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KATO'S INEQUALITY: A CHARACTERISATION OF GENERATORS OF POSITIVE SEMIGROUPS

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ABSTRACT

Let A be the generator of a strongly continuous semigroup on a Banach lattice F (satisfying some mild restrictions). It is shown that the semigroup consists of positive operators if and only if A satisfies an abstract version of Kato's inequality and the adjoint A' of A possesses a strictly positive subeigenvector. Domination of semigroups is also characterised by an inequality for their generators.

Introduction

Although the theory of positive semigroups has progressed rapidly during the last few years, an intrinsic characterisation of generators of positive semigroups has not so far been given. The problem is obvious from the general theory: since the infinitesimal generator determines a semigroup uniquely, one expects to find a condition on the generator which describes the positivity of the semigroup. From a practical point of view as well there seems to be a need for such a characterisation. In fact, it lies in the very nature of the theory that frequently the generator but not the semigroup is known explicitly. Since a variety of results (concerning spectral theory, asymptotics, perturbation theory, etc.) for positive semigroups is available today, it is important to find conditions on the generator which enable one to verify positivity (of the associated semigroup).

Characterisations of positivity together with additional properties are known. Phillips [23] characterised positive *contraction* semigroups by *dispersiveness* of the generator. The more general notion of *p-dissipativity* with respect to a half-norm p was introduced in [5] and allows one to treat contractivity in a very general sense (see also Batty and Robinson [8]).

A condition of a different kind is the following abstract version of Kato's inequality:

$$\langle (\operatorname{sign} f)Af, \phi \rangle \leqslant \langle |f|, A'\phi \rangle \qquad \text{(K)}$$
$$f \in D(A),\ 0 \leqslant \phi \in D(A').$$

Of course, this inequality is inspired by Kato's classical inequality for the Laplacian ([19]; see also [24,X.27]). It was Nagel who conjectured that some abstract version of this inequality is equivalent to positivity (cf. [21]). We confirm Nagel's conjecture in the following form. Let A be the generator of a semigroup on a Banach lattice E (which for simplicity is supposed to satisfy some mild restrictions). Then the semigroup is positive if and only if A satisfies (K) and the adjoint A' of A possesses a strictly positive subeigenvector ϕ (i.e. $\phi \in D(A')$ and $A'\phi \leqslant \lambda\phi$ for some $\lambda \in \mathrm{R}$). It will be shown by examples that the two conditions in this result are independent.

A related problem is to express in terms of the generator when one semigroup is dominated by another. This can be done in a similar manner by an inequality involving the 'signum operator'. It is remarkable that here it is not necessary to start with a generator. The inequality and a range condition are sufficient to obtain a semigroup.

In the last section we investigate a special kind of domination. *Disjointness-preserving semigroups* are described as those semigroups which are dominated by a lattice semigroup. This puts a new complexion on 'Kato's equality', which is known to characterise generators of lattice semigroups by a result of Nagel and Uhlig [21].

1. The characterisation

Let E be a σ-order complete real Banach lattice [26, II.§1]. We first describe the sign operator. Let $f \in E$. There exists a unique bounded operator 'sign f' which satisfies

$$|(\mathrm{sign}\, f)g| \leqslant |g| \quad (g \in E) \tag{1.1}$$
$$(\mathrm{sign}\, f)g = 0 \quad \text{if } f \perp g \tag{1.2}$$
$$(\mathrm{sign}\, f)f = |f|\,. \tag{1.3}$$

Here we understand by $f \perp g$ that f and g are *disjoint*, i.e. $\inf \{|f|, |g|\} = 0$.

If for $u \in E_+$ the band projection onto the band $u^{\perp\perp}$ generated by u is denoted by P_u, then

$$\mathrm{sign}\ f = P_{f^+} - P_{f^-}. \tag{1.4}$$

Example. Let $E = L^p(X, \mu)$ (where (X, μ) is a measure space and $1 \leqslant p \leqslant \infty$) and $f \in E$. Let $m \in L^\infty$ be given by

$$m(x) = \begin{cases} 1 \text{ if } f(x) > 0, \\ -1 \text{ if } f(x) < 0, \\ 0 \text{ if } f(x) = 0. \end{cases}$$

Then $(\mathrm{sign}\ f)g = m.g \quad (g \in E)$.

Now let $(T(t))_{t \geqslant 0}$ be a *semigroup* (by that we always mean a strongly continuous semigroup of linear operators) on E with generator A. We first consider necessary conditions for the positivity of the semigroup.

Proposition 1.1. *If $T(t) \geqslant 0$ $(t \geqslant 0)$ then Kato's inequality holds in the weak form, i.e.*

$$\langle (\mathrm{sign}\, f)Af, \phi \rangle \leqslant \langle |f|, A'\phi \rangle \tag{K}$$
$$(f \in D(A),\ 0 \leqslant \phi \in D(A')).$$

PROOF. Let $f \in D(A)$, $0 \leqslant \phi \in D(A')$. Then

$$\begin{aligned}
\langle (\operatorname{sign} f) Af, \phi \rangle &= \lim_{t \to 0} 1/t \langle (\operatorname{sign} f)(T(t)f - f), \phi \rangle \\
&= \lim_{t \to 0} 1/t \langle (\operatorname{sign} f) T(t)f - |f|, \phi \rangle \\
&\leqslant \lim_{t \to 0} 1/t \langle |T(t)f| - |f|, \phi \rangle \\
&\leqslant \lim_{t \to 0} 1/t \langle T(t)|f| - |f|, \phi \rangle \\
&= \lim_{t \to 0} \langle |f|, 1/t(T(t)'\phi - \phi) \rangle \\
&= \langle |f|, A'\phi \rangle . \square
\end{aligned}$$

Let $D(A')_+ = E'_+ \cap D(A')$. Consider the condition

$$\overline{D(A')_+}^{\sigma(E',E)} = E'_+ \tag{1.5}$$

(which is satisfied if the semigroup is positive). If (K) and (1.5) hold, then Kato's inequality holds in the strong form as well, whenever it makes sense, i.e.

$$(\operatorname{sign} f)Af \leqslant A|f| \qquad (\text{if } f, |f| \in D(A)). \tag{1.6}$$

However, it will be seen in section 3 that (K) and (1.5) are *not sufficient* for the positivity of the semigroup. So we consider another necessary condition.

Definition 1.2. A subset M' of E' is called *strictly positive* if for every $f \in E_+$, $\langle f, \phi \rangle = 0$ for all $\phi \in M'$ implies $f = 0$. An element ϕ of E'_+ is called strictly positive if the set $\{\phi\}$ is strictly positive.

Example 1.3. Let $E = L^p(X,\mu)$ $(1 \leqslant p \leqslant \infty)$, where (X,μ) is a σ-finite measure space. Then $\phi \in E' = L^q(X,\mu)$ (where $1/p + 1/q = 1$) is strictly positive if and only if $\phi(x) > 0$ μ—a.e. Note that strictly positive elements of E' always exist in this case.

Definition 1.4. Let B be an operator on a Banach lattice F and let $u \in F$. Then u is called a *positive subeigenvector* of B if

(a) $0 < u \in D(B)$ and

(b) $Bu \leqslant \lambda u$ for some $\lambda \in \mathbb{R}$.

Proposition 1.5. *If the semigroup $(T(t))_{t \geqslant 0}$ is positive, then there exists a strictly positive set M' of subeigenvectors of A' (the adjoint of the generator A). Moreover, if there exist strictly positive linear forms on E, then there exists a strictly positive subeigenvector of A'.*

PROOF. Let $\lambda > 0$ such that $R(\lambda,A) = (\lambda - A)^{-1}$ exists and $R(\lambda,A) \geqslant 0$. Let $N' \subset E'_+$ be strictly positive. Then $M' := \{R(\lambda,A)'\psi : \psi \in N'\} \subset D(A') \cap E'_+$. We show that M' is strictly positive. Indeed, let $f \in E_+$ such that $\langle f, \phi \rangle = 0$ for all $\phi \in M'$. Then $\langle R(\lambda,A)f, \psi \rangle = 0$ for all $\psi \in N'$. Hence $R(\lambda,A)f = 0$ since N' is strictly positive. Consequently, $f = 0$. The set M' consists of subeigenvectors of A'. In fact, let $\psi \in N'$, $\phi = R(\lambda,A)'\psi$. Then $A'\phi = \lambda\phi - \psi \leqslant \lambda\phi$. □

The following is our characterisation.

