Spectral Properties of Lamperti Operators

WOLFGANG ARENDT

1. Introduction. Weighted composition operators (that is, operators of the form $f \mapsto h \cdot f \circ \varphi$ defined on some Banach space E of complex valued functions on a set X, where φ is a mapping from X to X and h a complex valued function on X) have been studied for various domains. In particular, the spectra of these operators have been investigated by several authors. For example, the case where E is the disc algebra was considered by Kamowitz [4] and [5]; for $E = L^2(X)$, see Petersen [13] and the survey article of Nordgren [12] with the literature mentioned there; if $E = l^2(\mathbf{N})$ or $l^2(\mathbf{Z})$ weighted shift operators belong to this class of operators and Shields [16, Section 5] gives an account of their spectral properties.

We are interested in the case where E is a Banach lattice. Although the common Banach lattices (like $C(K)$ or $L^p(X)$) are function spaces, we prefer a more general approach. Weighted composition operators preserve orthogonality (in the lattice sense) and are order bounded. We call operators with these properties Lamperti operators. This definition makes sense in arbitrary Banach lattices and coincides with Charn-Huen Kan's terminology (on $L^p(X)$) [7]. In the literature, this kind of operators appeared first in connection with isometries on $L^p(X)$. Indeed, all isometries on $L^p(X)$ ($1 \leq p \leq \infty$, $p \neq 2$) are Lamperti operators. This was shown by Banach [2, page 175] for $X = [0,1]$ and by Lamperti [9] for the σ-finite case.

The spectrum of these operators, too, has been studied in the literature. For example, the monograph of Schaefer [14] contains an exposition for lattice homomorphisms (i.e. positive Lamperti operators), and Kitover [8] states several results for invertible orthogonality preserving operators.

In the present paper we discuss basically two properties of the spectra of Lamperti operators. In Section 3 we prove that an invertible Lamperti operator T on E, whose spectrum is contained in a sector of angle $2\pi/3$, is in the center of E (that is, T is a multiplication operator in the case $E = C(K)$ or $L^p(X)$, for example). This is a generalization of the main result of [15]. The proof we give here, however, uses a completely different idea and is considerably simpler.

Section 4 starts with a theorem (4.1) which asserts that certain spectral projections of a Lamperti operator have an ideal as image, and so one can decompose the operator under preservation of the Lamperti property. As a consequence, a relation between the spectrum of the operator and its modulus can be established (4.4). Another application is the spectral characterization of uniquely ergodic

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homeomorphisms which we give in Section 5.

We begin with a preliminary part (Section 2) where basic properties of Lamperti operators are put together.

As a general reference for terminology and theory of Banach lattices and positive operators we use the monograph of H. H. Schaefer [14]. Throughout this paper, the term "Banach lattice" stands for *complex Banach lattice* [14, II §11].

2. Lamperti operators. Let E, F be Banach lattices, $T: E \to F$ a linear mapping. T is called *order bounded* if for every $x \in E_+$ there exists $y \in F_+$ such that $|Tz| \leq y$ whenever $|z| \leq x$. If for every $x \in E_+$ the set $\{|Tz| : |z| \leq x\}$ has a supremum in F, then there exists a (unique) positive operator $|T|: E \to F$ such that

$$(2.1) \qquad |T|x = \sup\{|Tz| : |z| \leq x \quad \text{for all} \quad x \in E_+\}.$$

$|T|$ is called the *modulus* of T. If T has a modulus, then T is obviously order bounded. Conversely, if F is order complete and T is order bounded, then T possesses a modulus. Finally, note that every order bounded operator is continuous. See [14, IV §1 and V 7.3] for all this. T is a *lattice homomorphism* if $|Tz| = T|z|$ for all $z \in E$. A bijective lattice homomorphism is called a lattice isomorphism. If T is bijective, then T is a lattice isomorphism if and only if T and T^{-1} are positive. Therefore the adjoint of a lattice isomorphism is a lattice isomorphism.

Definition 2.1. T is a *Lamperti operator* if T is order bounded and $Tx \perp Ty$ whenever $x, y \in E$ such that $x \perp y$.

Note, if G is a Banach lattice, $x, y \in G$, we say that x and y are *orthogonal* if $|x| \wedge |y| = 0$. We express this symbolically by $x \perp y$.

Example 2.2. *1.* If X is a compact space we denote by $C(X)$ the Banach lattice of all continuous complex valued functions on X with the supremum norm. Let X, Y be compact spaces, $S: C(X) \to C(Y)$ a linear mapping. S is a Lamperti operator if and only if there exists a function $\varphi: Y \to X$ and $h \in C(Y)$ such that

$$(2.2) \qquad Sz(t) = h(t) \cdot z(\varphi(t)) \qquad (t \in Y) \quad \text{for all} \quad z \in C(X).$$

Obviously, $h = S1_X$ (where $1_X(s) = 1$ for all $s \in X$). Moreover, φ is uniquely determined and continuous on $Y_0 = \{t \in Y : h(t) \neq 0\}$. S has a modulus, given by

$$(2.3) \qquad |S|z(t) = |h(t)| \cdot z(\varphi(t)) \qquad (t \in Y) \quad \text{for all} \quad z \in C(X).$$

In particular,

$$(2.4) \qquad |S||z| = |Sz| = ||S|z| \qquad \text{for all} \quad z \in C(X).$$

Remark. If S is a lattice homomorphism this has been proved by Wolff [18]. The proof here is similar:

Proof. If S is given by (2.2) it is obvious that S is a Lamperti operator. To prove the converse suppose that S is a Lamperti operator. Let $t \in Y$ such that $\mu := S'\delta_t \neq 0$. We show that the support of $|\mu|$ is a singleton.

In fact, if this is not the case, then there exist u_1, $u_2 \in C(X)$ such that $u_1 \wedge u_2 = 0$ and $\langle u_1, |\mu| \rangle > 0$, $\langle u_2, |\mu| \rangle > 0$. Since $\langle u_1, |\mu| \rangle = \sup\{|\langle z, \mu \rangle| : |z| \leq u_1\}$, it follows that there exists $z_1 \in C(X)$ such that $|z_1| \leq u_1$ and $|\langle z_1, \mu \rangle| > 0$. Similarly, there exists z_2 such that $|z_2| \leq u_2$ and $|\langle z_2, \mu \rangle| > 0$. Hence, $|Sz_1(t)| = |\langle Sz_1, \delta_t \rangle| = |\langle z_1, \mu \rangle| > 0$ and $|Sz_2(t)| > 0$. This is a contradiction since $z_1 \perp z_2$.

Let $Y_0 = \{t \in Y : S'\delta_t \neq 0\}$. Then for every $t \in Y_0$ there exists exactly one $\varphi(t) \in X$ such that $\operatorname{supp}(S'\delta_t) = \{\varphi(t)\}$. Hence there exists $h(t) \in \mathbf{C}\setminus\{0\}$ such that $S'\delta_t = h(t)\delta_{\varphi(t)}$ $(t \in Y_0)$. Let $\varphi(t) \in X$ be arbitrary and $h(t) = 0$ if $t \in Y \setminus Y_0$. Then for every $z \in C(X)$, $Sz(t) = h(t) \cdot z(\varphi(t))$ for all $t \in Y$. In particular, $S1_X = h$. Therefore, $h \in C(Y)$. The fact that φ is continuous on Y_0 follows from an Urysohn argument.

