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Order Isomorphisms of Fourier-Stieltjes Algebras

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1. Introduction

The Fourier-Stieltjes algebra $B(G)$ of a locally compact group G is the space of all linear combinations of continuous positive definite functions on G. Whereas $B(G)$ has usually been investigated as a Banach algebra (with pointwise multiplication), we consider another structure. We view $B(G)$ as an ordered vector space in two senses; these orderings are determined by: the cone $P(G)$ of all continuous positive definite functions on G and the cone $B(G)_+$ of all pointwise positive functions on G.

The purpose of this article is to show that the "biordered space" $(B(G), P(G), B(G)_+)$ is a complete invariant for the locally compact group G. More precisely, if G_1 and G_2 are two locally compact groups, then G_1 is isomorphic to G_2 as a topological group if and only if there exists a bijective linear mapping T: $B(G_1) \to B(G_2)$ such that $TB(G_1)_+ = B(G_2)_+$ and $TP(G_1) = P(G_2)$.

In this way we have found a complete invariant for locally compact groups which is very simple to define. Only the structure of a topological group is necessary for its definition. In particular, the Haar measure is not needed.

The proofs, however, make use of Haar measure. In fact, they depend on Walter's theorem [9, Theorem 2], which says that two locally compact groups G_1 and G_2 are isomorphic if and only if there exists an *isometric* algebra isomorphism T from $B(G_1)$ onto $B(G_2)$. Here the norm of $B(G)$, which is defined via $L^1(G)$, is essential to obtain complete invariance.

So our first step towards the main result is to show that in Walter's theorem the condition that T be isometric can be replaced by the assumption that T is an order isomorphism for the positive definite ordering (Sect. 3). This seems to be of independent interest, since it shows that the "ordered algebra" $(B(G), \cdot, P(G))$ is a complete invariant (which is "Haar measure free").

The next step is to investigate order isomorphisms for the pointwise ordering. In Sect. 5 we obtain the following characterization: A bijective linear mapping T

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from $B(G_1)$ onto $B(G_2)$ satisfies $TB(G_1)_+ = B(G_2)_+$ if and only if there exists an invertible element h in $B(G_2)_+$ and a $*$-algebra isomorphism $V: B(G_1) \to B(G_2)$ such that $Tf = h \cdot Vf$ $(f \in B(G_1))$. Both steps put together show that, up to a multiplicative positive constant, biorderisomorphisms have to be isometric $*$-isomorphisms, so that Walter's result applies.

In a previous paper [1], we made an analogous investigation of the Fourier algebra $A(G)$. We would like to comment on how those results and their proofs are related to the present paper. In [1] we showed that the "biordered space" $(A(G), A(G) \cap P(G), A(G)_+)$ is a complete invariant of G as well. This space, however, is more complicated to define than $B(G)$ (it involves the norm of $B(G)$, since $A(G)$ is the closure of all functions in $B(G)$ which have compact support; see [3] for this and equivalent definitions of $A(G)$). The proofs are related in the following way: The first step described above is a generalization of the corresponding result for $A(G)$. To deal with order isomorphisms for the pointwise ordering, however, is a completely different matter in the two cases. Actually, for $A(G)$, we used the fact that the maximal ideal space of $A(G)$ can be identified with G. This is not true for $B(G)$ in general. Hence our proof for $A(G)$ does not carry over to the $B(G)$ case. On the other hand, the method in the present paper involves the multipliers of $B(G)$, which are given by the elements of $B(G)$ itself. The same method applies to the Fourier algebra of amenable groups, mainly because in that case the multiplier algebra of $A(G)$ coincides with $B(G)$. This fails in general, however (see also the remark following the proof of Theorem 5.2 below).

Finally, we point out that our main result can be applied to the group algebra. So we prove in Sect. 6 that $L^1(G)$, too, carries the natural structure of a biordered space which, as above, is a complete invariant of G.

2. Preliminaries

As a basic reference we use Eymard's fundamental article [3]. Throughout this paper G, G_1, G_2 are locally compact groups. We denote the set of all continuous positive definite functions on G by $P(G)$. It is a proper cone in the vector space $C^b(G)$ (of all continuous bounded complex valued functions on G), i.e.:

$$\mathbb{R}_+ P(G) \subset P(G)$$
$$P(G) + P(G) \subset P(G)$$
$$P(G) \cap (-P(G)) = \{0\}.$$

Moreover, $P(G) \cdot P(G) \subset P(G)$ and $1 \in P(G)$. Thus, $B(G) = \operatorname{span} P(G)$ is a unital subalgebra of $C^b(G)$. It is called the Fourier-Stieltjes algebra of G. $B(G)$ can be canonically identified with the dual space of $C^*(G)$, the enveloping C^*-algebra of $L^1(G)$, and is a Banach algebra with respect to the dual norm $\| \ \|$. This norm satisfies

$$\|u\|_\infty \leqq \|u\| \quad (u \in B(G))$$
$$\|p\| = \|p\|_\infty = p(e) \quad (p \in P(G))$$

where $\|u\|_\infty = \sup_{t\in G} |u(t)|$ and $e\in G$ denotes the unit element of G. $B(G)$ is an involutive Banach algebra for the involution $\bar{\ }$ given by

$$\bar{u}(t) = \overline{u(t)}$$

(where $\bar{z}$ denotes the complex conjugate of $z\in\mathbb{C}$).