Theorem 1.6. *The semigroup $(T(t))_{t \geqslant 0}$ is positive if and only if its generator A satisfies the following condition.*

There exists a strictly positive set M' of subeigenvectors of A' such that

$$\langle (\text{sign } f)\ Af, \phi \rangle \leqslant \langle |f|, A'\phi \rangle \text{ for all } f \in D(A),\ \phi \in M'. \tag{K}$$

Corollary 1.7. *Assume that E' contains strictly positive functionals. Then the semigroup is positive if and only if there exists a strictly positive subeigenvector ϕ of A' such that*

$$\langle (\text{sign } f)\ Af, \phi \rangle \leqslant \langle |f|, A'\phi \rangle \text{ for all } f \in D(A). \tag{K}$$

Remark. 1.8. For the application of our criterion the following improvement (of one direction of the characterisation) is important. If the condition (K) is merely satisfied for all $f \in D_0$, where D_0 is a core of A, then the semigroup is positive. This will be obvious from the proofs and [5, theorem 2.3].

Remark 1.9. In Theorem 1.6 and Corollary 1.7 one can replace inequality (K) by the inequality

$$\langle (P_{f^+})Af, \phi \rangle \leqslant \langle f^+, A'\phi \rangle. \tag{1.7}$$

Indeed, (1.7) for $-f$ gives $\langle (-P_{f^-})Af, \phi \rangle \leqslant \langle f^-, A'\phi \rangle$. Adding up both inequalities one obtains $\langle (\text{sign} f)Af, \phi \rangle \leqslant \langle |f|, A'\phi \rangle$. On the other hand, if A generates a positive semigroup, one sees by the obvious alterations in the proof of Proposition 1.1 that (1.7) holds for all $f \in D(A)$, $\phi \in D(A')_+$.

We conclude this section by formulating our result for a Banach lattice E which is not σ-order complete. Then the signum operator is not defined. But the bidual E'' of E is order complete. Denote by $q : E \to E''$ the evaluation mapping. Then Theorem 1.6 holds if the inequality (K) is replaced by

$$\langle (\text{sign } q(f))\ (q(Af)), \phi \rangle \leqslant \langle |f|, A'\phi \rangle.$$

In the case $E = C_0(X)$ a more concrete version is possible which we want to state explicitly.

Let X be a locally compact space and $E = C_0(X)$ the space of all real valued continuous functions on X which vanish at infinity. Note that E is not σ-order complete unless X is σ-Stonian. For $f \in C_0(X)$ we define the function sign f by

$$(\operatorname{sign} f)(x) = \begin{cases} 1 \text{ if } f(x) > 0, \\ -1 \text{ if } f(x) < 0, \\ 0 \text{ if } f(x) = 0. \end{cases}$$

Then (sign f) is a bounded Borel function.

If $\mu \in M(X) = C_0(X)'$ we set

$$\langle g, \mu \rangle := \int g(x)\, d\mu(x)$$

for every bounded Borel function g on X.

Theorem 1.10. *Let A be the generator of a semigroup on $C_0(X)$. The semigroup is positive if and only if there exists a strictly positive set M' of subeigenvectors of A' such that*

$$\langle (\operatorname{sign} f)Af, \mu \rangle \leqslant \langle |f|, A'\mu \rangle \textit{ for all } f \in D(A),\ \mu \in M'. \tag{K}$$

Remark. We point out that for compact X a simpler condition is equivalent to positivity, namely a minimum principle (see [5]). In fact, the space $C(X)$ (X compact) plays an exceptional role in this context since its positive cone contains interior points. For a comparison of Kato's inequality and the minimum principle we refer to [4].

2. The proofs

Our arguments are based on the results of [5] on p-contraction semigroups and p-dissipative operators (see also [8]).

Let F be a Banach space. A mapping $p : F \to \mathbb{R}$ is called a *sublinear functional* if

$$p(f+g) \leqslant p(f) + p(g) \qquad (f, g \in F) \tag{2.1}$$
$$p(\lambda f) = \lambda p(f) \qquad (f \in F, \lambda \in \mathbb{R}_+). \tag{2.2}$$

We call p a *half-norm* if in addition

$$p(f) + p(-f) > 0 \text{ for all } 0 \neq f \in F. \tag{2.3}$$

Then $\|f\|_p := p(f) + p(-f)$ defines a norm on F. (This is the motivation for the terminology.)

Example 2.1. (a) $p(f) = \|f\|$ defines a continuous half-norm on F.

(b) Let E be a real Banach lattice. $N(f) = \|f^+\|$ defines a continuous half-norm on E (the *canonical half-norm*).

(c) Let E be a real Banach lattice and $\phi \in E'$. Let $p(f) = \langle f^+, \phi \rangle$ ($f \in E$). Then p is a continuous sublinear functional. Moreover, p is a half-norm if and only if ϕ is strictly positive.

Remark 2.2. To every continuous half-norm p on F there corresponds a closed proper cone $F_p := \{f \in E : p(-f) \leqslant 0\}$. In Example 2.1 (a), we have $F_p = \{0\}$; in (b), $E_p = E_+$, and in (c), $E_p = E_+$ if ϕ is strictly positive.

Let p be a continuous sublinear functional on F. The *subdifferential dp* of p is defined as follows. Let $f \in F$; then

$$dp(f) = \{\phi \in F' : \langle g,\phi \rangle \leqslant p(g) \text{ for all } g \in F \text{ and } \langle f, \phi \rangle = p(f)\}. \tag{2.4}$$

It follows from the Hahn-Banach theorem that $dp(f) \neq \varnothing$ for all $f \in F$.

An operator A on F is called *p-dissipative* if for every $f \in D(A)$ there exists $\phi \in dp(f)$ such that $\langle Af, \phi \rangle \leqslant 0$.

Proposition 2.3. *Let A be the generator of a strongly continuous semigroup $(T(t))_{t \geqslant 0}$. Then the following are equivalent.*

(i) *$T(t)$ is p-contractive for all $t \geqslant 0$;*
i.e. $p(T(t)f) \leqslant p(f)$ $\quad (f \in E)$.

(ii) *A is p-dissipative.*

(iii) *There exists a core D_0 of A such that $A_{|D_0}$ is p-dissipative.*

Remark. Suppose that p is a continuous half-norm. If A satisfies the equivalent conditions of the proposition, then the semigroup is positive for the ordering induced by p (see Remark 2.2).

For the proof of Proposition 2.3 see [5, theorem 4.1] or [8,2.1.1].

Proposition 2.4. *Let A be a densely defined operator on E and $\phi \in D(A')_+$ such that $A'\phi \leqslant 0$. Denote by p the sublinear functional given by $p(f) = \langle f^+, \phi \rangle$. If*

$$\langle (\operatorname{sign} f)\, Af, \phi \rangle \leqslant \langle |f|, A'\phi \rangle \quad (f \in D(A)), \tag{K}$$

then A is p-dissipative.

PROOF. Let $f \in D(A)$. Let $P = I - P_{f^+} - P_{f^-}$, $Q = P_{f^+} + 1/2\, P$ and $\psi = Q'\phi$. We show that

$$\psi \in dp(f). \tag{2.5}$$

Let $g \in E$. Since $0 \leqslant Q \leqslant I$ we have $\langle g, \psi \rangle = \langle Qg, \phi \rangle \leqslant \langle Qg^+, \phi \rangle \leqslant \langle g^+, \phi \rangle = p(g)$. Moreover, $\langle f, \psi \rangle = \langle Qf, \phi \rangle = \langle P_{f^+}f + 1/2Pf, \phi \rangle = \langle f^+, \phi \rangle = p(f^+)$. So (2.5) follows by the definition of $dp(f)$.

