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Order Isomorphisms of Fourier Algebras

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If G is a locally compact group, we denote by $A(G)$ the Fourier algebra of G, by $A(G)_+$ the set of all pointwise positive functions in $A(G)$, and by $P(G)$ the set of all continuous positive definite functions on G. We show that two locally compact groups G_1 and G_2 are isomorphic if and only if there exists a bijective linear mapping T: $A(G_1) \to A(G_2)$ such that $T(A(G_1)_+) = A(G_2)_+$ and $T(A(G_1) \cap P(G_1)) = A(G_2) \cap P(G_2)$.

1. INTRODUCTION

Let $A(G)$ be the Fourier algebra of a locally compact group G [1]. $A(G)$ is a Banach algebra, but it is also an ordered vector space for two different order relations, namely, the *pointwise ordering*, and the *positive definite ordering*, which is defined by the cone of all positive definite functions in $A(G)$.

Let G_1 and G_2 be two locally compact groups. If $\alpha: G_2 \to G_1$ is a topological group isomorphism or antiisomorphism and c a positive real number, consider the mapping T: $A(G_1) \to A(G_2)$ defined by

$$Tf = c \cdot f \circ \alpha \tag{1.1}$$

for all $f \in A(G_1)$. It is obvious that T is an order isomorphism for the two orderings we described above.

Our main result asserts that the converse is also true: If T: $A(G_1) \to A(G_2)$ is a bijective linear mapping such that T and T^{-1} are positive for both order relations, then T has the form (1.1) (in particular, T is a positive multiple of an algebra isomorphism). Thus the pointwise and the positive definite orderings in $A(G)$ determine the group G up to isomorphism.

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For *abelian* groups our result has the following interpretation by Pontryagin duality. Let G_1 and G_2 be abelian locally compact groups, and let $\hat{}$ denote Fourier transformation. If T: $L^1(G_1) \to L^1(G_2)$ is a bijective linear mapping such that $Tf \geqslant 0$ (pointwise a.e.) if and only if $f \geqslant 0$ (pointwise a.e.), and $(Tf)\hat{} \geqslant 0$ (pointwise) if and only if $\hat{f} \geqslant 0$ (pointwise), then there exist $c > 0$ and a topological group isomorphism α: $G_2 \to G_1$ such that $Tf = c \cdot f \circ \alpha$ for all $f \in L^1(G_1)$ (Corollary 4.4). In a subsequent article, which deals with order isomorphisms of Fourier–Stieltjes algebras, we shall prove a generalization of this corollary to arbitrary locally compact groups, thus obtaining the full dual of the main theorem of the present article.

Let us indicate the way which leads to this theorem. Our starting point is the following result of Walter [4, Theorem 2]. Let T: $A(G_1) \to A(G_2)$ be an *isometric algebra* isomorphism. Then there exist $s \in G_1$ and a mapping α: $G_2 \to G_1$ which is a topological isomorphism or antiisomorphism such that $Tf(t) = f(s\alpha(t))$ ($t \in G_2$) for all $f \in A(G_1)$.

Our first step is to show that the condition of an algebra isomorphism T being isometric (as in Walter's theorem) follows from the assumption that T is an *order isomorphism for the positive definite ordering* (Theorem 3.2). This result is of independent interest. It can be considered as a dual version of a theorem of Kawada [2, Theorem 1].

In a second step we also replace the multiplicativity of T by an order theoretic notion. More precisely, we prove that an *order isomorphism for the pointwise ordering* is a weighted composition operator (Proposition 4.2). A combination of the two steps with Walter's result yields the desired conclusion (Theorem 4.3).

From an inspection of both Walter's paper and our proofs (notice in particular the repeated use of [3]) it is clear that operator algebra methods play a crucial role, as is often the case in non-commutative harmonic analysis (cf. [1]).

2. Preliminaries

Our general reference is Eymard's fundamental work [1]. Throughout this article G, G_1, and G_2 are locally compact groups. The spaces $L^p(G)$ ($p = 1, 2, \infty$) are defined as usual with respect to the left Haar measure on G. By λ: $G \to \mathscr{L}(L^2(G))$ we denote the left regular representation of G; i.e., for $s \in G$, $\lambda_s \in \mathscr{L}(L^2(G))$ is defined by $\lambda_s f(t) = f(s^{-1}t)$ ($t \in G$) for all $f \in L^2(G)$. We use e for the unit of G and I for the identity operator on $L^2(G)$. Obviously $\lambda_e = I$. By $VN(G)$ we denote the von Neumann algebra generated in $\mathscr{L}(L^2(G))$ by all translation operators λ_s ($s \in G$).