Example 2.2. 2. Let (X, Σ, μ) be a measure space, $\varphi : X \to X$ a measure preserving transformation and $h \in L^\infty(X)$. Let S be defined on $E = L^p(X)$ $(1 \leq p \leq \infty)$ by $Sz(t) = h(t) \cdot z(\varphi(t))$ $(t \in X)$ for all $z \in E$. S is a Lamperti operator.

Lemma 2.3. *Let G be a Banach lattice and $x, y \in G$. Then $x \perp y$ if and only if $|x + cy| = |x - cy|$ for $c = 1, i$.*

Proof. The lemma is true for $G = \mathbf{C}$ and follows from this for $G = C(X)$ (X compact). The general case can be reduced to the latter because, if $x, y \in G$, then $x, y \in G_u$ where $u = x + y$, and G_u (the principal ideal defined by u) is isomorphic to some space $C(X)$ [14, II 7.4 and §11].

Theorem 2.4. *The following assertions are equivalent.*

(i) T is a Lamperti operator.

(ii) T is order bounded and $|Tx| = |Ty|$ whenever $x, y \in E$ such that $|x| = |y|$.

(iii) $|x| \leq |y|$ implies $|Tx| \leq |Ty|$ for all $x, y \in E$.

(iv) There exists a lattice homomorphism $S : E \to F$ such that $|Tz| \leq S|z|$ for all $z \in E$.

(v) $|T|$ exists and satisfies $|Tz| = \big| |T| z \big| = |T|\,|z|$ for all $z \in E$. In particular, $|T|$ is a lattice homomorphism.

Proof. (i) implies (v). Let $x \in E_+$. Since T is order bounded, there exists $y \in F_+$ such that $TE_x \subset F_y$. There exist compact spaces X and Y such that E_x is isomorphic to $C(X)$ and F_y to $C(Y)$ [14, II 7.4 and §11]. Under this identification, the restriction of T to $C(X)$ defines an operator $T_0 : C(X) \to C(Y)$ which is again a Lamperti operator. Therefore T_0 has the form (2.2). It follows that $|T_0|x = \sup\{|Tz| : |z| \leq x\}$ exists in F_y. Since F_y is an ideal in F, $|T_0|x$ is also the supremum of $\{|Tz| : |z| \leq x\}$ in F. This proves that $|T|$ exists. Moreover, the restriction of $|T|$ to E_x coincides with $|T_0|$. Hence (v) follows from (2.4).

(v) implies trivially (iv) (take $S = |T|$).

(iv) implies (i). If (iv) is true, then T is order bounded. Let $x, y \in E$ such that $x \perp y$. Then, $|Tx| \wedge |Ty| \leq S|x| \wedge S|y| = S(|x| \wedge |y|) = 0$. Therefore, $Tx \perp Ty$.

(v) implies (iii). Let $|x| \leq |y|$. Then $|Tx| = |T|\,|x| \leq |T|\,|y| = |Ty|$.

(iii) implies trivially (ii).

(ii) implies (i). Let $x, y \in E$ such that $x \perp y$. Then for $c = 1, i$, $|x + cy| = |x - cy|$ (by 2.3). Hence (ii) implies that $|Tx + cTy| = |Tx - cTy|$ for $c = 1, i$, so that $Tx \perp Ty$ (by 2.3).

It is an open problem whether in Definition 2.1 the condition of T being order bounded is redundant. This is at least the case if T is bounded and E is σ-order complete (as for example $E = L^p(X)$, $1 \leq p \leq \infty$; cf. [7]).

Theorem 2.5. *Suppose that E is σ-order complete and T is bounded. If $x \perp y$ implies $Tx \perp Ty$ for all $x, y \in E$, then T is a Lamperti operator.*

Proof. We show that $|T|$ exists.

1. Let $x, y \in E_+$. Then $|Tx + Ty| = |Tx| + |Ty|$. In fact, let $u = x + y$. There exists a compact quasi-Stonian space X such that E_u is isomorphic to $C(X)$ [14, II 7.4 and §11]. Therefore x (respectively y) can be (simultaneously) approximated in $C(X)$ (and so in E) by a function f (respectively g) of the form

$$f = \sum_{i=1}^{n} \alpha_i 1_{A_i} \qquad g = \sum_{i=1}^{n} \beta_i 1_{A_i}$$

where $\{A_i : i = 1,\ldots,n\}$ is a partition of X in open and closed subsets and $\alpha_i, \beta_i \in \mathbf{R}_+$ $(i = 1,\ldots,n)$. Note: for $A \subset X$, 1_A denotes the characteristic function of A. Since $1_{A_i} \perp 1_{A_j}$ for $i \neq j$, we have by hypothesis $T1_{A_i} \perp T1_{A_j}$. Consequently,

$$\begin{aligned} |T(f+g)| &= \left| \sum_{i=1}^{n} (\alpha_i + \beta_i) T1_{A_i} \right| = \sum_{i=1}^{n} (\alpha_i + \beta_i)|T1_{A_i}| \\ &= \sum_{i=1}^{n} \alpha_i |T1_{A_i}| + \sum_{i=1}^{n} \beta_i |T1_{A_i}| \\ &= \left| \sum_{i=1}^{n} \alpha_i T1_{A_i} \right| + \left| \sum_{i=1}^{n} \beta_i T1_{A_i} \right| \\ &= |Tf| + |Tg|. \end{aligned}$$

Since T is continuous, this implies $|T(x + y)| = |Tx| + |Ty|$.

2. The map $x \mapsto |Tx|$ from E_+ to F_+ is additive by 1 and obviously positive homogeneous. Let $S : E \to F$ be the linear extension of this mapping. Then

$$\sup_{\theta \in [0,2\pi)} \operatorname{Re}(e^{i\theta} T) = S$$

where the supremum is taken in the ordered Banach space $\mathscr{L}(E,F)$ of all bounded operators from E to F. This implies that $|T|$ exists, and actually coincides with S [14, V 7.3].

Lemma 2.6. *If T is a Lamperti operator, T' has a modulus and $|T'| = |T|'$.*

Proof. Let $y' \in F'_+$. One always has

$$\sup\{|T'z'| : |z'| \leq y'\} \leq |T|'y'. \tag{2.5}$$

Let $x \in E_+$. Since T is a Lamperti operator, by 2.4 (v), we get

$$\langle x, \sup\{|T'z'| : |z'| \le y'\}\rangle \ge \sup\{|\langle x, T'z'\rangle| : |z'| \le y'\}$$
$$= \sup\{|\langle Tx, z'\rangle| : |z'| \le y'\} = \langle |Tx|, y'\rangle = \langle |T|x, y'\rangle = \langle x, |T|'y'\rangle.$$

Since $x \in E_+$ is arbitrary, this shows that actually the equality is valid in (2.5). This proves the lemma.

Proposition 2.7. *Suppose that T is a Lamperti operator. T is invertible if and only if $|T|$ is invertible. Moreover, if T is invertible, then T^{-1} and T' are Lamperti operators and $|T^{-1}| = |T|^{-1}$.*

Proof. 1. Suppose that T is bijective. Then $|T|$ is injective. In fact, $|T|z = 0$ implies $0 = ||T|z| = |Tz|$ (by 2.4 (v)). Therefore, $z = 0$ since T is injective. $|T|$ is surjective. In fact, if $y \in F_+$ there exists $x \in E$ such that $Tx = y$. Hence, $y = |Tx| = |T||x|$. We conclude, $|T|E_+ = F_+$ which implies $|T|E = F$.