The proof will be finished when we have shown that

$$\langle Af, \psi \rangle \leqslant 0. \tag{2.6}$$

One has trivially

$$\langle (P_{f^+} + P_{f^-} + P)\, Af, \phi \rangle = \langle f, A'\phi \rangle. \tag{2.7}$$

Addition of (2.7) and (K) gives

$$\langle (2P_{f^+} + P)\, Af, \phi\rangle \leqslant \langle f^+, A'\phi\rangle \leqslant 0.$$

Hence $\langle Af, \psi\rangle = \langle QAf, \phi\rangle \leqslant 0.$ $\square$

Proof of Theorem 1.6. Propositions 1.1 and 1.5 give one implication. In order to show the other, assume that the condition in Theorem 1.6 is satisfied. We have to show that $T(t) \geqslant 0$ for all $t \geqslant 0$.

Let $\phi \in M'$. Consider the half-norm $p(f) = \langle f^+, \phi\rangle$ and the operator $B = A - \lambda$, where $\lambda \in \mathrm{R}$ is such that $A'\phi \leqslant \lambda\phi$. Then B satisfies $B'\phi \leqslant 0$ and (K) as well. So it follows from Proposition 2.4 that B is p-dissipative. Since B generates the semigroup $(e^{-\lambda t}T(t))_{t\geqslant 0}$, we obtain from Proposition 2.3 that $p(e^{-\lambda t}T(t)f) \leqslant p(T(t)f)$, $(f \in E,\, t \geqslant 0)$. Hence,

$$\langle (T(t)f)^+, \phi\rangle \leqslant e^{\lambda t}\, \langle f^+, \phi\rangle \quad (f \in E,\ t \geqslant 0). \tag{2.8}$$

Now let $t > 0$ and $f \leqslant 0$. Then $f^+ = 0$, so it follows from (2.8) that $\langle (T(t)f)^+, \phi\rangle \leqslant 0$. Since $\phi \in M'$ is arbitrary and M' is strictly positive, it follows that $(T(t)f)^+ = 0$; i.e. $T(t)f \leqslant 0$. This implies that $T(t) \geqslant 0$. $\square$

The proof of Theorem 1.10 is identical to the proof given above if the symbols $(\mathrm{sign} f)$, P_{f^+}, etc., are interpreted as Borel functions.

Remark 2.5.

(a) Proposition 1.1, which gives one implication of Theorem 1.6, had been proved (in a different way) in [3, remark 3.9]. The other implication of Theorem 1.6 has been obtained independently by Schep [30] with a different method of proof. In particular, Schep's argument seems not to apply for the case where condition (K) is only known to hold on a core of A (cf. Remark 1.8).

(b) Using Proposition 2.4 one can show with the help of the proof of [5, theorem 2.4] that a densely defined operator which satisfies the conditions of Theorem 1.6 is closable (cf. Theorem 4.4).

Remark 2.6. The proof of Theorem 1.6 shows the following. If A is the generator of a positive semigroup and E' contains strictly positive linear forms, then there exists a continuous half-norm p on E and $w \in \mathrm{R}$ such that $A - w$ is p-dissipative. We stress that p cannot be replaced by the norm, since in general none of the semigroups $(e^{-wt}T(t))_{t\geqslant 0}$ $(w \in \mathrm{R})$ is contractive for the norm (cf. [7] and [12]).

3. Examples and discussion

The examples in this section are chosen in order to show that the two ingredients of our characterisation in Theorem 1.6 (namely, Kato's inequality and the existence of a strictly positive set of subeigenfunctionals of A') are independent conditions. But these examples also illustrate how our criterion is handled for concrete operators.

As a first example we consider the first derivative with boundary conditions on $E = L^p[0, 1]$ $(1 \leqslant p < \infty)$. By $AC[0, 1]$ we denote the space of all absolutely continuous functions on $[0, 1]$. Let $A_{\max}$ be given by

$$D(A_{\max}) = \{f \in AC[0, 1] : f' \in L^p[0, 1]\}$$
$$A_{\max} f = f' \qquad\qquad (f \in D(A_{\max})).$$

Lemma 3.1. *Let $f \in AC[0, 1]$. Then $|f| \in AC[0, 1]$ and*

$$|f|' = (\operatorname{sign} f) \cdot f' \quad \text{(a.e.)}.$$

This is easy to prove. As a consequence of the lemma, $D(A_{\max})$ is a sublattice of E and

$$(\operatorname{sign} f) A_{\max} f = A_{\max} |f| \quad (f \in D(A_{\max})). \tag{3.1}$$

For $\lambda > 0$ one has

$$\ker (\lambda - A_{\max}) = \mathbb{R} \cdot e_\lambda \text{ where } e_\lambda(x) = e^{\lambda x}. \tag{3.2}$$

Hence $A_{\max}$ is not a generator. We impose the following boundary conditions.

Let $d \in \mathbb{R}$. Consider the restriction A_d of $A_{\max}$ with the domain

$$D(A_d) = \{f \in D(A_{\max}) : f(1) = d f(0)\}.$$

Then A_d is the generator of the semigroup $(T_d(t))_{t \geqslant 0}$ given by

$$T_d(t) f(x) = d^n f(x + t - n) \text{ if } x + t \in [n, n + 1) \quad (n \in \mathbb{N}). \tag{3.3}$$

This is not difficult to prove. Actually (3.3) defines a group if $d \neq 0$ and if we let $t \in \mathbb{R}$, $n \in \mathbb{Z}$. For $d = 0$ one obtains the nilpotent shift semigroup on E. One sees from (3.3) that the semigroup $(T_d(t))_{t \geqslant 0}$ is positive if and only if $d \geqslant 0$.

Let us fix $d < 0$. Let $A = A_d$ and $T(t) = T_d(t)$ for $t \geqslant 0$. Then $(T(t))_{t \geqslant 0}$ is a semigroup which is *not positive*. Nevertheless its generator A satisfies Kato's inequality. Even the equality is valid; i.e.

$$\langle (\operatorname{sign} f) Af, \phi \rangle = \langle |f|, A' \phi \rangle \text{ for all } f \in D(A),\ 0 \leqslant \phi \in D(A'). \tag{3.4}$$

Proof. It is not difficult to see that

$$\begin{aligned} D(A') &= \{\phi \in AC[0, 1] : \phi' \in L^q[0, 1],\ \phi(0) = d\phi(1)\} \\ A'\phi &= -\phi' \text{ for all } \phi \in D(A') \end{aligned} \tag{3.5}$$

where $1/p + 1/q = 1$. Let $\phi \in D(A')_+$. Since $d < 0$, it follows that $\phi(0) = \phi(1) = 0$. Hence for $f \in D(A)$,

$$\begin{aligned} \langle (\operatorname{sign} f) Af, \phi \rangle &= \langle (\operatorname{sign} f) f', \phi \rangle = \langle |f|', \phi \rangle \\ &= \int_0^1 |f|'(x) \phi(x)\, dx \\ &= |f| \phi \big|_0^1 - \int_0^1 |f(x)| \phi'(x)\, dx \\ &= |f(1)| \phi(1) - |f(0)| \phi(0) + \langle |f|, A' \phi \rangle \\ &= \langle |f|, A' \phi \rangle. \quad \square \end{aligned}$$

Remark 3.2. The equality (3.4) does not hold for all $\phi \in D(A')$, however. In fact, this would imply that $|f| \in D(A)$ and $(\operatorname{sign} f)\, Af = A\,|f|$ for all $f \in D(A)$. Thus by [21, 3.5] (or Corollary 5.6) the semigroup would consist of lattice homomorphisms. The reason why in this example the *equality* holds will be explained from a more general point of view in section 5 (see Proposition 5.9).

Even though the semigroup $(T(t))_{t \geqslant 0}$ is *not* positive its generator A has other surprising properties besides (3.4). For instance, the positive cones $D(A)_+ := D(A) \cap E_+$ and $D(A')_+ := D(A') \cap E'_+$ satisfy

$$\overline{D(A)_+} = E_+ \text{ and } \overline{D(A')_+}^{\sigma(E',\, E)} = E'_+. \tag{3.6}$$

Thus the question following remark 3.10 in [3] (resp. problem 1.5 in [4]) has a negative answer.

