $u \in \mathrm{D}(A) \cap \mathrm{int}(X_+)$. Property (P) as well as the hypothesis of the proposition hold for A if and only if they hold for $(A - \lambda)$ $(\lambda \in \mathbf{R})$. Since $Au \leqslant \lambda u$ for some $\lambda \in \mathbf{R}$, we can assume that $Au \leqslant 0$. We show that A is Φ_u-dissipative. Then (P) follows from 2.5 and 6.4.

Let $x \in \mathrm{D}(A)$. We have to show that there exist arbitrarily small $t > 0$ such that $\Phi_u(x - tAx) \geqslant \Phi_u(x)$. Let $t > 0$ be such that $\lambda = t^{-1} \in \rho(A)$ and $\mathrm{R}(\lambda, A) \geqslant 0$. Let $u' = (\lambda - A)u$. Since $Au \leqslant 0$, we have $\lambda u \leqslant u'$. Hence

$$\lambda \mathrm{R}(\lambda, A)u \leqslant \mathrm{R}(\lambda, A)u'. \tag{6.4}$$

Now let $a > \Phi_u(x - tAx)$. Then $au \geqslant x - tAx$; consequently, $\lambda au \geqslant \lambda x - Ax$. Since $R(\lambda, A) \geqslant 0$, this implies

$$x \leqslant a\lambda R(\lambda, A)u \leqslant aR(\lambda, A)u' = au \qquad \text{(by (6.4))}.$$

Hence $\Phi_u(x) \leqslant a$. Since $a > \Phi_u(x - tAx)$ was arbitrary, we conclude that $\Phi_u(x) \leqslant \leqslant \Phi_u(x - tAx)$.

LEMMA 6.7. *Let A satisfy* (P). *If $\lambda \in \rho(A)$ is such that* $R(\lambda, A) \geqslant 0$, *then there exists $u \in D(A) \cap \operatorname{int}(X_+)$ with $Au \leqslant \lambda u$.*

Proof. Let $u' \in D(A) \cap \operatorname{int}(X_+)$ and let $u = R(\lambda, A)u'$.

We first show that $u \in \operatorname{int}(X_+)$. By [12, Corollary 1.4], this is equivalent to: $\langle u, f \rangle > 0$ for all $f \in X_+^* \setminus \{0\}$. So let $f \in X_+^*$, $f \neq 0$. Since $u' \in \operatorname{int}(X_+)$, it follows that $\langle u', f \rangle > 0$. If $\langle u, f \rangle = 0$, then $\langle Au, f \rangle \geqslant 0$ by (P). Since $\lambda u - Au = u'$, this implies that $0 = \lambda\langle u, f \rangle = \langle u', f \rangle + \langle Au, f \rangle > 0$, which is absurd. So $\langle u, f \rangle > > 0$, as claimed.

Since $(\lambda - A)u = u'$, $Au \leqslant \lambda u$.

Now we are able to prove the two theorems of Section 5.

Proof of Theorem 5.1. (1) implies (2). Let $u \in D(A) \cap \operatorname{int}(X_+)$. There exists $\lambda_0 \in \mathbf{R}$ such that $Au \leqslant \lambda_0 u$. Since A satisfies (1) (resp. (2)) if and only if $A - \lambda_0$ satisfies (1) (resp. (2)), we can assume that $\lambda_0 = 0$. So $Au \leqslant 0$ and A is Φ_u-dissipative by 6.3. Let $\lambda > 0$ be such that $(\lambda - A)D(A) = X$. We claim that $\lambda \in \rho(A)$ and $R(\lambda, A) \geqslant 0$. This will show that (2) follows from (1). The operator $(\lambda - A)$ is injective by 2.2. So $\lambda - A)$ is bijective. Thus, in order to show that $\lambda \in \rho(A)$, it is enough to show that A is closed. By Theorem 2.4, A is closable and its closure $\tilde{A}$ is Φ_u-dissipative. Moreover, $(\lambda - A)D(A) = X = (\lambda - \tilde{A})D(\tilde{A})$. Since $\lambda - \tilde{A}$ is injective (by 2.2), this implies that $D(\tilde{A}) = D(A)$. Hence A is closed. It remains to show that $R(\lambda, A) \geqslant 0$. Let $y \geqslant 0$. We have to show that $x := R(\lambda, A)y \geqslant 0$. Since A is Φ_u-dissipative, we have $\Phi_u(-x) \leqslant \Phi_u(-x + \lambda^{-1}Ax)$; hence $\lambda\Phi_u(-x) \leqslant \Phi_u(-\lambda x + Ax) = \Phi_u(-y) = = 0$. Thus $x \geqslant 0$.