$A(G)$ is the Fourier algebra of G (see [1, Chap. 3] for the definition). Recall that $A(G)$ is a subalgebra of $C_0(G)$, the algebra of all continuous complex valued functions on G which vanish at infinity. With its own norm it is a regular (commutative) Banach algebra [1, 3.2], and its Gelfand space coincides with G (the points of G correspond to multiplicative linear functionals by evaluation [1, 3.34]).

The dual space of $A(G)$ can be identified with $VN(G)$ [1, 3.10], the duality being given by

$$\langle f, \lambda_s \rangle = f(s) \qquad (f \in A(G), s \in G) \tag{2.1}$$

[1, 3.14]. In particular, $\{\lambda_s \mid s \in G\}$ is the set of all multiplicative linear functionals on $A(G)$.

$A(G)$ is an ordered vector space for two order relations. The first one is the *pointwise ordering* defined by

$$f \geqslant 0 \qquad \text{if and only if } f(s) \geqslant 0 \text{ for all } s \in G.$$

We denote the corresponding positive cone by $A(G)_+$. The second one is the ordering defined by the cone $P(G) \cap A(G)$, where $P(G)$ denotes the cone of all continuous positive definite functions on G. We refer to this order relation as to the *positive definite ordering*, and for $f \in A(G)$ we write

$$f \gg 0 \qquad \text{if and only if } f \in P(G).$$

To these orderings there correspond dual orderings in $VN(G)$, which we denote again by $\geqslant$ (resp. $\gg$); i.e., if $x \in VN(G)$, let $x \geqslant 0$ (resp. $x \gg 0$) if and only if $\langle f, x \rangle \geqslant 0$ for all $f \in A(G)_+$ (resp. for all $f \in P(G) \cap A(G)$). Obviously,

$$\lambda_s \geqslant 0 \qquad \text{for all } s \in G. \tag{2.2}$$

Notice that for $x \in VN(G)$ one has $x \gg 0$ if and only if x is positive as an operator on the Hilbert space $L^2(G)$, i.e., if and only if x is hermitian and its spectrum is positive. In particular, since λ_s is unitary for all $s \in G$, one has

$$\lambda_s \gg 0 \qquad \text{if and only if } s = e. \tag{2.3}$$

3. A Dual Version of Kawada's Theorem

Lemma 3.1. *Let $T\colon A(G_1) \to A(G_2)$ be a linear mapping which is positive for the positive definite ordering, i.e., $T(A(G_1) \cap P(G_1)) \subset A(G_2) \cap P(G_2)$. Then T is continuous.*

Proof. (The lemma can be formulated and proved for positive linear mappings between preduals of von Neumann algebras and is probably well known in that generality, but for completeness we give the proof in the above special case.) First we show that there exists a positive real number c such that $\|Tf\| \leqslant c\|f\|$ for all $f \in A(G_1) \cap P(G_1)$. If not, one can find $f_n \in A(G_1) \cap P(G_1)$ such that $\|f_n\| \leqslant 2^{-n}$ and $\|Tf_n\| \geqslant n$ for all $n \in \mathbb{N}$. Put $f = \sum_{n=1}^{\infty} f_n$. Then $Tf \gg Tf_n \gg 0$ for all $n \in \mathbb{N}$. Hence

$$\|Tf\| = \langle Tf, I\rangle \geqslant \langle Tf_n, I\rangle = \|Tf_n\| \geqslant n$$

for all $n \in \mathbb{N}$, which is impossible. The lemma then follows from the fact that every $f \in A(G_1)$ can be written as $f = f_1 - f_2 + i(f_3 - f_4)$, where $f_j \in A(G_1) \cap P(G_1)$ and $\|f_j\| \leqslant \|f\|$ $(j = 1, 2, 3, 4)$. ∎

THEOREM 3.2. *Let T: $A(G_1) \to A(G_2)$ be an algebra isomorphism and an order isomorphism for the positive definite ordering. Then there exists a mapping α: $G_2 \to G_1$ which is a topological group isomorphism or antiisomorphism such that $Tf = f \circ \alpha$ for all $f \in A(G_1)$.*