The cone $P(G)$ defines an order relation on $B(G)$. We refer to it as the *positive definite ordering.*

There is another order relation on $B(G)$ which is defined by the cone $B(G)_+$ of all pointwise positive functions in $B(G)$. That is, for $u\in B(G)$, $u\in B(G)_+$ if and only if $u(t)\geqq 0$ for all $t\in G$.

We refer to this order relation as to the *pointwise ordering* and write

$$u\leqq v \quad \text{if and only if} \quad v-u\in B(G)_+ .$$

Of course, $B(G)_+$ is a proper cone. It is also *generating*, that is, $\operatorname{span} B(G)_+ = B(G)$. In fact, if $u\in B(G)$, then $\operatorname{Re} u=\frac{1}{2}(u+\bar{u})\in B(G)$ and $\operatorname{Im} u=\frac{1}{2i}(u-\bar{u})\in B(G)$. Since $u=\operatorname{Re} u+i\operatorname{Im} u$, it is enough to show that $v\in\operatorname{span} B(G)_+$, whenever v is a real valued function in $B(G)$. Since $-\|v\|_\infty 1\leqq v\leqq \|v\|_\infty 1$, it follows that $v_1= v+\|v\|_\infty 1\in B(G)_+$ and $v_2=\|v\|_\infty 1-v\in B(G)_+$. So $v=\frac{1}{2}(v_1-v_2)\in\operatorname{span} B(G)_+$.

Corresponding to these two orderings are the notions of positive linear mappings: A linear mapping $T: B(G_1)\to B(G_2)$ is *positive for the positive definite ordering* (resp., *the pointwise ordering*) if $TP(G_1))\subset P(G_2)$ (respectively, $TB(G_1)_+\subset B(G_2)_+$). T is called an *order isomorphism for the positive definite ordering* (respectively, *the pointwise ordering*) if T is bijective and $TP(G_1)=P(G_2)$ (respectively, $TB(G_1)_+ = B(G_2)_+$). Finally, we say that T is a *biorder isomorphism* if T is an order isomorphism for the positive definite and the pointwise ordering.

3. Kawada's Theorem for the Fourier-Stieltjes Algebra

Kawada showed that the ordered algebra $L^1(G)$ is a complete invariant for G. More precisely: G_1 and G_2 are isomorphic if and only if there exists an algebra isomorphism T from $L^1(G_1)$ onto $L^1(G_2)$ such that $f\geqq 0$ (a.e.) if and only if $Tf\geqq 0$ (a.e.) for all $f\in L^1(G_1)$ [6, Theorem 1]. The purpose of this section is to establish an analogue of Kawada's theorem for the Fourier-Stieltjes algebra. This amounts to replacing, in Walter's theorem, the assumption that the algebra isomorphism T is isometric by the assumption that T is an order isomorphism for the positive definite ordering. Actually, we find a device (Lemma 3.1) which also allows the deduction of a "Kawada-type-theorem" from a "Wendel-type-theorem" in other situations (here we make allusion to Wendel's classical theorem [10, Theorem 1] characterizing isometric isomorphisms of group algebras).

Let $\mathfrak{A}$ be a von Neumann algebra and R its predual. By $\mathfrak{A}_+$ we denote the usual positive cone (of all hermitian elements with positive spectrum) in $\mathfrak{A}$. We denote by $R_+\subset R$ the cone of all positive, normal functionals on $\mathfrak{A}$. Then $\mathfrak{A}_+$ is the dual cone of R_+, i.e.

$$\mathfrak{A}_+ = \{x \in \mathfrak{A} | \langle f, x \rangle \geqq 0 \quad \text{for all} \quad f \in R_+\}.$$

Let 1 be the algebra unit in $\mathfrak{A}$. Then

$$\|f\| = \langle f, 1 \rangle \quad \text{for all} \quad f \in R_+. \tag{3.1}$$

Assume now in addition, that R is a Banach algebra. We will consider the condition that 1 is a multiplicative linear functional on R, that is

$$\langle f \cdot g, 1 \rangle = \langle f, 1 \rangle \langle g, 1 \rangle \quad (f, g \in R). \tag{3.2}$$

Since $\operatorname{span} R_+ = R$, (3.2) is equivalent to

$$\|f \cdot g\| = \|f\| \cdot \|g\| \quad \text{for all} \quad f, g \in R_+. \tag{3.3}$$

Lemma 3.1. *For $i = 1, 2$, let R_i be a Banach algebra and the predual of a von Neumann algebra $\mathfrak{A}_i$ such that the unit element, 1, of $\mathfrak{A}_i$ is a multiplicative linear functional on R_i. Let $T: R_1 \to R_2$ be an algebra isomorphism such that*

$$T(R_1)_+ = (R_2)_+.$$

Denote the adjoint of T by T'. Then $T'1 = 1$ and T is an isometry.