It follows from 6.6 that (2) implies (1).

Proof of Theorem 5.3. If A is the generator of a positive C_0-semigroup, (2) is trivially satisfied (cf. [9, 11.7.2]). So, let us assume (2) (equivalently (1)). Let $u \in D(A) \cap \operatorname{int}(X_+)$, $Au \leqslant \lambda u$. Let $B = A - \lambda$. Then $Bu \leqslant 0$ and B satisfies (P) because (1) is valid. By 6.3, B is Φ_u-dissipative. Since Φ_u satisfies (C4) by 6.1, it follows from Remark 4.2 that B generates a positive, bounded C_0-semigroup $\{e^{tB}; t > 0\}$. Hence A generates the semigroup $\{e^{\lambda t}e^{tB}; t > 0\}$ which is positive and has type $\leqslant \lambda$. This shows that $\omega \leqslant \inf\{\lambda; Au \leqslant \lambda u$ for some $u \in D(A) \cap \operatorname{int}(X_+)\}$. Now, (5.2) follows from (5.1) and Lemma 6.7.

REMARK 6.8. Let $\mathfrak{A}$ be a C^*-algebra with unit u. Denote by X the set of all hermitian elements of $\mathfrak{A}$. X is an ordered Banach space and u an interior point of its positive cone X_+. Let Φ_u be the half-norm (6.1). Obviously, $\Phi_u(x) = \|x^+\|$ $(x \in X)$. Let us also consider the half-norm Φ given by $\Phi(x) = \|x\|$. Then $\Phi(x) = \Phi_u(x) \vee \Phi_u(-x)$. This shows in particular that X_+ is a normal cone (cf. Remark 4.2). Hence Theorem 5.3 is valid for densely defined operators on X.

We want to compare Φ_u and Φ and the corresponding notions of dissipativity. For the subdifferentials we have the following relation:

$$\mathrm{d}\Phi_u(x) = \mathrm{d}\Phi(x^+) \quad \text{for all } x \in X \text{ such that } x^+ \neq 0. \tag{6.5}$$

[In fact, let $f \in \mathrm{d}\Phi_u(x)$. Then $f \geqslant 0$ (by 3.4). Hence $\langle z, f\rangle \leqslant \langle |z|, f\rangle \leqslant \Phi(z)$ for all $z \in X$. Moreover,

$$\langle x^-, f\rangle = \langle x^+, f\rangle - \langle x, f\rangle = \langle x^+, f\rangle - \|x^+\| \leqslant \|x^+\| - \|x^+\| = 0.$$