Proof. T is continuous by Lemma 3.1; let T': $VN(G_2) \to VN(G_1)$ denote its adjoint. First observe that $T'I = I$. Indeed, since T is an algebra homomorphism, $T'I$ is a (nonzero) multiplicative linear functional on $A(G_1)$. Therefore $T'I = \lambda_s$ for some $s \in G_1$. But $T'I \gg 0$ because $I \gg 0$; hence $s = e$ (2.3). By [3, Corollary 1] it follows that T' is a contraction. By the same argument $(T')^{-1} = (T^{-1})'$ is a contraction. Hence T' is an isometry. Since T is then an isometric algebra isomorphism, by [4, Theorem 3] there exist $s \in G_1$ and a topological group isomorphism or antiisomorphism α: $G_2 \to G_1$ such that $(Tf)(t) = f(s\alpha(t))$ $(t \in G_2)$ for all $f \in A(G_1)$. Using (2.1) one checks immediately that $T'\lambda_e = \lambda_s$; hence the first observation of the proof implies $s = e$. ∎

Remark 3.3. In particular, Theorem 3.2 gives a dual version of Kawada's theorem [2, Theorem 1], which asserts the following: G_1 and G_2 are isomorphic as locally compact groups if and only if there exists an algebra isomorphism T: $L^1(G_1) \to L^1(G_2)$ such that $Tf \geqslant 0$ a.e. if and only if $f \geqslant 0$ a.e. for all $f \in L^1(G_1)$.

COROLLARY 3.4. *Let α: $G_2 \to G_1$ be a mapping such that $f \mapsto f \circ \alpha$ defines a bijection from $A(G_1) \cap P(G_1)$ onto $A(G_2) \cap P(G_2)$, then α is a topological group isomorphism or antiisomorphism.*

4. ORDER ISOMORPHISMS OF FOURIER ALGEBRAS

Recall [1; 4.4, 4.5] the definition of the support of an operator $x \in VN(G) = A(G)'$:

supp $x = \{s \in G \mid$ for every neighborhood U of s there exists $f \in A(G)$ with support contained in U such that $\langle f, x\rangle \neq 0\}$.

LEMMA 4.1. *Let* $x \in VN(G)$, $x \geqslant 0$. *Then* $\operatorname{supp} x = \{s \in G \mid$ *for every neighborhood* U *of* s *there exists* $f \in A(G)_+$ *with support contained in* U *such that* $\langle f, x \rangle > 0\}$.

Proof. Let $s \in G$ and suppose that there exists a neighborhood U of s such that $\langle g, x \rangle = 0$ for all $g \in A(G)_+$ with $\operatorname{supp} g \subset U$. It suffices to show that $s \notin \operatorname{supp} x$. Let K be a compact neighborhood of s contained in U. Using [1, 3.2] one finds a function $g \in A(G)_+$ such that $\operatorname{supp} g \subset U$ and $g(t) = 1$ for all $t \in K$. Now let $f \in A(G)$ be real valued and such that $\operatorname{supp} f \subset K$. Then $-\|f\|_\infty \, g \leqslant f \leqslant \|f\|_\infty g$. Since $x \geqslant 0$ it follows that $-\|f\|_\infty \langle g, x \rangle \leqslant \langle f, x \rangle \leqslant \|f\|_\infty \langle g, x \rangle$. But by our assumption $\langle g, x \rangle = 0$. Hence $\langle f, x \rangle = 0$. As the real and the imaginary part of an arbitrary $f \in A(G)$ still belong to $A(G)$ [1, 3.8], we have proved that $\operatorname{supp} f \subset K$ implies $\langle f, x \rangle = 0$, and hence that $s \notin \operatorname{supp} x$. ∎

PROPOSITION 4.2. *Let* $T: A(G_1) \to A(G_2)$ *be a continuous linear bijection. If* T *is an order isomorphism for the pointwise ordering, i.e.,* $T(A(G_1)_+) = A(G_2)_+$, *there exists a bijective mapping* $\alpha: G_2 \to G_1$ *and a function* $c: G_2 \to (0, \infty)$ *such that*

$$Tf(t) = c(t)\, f(\alpha(t)) \qquad (t \in G_2) \tag{4.1}$$

for all $f \in A(G_1)$. *Moreover,*

$$T^{-1}g(s) = [1/c(\alpha^{-1}(s))]\; g(\alpha^{-1}(s)) \qquad (s \in G_1) \tag{4.2}$$

for all $g \in A(G_2)$.