Proof. First of all, T is continuous as a consequence of the positivity condition (see, e.g., the proof of [1, 3.1]). For $i = 1, 2$ let M_i denote the set of all multiplicative linear functionals on R_i. If $x \in M_i$, then x is bounded and $\|x\| \leqq 1$ [5, VIII, 2.8 Lemma]. Hence $x \leqq 1$ for every $x \in M_i \cap (\mathfrak{A}_i)_+$. By our assumption, it follows that $1 = \max\{x | x \in (\mathfrak{A}_i)_+ \cap M_i\}$. Since T is an algebra isomorphism, we have $T'M_2 = M_1$; and since $T'(\mathfrak{A}_2)_+ = (\mathfrak{A}_1)_+$, it follows that $T'((\mathfrak{A}_2)_+ \cap M_2) = (\mathfrak{A}_1)_+ \cap M_1$
. Consequently,

$$\begin{aligned} T'1 &= T'(\max\{x | x \in (\mathfrak{A}_2)_+ \cap M_2\}) = \max\{T'x | x \in (\mathfrak{A}_2)_+ \cap M_2\} \\ &= \max\{x | x \in (\mathfrak{A}_1)_+ \cap M_1\} = 1. \end{aligned}$$

It follows from [7, Corollary 1] that T' is a contraction. Similarly, $(T')^{-1}$ is contractive. Thus T' is an isometry, and consequently T is an isometry, as well.

Theorem 3.2. *Let $T: B(G_1) \to B(G_2)$ be an algebra isomorphism and an order isomorphism for the positive definite ordering. Then there exists a topological group isomorphism or antiisomorphism $\alpha: G_2 \to G_1$ such that*

$$Tf = f \circ \alpha \quad \textit{for all} \quad f \in B(G_1).$$

Proof. $R = B(G)$ is a Banach algebra which satisfies the conditions considered in Lemma 3.1. In fact, $B(G)$ is the dual of $C^*(G)$. Hence its dual is a von Neumann algebra because it is the bidual of a C^*-algebra. Moreover, $R_+ = P(G)$ and for $u \in P(G)$, $\langle u, 1 \rangle = \|u\| = u(e)$. Hence 1 is a multiplicative functional on $B(G)$. So we can apply 3.1 to conclude that T is an isometry. It follows from Walter's theorem [9, Theorem 2] that there exists a topological group isomorphism or antiisomorphism $\alpha: G_2 \to G_1$ and $s \in G_1$ such that $Tf(t) = f(s\alpha(t))$. In particular, $f(e) = \langle f, 1 \rangle = \langle f, T'1 \rangle = f(s)$ for all $f \in B(G_1)$. This implies that $s = e$, so that T has the desired form.

In the same way, one can obtain Kawada's theorem [6, Theorem 1] from Wendel's theorem [10, Theorem 1] and also our earlier result [1, Theorem 3.2] (a Kawada type theorem for $A(G)$) from [9, Theorem 3] (a Wendel type theorem for $A(G)$) with the help of Lemma 3.1.

The situation for $M(G)$ is similar. We describe it explicitly: $M(G)$ denotes the Banach space of all bounded regular complex Borel measures on G. With convolution as multiplication, $M(G)$ is a Banach algebra. For $\mu \in M(G)$, we write $\mu \geqq 0$ if μ is a positive measure in the usual sense. $M(G)$ is the predual of a von Neumann algebra and (3.3) is satisfied. So Lemma 3.1 allows to deduce the following result from Johnson's theorem [4, Theorem 3].

Theorem 3.3. *Let $T: M(G_1) \to M(G_2)$ be an algebra isomorphism such that*

$$\mu \geqq 0 \quad \textit{if and only if} \quad T\mu \geqq 0$$

($\mu \in M(G_1)$). *Then there exists a topological group isomorphism α: $G_2 \to G_1$ such that T has the form*

$$T\mu(K) = \mu(\alpha(K))$$

for all $\mu \in M(G_1)$ and all compact subsets K of G_2.

Remark. It is possible to prove 3.3 directly via Kawada's theorem: A subspace J of $M(G)$ is called an *order ideal* if $|\nu| \leqq |\mu|$, where $\mu \in J$, implies that $\nu \in J$. Now $L^1(G)$ is the smallest element (for inclusion) in the set of all closed, non-zero order and algebra ideals of $M(G)$. [Indeed, let J be a closed, non-zero order and algebra ideal in $M(G)$. Suppose, there exists $t_0 \in G$ such that $f(t_0) = 0$ for all $f \in J_0 := J \cap L^1(G) \cap C^b(G)$. Then $f(e) = (\delta_{t_0} * f)(t_0) = 0$ for all $f \in J_0$. Hence if $\mu \in J$, then $\langle f, \mu \rangle = \check{f} * \mu(e) = 0$ for all continuous functions f on G with compact support (where $\check{f}(t) = f(t^{-1})$ (for $t \in G$)), i.e. $\mu = 0$, a contradiction. Thus for every $t \in G$ there exists $f \in J_0$ such that $f(t) \neq 0$. Since J is a closed order ideal, it follows that $L^1(G) \subset J$.] Thus, if $T: M(G_1) \to M(G_2)$ is an order and algebra isomorphism, then $TL^1(G_1) = L^1(G_2)$ and we can apply Kawada's theorem to conclude that G_1 and G_2 are isomorphic. Moreover, T is uniquely determined by its restriction to $L^1(G_1)$. So the precise form of T given in 3.3 can be obtained once it is established for the restriction. This is done through Wendel's modification [10, Theorem 2] of Kawada's result.