Moreover, (3.1) shows that A satisfies Kato's inequality (in the strong sense) *formally*. In order to formulate this more precisely, observe that it follows from (3.2) that $D(A_{\max}) = D(A) + \mathbb{R} \cdot e_\lambda$ (where $0 < \lambda \in \rho(A)$). Thus the extension $A_{\max}$ of A satisfies the following.

$$A_{\max} \text{ is closed.} \tag{3.7}$$

$$D(A_{\max}) \text{ is a sublattice of } E. \tag{3.8}$$

$$D(A) \text{ has codimension one in } D(A_{\max}). \tag{3.9}$$

$$(\operatorname{sign} f)\, Af = A_{\max}\, |f| \text{ for all } f \in D(A). \tag{3.10}$$

It is also remarkable that there exists a dense sublattice $D_0 := \{f \in D(A) : f(0) = f(1) = 0\}$ of E which is included in $D(A)$. But D_0 is not a core of A (this would imply the positivity of the semigroup by [4, theorem 3.4] if $|d| \leqslant 1$).

Since $(T(t))_{t \geqslant 0}$ is not positive but (3.4) holds, it follows from Theorem 1.6 that there exists no strictly positive subeigenvector of A'. In fact, more is true.

$$0 \leqslant \phi \in D(A'),\ A'\phi \leqslant \mu\phi \text{ for some } \mu \in \mathbb{R} \text{ implies } \phi = 0. \tag{3.11}$$

PROOF. Suppose that $0 \leqslant \phi \in D(A')$ such that $-\phi' = A'\phi \leqslant \mu\phi$. We can assume that $0 \leqslant \mu$. Let $\psi(x) = \phi(1-x)$. Then $\psi'(x) = -\phi'(1-x) \leqslant \mu\phi(1-x) = \mu\psi(x)$. Since $\psi(0) = 0$, we get

$$\psi(x) = \int_0^x \psi'(y)\, dy \leqslant \mu \int_0^x \psi(y)\, dy \quad (x \in [0, 1]).$$

It follows from Gronwall's lemma that $\psi \leqslant 0$. Hence $\phi = \psi = 0$. □

In view of the preceding example one might presume that the existence of a strictly positive set of subeigenvectors of the adjoint of the generator actually implies the positivity of the semigroup. This is not the case. To give an example consider $E = L^2(\mathbb{R})$ and the operator B given by

$$Bf = f^{(3)} \text{ with domain}$$

$$D(B) = \{f \in C^2(\mathbb{R}) : f', f'' \in L^2(\mathbb{R}), f'' \in AC(\mathbb{R}), f^{(3)} \in L^2(\mathbb{R})\}.$$

Then B is the generator of a unitary group $(U(t))_{t\in\mathbb{R}}$. In particular, B is skew-adjoint, i.e. $B' = -B$. We claim that

$$B' \text{ has a strictly positive subeigenvector } \phi. \tag{3.12}$$

PROOF. Let $\lambda > 0$ and

$$\phi(x) = \begin{cases} e^{-\lambda x} & \text{for } x \geqslant 1, \\ g(x) & -1 < x < 1, \\ e^{\lambda x} & \text{for } x \leqslant -1, \end{cases}$$

where $g \in C^3[-1, 1]$ such that $g(x) > 0$ for all $x \in [-1, 1]$ and such that $\phi \in C^3(\mathbb{R})$. Moreover, choose g such that $g(0) = 1$ and $g'(0) = g''(0) = 0$. Since $g, g^{(3)} \in C(\mathbb{R})$ and $\inf\{g(x) : x \in [-1, 1]\} > 0$ there exists $\mu \geqslant \lambda^3$ such that $-g^{(3)}(x) \leqslant \mu g(x)$ for all $x \in [-1, 1]$. Consequently,

$$-\phi^{(3)}(x) = \left.\begin{cases} \lambda^3 e^{-\lambda x} & (x \geqslant 1) \\ -g^{(3)}(x) & (x \in [-1, 1]) \\ -\lambda^3 e^{\lambda x} & (x < -1) \end{cases}\right\} \leqslant \mu\phi(x).$$

Hence $B'\phi = -\phi^{(3)} \leqslant \mu\phi$. □

But the semigroup $(U(t))_{t\geqslant 0}$ is *not* positive. In fact, we show that there exists $f \in D(B)$ such that

$$\langle(\operatorname{sign} f)\, Bf, \phi\rangle > \langle |f|, B'\phi\rangle. \tag{3.13}$$

PROOF. Let $f \in D(B)$ be such that $f(x) = e^{-x}\sin x$ in a neighbourhood of 0 and $f(x) > 0$ for $x > 0$ and $f(x) < 0$ for $x < 0$. Then

$$\langle(\operatorname{sign} f)\, Bf, \phi\rangle = -\int_{-\infty}^{0} f^{(3)}(x)\phi(x)\, dx + \int_{0}^{\infty} f^{(3)}(x)\,\phi(x)\, dx.$$

Hence,

$$\begin{aligned}
\langle |f|, B'\phi\rangle &= \int_{-\infty}^{0} (-f(x))\,(-\phi^{(3)}(x))\, dx + \int_{0}^{\infty} f(x)\,(-\phi^{(3)}(x))\, dx \\
&= -\int_{-\infty}^{0} f^{(3)}(x)\,\phi(x)\, dx + \int_{0}^{\infty} f^{(3)}(x)\,\phi(x)\, dx \\
&\quad + [f''\phi]\,|_{-\infty}^{0} - [f''\phi]\,|_{0}^{\infty} \;(\text{since } \phi''(0) = \phi'(0) = (0)) \\
&= \langle(\operatorname{sign} f)\, Bf, \phi\rangle + 2f''(0)\,\phi(0) \\
&< \langle(\operatorname{sign} f)\, Bf, \phi\rangle
\end{aligned}$$

since $f''(0)\,\phi(0) = f''(0) = -2$. □

We now show, however, that B satisfies Kato's inequality for positive elements, i.e.

$$P_f\, Bf \leqslant Bf \text{ for all } f \in D(B)_+. \tag{3.14}$$

In fact, more is true. B is *local*, i.e.

$$f \perp g \text{ implies } Bf \perp g \text{ for all } f \in D(B),\ g \in L^2(\mathbb{R}). \tag{3.15}$$

PROOF. Let A be the generator of the translation group. Then A is local by [21, 3.3]. Hence $B = A^3$ is local as well. $\square$

So this example shows that even if there exists a strictly positive subeigenvector of the adjoint of the generator, Kato's inequality for positive elements alone does not suffice for the positivity of the semigroup.

Next we make some observations concerning positive subeigenvectors. Assume that A is the generator of a positive semigroup $(T(t))_{t\geq 0}$ on a Banach lattice E. Let $\phi \in D(A')_+$ and $\lambda \in \mathbb{R}$. Then

$$A'\phi \leqslant \lambda\phi \text{ if and only if } T(t)'\phi \leqslant e^{\lambda t}\phi \ (t \geqslant 0). \tag{3.16}$$

PROOF. If $T(t)'\phi \leqslant e^{\lambda t}\phi$ for all $t \geqslant 0$, then

$$A'\phi = \sigma(E', E) - \lim_{t\to 0} 1/t\,(T(t)'\phi - \phi) \leqslant \lim_{t\to 0} 1/t\,(e^{\lambda t}\phi - \phi) = \lambda\phi.$$

For the converse let $f \in E_+$. Then

$$\begin{aligned}\langle f, T(t)'\phi\rangle &= \langle f, \phi\rangle + \int_0^t \langle f, T(s)'A'\phi\rangle\, ds \\ &\leqslant \langle f, \phi\rangle + \lambda \int_0^t \langle f, T(s)'\phi\rangle\, ds.\end{aligned}$$

It follows from Gronwall's lemma that $\langle f, T(t)'\phi\rangle \leqslant e^{\lambda t}\langle f, \phi\rangle$. $\square$

Assume now that ϕ is a subeigenvector of A'. Then it follows from (3.16) that the ideal $J := \{f \in E : \langle\, |f|, \phi = 0\}$ is invariant under the semigroup. From this we conclude the following.

Proposition 3.3. *If the semigroup is irreducible* (*see* [29]), *then every positive subeigenvector of A' is strictly positive.*

Example. For $d > 0$ the semigroup $(T_d(t))_{t\geq 0}$ considered at the beginning of this section is irreducible. Thus every positive subeigenvector of A'_d is strictly positive.