Proof. As before T': $VN(G_2) \to VN(G_1)$ denotes the adjoint of T. Let $t \in G_2$, $x = T'\lambda_t \in VN(G_1)$. We show that $\operatorname{supp} x$ is a singleton. In fact, by [1, 4.6], $\operatorname{supp} x \neq \varnothing$, since $x \neq 0$. Suppose that there exist two different elements $s_1, s_2 \in \operatorname{supp} x$. Let U_j be a neighborhood of s_j ($j = 1, 2$) such that $U_1 \cap U_2 = \varnothing$. Since $x \geqslant 0$ (by (2.2) and the fact that T is positive for the pointwise ordering) it follows from Lemma 4.1 that there exist functions f, $g \in A(G_1)_+$ such that $\operatorname{supp} f \subset U_1$, $\operatorname{supp} g \subset U_2$, $\langle f, x \rangle > 0$ and $\langle g, x \rangle > 0$. Thus (using (2.1)) $Tf(t) = \langle Tf, \lambda_t \rangle = \langle f, T'\lambda_t \rangle = \langle f, x \rangle > 0$, and similarly $Tg(t) > 0$. Since $Tf \geqslant 0$ and $Tg \geqslant 0$ the regularity of $A(G_2)$ implies that one can find $k \in A(G_2)_+$, $k \neq 0$, such that $k \leqslant Tf$ and $k \leqslant Tg$. Put $h = T^{-1}k$. Since $T^{-1}(A(G_2)_+) \subset A(G_1)_+$ it follows that $0 \leqslant h \leqslant f$ and $h \leqslant g$. But $h \neq 0$, so there exists $s \in G_1$ such that $0 < h(s) \leqslant f(s)$, $h(s) \leqslant g(s)$. Hence $s \in U_1 \cap U_2$, a contradiction.

Thus for every $t \in G_2$ there exists $\alpha(t) \in G_1$ such that $\operatorname{supp} T'\lambda_t = \{\alpha(t)\}$. By [1, 4.9], there exists $c(t) \in \mathbb{C}$ such that $T'\lambda_t = c(t)\lambda_{\alpha(t)}$. As T' is injective and T is positive for the pointwise ordering we have $c(t) > 0$. If $f \in A(G_1)$, then $Tf(t) = \langle f, T'\lambda_t \rangle = c(t)\, f(\alpha(t))$ for all $t \in G_2$. Hence T is of the form (4.1).

Similarly, there exist d: $G_1 \to (0, \infty)$ and β: $G_1 \to G_2$ such that $(T')^{-1}\lambda_s = d(s)\,\lambda_{\beta(s)}$ for all $s \in G_1$. Hence

$$\lambda_s = T'(T')^{-1}\lambda_s = c(\beta(s))\, d(s)\lambda_{\alpha(\beta(s))}$$

for all $s \in G_1$, and

$$\lambda_t = (T')^{-1}T'\lambda_t = d(\alpha(t))\, c(t)\lambda_{\beta(\alpha(t))}$$

for all $t \in G_2$. Thus α is bijective and $\beta = \alpha^{-1}$ and $d(s) = 1/c(\alpha^{-1}(s))$ $(s \in G_1)$. ∎

We are now able to prove the main theorem.

THEOREM 4.3. *Let T: $A(G_1) \to A(G_2)$ be a linear order isomorphism for both the pointwise ordering and the positive definite ordering. Then there exist a positive real number c and a mapping α: $G_2 \to G_1$ which is a topological group isomorphism or antiisomorphism such that $Tf = c \cdot f \circ \alpha$ for all $f \in A(G_1)$.*

Proof. It follows from Lemma 3.1 that T is continuous. By Proposition 4.2, there exist a bijective mapping α: $G_2 \to G_1$ and a mapping c: $G_2 \to (0, \infty)$ such that $Tf(t) = c(t)\, f(\alpha(t))$ $(t \in G_2)$ for all $f \in A(G_1)$. Since T' is positive in the von Neumann algebra sense it follows that $c(e)\lambda_{\alpha(e)} = T'\lambda_e \gg 0$. Thus $\lambda_{\alpha(e)} \gg 0$. Consequently $\alpha(e) = e$ (2.4). Moreover we have $\|T'\| = \|T'I\|$ (this follows from [3, Corollary 1]). As $\|\lambda_t\| = 1$, one obtains that $c(t) = \|c(t)\lambda_{\alpha(t)}\| = \|T'\lambda_t\| \leqslant \|T'\| = \|T'\lambda_e\| = c(e)$ for all $t \in G_2$. The same argument applied to T^{-1} shows that $1/c(\alpha^{-1}(s)) \leqslant 1/c(e)$ for all $s \in G_1$. Since α is bijective we conclude that $c(e) \leqslant c(t) \leqslant c(e)$ for all $t \in G_2$; that is, $c(t) = c(e)$ for all $t \in G_2$.