4. Multipliers

By $M(B(G))$ we denote the space of all multipliers of $B(G)$, i.e. (since $B(G)$ has a unit)

$$M(B(G)) = \{M_h | h \in B(G)\}$$

where $M_h: B(G) \to B(G)$ is defined by $M_h f = h \cdot f$ $(f \in B(G))$.

Lemma 4.1. *Let $M: B(G) \to B(G)$ be a linear mapping. Then M is a multiplier if and only if for every $f \in B(G)$ and $t \in G$, $f(t) = 0$ implies $Mf(t) = 0$.*

Proof. The condition is clearly necessary. To prove sufficiency assume that $f(t)=0$ implies $Mf(t)=0$ $(f\in B(G), t\in G)$. Let $h=M1$ and $g\in B(G)$. We show that $Mg=h\cdot g$. Let $t\in G$. Then $(g-g(t)1)(t)=0$. Hence by our assumption,

$$Mg(t)-g(t)h(t)=M(g-g(t)1)(t)=0.$$

That is,

$$Mg(t)=h(t)g(t) \quad \text{for every} \quad t\in G.$$

We give now an order theoretical characterization of multipliers which are positive for the pointwise ordering.

Proposition 4.2. *Let $M: B(G))\to B(G)$ be a linear mapping satisfying $MB(G)_+ \subset B(G)_+$. Then M is a multiplier if and only if there exists $c\geqq 0$ such that*

$$Mf\leqq cf \quad \text{for all} \quad f\in B(G)_+. \tag{4.1}$$

Proof. If $M=M_h$ for some $h\in B(G)$, then $h=M1\in B(G)_+$. Since $h\leqq \|h\|_\infty 1$, it follows that $Mf=hf\leqq \|h\|_\infty f$ for all $f\in B(G)_+$. This establishes one implication. To prove the other one we assume that (4.1) is valid. Let $g\in B(G), t\in G$ such that $g(t)=0$. In order to apply 4.1 we have to prove that $Mg(t)=0$. Consider the scalar product

$$[k,h]=M(k\cdot \bar{h})(t) \quad (k,h\in B(G)).$$

Applying the Cauchy-Schwarz inequality to $k=g$ and $h=1$ we obtain:

$$|Mg(t)|^2=|[g,1]|^2\leqq [g,g][1,1]=M(|g|^2)(t)[1,1]. \tag{4.2}$$

Since $g(t)=0$, it follows from (4.1) that $M(|g|^2)(t)\leqq c|g|^2(t)=c|g(t)|^2=0$, and so by (4.2) we conclude that $Mg(t)=0$.

Next, we characterize isometric multipliers and multipliers which are order isomorphisms for the positive definite ordering.

Lemma 4.3. *Let $h\in B(G)$. If $\|h\|=|h(t)|\neq 0$ for all $t\in G$, then $h/h(e)$ is an character (i.e. a homomorphism from G into $\mathbb{C}$).*

Proof. By [3, (2.14)] there exist a unitary representation $\pi: G\to \mathscr{L}(H)$ in some Hilbert space H and $\xi,\eta\in H$ such that $h(t)=(\pi(t)\xi,\eta)$ and $\|h\|=\|\xi\|\cdot\|\eta\|$. Hence

$$\|h\|=|h(t)|=|(\pi(t)\xi|\eta)|\leqq \|\xi\|\cdot\|\eta\|=\|h\| \quad (t\in G).$$

Thus the equality in the Cauchy-Schwarz inequality holds, and so there exists $\lambda(t)\in\mathbb{C}$ such that $\pi(t)\xi=\lambda(t)\eta$. In particular, $\xi=\lambda(e)\eta$ so that $\pi(t)\xi=(\lambda(t)/\lambda(e))\xi$. Hence $\chi=\dfrac{\lambda}{\lambda(e)}$ is a character and $h=c\cdot\chi$, where $c=(\xi|\eta)=h(e)$.