Since $f \geqslant 0$, it follows that $\langle x^-, f\rangle = 0$. Hence $\langle x^+, f\rangle = \langle x, f\rangle = \|x^+\|$. We have proved that $f \in \mathrm{d}\Phi(x^+)$. Conversely, let $f \in \mathrm{d}\Phi(x^+)$. Let $f = f^+ - f^-$ be the Jordan decomposition of f; that is, $f^+, f^- \in X^*_+$, $\|f\| = \|f^+\| + \|f^-\|$. Then

$$\langle x^+, f^-\rangle = \langle x^+, f^+\rangle - \langle x^+, f\rangle = \langle x^+, f^+\rangle - \|x^+\| \leqslant 0,$$

so that $\langle x^+, f^-\rangle = 0$. Hence $\langle x^+, f^+\rangle = \|x^+\|$, and it follows that $\|f^+\| = 1$ if $x^+ \neq 0$. But then $\|f^-\| = 0$ and we conclude that $f \geqslant 0$. Since $\|f\| \leqslant 1$, it follows that $\langle z, f\rangle \leqslant \langle z^+, f\rangle \leqslant \Phi_u(z)$ for all $z \in X$. Hence $f \in \mathrm{d}\Phi_u(x)$.]

Let A be an operator on X with dense domain $\mathrm{D}(A)$. It follows from (6.5) that A is Φ_u-dissipative if and only if

$$\text{for every } x \in \mathrm{D}(A) \text{ there exists } f \in \mathrm{d}\Phi(x^+) \text{ such that } \langle Ax, f\rangle \leqslant 0. \tag{6.6}$$

(6.6) is Jørgensen's definition of dispersiveness [10]. Thus Theorem 2.5 implies the equivalence of (i) and (ii) in [10] (namely dispersiveness and strict dispersiveness). We now compare dissipativity (that is Φ-dissipativity) with Φ_u-dissipativity, assuming that $u \in \mathrm{D}(A)$.

The following assertions are equivalent:

(i) A is Φ_u-dissipative

(ii) A is dissipative and satisfies (P)

(iii) $Au \leqslant 0$ and A satisfies (P).

[In fact, if A is Φ_u-dissipative, then

$$\Phi(x) = \Phi_u(x) \vee \Phi_u(-x) \leqslant \Phi_u(x - tAx) \vee \Phi_u(-(x - tAx)) = \Phi(x - tAx)$$

for all $x \in \mathrm{D}(A)$ and $t \geqslant 0$. Hence A is dissipative. Moreover, A satisfies (P) by 6.4 and 2.5. We have proved that (i) implies (ii). If A is dissipative, then $Au \leqslant 0$. [In fact, let $f \geqslant 0$, $\|f\| = 1$. Then $\langle u, f \rangle = 1 = \Phi(u)$. Hence $f \in \mathrm{d}\Phi(u)$. Since A is strictly dissipative by 2.5, it follows that $\langle Au, f \rangle \leqslant 0$. Consequently, $Au \leqslant 0$. This proves that (iii) follows from (ii). Finally, Lemma 6.3 gives the remaining implication.]

To prove another relation, assume that A is dissipative. If $Au \geqslant 0$, then A satisfies (P).

[In fact, suppose $x \in X_+ \cap \mathrm{D}(A)$ and $f \in X_+^*$, $\|f\| = 1$, such that $\langle x, f \rangle = 0$. Let $y = \|x\|u - x$. Then $y \in \mathrm{D}(A)$, and $f \in \mathrm{d}\Phi(y)$ because

$$\Phi(y) \geqslant \langle y, f \rangle = \|x\|\langle u, f \rangle - \langle x, f \rangle = \|x\| - 0 \geqslant \|y\| = \Phi(y).$$

Hence

$$0 \geqslant \langle Ay, f \rangle = \|x\|\langle Au, f \rangle - \langle Ax, f \rangle$$

and so $\langle Ax, f \rangle \geqslant \|x\|\langle Au, f \rangle \geqslant 0$; that is, (P) holds.]

In particular, we get the following conclusion (using the fact that (iii) implies (i)):

If $Au = 0$, then A is dissipative if and only if A is Φ_u-dissipative (cf. [10, Corollary]). Of course, if A is the generator of a C_0-semigroup, this follows from the well known fact that a bounded *-preserving operator T on $\mathfrak{A}$ with $Tu = u$ is positive if and only if T is contractive.

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