The existence of positive subeigenvectors is related to the Krein-Rutman theorem. If A has a compact resolvent and $\sigma(A) \neq \varnothing$, then the Krein-Rutman theorem asserts that there exists a positive eigenvector of A' (and A) for the eigenvalue $s(A) := \sup\{Re\,\lambda : \lambda \in \sigma(A)\}$. It is easy to see that A_d has compact resolvent and $\sigma(A_d) \neq \varnothing$ for $d \neq 0$. Thus A'_d has a positive eigenvector if and only if $d > 0$.

4. Domination

Frequently it is useful to be able to compare two semigroups on a Banach lattice with respect to the ordering. In this section we assume that E is a σ-order complete *complex* Banach lattice [26, II. §11]. Let $(T(t))_{t\geq 0}$ be a *positive* semigroup with generator A and $(S(t))_{t\geq 0}$ a semigroup with generator B. We say $(T(t))_{t\geq 0}$ *dominates* $(S(t))_{t\geq 0}$ if

$$|S(t)f| \leqslant T(t)|f| \text{ for all } f \in E,\ t > 0. \tag{4.1}$$

We first observe that domination of the semigroup is equivalent to domination of the resolvents. More precisely, (4.1) holds if and only if

$$|R(\lambda, B)f| \leqslant R(\lambda, A)|f| \quad (f \in E) \text{ for large real } \lambda. \tag{4.2}$$

Proof. (4.2) follows from (4.1) since the resolvent is given by the Laplace transform of the semigroup. Conversely, if (4.2) holds, then

$$\begin{aligned} |S(t)f| &= \lim_{n\to\infty} |((n/t)R(n/t, B))^n f| \\ &\leqslant \lim_{n\to\infty} ((n/t)R(n/t, A))^n |f| \\ &= T(t)|f| \quad (t \geqslant 0, f \in E). \quad \square \end{aligned}$$

One can describe domination by an inequality for the generators in a manner analogous to the characterisation of positive semigroups in section 1; however, no positive subeigenvectors are needed here.

We briefly want to explain the sign operator in a complex Banach lattice. Let $f \in E$. There exists a unique operator $S \in \mathscr{L}(E)$ satisfying

$$Sf = |f| \tag{4.3}$$

$$|Sg| \leqslant |g| \quad (g \in E) \tag{4.4}$$

$$Sg = 0 \quad \text{if } g \perp f \tag{4.5}$$

(see [21, 2.1]).

Example 4.1. Let $E = L^p(X, \mu)$ $(1 \leqslant p < \infty)$ and $f \in E$. Then

$$(\operatorname{sign} f)(x) = \begin{cases} \overline{f(x)}/|f(x)| & \text{if } f(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

defines a function in L^∞. The operator S is given by

$$Sg = (\operatorname{sign} f) \cdot g \quad (g \in E).$$

We define $\operatorname{sign} f := S \in \mathscr{L}(E)$. Thus in the case $E = L^p$ we identify the function $(\operatorname{sign} f)$ and the multiplication operator it defines.

Remark 4.2. If $(T(t))_{t\geqslant 0}$ is a positive semigroup on a σ-order complete *complex* Banach lattice, then its generator satisfies Kato's inequality in the form (K) if 'sign f' is interpreted as above. This can be proved as Proposition 1.1. However, for the characterisation of positive semigroups one can restrict oneself to the real case by making use of the following observation.

Let E be a complex Banach lattice. Denote by E_{R} the real Banach lattice associated with E. Then $E = E_{\mathrm{R}} + iE_{\mathrm{R}}$; i.e. for $f \in E$ there exist unique elements $Re f$, $Im f$ of E_{R} such that $f = Re f + i\, Im f$. Let $\bar{f} = Re f - i\, Im f$.

Let $(S(t))_{t\geqslant 0}$ be a semigroup on E with generator A. We say that $(S(t))_{t\geqslant 0}$ is *real* if $S(t)E_{\mathrm{R}} \subset E_{\mathrm{R}}$ for all $t \geqslant 0$. It is easy to describe this in terms of the generator. We say that A is *real* if $f \in D(A)$ implies $\bar{f} \in D(A)$ and $A\bar{f} = \overline{Af}$. Then

$$(S(t))_{t\geqslant 0} \text{ is real if and only if } A \text{ is real.} \tag{4.6}$$

Theorem 4.3. *Let $(T(t))_{t\geqslant 0}$ be a positive semigroup with generator A and $(S(t))_{t\geqslant 0}$ a semigroup with generator B. The following assertions are equivalent.*

(i) $|S(t)f| \leqslant T(t)|f|$ *for all* $f \in E,\ t > 0$.

(ii) $Re\ \langle(\operatorname{sign} f)\ Bf,\ \phi\rangle \leqslant \langle\, |f|,\ A'\phi\rangle$ *for all* $f \in D(B),\ \phi \in D(A')_+$.

The author learnt Theorem 4.3 from T. Kato. There are similar results due to Simon [31], [32], and Hess, Schrader and Uhlenbrock [16]. Our aim is to generalise Theorem 4.3 by replacing the condition that B is a generator by a range condition. The precise formulation is the following.

Theorem 4.4. *Let $(T(t))_{t\geqslant 0}$ be a positive semigroup with generator A. Let B be a densely defined operator such that*

$$Re\ \langle(\operatorname{sign} f)\ Bf, \phi\rangle \leqslant \langle\, |f|, A'\phi\rangle \ \text{ for all } f \in D(B), \phi \in D(A')_+. \tag{4.7}$$

Then B is closable. Moreover, if $(\lambda - B)D(B)$ is dense in E for some $\lambda >$ max $\{0, s(A)\}$, then $B^{\#}$ (the closure of B) generates a semigroup which is dominated by $(T(t))_{t\geqslant 0}$.

We will use the following notion. Let A be the generator of a positive semigroup. The *spectral bound* $s(A)$ is defined by $s(A) := \sup\ \{Re\ \lambda : \lambda \in \sigma(A)\}$. Note that $R(\lambda, A) \geqslant 0$ for all $\lambda \geqslant s(A)$.

PROOF OF THEOREM 4.4. 1. We show that B is closable. Let $u_n \in D(B)$ such that $u_n \to 0$ and $Bu_n \to v$. We have to show that $v = 0$. Considering $A - \mu$ and $B - \mu$ for some $\mu > s(A)$ instead of A and B we may assume that $s(A) < 0$. Then there exists a strictly positive set $M' \subset E'$ such that

$$\phi \in D(A') \text{ and } A'\phi \leqslant 0 \text{ for all } \phi \in M' \tag{4.8}$$

(see the proof of Proposition 1.5).

Let $\phi \in M'$ and p be the sublinear functional given by $p(f) = \langle |f|, \phi\rangle$. We show that B is p-dissipative. Let $f \in D(B)$, $\psi = (\operatorname{sign} f)'\phi$. Then it is easy to see that

$$\psi \in dp(f) := \{\psi \in E' : Re\ \langle g, \psi\rangle \leqslant p(g)\ (g \in E);\ \langle f, \psi\rangle = p(f)\}.$$

Moreover, by (4.7) and (4.8) one obtains that

$$Re\ \langle Bf, \psi\rangle = Re\ \langle(\operatorname{sign} f)\ Bf, \phi\rangle \leqslant \langle\, |f|, A'\phi\rangle \leqslant 0.$$

Thus B is p-dissipative; i.e.

$$p((\lambda - B)f) \geqslant \lambda p(f) \text{ for all } f \in E, \lambda > 0.$$

By the proof of [5, theorem 2.4] one sees that $p(v) = 0$; i.e. $\langle\, |v|, \phi\rangle = 0$. Since $\phi \in M'$ was arbitrary we conclude that $v = 0$.

2. Let $\lambda > \lambda_0 :=$ max $\{s(A), 0\}$. We show that for $f \in D(B)$,

$$g = (\lambda - B)f \text{ implies } |f| \leqslant R(\lambda, A)\ |g|. \tag{4.9}$$

Let $\psi \in E'_+$. We have to show that $\langle\, |f|, \psi\rangle \leqslant \langle R(\lambda, A)\ |g|, \psi\rangle$.