Proposition 4.4. *Let $h\in B(G)$. M_h is isometric if and only if $h=c\cdot\chi$, where χ is a continuous character on G and $c\in\mathbb{C}$, $|c|=1$.*

Proof. If χ is a continuous character, then $\chi\in P(G)$. Hence $\|\chi\|=\chi(e)=1$. Similarly, $\|\chi^{-1}\|=1$. Thus $\|M_\chi\|=\|(M_\chi)^{-1}\|=1$; i.e., M_χ is an isometry. Conversely, suppose that $M=M_h$ is an isometry, where $h\in B(G)$. Then $\|h\|=\|M1\|=\|1\|=1$. For $t\in G$, let $\delta_t\in B(G)'$ be defined by $\langle u,\delta_t\rangle=u(t)$ $(u\in B(G))$. Then $\|\delta_t\|=1$. Hence

$|h(t)| = \|h(t)\delta_t\| = \|M'\delta_t\| = \|\delta_t\| = 1$ $(t \in G)$ (here M' denotes the adjoint of M). It follows from 4.3 that $h/h(e)$ is a character. Hence h has the desired form.

Proposition 4.5. *Let $h \in B(G)$. M_h is an order isomorphism for the positive definite ordering if and only if*

$$h = c \cdot \chi, \text{ where } \chi \text{ is a continuous character and } c > 0. \tag{4.3}$$

Proof. Suppose that M_h is an order isomorphism for the positive definite ordering. Then $h = M_h 1 \in P(G)$ and $\frac{1}{h} = M_h^{-1} 1 \in P(G)$. Let $\chi = h/h(e)$. Since $\chi \in P(G)$, we have $|\chi(t)| \leqq \chi(e) = \|\chi\| = 1$. Moreover, $1/\chi \in P(G)$ so that $|1/\chi(t)| \leqq 1/\chi(e) = 1$ for all $t \in G$. Consequently, $|\chi(t)| = 1 = \|\chi\|$ for all $t \in G$. It follows from 4.3 that χ is a character. Hence $h = h(e)\chi$ has the form (4.3). The converse implication is obvious.

5. Order Isomorphisms for Fourier-Stieltjes Algebras

Proposition 5.1. *Let $V: B(G_1) \to B(G_2)$ be a $*$-algebra isomorphism (that is, V is a bijective linear mapping such that $\overline{Vf} = V\bar{f}$ and $V(f \cdot g) = Vf \cdot Vg$ for all $f, g \in B(G)$). Then V is an order isomorphism for the pointwise ordering.*

Proof. Denote by $A(G_i)$ the Fourier algebra of G_i $(i = 1, 2)$ (see [3] for the definition). We show first that $Vf \geqq 0$ whenever $f \in A(G_1)$ and $f \geqq 0$. In fact, for a given $t \in G_2$, the map $f \to Vf(t)$ is a multiplicative linear functional on $A(G_1)$. Let Y be the set of all $t \in G_2$ such that this functional does not vanish identically on $A(G_1)$. Then by [3, (3.34)], for all $t \in Y$ there exists $\alpha(t) \in G_1$ such that $Vf(t) = f(\alpha(t))$. Since $Vf(t) = 0$ for $t \notin Y$, it follows that $Vf \geqq 0$ whenever $f \in A(G_1)$ and $f \geqq 0$. Replacing V with V^{-1} we also have

$$V^{-1} g \geqq 0 \quad \text{for} \quad g \in A(G_2), \quad g \geqq 0. \tag{5.1}$$

We show now that V is positive for the pointwise ordering. If this is not true, then there exists $f \in B(G_1)_+$ such that $Vf(t) < 0$ for some $t \in G_2$ (observe that $Vf\,(= V\bar{f} = \overline{Vf})$ is real). Choose $h \in A(G_2)$, $h \geqq 0$, such that $h(t) \neq 0$ and h vanishes outside $\{s \in G_2 | Vf(s) < 0\}$ (use [3, (3.2)]). Then $0 \neq Vf \cdot h \leqq 0$. Since $A(G_2)$ is an ideal in $B(G_2)$, $Vf \cdot h \in A(G_2)$. So it follows from (5.1) that $0 \neq f \cdot V^{-1}h = V^{-1}(Vf \cdot h) \leqq 0$. Since $V^{-1}h \geqq 0$ (again by (5.1)), this contradicts $f \geqq 0$. Replacing V with V^{-1} we obtain that V^{-1}, too, is positive for the pointwise ordering; so V is an order isomorphism for the pointwise ordering.

We can now characterize order isomorphisms for the pointwise ordering.

Theorem 5.2. *Let $T: B(G_1) \to B(G_2)$ be a linear mapping. T is an order isomorphism for the pointwise ordering if and only if T has the form*

$$Tf = h \cdot Vf \quad (f \in B(G_1)), \tag{5.2}$$

where $V: B(G_1) \to B(G_2)$ is a $*$-algebra isomorphism, $h \in B(G_2)_+$ and h is invertible in $B(G_2)$ (that is, $h(t) > 0$ for all $t \in G_2$ and $\frac{1}{h} \in B(G_2)$).

Remark. Order isomorphisms for the pointwise ordering are continuous. In fact, if $S: B(G_1) \to B(G_2)$ is linear and positive, then S is continuous for the supremum norm (since $-1 \leqq u \leqq 1$ implies $-S1 \leqq Su \leqq S1$). So it follows from the closed graph theorem that S is continuous.