2. Suppose that $|T|$ is invertible. We show that T is invertible.

$$\|Tz\| = \||Tz|\| = \||T|\,|z|\| \ge \||T|^{-1}\|^{-1}\|z\| \tag{2.6}$$

holds for all z (by 2.4 (v)). Hence T is injective. Moreover, $|T|$ is a lattice isomorphism; hence $|T|'$ is a lattice isomorphism. By Lemma 2.6, $|T'| = |T|'$, so it follows from 2.4 (iv) that T' is a Lamperti operator. Moreover, $|T'|$ is invertible so that T' is injective (by (2.6) applied to T'). Therefore, $(TE)^0 = \operatorname{Ker} T' = 0$. This implies that $(TE)^- = (TE)^{00} = F$, that is, T has dense image. From (2.6) it follows that T is invertible.

Let $x \in F_+$. Then,

$$|T|^{-1}x = \sup\{|T|^{-1}y : 0 \le y \le x\} = \sup\{|z| : z \in E, |Tz| = |T|\,|z| \le x\}$$
$$= \sup\{|T^{-1}w| : |w| \le x\}.$$

Hence, $|T^{-1}|$ exists and $|T^{-1}| = |T|^{-1}$. It follows that $|T^{-1}|$ is a Lamperti operator since $|T^{-1}| = |T|^{-1}$ is a lattice homomorphism.

3. Lamperti operators with spectrum in a sector. Throughout this section E denotes a Banach lattice. $\mathscr{L}(E)$ is the Banach algebra of all bounded operators on E. The *center* of E is the closed subalgebra of $\mathscr{L}(E)$ consisting of all operators T on E such that $|Tz| \le c|z|$ for all $z \in E$ and some $c \ge 0$. If $T \in Z(E)$, then T is a Lamperti operator and $|T| \in Z(E)$. (In fact, if $T \in Z(E)$, then $T'' \in Z(E'')$. Hence T'' preserves orthogonality. It follows from 2.5 that T'' is a Lamperti operator, and so T'' satisfies 2.4 (iii). But then T also satisfies 2.4 (iii); that is, T is a Lamperti operator. It can be seen from 2.4 (v) that $|T| \in Z(E)$.)

We quote two theorems which describe the principal properties of the center.

Theorem 3.1. *There exists a compact space X and an isometric algebra isomorphism Φ from $C(X)$ onto $Z(E)$ such that $|\Phi(f)| = \Phi(|f|)$ for all $f \in C(X)$. In particular, $Z(E)$ is a commutative Banach algebra and a Banach lattice (with norm, multiplication and order induced by $\mathscr{L}(E)$) and Φ is a lattice isomorphism.*

The proof of 3.1 can be found in [11] for real Banach lattices. The complex case is an easy deduction from the real one.

Recall, a projection P on E is called a *band projection* if PE and $\mathrm{Ker}(P)$ are orthogonal bands (i.e. $Px \perp (y - Py)$ for all $x,y \in E$). P is a band projection if and only if $0 \leq P \leq I$ [14, II 2.9]. Hence every band projection is in the center. In particular, if $T \in Z(E)$, then T commutes with every band projection. The converse is also true if E is σ-order complete [11, 7.12 and 4.2].

Theorem 3.2. *Suppose that E is σ-order complete and let T be a bounded operator on E. Then $T \in Z(E)$ if and only if $TP = PT$ for every band projection P on E.*

If A is a Banach algebra and $a \in A$ we denote by $\sigma_A(a)$ the spectrum of a in A. We write simply $\sigma(a)$ if no confusion can arise. For a bounded operator S on a Banach space G, $\sigma(S)$ always denotes the spectrum of the operator S, that is, the spectrum of S in the Banach algebra $\mathcal{L}(G)$.

Corollary 3.3. *$Z(E)$ is a full subalgebra of $\mathcal{L}(E)$ (that is, $\sigma_{Z(E)}(T) = \sigma(T)$ for all $T \in Z(E)$).*

Proof. Let $T \in Z(E)$ be invertible in $\mathcal{L}(E)$. It has to be shown that $T_{-1} \in Z(E)$. Considering T' if necessary we can assume that E is order complete. Let P be a band projection. Since $PT = TP$, we have $T^{-1}P = T^{-1}PTT^{-1} = T^{-1}TPT^{-1} = PT^{-1}$. It follows from 3.2 that $T^{-1} \in Z(E)$.

Example 3.4. 1. Let $E = C(X)$ (X compact). Then $\Phi: C(X) \to Z(E)$ given by $\Phi(f)g = f \cdot g$ for all $g \in E, f \in C(X)$ is an isometric, algebraic and lattice isomorphism from $C(X)$ onto $Z(E)$.

Example 3.4. 2. Let $E = L^p(X)$, $1 \leq p \leq \infty$, (X,Σ,μ) a σ-finite measure space. Define $\Phi: L^\infty(X) \to Z(E)$ by $\Phi(f)g = fg$ ($f \in L^\infty(X), g \in E$). Φ is an isometric, algebraic and lattice isomorphism from $L^\infty(X)$ onto $Z(E)$ (see [11] or [19]).

Let us fix the following notation. For $r_0 > 0$, θ_0, $\theta_1 \in [0,2\pi)$, we call the subset

$$\Delta(r_0,\theta_0,\theta_1) = \{re^{i\theta} : 0 < r \leq r_0, \theta_0 \leq \theta < \theta_0 + \theta_1\}$$

of $\mathbf{C}\backslash\{0\}$ a *sector*. θ_1 is the *angle of the sector*.

The following is the main theorem of this section.

Theorem 3.5. *Let T be a Lamperti operator on E. If $\sigma(T)$ is included in a sector of angle $2\pi/3$, then $T \in Z(E)$.*

At first, let us note some consequences of Theorem 3.5.

Corollary 3.6. *Let T be a Lamperti operator on E. Then*
1. $\sigma(T) \subset (0,\infty)$ if and only if $T \in Z(E)$ and $T \geq \varepsilon I$ for some $\varepsilon > 0$;
2. $\sigma(T) = \{1\}$ if and only if $T = I$.

Proof. By 3.5, $\sigma(T) \subset (0,\infty)$ implies that $T \in Z(E)$. Denote by $\Phi: C(X) \to Z(E)$ the isomorphism of 3.1. Since Φ is an algebraic isomorphism and because of 3.3, $\sigma(\Phi^{-1}(M)) = \sigma_{Z(E)}(M) = \sigma(M)$ for all $M \in Z(E)$. But in the Banach algebra $C(X)$ the spectrum of $f \in C(X)$ coincides with $f(X)$. Thus, if $M \in Z(E)$, then $\sigma(M) \subset (0,\infty)$ if and only if $\Phi^{-1}(M) \geq \varepsilon 1_X$ for some $\varepsilon > 0$, and the latter is equivalent to $M \geq \varepsilon I$ for some $\varepsilon > 0$ because Φ is a lattice isomorphism satisfying $\Phi(1_X) = I$. Similarly, if $\sigma(M) = \{1\}$, then $\Phi^{-1}(M) = 1_X$. Consequently, $M = \Phi(1_X) = I$.

Corollary 3.6.2 might be considered as an analog of a result of Kamowitz and Scheinberg [6], which says that an automorphism U on a semisimple commutative Banach algebra A satisfying $\sigma(U) = \{1\}$ is the identity operator.

In order to prove Theorem 3.5 we note the following easy special case of 3.5 *and* the just-mentioned result of Kamowitz and Scheinberg.

Lemma 3.7. *Let X be compact and U an automorphism (equivalently, a lattice isomorphism satisfying $U1_X = 1_X$) on $C(X)$. Then $\sigma(U) = \{1\}$ implies that $U = I$.*

As we said before, the lemma follows from [6]. But of course, it is also possible to give a short direct proof of this easy case.