Let $\phi = R(\lambda, A)'\psi \in D(A')_+$. Then by (4.7)

$$\begin{aligned}\langle |f|, \psi\rangle &= \langle |f|, (\lambda - A')\phi\rangle = Re\ \langle(\text{sign} f)(\lambda f), \phi\rangle - \langle |f|, A'\phi\rangle \\ &= Re\ \langle(\text{sign} f)(\lambda - B)f, \phi\rangle + Re\ \langle(\text{sign} f)\, Bf, \phi\rangle - \langle |f|, A'\phi\rangle \\ &\leqslant Re\ \langle(\text{sign} f)(\lambda - B)f, \phi\rangle = Re\ \langle(\text{sign} f)\, g, \phi\rangle \\ &\leqslant \langle |g|, \phi\rangle = \langle |g|, R(\lambda, A)'\,\psi\rangle = \langle R(\lambda, A)\,|g|, \psi\rangle.\end{aligned}$$

It follows from (4.9) that for $\lambda > \lambda_0$ and $f \in D(B^*)$

$$g = (\lambda - B^*)f \text{ implies } |f| \leqslant R(\lambda, A)\,|g|. \tag{4.10}$$

In particular, $(\lambda - B^*)$ is injective for $\lambda > \lambda_0$. Moreover,

$$\begin{gathered}|R(\lambda, B^*)\,g| \leqslant R(\lambda, A)\,|g| \text{ for all } g \in E \\ \text{whenever } \lambda_0 < \lambda \in \rho(B^*).\end{gathered} \tag{4.11}$$

Assume now that there exists $\mu > \lambda_0$ such that $(\mu - B)\,D(B)$ is dense in E. Then $(\mu - B^*)\,D(B^*) = E$. (Indeed, let $h \in E$. There exists $f_n \in D(B)$ such that $g_n := (\mu - B)f_n \to h$. By (4.9) it follows that $|f_n - f_m| \leqslant R(\lambda, A)\,|g_n - g_m|$. Thus (f_n) is a Cauchy sequence. Let $f = \lim_{n\to\infty} f_n$. Then $f \in D(B^*)$ and $(\mu - B^*)f = h$.) Thus $\mu \in \rho(B^*)$.

Let $\lambda_0 < \lambda \in \rho(B^*)$. Then it follows from (4.11) that $\| R(\lambda, B^*)\| \leqslant \| R(\lambda, A)\| \leqslant \| R(\lambda_0, A)\| := c$. Hence, $\text{dist}\,(\lambda, \sigma(B^*)) = r(R(\lambda, B^*))^{-1} \geqslant \| R(\lambda, B^*)\|^{-1} \geqslant 1/c$. This implies that $[\lambda_0, \infty) \subset \rho(B^*)$. Moreover, it follows from (4.11) that

$$|R(\lambda, B^*)^n f| \leqslant R(\lambda, A)^n\,|f| \quad (f \in E,\ n \in \mathrm{N}). \tag{4.12}$$

Let $w > \omega(A)$ (the type of $(T(t))_{t\geqslant 0}$). Then it follows from (4.12) that

$$\|(\lambda - w)^n R(\lambda, B^*)^n\| \leqslant \|(\lambda - w)^n R(\lambda, A)^n\| \text{ for all } \lambda > w,\ n \in \mathrm{N}.$$

So by the Hille-Yosida theorem, B^* is the generator of a semigroup $(S(t))_{t\geqslant 0}$. Finally, the domination follows from (4.11). ☐

Proof of Theorem 4.3. One direction follows from Theorem 4.4. The other can be proved in a way similar to Proposition 1.1. ☐

Example 4.5. As an illustration of Theorem 4.3 we consider the complex version of the first example of section 1.

Let $E = L^p[0, 1]$. For $d \in \mathrm{C}$ let $A_d\, f = f'$ with domain $D(A_d) = \{f \in AC[0, 1] : f' \in L^p[0, 1],\ f(1) = df(0)\}$. Then A_d generates a semigroup $(T_d(t))_{t\geqslant 0}$. Let $|d| \leqslant c$. Then $(T_d(t))_{t\geqslant 0}$ is dominated by $(T_c(t))_{t\geqslant 0}$. This can be seen by Theorem 4.3 as follows. Let $f \in D(A_d)$, $0 \leqslant \phi \in D(A'_c)$. Then $\phi(0) = c\phi(1)$. Hence

$$\begin{aligned}Re\ \langle(\text{sign} f)A_d f, \phi\rangle &= Re\ \langle(\text{sign} f)f', \phi\rangle = \langle|f|', \phi\rangle \\ &= \langle|f|, -\phi'\rangle + (|f(x)|\,\phi(x))\,|_0^1 \\ &= \langle|f|, (A_c)'\phi\rangle + |f(1)|\,\phi(1) - |f(0)|\,\phi(0) \\ &= \langle|f|, (A_c)'\,\phi\rangle + |f(0)|\,\phi(1)(|d| - c) \\ &\leqslant \langle|f|, (A_c)'\,\phi\rangle.\end{aligned}$$

Of course, in this example domination can also be verified by inspection of the semigroups.

Example 4.6. Let $(T(t))_{t\geqslant 0}$ be a positive semigroup with generator A. Let $M \in Z(E)$ (the *centre* of E (see [33, chapter 20]). For example, if $E = L^p(X, \mu)$ (where (X, μ) is a σ-finite measure space and $1 \leqslant p \leqslant \infty$) then M is the multiplication operator defined by a function in $L^\infty(X, \mu)$.

Let $B = A + M$. Then B generates a semigroup $(S(t))_{t\geqslant 0}$. Assume that $Re\, M \leqslant 0$. Let $f \in D(B)$ and $\phi \in D(A')_+$. Then

$$\begin{aligned} Re\,\langle(\operatorname{sign} f)\, Bf, \phi\rangle &= Re\,\langle(\operatorname{sign} f)\, Af, \phi\rangle + Re\,\langle(\operatorname{sign} f)\, Mf, \phi\rangle \\ &= Re\,\langle(\operatorname{sign} f)\, Af, \phi\rangle + Re\,\langle M\,|f|, \phi\rangle \\ &\leqslant \langle |f|, A'\phi\rangle. \end{aligned}$$

Thus, by Theorem 4.3, $(S(t))_{t\geqslant 0}$ is dominated by $(T(t))_{t\geqslant 0}$.

Domination and positivity are characterised simultaneously as follows.

Proposition 4.7. *Let E be a σ-order complete real Banach lattice. Let $(T(t))_{t\geqslant 0}$ be a positive semigroup with generator A and let $(S(t))_{t\geqslant 0}$ be a semigroup with generator B. The following are equivalent.*

(i) $0 \leqslant S(t) \leqslant T(t)$ *for all* $t \geqslant 0$.

(ii) $\langle P_{f^+} Bf, \phi\rangle \leqslant \langle f^+, A'\phi\rangle$ *for all* $f \in D(B)$, $\phi \in D(A')_+$.

(iii) $\langle P_{f^+} Bf, \phi\rangle \leqslant \langle f^+, A'\phi\rangle$ *for all* $f \in D_0$, $\phi \in D(A')_+$, *where* D_0 *is a core of* B.

Remark 4.8. Condition (ii) implies (4.7) (cf. Remark 1.9).

PROOF. One proves as in Proposition 1.1 that (i) implies (ii). It is trivial that (ii) implies (iii). Assume that (iii) holds. Let $\lambda > \lambda_0 = \max\{s(A), s(B), 0\}$. In a similar way as (4.10) one shows that for all $f \in D_0$

$$\lambda f - Bf = g \text{ implies } f^+ \leqslant R(\lambda, A)g^+. \tag{4.13}$$

Since D_0 is a core it follows that (4.13) also holds for all $f \in D(B)$. This implies that $(R(\lambda, B)g)^+ \leqslant R(\lambda, A)g^+$ for all $g \in E$, $\lambda > \lambda_0$. Consequently, $0 \leqslant R(\lambda, B) \leqslant R(\lambda, A)$ for all $\lambda > \lambda_0$. Hence (i) holds. □

Finally, if it is known that the semigroup $(S(t))_{t\geqslant 0}$ also is positive, domination can be characterised as follows.

Proposition 4.9. *Let E be a real Banach lattice, $(T(t))_{t\geqslant 0}$ a positive semigroup with generator A and $(S(t))_{t\geqslant 0}$ a positive semigroup with generator B. Consider the following conditions.*

(i) $S(t) \leqslant T(t)$ $(t \geqslant 0)$.

(ii) $\langle Bf, \phi\rangle \leqslant \langle f, A'\phi\rangle$ *for all* $f \in D(B)_+$, $\phi \in D(A')_+$.