Proof. It is clear from 5.1 that every operator of the form (5.2) is an order isomorphism for the pointwise ordering. To prove the converse assume that $T: B(G_1) \to B(G_2)$ is an order isomorphism for the pointwise ordering. Let $f \in B(G_1)_+$. $M = TM_f T^{-1}$ is a linear mapping from $B(G_2)$ into $B(G_2)$. We show that M is a multiplier. In fact, M is positive for the pointwise ordering. Moreover, since $0 \leqq f \leqq \|f\|_\infty 1$, it follows that $f \cdot T^{-1} g \leqq \|f\|_\infty T^{-1} g$, and consequently $Mg = T(f \cdot T^{-1} g) \leqq \|f\|_\infty g$ for all $g \in B(G_2)_+$. It follows from 4.2 that $M \in M(B(G_2))$. Since span $B(G_1)_+ = B(G_1)$ we conclude that

$$TM_f T^{-1} \in M(B(G_2)) \quad \text{for every} \quad f \in B(G_1). \tag{5.3}$$

So there exists a mapping $V: B(G_1) \to B(G_2)$ such that

$$TM_f T^{-1} = M_{Vf} \quad \text{for all} \quad f \in B(G_1), \tag{5.4}$$

with the property that $Vf \geqq 0$ if $f \geqq 0$.

From (5.4) follows that V is a $*$-algebra isomorphism. In fact, for $g, h \in B(G_1)$,

$$M_{V(f \cdot g)} = TM_{f \cdot g} T^{-1} = TM_f M_g T^{-1} = TM_f T^{-1} TM_g T^{-1} = M_{Vf} \cdot M_{Vg} = M_{(Vf)(Vg)}.$$

This implies that $Vf \cdot Vg = V(f \cdot g)$. Similarly, $V(\alpha f + \beta g) = \alpha Vf + \beta Vg$, $\overline{Vf} = V\bar{f}$ $(\alpha, \beta \in \mathbb{C})$; and it is easy to see by exchanging T and T^{-1} that V is bijective.

Finally, let $h = T1$. We show that $Tf = h \cdot Vf$ for all $f \in B(G_1)$. Indeed, $Tf = TM_f 1 = TM_f T^{-1} T1 = M_{Vf} h = h \cdot Vf$ (by (5.4)). Of course, $h = T1 \geqq 0$. Let $g = T^{-1} 1$. Then $h \cdot Vg = Tg = 1$. Hence h is invertible in $B(G_2)$ and $\frac{1}{h} = Vg \in B(G_2)$.

Remark. The above theorem is the analogue of [1, Proposition 4.2], where it is proved that if $T: A(G_1) \to A(G_2)$ is a continuous order isomorphism for the pointwise ordering, then there exists a bijection $\alpha: G_2 \to G_1$ and a function $h: G_2 \to (0, \infty)$ such that $Tf = h \cdot f \circ \alpha$ for all $f \in A(G_1)$ (it is easy to show that α is actually a homeomorphism and h is continuous). To see the analogy, one has to observe that $f \circ \alpha \in M(A(G_2))$ (the algebra of multipliers of $A(G_2)$, considered as functions on G_2) whenever $f \in A(G_1)$. Indeed, if $g \in A(G_2)$, there exists $g_1 \in A(G_1)$ such that $g = Tg_1$, and $(f \circ \alpha) \cdot g = T(f \cdot g_1) \in A(G_2)$. Consequently the conclusion of [1, Proposition 4.2] can be restated so as to assert the existence of an injective $*$-homomorphism $V: A(G_1) \to M(A(G_2))$, $f \to f \circ \alpha$, and a function h as above such that $Tf = h \cdot Vf$ for all $f \in A(G_1)$. In this form, Proposition 4.2 of [1] can be proved along the same lines as Theorem 5.2 above.

If moreover G_1 and G_2 are amenable, V actually maps $A(G_1)$ onto $A(G_2)$, and $h \in B(G_2)$.

The following is the main theorem.

Theorem 5.3. *Let $T: B(G_1) \to B(G_2)$ be a biorder isomorphism. Then there exists a topological group isomorphism or antiisomorphism $\alpha: G_2 \to G_1$ and a constant $c > 0$*

such that

$$Tf = c \cdot f \circ \alpha$$

for all $f \in B(G_1)$.

Proof. By 5.2, $T = M_h \cdot V$ where V is a $*$-algebra isomorphism and $h \in B(G_2)_+$ is invertible in $B(G_2)$. Moreover, $M_{Vg} = TM_gT^{-1}$ for all $g \in B(G_1)$ (by (5.4)). Let $g \in P(G_1)$. Then M_g and consequently $M_{Vg} = TM_gT^{-1}$ are positive linear mappings for the positive definite ordering. This implies that $Vg = M_{Vg}1 \in P(G_2)$. Hence V is positive for the positive definite ordering. The same argument applied to V^{-1} shows that V is an order isomorphism for the positive definite ordering. It follows from Theorem 3.2 that there exists a topological group isomorphism or anti-isomorphism $\alpha: G_2 \to G_1$ such that $Vf = f \circ \alpha$ for all $f \in B(G_1)$. Moreover, $M_h = TV^{-1}$ is an order isomorphism for the positive definite ordering. So $h/h(e)$ is a character by 4.5. Since $h \geqq 0$, it follows that $h(t)/h(e) = 1$ for all $t \in G_2$. Thus, with $c = h(e)$, $Tf = c \cdot f \circ \alpha$ for all $f \in B(G_1)$.

