Math. Z. 185, 567–571 (1984)

Mathematische Zeitschrift

Factorization by Lattice Homomorphisms

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The purpose of the present note is to prove the following: Let V be a lattice homomorphism from an order complete vector lattice E into E. Then given a positive linear mapping S on E every positive linear mapping T dominated by $S \circ V$ has a factorization $T = S_1 \circ V$ where S_1 is a positive linear mapping on E dominated by S.

In the special case where S is the identity mapping this can be considered a Radon-Nikodym type theorem and has been proved by Luxemburg and Schep [12, 4.2] (see also [11, 4.1] and [6, Satz 9]). Here we don't use the theory of components (as in [12]) but give an elementary proof via the vector lattice valued Hahn Banach theorem.

Two different applications of our result are given. The first concerns injective Banach lattices. We show that the injective objects in the category of all Banach lattices are the same no matter whether the positive contractions or the regular operators with contractive modulus are considered as morphisms. As another easy consequence we obtain that the o-spectrum (see [3, 14]) of a lattice homomorphism is cyclic. Our notations follow Schaefer's monograph [13].

1. Factorization by Lattice Homomorphisms

Let E, F, G be vector lattices, E order complete, and $V\colon F \to G$ be a lattice homomorphism.

Theorem 1.1. *Given a positive linear mapping $S\colon G \to E$, every positive linear $T\colon F \to E$ which satisfies $T \leqq S \circ V$ admits a factorization*

$$T = S_1 \circ V$$

where $S_1\colon G \to E$ is linear and $0 \leqq S_1 \leqq S$.

Proof. Let $G_1 = VF$. G_1 is a sublattice of G. Define $S_0\colon G_1 \to E$ by $S_0 Vx = Tx$ $(x \in F)$. S_0 is well defined. In fact, let $Vx = 0$. Then $|Tx| \leqq T|x| \leqq SV|x| = S|Vx|$

$=0$, and so $Tx=0$. Consequently, if $Vx_1=Vx_2$, then $T(x_1-x_2)=0$, hence $S_0 Vx_1=S_0 Vx_2$. S_0 is obviously a linear mapping.

Let $p: G\to E$ be defined by $p(y)=Sy^+$. p is sublinear. In fact, let $y_1, y_2\in G$. Then $(y_1+y_2)^+\leqq y_1^+ + y_2^+$. Hence $p(y_1+y_2)=S(y_1+y_2)^+\leqq Sy_1^+ + Sy_2^+ = p(y_1) + p(y_2)$. Moreover, for $y\in G$, $\lambda\in\mathbb{R}$, $\lambda\geqq 0$, $p(\lambda y)=S(\lambda y)^+=\lambda S y^+=\lambda p(y)$. We show next that

$$S_0 y\leqq p(y) \quad (y\in G_1). \tag{1.1}$$

Let $y=Vx$ $(x\in F)$. Then $S_0 y=Tx\leqq Tx^+\leqq SVx^+=S(Vx)^+=Sy^+=p(y)$. By the vector lattice valued Hahn Banach theorem [13, II 7.9] S_0 has a linear extension $S_1: G\to E$ which satisfies

$$S_1 y\leqq p(y) \quad (y\in G). \tag{1.2}$$

S_1 has the desired properties. In fact, $S_1 Vx=S_0 Vx=Tx$ $(x\in F)$ by the definition of S_0. Moreover, S_1 is positive. In fact, let $y\in G_+$. Then $-S_1 y=S_1(-y)\leqq p(-y)=S(-y)^+=0$. Hence $S_1 y\geqq 0$. Finally, for $y\in G_+$, $S_1 y\leqq p(y)=Sy$ (by (1.2)). Hence $S_1\leqq S$.

A positive linear mapping $R: G\to F$ is called *interval preserving* if R maps order intervals onto order intervals; i.e. if $R[x,y]=[Rx,Ry]$ for all $x,y\in G$, $x\leqq y$.

A linear mapping $T: F\to E$ is called *regular* if T is difference of positive linear mappings. The space $\mathscr{L}^r(F,E)$ of all regular linear mappings from F into E is an order complete vector lattice [13, IV, §1] (recall that E is assumed to be order complete).