(iii) $Bf \leqslant Af$ *for* $0 \leqslant f \in D(A) \cap D(B)$.

Then (i) *and* (ii) *are equivalent and imply* (iii). *Moreover, if* $D(A) \subset D(B)$ *or* $D(B) \subset D(A)$, *then* (iii) *implies* (i).

PROOF. Assume that (i) holds. Then for $f \in D(B)_+$, $\phi \in D(A')_+$,

$$\begin{aligned}\langle Bf, \phi\rangle &= \lim_{t\to 0} 1/t \langle S(t)f - f, \phi\rangle \leqslant \lim_{t\to 0} 1/t \langle T(t)f - f, \phi\rangle \\ &= \langle f, A'\phi\rangle.\end{aligned}$$

So (ii) holds; (iii) is proved similarly.

Now assume (ii). Let $\lambda > \max\{s(A), s(B)\}$. Let $g \in E_+$, $\psi \in E'_+$. Then $\langle R(\lambda, B)g - R(\lambda, A)g, \psi\rangle = \langle R(\lambda, A)g, \lambda R(\lambda, B)'\psi - \psi\rangle - \langle \lambda R(\lambda, A)g - g, R(\lambda, B)'\psi\rangle = \langle f, B'\phi\rangle - \langle Af, \phi\rangle \leqslant 0$ where $f = R(\lambda, A)g \in D(A)_+$ and $\phi = R(\lambda, B)'\psi \in D(B')_+$. Hence $R(\lambda, B) \leqslant R(\lambda, A)$ and (i) follows.

Finally, we prove that (iii) implies (i) if $D(B) \subset D(A)$, say. Let $\lambda > \max\{s(A), s(B)\}$. Then $(A - B)R(\lambda, B)$ is a positive operator. Hence $R(\lambda, A) - R(\lambda, B) = R(\lambda, A)(A - B)R(\lambda, B) \geqslant 0$. This implies (i). □

Example 4.10. Let B be the generator of a positive semigroup $S(t))_{t\geqslant 0}$, C a bounded positive operator. Then $A = B + C$ with $D(A) = D(B)$ is the generator of a semigroup $(T(t))_{t\geqslant 0}$. It can be seen from the product formula (see e.g. [9]) that $(T(t))_{t\geqslant 0}$ is positive. Since $Bf \leqslant Af$ for all $f \in D(B)_+$, it follows from Proposition 4.9 that $S(t) \leqslant T(t)$ for all $t \geqslant 0$.

The preceding results can be applied to the perturbation by multiplication operators. Let (X, μ) be a σ-finite measure space and $E = L^p(X, \mu)$ $(1 \leqslant p < \infty)$. Consider a positive semigroup $(T(t))_{t\geqslant 0}$ with generator A. Let $m: X \to \mathbb{R}$ *be a measurable function such that* $m(x) \leqslant 0$ for all $x \in X$. Let $D(m) = \{f \in E : F\cdot m \in E\}$. Define the operator B with domain $D(B) = D(A) \cap D(m)$ by $Bf = Af + m\cdot f$ $(f \in D(B))$.

Theorem 4.11. *If there exists a quasi-interior subeigenvector u of A such that $u \in D(m)$, then B is closable and the closure B^* of B is the generator of a positive semigroup $(S(t))_{t\geqslant 0}$ which is dominated by $(T(t))_{t\geqslant 0}$.*

For the proof of the theorem we need the following lemma.

Lemma 4.12. *Let A and B be generators of positive semigroups $(T(t))_{t\geqslant 0}$ (resp., $S(t)_{t\geqslant 0}$). If $(T(t))_{t\geqslant 0}$ dominates $(S(t))_{t\geqslant 0}$, then $s(B) \leqslant s(A)$.*

PROOF OF LEMMA 4.12. Let $\lambda > s(A)$. Then for all $\mu \leqslant \max\{\lambda, s(B)\}$ one has $0 \leqslant R(\mu, B) \leqslant R(\lambda, A)$, and so $\operatorname{dist}(\mu, \sigma(B)) \geqslant \| R(\mu, B)\|^{-1} \geqslant \| R(\lambda, A)\|^{-1}$. This implies that $[\lambda, \infty) \subset \rho(\mathrm{B})$. □

PROOF OF THEOREM 4.11. There exists $\mu > 0$ such that $Au \leqslant \mu u$. Let $\lambda > \max\{s(A), \mu\}$. Then $\lambda R(\lambda, A)u = AR(\lambda, A)u + u \leqslant \mu R(\lambda, A)u + u$. Hence $R(\lambda, A)u \leqslant cu$ where $c > 0$. It follows that $R(\lambda, A)E_u \subset E_u \cap D(A) \subset D(B)$. Hence $D(B)$ is dense.

Let $f \in D(B)$, $\phi \in D(A')_+$. Then

$$\langle P_{f^+} Bf, \phi\rangle \leqslant \langle f^+, A'\phi\rangle. \tag{4.14}$$

In fact, $\langle P_{f^+} Bf, \phi \rangle = \langle P_{f^+} Af, \phi \rangle + \langle P_{f^+} m{\cdot}f, \phi \rangle$

$$= \langle P_{f^+} Af, \phi \rangle + \langle m{\cdot}f^+, \phi \rangle$$
$$\leqslant \langle P_{f^+} Af, \phi \rangle$$
$$\leqslant \langle f^+, A'\phi \rangle \quad \text{(by (1.7))}.$$

But (4.14) implies (4.7). So it follows from Theorem 4.4 that B is closable. Moreover, if we can show that $(\lambda - B^*)\, D(B^*)$ is dense in E, it follows that B^* is the generator of a semigroup $(S(t))_{t\geqslant 0}$. In that case (4.14) implies by Proposition 4.7 that $(S(t))_{t\geqslant 0}$ is dominated by $(T(t))_{t\geqslant 0}$.

We show now that $(\lambda - B^*)\, D(B^*)$ is dense in E. Let $m_n = \sup\{m, -nl_X\}$ $(n \in \mathrm{N})$ and $B_n = A + m_n$. Then B_n is the generator of a positive semigroup and it follows from Proposition 4.9 that $0 \leqslant R(\lambda, B_{n+1}) \leqslant R(\lambda, B_n) \leqslant R(\lambda, A)$ for all $n \in \mathrm{N}$, $\lambda > s(A)$. (Note that $s(B_n) \leqslant s(A)$ by Lemma 4.12.) Let $0 \leqslant f \in E_u$. Let $g_n = R(\lambda, B_n)f$. Then $g = \inf_{n\in\mathrm{N}} g_n = \lim_{n\to\infty} g_n$ exists. Moreover $g_n \in D(B)$ and $\lim_{n\to\infty} (\lambda - B)g_n = f + \lim_{n\to\infty} (B_n - B)g_n = f$, since $|(B_n - B)g_n| \leqslant (m_n - m)\,|g_n| = (m_n - m)\,|R(\lambda, B_n)f| \leqslant (m_n - m)\, R(\lambda, A)\,|f| \leqslant c'\,(m_n - m)u$. But $\lim_{n\to\infty} (m_n - m)u = 0$ since $u \in D(m)$. Thus $g \in D(B^*)$ and $(\lambda - B^*)g = f$. We have shown that $E_u \subset (\lambda - B^*)D(B^*)$. Hence $(\lambda - B^*)\, D(B^*)$ is dense in E. $\square$

Example 4.13. If $D(A) \subset L^\infty(X, \mu)$ and $m \in L^p(X, \mu)$, then the hypotheses of Theorem 4.11 are satisfied.

5. Semigroups of disjointness-preserving operators

In this section we consider a special case of domination. Let E be a complex Banach lattice. A bounded operator S on E is called *disjointness-preserving* if

$$f \perp g \text{ implies } Sf \perp Sg \;(f, g \in E). \tag{5.1}$$

Note that an operator S is a lattice homomorphism [26, II. 2.4] if and only if S is positive and disjointness-preserving.

In the following we will consider *disjointness-preserving semigroups* (by this we mean semigroups of disjointness-preserving operators). An example is the semigroup $(T_d(t))_{t\geqslant 0}$ defined in section 3.