We will prove Theorem 3.5 by a reduction to 3.7. Let us describe this procedure which seems to be of independent interest.

Recall, $Z(E)$ is a commutative Banach algebra with unit. We denote by Aut($Z(E)$) its automorphism group. By 3.1, $Z(E)$ is also a Banach lattice and it actually follows from 3.1 that every $U \in$ Aut($Z(E)$) is isometric and a lattice isomorphism (in fact, keeping the notation of 3.1, $\Phi^{-1}U\Phi$ is an automorphism on $C(X)$; so there exists a homeomorphism φ on X such that $\Phi^{-1}U\Phi f = f \circ \varphi$ for all $f \in C(X)$; hence $\Phi^{-1}U\Phi$ is an isometric lattice isomorphism and so the same is true for U since Φ is an isometric lattice isomorphism itself).

By 2.7, the inverse of a Lamperti operator is a Lamperti operator, and it follows immediately from the definition that the composition of two Lamperti operators is a Lamperti operator. Thus, the invertible Lamperti operators form a group which we denote by $G(E)$.

Proposition 3.8. *The mapping $\sim : T \mapsto \tilde{T}$, where*

$$\tilde{T}(M) = TMT^{-1} \qquad (M \in Z(E))$$

defines a group homomorphism from $G(E)$ into Aut($Z(E)$). *If E is σ-order complete, then* Ker $\sim = G(E) \cap Z(E)$.

Proof. 1. Let $T \in G(E)$. If $M \in Z(E)$, TMT^{-1} is a Lamperti operator. By definition of $Z(E)$, there exists $c > 0$ such that $|M| \leq cI$. Hence, $|TMT^{-1}| \leq |T|\,|M|\,|T|^{-1} \leq cI$. Consequently, $TMT^{-1} \in Z(E)$. Thus, $\tilde{T}$ maps $Z(E)$ into $Z(E)$. $\tilde{T}$ is an automorphism since $\tilde{T}(MN) = TMNT^{-1} = TMT^{-1}TNT^{-1} = \tilde{T}(M)\tilde{T}(N)$ for all $M, N \in Z(E)$ and $\tilde{T}$ has the inverse $(T^{-1})^{\sim}$.

2. $\sim$ is a group homomorphism. In fact, let $S, T \in G(E)$. Then $(ST)^{\sim}(M) = STM(ST)^{-1} = STMT^{-1}S^{-1} = \tilde{S}(\tilde{T}(M))$ for all $M \in Z(E)$. Thus, $(ST)^{\sim} = S^{\sim}T^{\sim}$. Moreover, $(\tilde{T})^{-1} = (T^{-1})^{\sim}$.

3. Since $Z(E)$ is commutative, $Z(E) \cap G(E) \subset \operatorname{Ker} \sim$. Conversely, let $T \in \operatorname{Ker} \sim$. Then $\tilde{T} = I$. Hence, $TP = TPT^{-1}T = \tilde{T}(P)T = PT$ for all $P \in Z(E)$. If E is σ-complete this implies $T \in Z(E)$ by Theorem 3.2.

Example 3.9. Let $E = C(X)$, X compact (respectively $E = L^p(X), (X,\Sigma,\mu)$ a σ-finite measure space, $1 \leq p \leq \infty$). Let $T: E \to E$ be given by $Tz = h \cdot z \circ \varphi$ for all $z \in E$ where $\varphi: X \to X$ is a homeomorphism (respectively an invertible measure preserving transformation) and $h: X \to \mathbf{C}$ is continuous (respectively measurable) such that $|h(t)| \geq \varepsilon > 0$ for all $t \in X$ (respectively almost all $t \in X$) for some $\varepsilon > 0$.

T is an invertible Lamperti operator on E. If $Z(E)$ is identified with $C(X)$ (respectively $L^\infty(X)$) as in 3.4, then $\tilde{T}f = f \circ \varphi$ for all $f \in C(X)$ (respectively $f \in L^\infty(X)$).

For the proof of 3.5 we denote the approximate point spectrum of a bounded operator S by $A\sigma(S)$. Note that the topological boundary of $\sigma(S)$ is included in $A\sigma(S)$.

Proof of Theorem 3.5. Considering T' instead of T if necessary, we can assume that E is order complete.

By hypothesis, there exists $\theta_0 \in [0,2\pi)$ such that $\sigma(T) \subset \Delta(r(T),\theta_0,2\pi/3)$. Replacing T with a suitable scalar multiple of T, we can assume that $\theta_0 = 0$.

Let R, $L: \mathscr{L}(E) \to \mathscr{L}(E)$ be defined by $R(S) = ST^{-1}$ and $L(S) = TS$ ($S \in \mathscr{L}(E)$). By [3, §2 Proposition 19], $\sigma(R) = \sigma(T^{-1})$ and $\sigma(L) = \sigma(T)$. Since $R \circ L = L \circ R$, we have

$$\begin{aligned}\sigma(R \circ L) &\subset \sigma(L) \cdot \sigma(R) = \sigma(T) \cdot \sigma(T)^{-1} \\ &\subset \Delta\left(r(T),0,\frac{2\pi}{3}\right) \Delta\left(r(T),0,\frac{2\pi}{3}\right)^{-1} \\ &\subset \left\{re^{i\theta} : 0 < r, -\frac{2\pi}{3} < \theta < \frac{2\pi}{3}\right\}.\end{aligned}$$

Moreover, $(R \circ L)Z(E) \subset Z(E)$ (by 3.8) and $\tilde{T}$ is the restriction of $R \circ L$ to $Z(E)$. Hence, $A\sigma(\tilde{T}) \subset \sigma(R \circ L)$. But $\sigma(\tilde{T}) \subset \{z \in \mathbf{C} : |z| = 1\}$ because $\tilde{T}$ is an isometry. Hence,

$$\sigma(\tilde{T}) = A\sigma(\tilde{T}) \subset \left\{e^{i\theta}; \frac{-2\pi}{3} < \theta < \frac{2\pi}{3}\right\}.$$

Since $z \in \sigma(\tilde{T})$ implies that $z^n \in \sigma(\tilde{T})$ for all $n \in \mathbf{Z}$ (by [14, V 4.4] because $\tilde{T}$ is a lattice homomorphism or by [6, Theorem 3] because $\tilde{T}$ is an automorphism), it follows that $\sigma(\tilde{T}) = \{1\}$. From 3.7 we conclude that $\tilde{T} = I$. Hence, $T \in Z(E)$ (by 3.8).

4. A spectral decomposition theorem. Let E be a Banach space and T a bounded operator on E. We denote by $r(T)$ the spectral radius of T and by $r_m(T)$ the real number $r_m(T) = \inf\{|\lambda| : \lambda \in \sigma(T)\}$. Note that $r_m(T) = r(T^{-1})^{-1}$ if T is invertible. For $\lambda \in \rho(T)$ $(= \mathbf{C}\setminus\sigma(T))$, we denote by $R(\lambda,T) = (\lambda - T)^{-1}$ the resolvent of T in λ. If $|\lambda| > r(T)$, then $\lambda \in \rho(T)$ and $R(\lambda,T)$ is given by the Neumann's series $R(\lambda,T) = \Sigma_{n=0}^{\infty} T^n/\lambda^{n+1}$.