Theorem 5.4. *Let* $T: B(G_1) \to B(G_2)$ *be an isometric order isomorphism for the pointwise ordering. Then there exists a topological group isomorphism or anti-isomorphism* $\alpha: G_2 \to G_1$, *and* $s \in G_1$, *such that*

$$Tf(t) = f(s \cdot \alpha(t)) \; (t \in G_2) \quad \textit{for all} \quad f \in B(G_1).$$

Proof. By 5.2 there exists a $*$-algebra isomorphism $V: B(G_1) \to B(G_2)$ and $h \in B(G_2)$ such that $T = M_hV$. Moreover, by (5.4), $TM_fT^{-1} = M_{Vf}$ for all $f \in B(G_1)$. Hence, $\|Vf\| = \|M_{Vf}\| = \|TM_fT^{-1}\| = \|M_f\| = \|f\|$ for all $f \in B(G_1)$; i.e. V is an isometry. By Walter's theorem [9, Theorem 2], there exists a topological group isomorphism or antiisomorphism $\alpha: G_2 \to G_1$ and $s \in G$, such that $Vf(t) = f(s \cdot \alpha(t))$ $(t \in G_2)$ for all $f \in B(G_1)$. Since V and T^{-1} are isometries, $M_h = TV^{-1}$ is also isometric. It follows from 4.4 that $h = c \cdot \chi$, where χ is a character and $c \in \mathbb{C}$, $|c| = 1$. Moreover, because of 5.1 V^{-1} is an order isomorphism for the pointwise order; and consequently, so is M_h. This implies that $h = 1$ so that $T = V$.

Taken together with the Theorems 3.2 and 5.3, the preceding result allows us to formulate the following conclusion, which summarizes the remarkable interplay between the algebraic, metric, and order structure of the Fourier-Stieltjes algebra: In Walter's theorem one can replace at will the multiplicative structure of $B(G)$ by the pointwise ordering, and/or its norm by the positive definite ordering.

6. Order Isomorphisms of Group Algebras

The space $L^1(G)$ is defined as usual with respect to a fixed left Haar measure. $L^1(G)$ is an involutive Banach algebra with convolution as multiplication and the involution * defined by

$$f^*(t) = \Delta(t)^{-1} \overline{f(t^{-1})} \qquad (f \in L^1(G)),$$

where Δ denotes the modular function of G. Again, there exist two orderings on $L^1(G)$. The *positive definite ordering* is defined by its involutive Banach algebra structure; i.e. by the closed convex cone $L^1(G)_p = \overline{co}\{g^* * g \mid g \in L^1(G)\}$.

By [2, 13.4.5] the dual cone of $L^1(G)_p$ in $L^\infty(G)$ is $P(G)$; that is

$$P(G) = \{p \in L^\infty(G) | \langle f, p\rangle \geqq 0 \quad \text{for all} \quad f \in L^1(G)_p\}. \tag{6.1}$$

Remark 6.1. If G is abelian, then $L^1(G)_p = \{f \in L^1(G) | \hat{f}(\gamma) \geqq 0$ for all $\gamma \in \hat{G}\}$, where $\hat{G}$ denotes the dual group of G.

The *pointwise ordering* on $L^1(G)$ is defined by the positive cone $L^1(G)_+ = \{f \in L^1(G) | f(t) \geqq 0$ for almost every $t \in G\}$.

Corresponding to these two orderings we have again the notion of order isomorphisms: Let $T: L^1(G_1) \to L^1(G_2)$ be a bijective linear mapping. T is an *order isomorphism for the positive definite ordering* (respectively *for the pointwise ordering*) if $TL^1(G_1)_p = L^1(G_2)_p$ (respectively $TL^1(G_1)_+ = L^1(G_2)_+$). T is called a *biorder isomorphism* if T is an order isomorphism for both orderings.

Let $\alpha: G_2 \to G_1$ be a topological group isomorphism. Because of the uniqueness of the Haar measure, for every measurable function f on G_1, $f \in L^1(G_1)$ if and only if $f \circ \alpha \in L^1(G_2)$ and there exists a unique constant $c(\alpha) > 0$ such that

$$c(\alpha) \int_{G_2} f(\alpha(t))dt = \int_{G_1} f(s)ds \quad \text{for all} \quad f \in L^1(G_1). \tag{6.2}$$

From this it is easy to see that the mapping

$$V_\alpha: L^1(G_1) \to L^1(G_2), \; V_\alpha f = c(\alpha) f \circ \alpha \tag{6.3}$$

is a $*$-algebra isomorphism. Hence V_α is an order isomorphism for the positive definite ordering (this is immediate from the definition of this ordering). Moreover, V_α is obviously an order isomorphism for the pointwise ordering. Thus V_α is a biorder isomorphism.