Remark 5.1. In [2] we called order bounded disjointness-preserving operators *Lamperti operators,* and it was shown that on a σ-order complete Banach lattice every disjointness-preserving operator is automatically order bounded. More recently Abramovich [1] showed that the assumption of σ-order continuity can be omitted and de Pagter [22] gave a simplified proof of this fact.

If $S \in \mathscr{L}(E)$ is disjointness-preserving, then the *modulus* $|S|$ of S exists. $|S|$ is a lattice homomorphism and is related to S by

$$|Sf| = |S|\,|f| \quad (f \in E). \tag{5.2}$$

Proposition 5.2. *Let $(S(t))_{t\geqslant 0}$ be a disjointness-preserving semigroup. Let $T(t) = |S(t)|$ $(t \geqslant 0)$. Then $(T(t))_{t\geqslant 0}$ is a strongly continuous semigroup.*

PROOF. Let $0 \leqslant s, t$ and $f \in E_+$. Then by (5.2),

$$T(s)T(t)f = T(s) \mid S(t)f \mid = \mid S(s)S(t)f \mid = \mid S(s+t)f \mid = T(s+t)f.$$

Since span $E_+ = E$, it follows that $(T(t))_{t\geqslant 0}$ is a semigroup. Moreover, for $f \in E_+$, $\lim_{t\to 0} T(t)f = \lim_{t\to 0} \mid S(t)f \mid = |f| = f$. This implies that $(T(t))_{t\geqslant 0}$ is strongly continuous. □

Remark. Derndinger [11] investigates the modulus of a semigroup in other cases.

Example 5.3. Let $d \in \mathrm{C}$ and $S(t) = T_d(t)$ be given by (3.3). Then $T(t) = T_{|d|}(t)$ $(t \geqslant 0)$.

Proposition 5.4. *Let B be the generator of a disjointness-preserving semigroup $(S(t))_{t\geqslant 0}$. Then B is local; i.e.*

$$Bf \perp g \text{ if } f \in D(B),\ g \in E \text{ such that } f \perp g. \tag{5.3}$$

The proof of [21, 3.3] can be adapted in an obvious way.

We now describe the relation between the generator of a disjointness-preserving semigroup and the generator of the modulus semigroup.

Theorem 5.5 *Assume that E is a complex Banach lattice with order continuous norm. Let $(S(t))_{t\geqslant 0}$ be a semigroup with generator B. The following assertions are equivalent.*

(i) *$(S(t))_{t\geqslant 0}$ is disjointness-preserving.*

(ii) *There exists a semigroup $(T(t))_{t\geqslant 0}$ with local generator A such that*

$$f \in D(B) \text{ implies } |f| \in D(A) \text{ and } Re\,((\operatorname{sign} f)\ Bf) = A|f|. \tag{5.4}$$

Moreover, if these equivalent conditions are satisfied, then $T(t) = \mid S(t) \mid$ $(t \geqslant 0)$.

Remark. The relation (5.4) is equivalent to

$$\langle Re\,((\operatorname{sign} f)\ Bf), \phi \rangle = \langle |f|, A'\phi \rangle \quad (f \in D(B),\ \phi \in D(A')).$$

In the case when A generates a positive semigroup, this is condition (4.7) in Theorem 4.4 with the inequality replaced by the equality. It is remarkable that, in contrast to the situation considered in Theorem 4.4, here condition (ii) implies the positivity of $(T(t))_{t\geqslant 0}$.

PROOF. This is an adaptation of the proof of [21, theorem 3.4] given by Nagel and Uhlig. Assume that (i) holds. Let $f \in D(B)$. Then $S(t)f$ is differentiable in t. By the chain rule [21, 3.1] $T(t)\,|f| = \mid S(t)f \mid$ is also differentiable and $d/dt_{|t=0}\, T(t)\,|f| = Re(\operatorname{sign} f)Bf$ (by [21, 2.2] and Proposition 5.4). Hence $|f| \in D(A)$ and $A|f| = Re(\operatorname{sign} f)Bf$. Conversely, assume that (ii) holds. Let $s > 0, f \in E$. We show that $\mid S(s)f \mid = T(t)\,|f|$. This implies that $S(s)$ is disjointness-preserving and $\mid S(s) \mid = T(s)$ (by [2, theorem 2.4]). Since $D(B)$ is dense we can assume that $f \in D(B)$. Let $\xi(t) = T(s-t) \mid S(t)f \mid$ $(t \in [0, s])$. Then using again [21, 3.1], [21, 2.2] and that A is local one obtains $d/dt\ \xi(t) = -AT(s-t) \mid S(t)f \mid + T(s-t)\, Re((\operatorname{sign} S(t)f)\, BS(t)f) = 0$ by the assumption (ii). Hence $\xi(0) = \xi(s)$, i.e. $\mid S(s)f \mid = T(s)\ |f|$. □

For the case when $S(t) = T(t)$ $(t \geqslant 0)$ we obtain the following.

Corollary 5.6. (Nagel and Uhlig [21, 3.4]). *Let $(T(t))_{t \geqslant 0}$ be a semigroup with generator A. The following assertions are equivalent.*

(i) *$T(t)$ is a lattice homomorphism for all $t \geqslant 0$.*

(ii) *$f \in D(A)$ implies $Re\ f$, $|f| \in D(A)$ and $Re((\mathrm{sign}\ f)Af) = A|f|$.*

Example 5.7. Let $E = L^p(X, \mu)$ (where (X, μ) is a σ-finite measure space and $1 \leqslant p < \infty$) and A_0 be the generator of a semigroup of lattice homomorphisms. Let $h \in L^\infty$ and $B = A_0 + h$ (i.e. B is given by $Bf = A_0 f + h \cdot f$ for $f \in D(B) = D(A_0)$). Let $A = A_0 + Re\ h$. Since A_0 generates a semigroup of lattice homomorphisms, we have $|f| \in D(A_0)$ whenever $f \in D(A_0)$ and $Re((\mathrm{sign}\ f)A_0 f) = A_0|f|$. Hence $Re((\mathrm{sign}\ f)Bf) = Re((\mathrm{sign}\ f)A_0 f) + ((Re\ h) \cdot |f|) = A_0|f| + (Re\ h)|f| = A\ |f|$ for all $f \in D(B)$. Thus it follows from Theorem 5.5 that B generates a disjointness-preserving semigroup whose modulus semigroup is generated by A.

Next we describe in terms of the domain of the generator when a disjointness-preserving semigroup is positive.

Proposition 5.8. *Let E be a complex Banach lattice with order continuous norm and B be the generator of a disjointness-preserving semigroup $(S(t))_{t \geqslant 0}$. The semigroup is positive if and only if B is real and* span $D(B)_+ = D(B)$.

PROOF. The conditions are clearly necessary. In order to prove sufficiency, we can assume that E is real. Denote by A the generator of $(T(t))_{t \geqslant 0}$, where $T(t) = |\ S(t)\ |$. Let $f \in D(B)_+$. Since B is local we have $Bf = P_f\ Bf = (\mathrm{sign} f)\ Bf = A|f| = Af$. By assumption, span $D(B)_+ = D(B)$. Thus it follows that $B \subset A$. This implies that $B = A$ since $\rho(B) \cap \rho(A) \neq \varnothing$. $\square$

Finally, we show that for generators of disjointness-preserving semigroups Kato's inequality holds in the reverse sense.

Proposition 5.9. *Let B be the generator of a disjointness-preserving semigroup $(S(t))_{t \geqslant 0}$ on a real Banach lattice E with order continuous norm. Then*

$$\langle(\mathrm{sign} f)Bf, \phi\rangle \geqslant \langle\ |f|, B'\phi\rangle \quad \textit{for all } f \in D(B), \phi \in D(B')_+. \tag{5.5}$$

PROOF. Let $T(t) = |\ S(t)\ |$ and denote by A the generator of $(T(t))_{t \geqslant 0}$. *Let $f \in D(B)$, $\phi \in D(B')_+$.* Then $\langle(\mathrm{sign} f)Bf, \phi\rangle = \langle A|f|, \phi\rangle = \lim_{t \to 0} (1/t) \langle T(t)\ |f| - |f|, \phi\rangle \geqslant \lim_{t \to 0} 1/t\ \langle S(t)\ |f| - |f|, \phi\rangle = \langle\ |f|, B'\phi\rangle$. $\square$

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