Corollary 1.2. *The right composition operator $R_V: \mathscr{L}^r(G,E)\to\mathscr{L}^r(F,E)$ defined by $R_V S=S\circ V$ is interval preserving.*

Remark. a) If $E=\mathbb{R}$, then R_V is the adjoint of V. In that case 1.2 has been proved by Lotz [10, 1.2].

b) Other relations between order properties of a positive linear mapping and the associated composition operator R_V have been investigated recently by Aliprantis, Burkinshaw and Kranz [1].

Corollary 1.3. *The space $\mathscr{L}^r(G,E)\circ V := \{S\circ V; S\in\mathscr{L}^r(G,E)\}$ is a (lattice-)ideal in $\mathscr{L}^r(F,E)$.*

We proceed with a dual version of 1.1. For the sake of simplicity we assume that E, F, and G are Banach lattices.

Theorem 1.4. *Assume that G has order-continuous norm. Let $U: G\to F$ be an interval preserving positive operator. Then, given a positive operator $S: E\to G$, every positive operator $T: E\to F$ which satisfies $T\leqq U\circ S$ admits a factorization*

$$T=U\circ S_1$$

where $S_1: E\to G$ is a positive operator such that $S_1\leqq S$.

Proof. $T\leqq U\circ S$ implies that $T'\leqq S'\circ U'$. Since U' is a lattice homomorphism by [10, 1.2], it follows from 1.1 that there exists a linear mapping $R: G'\to E'$ such that $0 \leqq R \leqq S'$ and $T' = R \circ U'$. Consequently, $T'' = U'' \circ R'$ and $0 \leqq R' \leqq S''$. Since G is an ideal in its bidual [13, II 5.10] and $S''E = SE \subseteq G$, the last inequality implies that $R'E \subset G$. Define S_1 as the restriction of R' to E. Then $0 \leqq S_1 \leqq S$ and $T = U \circ S_1$.

Question. Can the assumption that G has order continuous norm be omitted? A partial answer is known: It can if $G = E$ and S is the identity operator and V is order continuous [12, 3.1].

2. Injective Banach Lattices

Injective Banach lattices were introduced and investigated by Lotz [10]; Cartwright [5] and Haydon [8] obtained characterizations.

Definition 2.1. A Banach lattice E is called *injective*, if for every isometric lattice homomorphism $V: F \to G$ (F, G Banach lattices) and for every positive operator $T: F \to E$ there exists a positive operator $\hat{T}: G \to E$ such that $T = \hat{T} \circ V$ and $\|\hat{T}\| = \|T\|$.

Using the language of categories, injective Banach lattices are the injective objects in the category of all Banach lattices with the positive contractions as morphisms. As a consequence of the factorization Theorem 1.1 we are going to show that the injective objects are the same if the regular operators with contractive modulus are taken as morphisms instead of the positive contractions. This result was mentioned by Schaefer [15]. It can be used to give characterizations of Banach lattices with injective dual space via the projective tensor product for Banach lattices [15, 4.1].

Theorem 2.2 *Let E be an order complete Banach lattice. The following assertions are equivalent:*

(i) *E is injective.*

(ii) *For every isometric lattice homomorphism $V: F \to G$ (F, G Banach lattices) and every regular operator $T: F \to E$ there exists a regular operator $\hat{T}: G \to E$ such that $T = \hat{T} \circ V$ and $\|\hat{T}\|_r = \|T\|_r$.*

Recall that if E, F are Banach lattices and E is order complete, then $\mathscr{L}^r(F, E)$ is an order complete Banach lattice with respect to the r-norm defined by $\|T\|_r := \||T|\|$ $(T \in \mathscr{L}^r(F, E))$, where $|T|$ denotes the modulus of T [13, IV § 1].

Proof. For every isometric lattice homomorphism $V: F \to G$ consider the mapping $R_V: \mathscr{L}^r(G, E) \to \mathscr{L}^r(F, E)$ defined by $R_V T = T \circ V$ $(T \in \mathscr{L}^r(G, E))$. By 1.2 R_V is interval preserving.