The mapping $W: L^1(G_2) \to L^1(G_2)$ defined by $Wf = \Delta_2^{-1} \check{f}$ $(f \in L^1(G_2))$, where $\check{f}(t) = f(t^{-1})$ and Δ_2 denotes the modular function of G_2, is a biorder isomorphism, too. Indeed, it is obviously an order isomorphism for the pointwise ordering. Moreover, the adjoint $W': L^\infty(G_2) \to L^\infty(G_2)$ of W is given by $W'f = \check{f}$. Hence $W'P(G_2) = P(G_2)$. It follows from (6.1) that W is an order isomorphism for the positive definite ordering. Thus,

$$T = WV_\alpha \text{ is a biorder isomorphism}. \tag{6.4}$$

The following theorem shows that up to a positive constant every biorder isomorphism has the form (6.3) or (6.4).

Theorem 6.2. *Let $T: L^1(G_1) \to L^1(G_2)$ be a biorder isomorphism. Then there exists a topological group isomorphism $\alpha: G_2 \to G_1$ and a constant $c > 0$ such that*

$$Tf(t) = c \cdot f(\alpha(t)) \quad (t \in G_2, f \in L^1(G_1)) \tag{6.5}$$

or

$$Tf(t) = c \cdot \Delta_2(t)^{-1} f(\alpha(t^{-1})) \quad (t \in G_2, f \in L^1(G_1)). \tag{6.6}$$

Note. Concerning the formulation of (6.6), one should observe that $\alpha: G_2 \to G_1$ is a topological group isomorphism if and only if $\beta: G_2 \to G_1$ defined by $\beta(t) = \alpha(t^{-1})$ is a topological group antiisomorphism.

Proof. By [8, II, 5.3], T is continuous. Denote by $T': L^\infty(G_2) \to L^\infty(G_1)$ the adjoint of T. Since T is an order isomorphism for the positive definite ordering, it follows from (6.1) that $T'P(G_2) = P(G_1)$. Hence $T'B(G_2) = B(G_1)$ and so the restriction S of T' to $B(G_2)$ is an order isomorphism for the positive definite ordering from $B(G_2)$ onto $B(G_1)$. Since T is an order isomorphism for the pointwise ordering, S is so, too. By Theorem 5.2, there exists a topological group isomorphism $\beta: G_1 \to G_2$ and a constant $c_1 > 0$ such that S is given by

$$Su(s) = c_1 u(\beta(s)) \quad (s \in G_1, u \in B(G_2)) \tag{6.7}$$

or

$$Su(s) = c_1 u(\beta(s^{-1})) \quad (s \in G_1, u \in B(G_2)). \tag{6.8}$$

If (6.7) is valid, then by (6.2),

$$\begin{aligned}\langle Tf, u\rangle &= \langle f, Su\rangle = c_1 \int_{G_1} f(s)u(\beta(s))ds \\ &= c_1 \cdot \frac{1}{c(\beta)} \int_{G_2} f(\beta^{-1}(t))u(t)dt\end{aligned}$$

for all $f \in L^1(G_1)$, $u \in B(G_2)$. Since $B(G_2)$ separates $L^1(G_2)$, it follows that $Tf = (c_1/c(\beta)) \cdot f \circ \beta^{-1}$ for all $f \in L^1(G_1)$. Hence (6.5) is valid for $c = c_1/c(\beta)$ and $\alpha = \beta^{-1}$.

If (6.8) holds, then by (6.2) again

$$\begin{aligned}\langle Tf, u\rangle &= c_1 \int_{G_1} f(s)u(\beta(s^{-1}))ds = c_1 \int_{G_1} f(s)\check{u}(\beta(s))ds \\ &= c_1 \cdot \frac{1}{c(\beta)} \int_{G_2} f(\beta^{-1}(t))\check{u}(t)dt \\ &= \frac{c_1}{c(\beta)} \int_{G_2} \Delta_2(t)^{-1} f(\beta^{-1}(t^{-1}))u(t)dt\end{aligned}$$

for all $f \in L^1(G_1)$, $u \in B(G_2)$. Hence $Tf(t) = (c_1/c(\beta))\Delta_2(t)^{-1} f(\beta^{-1}(t^{-1}))$ for all $f \in L^1(G_1)$, and almost all $t \in G_2$. Thus T has the form (6.6) with $c = c_1/c(\beta)$ and $\alpha = \beta^{-1}$.

Remark 6.3. Theorem 6.2 implies that the "biordered space" ($L^1(G)$, $L^1(G)_+$, $L^1(G)_p$) is a complete invariant for the locally compact group G.

Remark 6.4. For abelian groups, Theorem 6.2 has been proved in [1] (in fact, by Remark 6.1, Theorem 6.2 is another formulation of [1, 4.3]).

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