A spectral subset σ_1 of $\sigma(T)$ is by definition an open and closed subset of $\sigma(T)$. To such a set the spectral projection P given by

$$P = \frac{1}{2\pi i}\int_c R(\lambda,T)\,d\lambda$$

is canonically associated, where c is the positively oriented boundary of a Cauchy domain having σ_1 in its interior and $\sigma_2 := \sigma(T)\setminus\sigma_1$ in its exterior. *P reduces T*, that is, $PT = TP$, or equivalently, PE and Ker P are invariant under T. If T_1 (respectively, T_2) denotes the restriction of T to PE (respectively, Ker P), then $\sigma(T_i) = \sigma_i$ $(i = 1,2)$.

From now on we assume that E is a Banach lattice. Then in general, order properties of T are lost in the spectral decomposition. PE and Ker P do not need to be sublattices. The next theorem, however, gives a positive result.

For $s \in [0,\infty)$, we let $\Gamma_s = \{z \in \mathbf{C} : |z| = s\}$, $\Gamma = \Gamma_1$ Let $T \in \mathscr{L}(E)$. If $\Gamma_s \cap \sigma(T) = \varnothing$ and $r_m(T) < s < r(T)$, let

$$\sigma_s(T) = \{z \in \sigma(T) : |z| \leq s\}.$$

$\sigma_s(T)$ is a spectral subset of $\sigma(T)$.

Theorem 4.1. *Let T be a Lamperti operator on E. Suppose that there exists* $s \in (r_m(T),r(T))$ *such that* $\Gamma_s \cap \sigma(T) = \varnothing$. *Then the spectral projection belonging to* $\sigma_s(T)$ *has an ideal as image.*

Proof. Let $P = (1/2\pi i)\int_{\Gamma_s} R(\lambda,T)\,d\lambda$, $E_1 = PE$ and denote by T_1 the restriction of T to E_1. Since $r(T_1) < s$, we have for $x \in E_1$

$$R(s,T)x = R(s,T_1)x = \sum_{n=0}^{\infty}\left(\frac{T_1^n}{s^{n+1}}\right)x = \sum_{n=0}^{\infty}\left(\frac{T^n}{s^{n+1}}\right)x$$

and

$$\sum_{n=0}^{\infty}\left\|\frac{T^n x}{s^{n+1}}\right\| = \sum_{n=0}^{\infty}\left\|\frac{T_1^n x}{s^{n+1}}\right\| < \infty.$$

Since T^n is a Lamperti operator, it follows from 2.4 (iii) that $\|T^n y\| \leq \|T^n x\|$ for $|y| \leq |x|$ $(n \in \mathbf{N})$. Hence, for all $\lambda \in \Gamma_s$ and all $y \in E$ such that $|y| \leq |x|$,

$$\sum_{n=0}^{\infty}\left\|\frac{T^n y}{\lambda^{n+1}}\right\| \leq \sum_{n=0}^{\infty}\left\|\frac{T^n x}{s^{n+1}}\right\| < \infty.$$

Moreover,

$$(\lambda - T)\sum_{n=0}^{\infty}\frac{T^n y}{\lambda^{n+1}} = \lim_{m\to\infty}(\lambda - T)\sum_{n=0}^{m}\frac{T^n y}{\lambda^{n+1}}$$

$$= \lim_{m\to\infty}\left(y - \frac{T^{m+1}y}{\lambda^{n+1}}\right) = y;$$

that is,

$$R(\lambda,T)y = \sum_{n=0}^{\infty} \frac{T^n y}{\lambda^{n+1}} \qquad (\lambda \in \Gamma_s, |y| \le |x|).$$

Consequently, for $y \in E$ satisfying $|y| \le |x|$,

$$Py = \frac{1}{2\pi i}\int_{\Gamma_s} \sum_{n=0}^{\infty} \frac{T^n y}{\lambda^{n+1}}\, d\lambda = \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{\Gamma_s} \frac{1}{\lambda^{n+1}}\, d\lambda\, T^n y = y$$

(since the series is uniformly convergent for $\lambda \in \Gamma_s$). We have proved that $y \in E_1$ if $|y| \le |x|$ for some $x \in E_1$; that is, E_1 is an ideal.

The spectral projection in the theorem need not be positive, even if $T \ge 0$. (For example, let $E = \mathbf{C}^2$ and let $T \in \mathscr{L}(E)$ be represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $\sigma(T) = \{0,1\}$. Let $s = 1/2$. Then $\sigma_s(T) = \{0\}$. For $\lambda \in \rho(T)$,

$$R(\lambda,T) = \begin{pmatrix} 1/(\lambda - 1) & 0 \\ 1/(\lambda - 1) - 1/\lambda & 1/\lambda \end{pmatrix}.$$

Hence,

$$P = \frac{1}{2\pi i}\int_{\Gamma_s} R(\lambda,T)\, d\lambda = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}$$

is not positive.) The situation is different if T' is also a Lamperti operator, as the following corollary shows (cf. [8, Theorem 2]).

Corollary 4.2. *Let $T \in \mathscr{L}(E)$. If T and T' are Lamperti operators and if $s \in (r_m(T), r(T))$ such that $\Gamma_s \cap \sigma(T) = \varnothing$, then the spectral projection associated with $\sigma_s(T)$ is a band projection.*

Proof. Let $P = (1/2\pi i)\int_{\Gamma_s} R(\lambda,T)\,d\lambda$. By 4.1, PE is an ideal. Since $P' = (1/2\pi i)\int_{\Gamma_s} R(\lambda,T')\,d\lambda$, P' is the spectral projection associated with $\sigma_s(T')$. Consequently, $J := P'E'$ is an ideal of E' (by 4.1 again). Hence $\operatorname{Ker} P = J^0$ is an ideal (by [14, II 4.8 Corollary]). This implies that P is a band projection [14, II 2.7 and 2.8].

If J is a closed ideal which is invariant under a bounded operator S on E we denote by $S_{|J}$ the restriction of S to J and by S_J the operator on E/J induced by S (that is, $S_J(x + J) = Sx + J$ for all $x + J \in E/J$).

Lemma 4.3. *Let T be a Lamperti operator on E. Let J be a closed ideal of E. Then $TJ \subset J$ if and only if $|T|J \subset J$. If $TJ \subset J$, then $T_{|J}$ and T_J are Lamperti operators. Moreover, $|T_{|J}| = |T|_{|J}$ and $|T_J| = |T|_J$.*

Proof. The first assertion follows immediately from 2.4 (v). Let J be a closed ideal in E which is invariant under T (hence under $|T|$). We show that T_J is a Lamperti operator and that $|T_J| = |T|_J$. Let $z \in E$. Then

$$|T_J(z + J)| = |Tz + J| = |Tz| + J = |T||z| + J$$
$$= |T|_J(|z| + J) = |T|_J|z + J|.$$

Since $|T|_J$ is a lattice homomorphism, it follows from 2.4 (iv) that T_J is a Lamperti operator. Since for every $x \in E_+$, $|T_J|(x + J) = |T_J(x + J)| = |T|_J(x + J)$, we conclude that $|T_J| = |T|_J$.

To show the use of Lemma 4.3 in our context let us assume that we are in the situation of Theorem 4.1. Let $J = PE$ where P is the spectral projection associated with $\sigma_s(T)$. Then $T_{|J}$ and T_J are Lamperti operators and $\sigma(T_{|J}) = \sigma_s(T)$ and $\sigma(T_J) = \sigma(T)\backslash\sigma_s(T)$. In particular, $r(T_{|J}) < s < r_m(T_J)$. We have found a spectral decomposition of T into the two Lamperti operators $T_{|J}$ and T_J.

We will now use this to prove a relation between $\sigma(T)$ and $\sigma(|T|)$. To simplify the notation we let $|\sigma| = \{|z| : z \in \sigma\} \subset [0,\infty)$ if σ is a subset of $\mathbf{C}$.