Denote by B_1 (resp. B_2) the closed unit ball of $\mathscr{L}^r(G, E)$ (resp. $\mathscr{L}^r(F, E)$) and let $B_{1+} = \mathscr{L}^r(F, E)_+ \cap B_1$, $B_{2+} = \mathscr{L}^r(G, E)_+ \cap B_2$. Then (i) means that $R_V B_{1+} = B_{2+}$ and (ii) is equivalent to $R_V B_1 = B_2$. Using the fact that R_V is interval preserving it is not difficult to show that $R_V B_{1+} = B_{2+}$ if and only if $R_V B_1 = B_2$.

Remark 2.3. Theorem 2.2 remains valid if the assumption that E be order complete is omitted. In fact, every injective Banach lattice is order-complete since there exists a positive projection from the Dedekind completion $\bar{E}$ of E onto E. On the other hand, even if E is not assumed to be order complete (ii) makes sense if the r-norm is defined by $\|T\|_r = \inf\{\|T_1 + T_2\|;\ T = T_1 - T_2,\ T_1 \geqq 0,\ T_2 \geqq 0\}$ [12, p. 231]. But (ii) implies that there exists a regular projection P from $\bar{E}$ onto E with $\|P\|_r \leqq 1$. An unpublished argument by D.H. Fremlin shows that P is positive. So (ii) implies that E is actually order complete.

Remark 2.4. Also the non-metric version of 2.2 is valid. In fact, one can show in a similar way via 1.2 that the injective objects in the category of all Banach lattices are the same no matter whether the positive or regular operators are considered as morphisms. In particular, one sees by a suitable modification of the proof that [15, 4.3] remains valid if injectivity is interpreted in this sense. The relation between this non-metric and the metric notion (Definition 2.1) of injectivity has been investigated in [9].

3. The *o*-Spectrum of Lattice Homomorphisms

Let E be an order complete Banach lattice. Then $\mathscr{L}^r(E)$ is a Banach algebra with respect to the r-norm. The spectrum of $T \in \mathscr{L}^r(E)$ in this Banach algebra is called the *o-spectrum* of T and denoted by $\sigma_0(T)$. This notion is discussed in [2, 3], and [14].

Since $\mathscr{L}^r(E)$ is a subalgebra of $\mathscr{L}(E)$, one has $\sigma(T) \subset \sigma_0(T)$ (where $\sigma(T)$ is the spectrum of the operator T in the usual sense), but $\sigma(T)$ may differ from $\sigma_0(T)$. An example where T is a unitary operator in l^2 and $\sigma(T) \neq \sigma_0(T)$ is given in [14], and in [3] a positive, hermitian, compact operator T on $L^2[0,1]$ is found such that $\sigma_0(T) \neq \sigma(T)$. However, it is an open problem whether spectrum and o-spectrum of lattice homomorphisms are equal. As a consequence of the factorization Theorem 1.1 one can at least see that the o-spectrum of a lattice homomorphism shares nice geometric properties with its spectrum. More precisely, let V be a lattice homomorphism on E. Then it is known that $\sigma(V)$ is cyclic; that is, if $re^{i\theta} \in \sigma(V)$ ($r > 0$, $\theta \in [0, 2\pi[$), then $re^{in\theta} \in \sigma(V)$ for all $n \in \mathbb{Z}$ [13, V 4.4].

Theorem 3.1. *$\sigma_0(V)$ is cyclic.*

Proof. Consider the right multiplication operator $R_V\colon \mathscr{L}^r(E) \to \mathscr{L}^r(E)$ defined by $R_V S = S \circ V$. By 1.2 R_V is interval preserving. So by [10, 1.2], $(R_V)'$ is a lattice homomorphism. Hence $\sigma((R_V)')$ is cyclic. But $\sigma_0(V) = \sigma(R_V) = \sigma((R_V)')$ (by [4, §5, Prop. 4(ii)]).

Remark 3.2. The problem whether $\sigma(V)$ and $\sigma_0(V)$ are equal is discussed in more detail in [2] and [7] and partial answers are given. The best result is the following: If V and V' are lattice homomorphisms (in particular, if V is a lattice isomorphism), then $\sigma(V) = \sigma_0(V)$ [7, 5.19].

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Receied July 31, 1983