Define $c = c(e) > 0$. Then $c^{-1}Tf(t) = f(\alpha(t))$ $(t \in G_2)$ for all $f \in A(G_1)$. Hence $c^{-1}T$ is an algebra isomorphism. Thus Theorem 3.2 implies that α is a topological group isomorphism or antiisomorphism. ∎

We want to reformulate Theorem 4.3 in the abelian case. If G is an abelian locally compact group we denote the dual group by $\hat{G}$ and we use $\hat{f}$ for the Fourier transform of $f \in L^1(G)$. For $f \in L^1(G)$ we write $f \geqslant 0$ if and only if $f(t) \geqslant 0$ for almost all $t \in G$, and $f \gg 0$ if and only if $\hat{f}(\chi) \geqslant 0$ for all $\chi \in \hat{G}$. With these notations we obtain the following corollary.

COROLLARY 4.4. *Suppose that G_1 and G_2 are abelian. Let T: $L^1(G_1) \to L^1(G_2)$ be a bijective linear mapping such that*

(a) *$Tf \geqslant 0$ if and only if $f \geqslant 0$,*

(b) *$Tf \gg 0$ if and only if $f \gg 0$ $(f \in L^1(G_1))$. Then there exist a positive real number c and a topological group isomorphism α: $G_2 \to G_1$ such that $Tf = c \cdot f \circ \alpha$ for all $f \in L^1(G_1)$.*

Proof. For an arbitrary locally compact abelian group G one has $A(\hat{G}) = \{\hat{f} | f \in L^1(G)\}$ by [1, 3.6]. Moreover, $A(\hat{G})' = VN(\hat{G})$ can be identified with $L^\infty(G)$ in such a way that the duality between $A(\hat{G})$ and $L^\infty(G)$ is given by $\langle \hat{f}, \phi \rangle = \int f(t)\phi(t)\, dt$ for all $f \in L^1(G)$, $\phi \in L^\infty(G)$. Finally $\hat{f} \gg 0$ in the sense of Section 2 if and only if $\langle f, \phi \rangle \geqslant 0$ for all $\phi \in L^\infty(G)$ such that $\phi(t) \geqslant 0$ for locally almost all $t \in G$, and this in turn means that $f \geqslant 0$ as defined above ($f \in L^1(G)$).

To prove the corollary, observe that T induces a bijective linear mapping $\hat{T}$: $A(\hat{G}_1) \to A(\hat{G}_2)$ by $\hat{T}\hat{f} = (Tf)\hat{}$ for all $f \in L^1(G_1)$. Having in mind the identification made above, one sees that the conditions (a) and (b) exactly mean that $\hat{T}$ is an order isomorphism for the pointwise and the positive definite orderings of $A(\hat{G}_1)$ and $A(\hat{G}_2)$. By Theorem 4.3, there exist $c' > 0$ and a topological group isomorphism β: $\hat{G}_2 \to \hat{G}_1$ such that $\hat{T}\hat{f} = c' \cdot \hat{f} \circ \beta$ for all $f \in L^1(G_1)$. Using the Pontryagin duality theorem we get a topological group isomorphism α: $G_2 \to G_1$ such that $\beta(\chi)(s) = \chi(\alpha^{-1}(s))$ for all $\chi \in \hat{G}_2$, $s \in G_1$. It follows from the uniqueness of the Haar measure that there exists $c'' > 0$ such that for every $f \in L^1(G_1)$,

$$\int_{G_1} f(s)\, ds = c'' \int_{G_2} f(\alpha(t)\, dt.$$

Let $f \in L^1(G_1)$. For every $\chi \in \hat{G}_2$, one has

$$\begin{aligned}(Tf)\hat{}(\chi) = \hat{T}\hat{f}(\chi) = c'\hat{f}(\beta(\chi)) &= c' \int_{G_1} f(s)\, \overline{\beta(\chi)(s)}\, ds \\ &= c' \int_{G_1} f(s)\, \overline{\chi(\alpha^{-1}(s))}\, ds = c'c'' \int_{G_2} f(\alpha(t))\, \overline{\chi(t)}\, dt \\ &= c'c''(f \circ \alpha)\hat{}(\chi).\end{aligned}$$

By the injectivity of the Fourier transformation it follows that $Tf = c'c'' \cdot f \circ \alpha$. Finally we put $c = c'c'' > 0$. ∎

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