Theorem 4.4. *Let T be a Lamperti operator on E. Then*

$$|\sigma(T)| = \sigma(|T|) \cap [0,\infty).$$

Proof. *a)* We show that $r(T) = r(|T|)$. For every $n \in \mathbf{N}$, T^n is a Lamperti operator and $|T^n| = |T|^n$. Moreover, if S is a Lamperti operator, then $\|S\| = \||S|\|$ (this follows immediately from 2.4 (v)). Hence

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = \lim_{n\to\infty} \||T|^n\|^{1/n} = r(|T|).$$

b) We show that $r_m(T) = r_m(|T|)$. By 2.7, $0 = r_m(T)$ if and only if $0 = r_m(|T|)$. Suppose now that $r_m(T) > 0$. Then T^{-1} is a Lamperti operator and $|T^{-1}| = |T|^{-1}$ (by 2.7). Therefore it follows from a) that $r_m(T) = r(T^{-1})^{-1} = r(|T|^{-1})^{-1} = r_m(|T|)$.

c) We show that $\sigma(|T|) \cap (r_m(T),r(T)) = |\sigma(T)| \cap (r_m(T),r(T))$. Let $s \in (r_m(T),r(T))$ and suppose that $s \notin |\sigma(T)|$. Then $\Gamma_s \cap \sigma(T) = \emptyset$. By 4.1, there exists a closed ideal J of E which is invariant under T such that $r(T_{|J}) < s < r_m(T_J)$. It also follows from 4.3 and a) and b) that $|T|$ leaves J invariant and $r(|T|_{|J}) < s < r_m(|T|_J)$. Since $\sigma(|T|) \subset \sigma(|T|_{|J}) \cup \sigma(|T|_J)$, we conclude that $s \notin \sigma(|T|)$. Conversely, let $s \in (r_m(T),r(T))$ such that $s \notin \sigma(|T|)$. Since $|T|$ is a lattice homomorphism, $z \in \sigma(|T|)$ implies $|z| \in \sigma(|T|)$ [14, V 4.4]. Therefore $\Gamma_s \cap \sigma(|T|) = \emptyset$. By 4.1, there exists a closed ideal J which is invariant under $|T|$ such that $r(|T|_{|J}) < s < r_m(|T|_J)$. It follows from 4.3 and a) and b) that $r(T_{|J}) < s < r_m(T_J)$. Since $\sigma(T) \subset \sigma(T_{|J}) \cup \sigma(T_J)$, we conclude that $s \notin |\sigma(T)|$.

d) We prove the equality in the theorem. Let $s \in \sigma(|T|) \cap [0,\infty)$. If $s \in (r_m(T),r(T))$, then $s \in |\sigma(T)|$, by c). If $s = r(T)$ (respectively $s = r_m(T)$), then $s \in |\sigma(T)|$ because $\Gamma_{r(T)} \cap \sigma(T) \neq \emptyset$ (respectively, $\Gamma_{r_m(T)} \cap \sigma(T) \neq \emptyset$).

Conversely, let $s \in |\sigma(T)|$. Again, if $s \in (r_m(T), r(T))$, then $s \in \sigma(|T|)$ by c). If $s = r(T) = r(|T|)$, then $s \in \sigma(|T|)$ because $r(|T|) \in \sigma(|T|)$. Finally, let $s = r_m(T)$. If $r_m(T) = 0$, then $r_m(|T|) = 0$ by a). Therefore $s = 0 \in \sigma(|T|)$. If $r_m(T) > 0$, then $r_m(|T|) = r(|T|^{-1})^{-1}$. Since $r(|T|^{-1}) \in \sigma(|T|^{-1})$, it follows that $s = r(|T|^{-1})^{-1} \in \sigma(|T|^{-1})^{-1} = \sigma(|T|)$.

Definition 4.5. A bounded operator T on E is called *irreducible* if there exists no closed ideal $J \neq 0, E$ such that $TJ \subset J$. T is called *band irreducible* if there exists no band projection $P \neq 0, I$ such that $TP = PT$.

Corollary 4.6. *Let T be a Lamperti operator. If a) T is irreducible or b) T is band irreducible and T' is a Lamperti operator, then*

$$|\sigma(T)| = [r_m(T), r(T)].$$

Remark 4.7. The condition of T' being a Lamperti operator can be formulated in terms of $|T|$: If T is a Lamperti operator, then T' is a Lamperti operator if and only if $(|T|E)^-$ is an ideal. We omit the proof, even though it is not quite obvious. Note however, if T is an invertible Lamperti operator, then T' is a Lamperti operator by 2.7. Therefore the hypothesis 4.6 b) is satisfied in this case.

Example 4.8. 1. Let T be a Lamperti operator on $C(X)$ (X compact). By 2.2.1, there exists a function $\varphi: X \to X$ and $h \in C(X)$ such that $Tf = h \cdot f \circ \varphi$ for all $f \in C(X)$.

The closed ideals in $C(X)$ correspond one-to-one to closed subsets S of X by $S \mapsto I_S = \{f \in C(X) : f(s) = 0 \text{ for all } s \in S\}$. Let $X_0 = \{s \in X : h(s) \neq 0\}$. I_S is invariant under T if and only if $\varphi(S \cap X_0) \subset S$. We conclude from 4.6:

If there exists no closed subset $S \neq \emptyset, X$ of X such that $\varphi(S \cap X_0) \subset S$, then $|\sigma(T)| = [r_m(T), r(T)]$.

I_S is a projection band if and only if S is open and closed. Moreover, T is reduced by the corresponding band projection if and only if $\varphi^{-1}(S) \cap X_0 = S \cap X_0$. Using Remark 4.7, one can show that T' is a Lamperti operator if and only if φ is injective on X_0. So we conclude from 4.6:

If φ is injective on X_0 and there exists no open and closed subset $S \neq X, \emptyset$ of X such that $\varphi^{-1}(S) \cap X_0 = S \cap X_0$, then $|\sigma(T)| = [r_m(T), r(T)]$.

Example 4.8. 2. Let (X, Σ, μ) be a measure space and $\varphi: X \to X$ an invertible, measure preserving transformation. Let $h \in L^\infty(X)$ such that $|h(t)| \geq \varepsilon > 0$ for almost all $t \in X$ and some $\varepsilon > 0$. Let $E = L^p(X)$, $1 \leq p \leq \infty$ and define $T \in \mathcal{L}(E)$ by $Tf = h \cdot f \circ \varphi$ $(f \in E)$. T is an invertible Lamperti operator.

If P is a band projection on E, then there exists a measurable subset S of X such that $Pf = 1_S f$ $(f \in E)$ where 1_S denotes the characteristic function of S. P reduces T if and only if $\mu(\varphi^{-1}(S) \Delta S) = 0$. Therefore, T is band irreducible if and only if φ is ergodic. So 4.6 gives:

If φ is ergodic, then $|\sigma(T)| = [r_m(T), r(T)]$.

Example 4.8. 3. Let $E = l^p(\mathbf{N})$ (respectively $E = l^p(\mathbf{Z})$), $1 \le p \le \infty$, and T be a weighted shift operator on E given by $(Tx)_n = a_n x_{n+1}$ for all $n \in \mathbf{N}$ (respectively $n \in \mathbf{Z}$) and $x = (x_n) \in E$, where $(a_n) \in l^\infty(\mathbf{N})$ (respectively, $(a_n) \in l^\infty(\mathbf{Z})$). It is obvious that T is a Lamperti operator. Also T' is a Lamperti operator. This can be seen using 4.7 or, for $1 \le p < \infty$, from the explicit form of T'. Moreover, T is band irreducible if and only if $a_n \ne 0$ for all $n \in \mathbf{N}$ (respectively $n \in \mathbf{Z}$).

Finally, $\sigma(T)$ is rotationally invariant, that is, $\lambda \in \sigma(T)$ implies $\alpha\lambda \in \sigma(T)$ for all $\alpha \in \Gamma$. (In fact, let $\alpha \in \Gamma$. Define $U : E \to E$ by $(Ux)_n = \alpha^{-n} x_n$. Then $UTU^{-1} = \alpha T$. Hence $\sigma(T) = \sigma(UTU^{-1}) = \sigma(\alpha T) = \alpha\sigma(T)$).

We now apply 4.6 and get:

$$\sigma(T) = \{z \in \mathbf{C} : r_m(T) \le |z| \le r(T)\}.$$

(Of course, in the case $E = l^p(\mathbf{N}), r_m(T) = 0$.) A different proof of this result (for $p = 2$) can be found in [16].

5. Uniquely ergodic homeomorphisms. Throughout this section X denotes a compact space and φ a homeomorphism on X. Every $h \in C(X)$ defines a Lamperti operator T_h on $C(X)$ by means of

$$T_h f = h \cdot f \circ \varphi \qquad (f \in C(X)).$$

T_h is a lattice isomorphism if and only if h is strictly positive (that is, $h(s) > 0$ for all $s \in X$).

The purpose of this section is to compute $\sigma(T_h) \cap [0,\infty)$ in terms of h and φ. For $h \in C(X)_+$ let

$$p_n(h) = (h \cdot h \circ \varphi \cdot \ldots \cdot h \circ \varphi^{n-1})^{1/n} \qquad (n \in \mathbf{N})$$

Theorem 5.1. *Let $h \in C(X)$ be strictly positive. Then*

$$\begin{aligned}\sigma(T_h) \cap [0,\infty) &= \left(\bigcup_{m \in \mathbf{N}} \bigcap_{n \ge m} p_n(h)(X)\right)^- \\ &= \{r > 0 : \forall \varepsilon > 0 \quad \forall m \in \mathbf{N} \quad \exists n \ge m \\ &\qquad \textit{such that} \quad p_n(h)^{-1}(r - \varepsilon, r + \varepsilon) \ne \emptyset\}.\end{aligned}$$

For the proof we need the following lemma.

Lemma 5.2. *Let $h \in C(X)$ be strictly positive. Then*
a) $\|T_h^n\|^{1/n} = \sup\{p_n(h)(s) : s \in X\}$
b) $\|T_h^{-n}\|^{-1/n} = \inf\{p_n(h)(s) : s \in X\} \qquad (n \in \mathbf{N})$.

Proof. Since T_h^n is given by $(T_h)^n f = p_n(h)^n f \circ \varphi^n$ for all $f \in C(X)$, it follows that

$$\|T_h^n\|^{1/n} = \sup\{p_n(h)(s) : s \in X\} \qquad (n \in \mathbf{N}).$$

Let $n \in \mathbf{N}$. The inverse of T_h is given by

$$T_h^{-1} f = (1/h \circ \varphi^{-1}) f \circ \varphi^{-1} \qquad (f \in C(X)).$$

Therefore,

$$T_h^{-n} f = (1/(h \circ \varphi^{-1} \cdot \ldots \cdot h \circ \varphi^{-n})) f \circ \varphi^{-n} \qquad (f \in C(X)).$$

Consequently,

$$\begin{aligned}
\|T_h^{-n}\|^{-1/n} &= [\sup\{1/(h(\varphi^{-1}(t)) \cdot \ldots \cdot h(\varphi^{-n}(t)) : t \in X\}]^{-1/n} \\
&= [\sup\{1/(h(s) \cdot \ldots \cdot h(\varphi^{n-1}(s)) : s \in X\}]^{-1/n} \\
&= [1/(\inf\{p_n(h)(s) : s \in X\})]^{-1} \\
&= \inf\{p_n(h)(s) : s \in X\}.
\end{aligned}$$

Proof of Theorem 5.1. *Let* $h_n = p_n(h)$ $(n \in \mathbf{N})$. It is clear that

$$\begin{aligned}
M_1 := \left(\bigcup_{m \in \mathbf{N}} \bigcap_{n \geq m} h_n(X) \right)^{-} &= \{r > 0 : \forall \varepsilon > 0 \ \exists m \in \mathbf{N} \\
&\qquad \forall n \geq m \ h_n^{-1}(r - \varepsilon, r + \varepsilon) \neq \varnothing\} \\
&\subset \{r > 0 : \forall \varepsilon > 0 \ \forall m \in \mathbf{N} \ \exists n \geq m \\
&\qquad h_n^{-1}(r - \varepsilon, r + \varepsilon) \neq \varnothing\} \\
&=: M_2.
\end{aligned}$$

It has to be shown that $\sigma(T_h) \cap [0,\infty) = M_1 = M_2$.

1. We show that $\sigma(T_h) \cap [0,\infty) \subset M_1$. Let $r > 0$ such that $r \notin M_1$. Then there exists $\varepsilon > 0$ and an increasing sequence $(n_k)_{k \in \mathbf{N}}$ in $\mathbf{N}$ such that $h_{n_k}^{-1}(r - \varepsilon, r + \varepsilon) = \varnothing$ $(k \in \mathbf{N})$. Let $k \in \mathbf{N}$ be so large that

$$(r - \varepsilon)[(\max_{t \in X} h(t))/(\min_{t \in X} h(t))]_{1}/n_k < r$$

and

$$(r + \varepsilon)[(\min_{t \in X} h(t))/(\max_{t \in X} h(t))]_{1}/n_k > r.$$

Let

$$X_1 = \{t \in X : h_{n_k}(t) \leq r - \varepsilon\} = \{t \in X : h_{n_k}(t) < r\},$$

$$X_2 = \{t \in X : h_{n_k}(t) \geq r + \varepsilon\} = \{t \in X : h_{n_k}(t) > r\}.$$

Since h_{n_k} is continuous, X_1, X_2 are open and closed. Moreover, $X = X_1 \cup X_2$, $X_1 \cap X_2 = \varnothing$. We claim that $\varphi(X_i) \subset X_i$ $(i = 1,2)$. For $t \in X$,

$$\begin{aligned}
h_{n_k}(\varphi(t)) &= [h(\varphi(t)) \cdot \ldots \cdot h(\varphi^{n_k}(t))]^{1/n_k} \\
&= h_{n_k}(t) \cdot [h(\varphi^{n_k}(t))/h(t)]^{1/n_k}.
\end{aligned}$$

Thus, by the choice of k, $h_{n_k}(\varphi(t)) < r$ if $t \in X_1$ and $h_{n_k}(\varphi(t)) > r$ if $t \in X_2$.

Since φ is bijective, it follows that $\varphi(X_i) = X_i$ ($i = 1,2$). The projection band $J_i = \{f \in C(X): f(t) = 0 \text{ for all } t \notin X_i\}$ is invariant under T. Let T_i be the restriction of T to J_i ($i = 1,2$). By 5.2, if $J_1 \neq \{0\}$, then

$$r(T_1) \leq \|T_1^{n_k}\|^{1/n_k} = \sup\{h_{n_k}(t): t \in X_1\} < r,$$

and if $J_2 \neq \{0\}$, then

$$r_m(T_2) = r(T_2^{-1})^{-1} \geq \|T_2^{-n_k}\|^{-1/n_k} = \inf\{h_{n_k}(t): t \in X_2\} > r.$$

Thus, $r \notin \sigma(T_1) \cup \sigma(T_2) = \sigma(T)$ (where $\sigma(T_i) := \varnothing$ if $J_i = \{0\}$).

2. We show that $M_2 \subset \sigma(T) \cap (0,\infty)$. Let $r \in (0,\infty)$ such that $r \notin \sigma(T)$. We can assume that

$$r_m(T) < r < r(T) \qquad \text{(since by 5.2,}$$

$$r_m(T) = \lim_{n\to\infty}(\inf_{t\in X} h_n(t)) \quad \text{and} \quad r(T) = \lim_{n\to\infty}(\sup_{t\in X} h_n(t))).$$

By 4.2, there exists a projection band $J \neq 0$, E such that J and $J^\perp$ are invariant under T and such that $r(T_1) < r < r_m(T_2)$, where $T_1 = T_{|J}$, and $T_2 = T_{|J^\perp}$. There are open and closed subsets X_1, X_2 of X such that $X_1 \cup X_2 = \varnothing$, $X_1 \cap X_2 = X$ and

$$J = \{f \in C(X): f(t) = 0 \quad \text{for all} \quad t \notin X_1\}$$

$$J^\perp = \{f \in C(X): f(t) = 0 \quad \text{for all} \quad t \notin X_2\}.$$

By 5.2,

$$\lim_{n\to\infty}(\sup_{t\in X_1} h_n(t)) = r(T_1) < r$$

and

$$\lim_{n\to\infty}(\inf_{t\in X_2} h_n(t)) = r_m(T_2) > r.$$

This implies that $r \notin M_2$.

Corollary 5.3. *Let $h \in C(X)$ be strictly positive.*

(a) If $p_n(h)\circ\varphi^n = p_n(h)$ (in particular, if $h\circ\varphi^n = h$) for some $n \in \mathbf{N}$, then $\sigma(T_h) \cap [0,\infty) = p_n(h)(X)$.

(b) If $(p_n(h))_{n\in\mathbf{N}}$ converges uniformly to some function $k \in C(X)$, then $\sigma(T_h) \cap [0,\infty) = k(X)$.

(c) Let $r > 0$. Then $\sigma(T_h) \cap [0,\infty) = \{r\}$ if and only if $\lim_{n\to\infty} p_n(h) = r \cdot 1_X$ uniformly.

Proof. *(a)* For $n = 1$ the assertion follows immediately from 5.1. Let $n > 1$, $r > 0$. Then $r \in \sigma(T_h)$ if and only if $r^n \in \sigma((T_h)^n)$. But, since $(T_h)^n f = (h_n)^n \cdot f\circ\varphi_n$ ($f \in C(X)$), this is equivalent to $r^n \in h_n^n(X)$ by the case $n = 1$, and this means that $r \in h_n(X)$ (where $h_n = p_n(h)$).

(b) and (c) are immediate consequences of 5.1.

For $f \in C(X)$ let

$$M_n f = \frac{1}{n} \sum_{m=0}^{n-1} f \circ \varphi^m .$$

φ is called *uniquely ergodic* if for every $f \in C(X)$, the sequence $(M_n f)_{n \in \mathbf{N}}$ converges uniformly to a constant function (see [17]).

Corollary 5.4. *The following assertions are equivalent.*

(i) φ is uniquely ergodic.

(ii) $\sigma(T_h) \subset \{z \in \mathbf{C} : |z| = r(T_h)\}$ for every strictly positive $h \in C(X)$.

Proof. The mapping $f \to \exp(f)$, where $(\exp(f))(t) = e^{f(t)}$ for all $t \in X$, maps $C_{\mathbf{R}}(X)$ (the set of all real valued functions in $C(X)$) onto the set of all strictly positive functions in $C(X)$. Let $f \in C_{\mathbf{R}}(X)$. Since $\exp(M_n f) = p_n(\exp(f))$ for all $n \in \mathbf{N}$, we get for $c \in \mathbf{R} : \lim_{n \to \infty} M_n f = c 1_X$ uniformly if and only if $\lim_{n \to \infty} p_n(\exp(f)) = e^c \cdot 1_X$ uniformly, and by 5.3, the latter assertion is equivalent to $\sigma(T_{\exp(f)}) = \{e^c\}$. From this the corollary follows.

Remark 5.5. If X is infinite and φ is uniquely ergodic, then for strictly positive $h \in C(X)$ one actually has:

$$\sigma(T_h) = \{z \in \mathbf{C} : |z| = r(T_h)\}.$$

In fact, one can assume that $r(T_h) = 1$. Suppose the inclusion of 5.4 (ii) is proper. Since $\sigma(T_h)$ is a union of subgroups of Γ [14, V 4.4], it follows that $\sigma(T_h)$ consists of roots of the unity with a common upper bound p, say. Then $z^{p!} = 1$ for all $z \in \sigma(T_h)$. By 3.5, this implies that $T^{p!} = I$. In particular, $\varphi^{p!} = \mathrm{id}$. Therefore, $\lim_{n \to \infty} M_n(f) = M_{p!}(f)$ for all $f \in C(X)$. Since φ is uniquely ergodic, this implies that $M_{p!}(f)$ is a constant function for all $f \in C(X)$. This cannot be unless X is finite.

Example 5.6. Let $\varphi : \Gamma \to \Gamma$ be defined by $\varphi(z) = z z_0$, where $z_0 \in \Gamma$ such that $z_0^n \neq 1$ for all $n \in \mathbf{N}$. Then φ is uniquely ergodic [17, Chapter 5]. Let $E = L^\infty(\Gamma)$ (with respect to the Haar measure) and consider for each $h \in L^\infty(\Gamma)$ the operator S_h on E defined by $S_h f = h \cdot f \circ \varphi$ for all $f \in E$.

If h is continuous and strictly positive, then $r(S_h) = r_m(S_h)$.

In fact, denote by T_h the restriction of S_h to $C(\Gamma)$. By 5.4, $r(T_h) = r_m(T_h)$. One sees from the formula for the spectral radius that $r(S_h) = r(T_h)$ and $r_m(S_h) = r_m(T_h)$. Hence, $r(S_h) = r_m(S_h)$.

On the other hand, there exists $h \in L^\infty(\Gamma)$ such that $0 < r_m(S_h) < r(S_h)$.

In fact, there exists a compact Stonian space X and a lattice isomorphism $U : L^\infty(\Gamma) \to C(X)$ such that $U 1_\Gamma = 1_X$. Hence, $R = U S_{1_\Gamma} U^{-1}$ is a lattice isomorphism on $C(X)$ satisfying $R 1_X = 1_X$. Therefore, there exists a homeomorphism ψ on X such that $Rg = g \circ \psi$ for all $g \in C(X)$.

Since X is Stonian, it follows from [1, Section 4, Proposition 1] that ψ is *not* uniquely ergodic. By 5.4, it follows that there exists a strictly positive $k \in C(X)$ such that $r_m(R_k) < r(R_k)$, where $R_k = M_k R$, $M_k g = k \cdot g (g \in C(X))$. Since $M_k \in Z(C(X))$, it follows that $M := U^{-1} M_k U \in Z(E)$. By 3.4.2, there exists $h \in L^\infty(\Gamma)$ such that $Mf = h \cdot f$ for all $f \in L^\infty(\Gamma)$. Moreover, $U^{-1} R_k U = U^{-1} M_k U U^{-1} R U = M S_{1_\Gamma} = S_h$. Therefore, $r_m(S_h) = r_m(R_k) < r(R_k) = r(S_h)$.

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Universität Tübingen—D-7400 Tübingen, West Germany

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