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Resolvent Positive Operators and Integrated Semigroups

by

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The Hille-Yosida theorem yields the following characterization of generators of positive semigroups in terms of the resolvent.

Theorem. Let A be a densely defined operator on an ordered Banach space E. Then A generates a positive strongly continuous semigroup if and only if the following two conditions are satisfied.

a) There exists $w \in \mathbb{R}$ such that $(w,\infty) \subset \rho(A)$ and $R(\lambda,A) := (\lambda-A)^{-1}$ is positive for all $\lambda \in (w, \infty)$ (where $\rho(A)$ denotes the resolvent set of A).

b) $\sup \{||(\lambda - w)^n R(\lambda,A)^n||: \lambda > w, n \in \mathbb{N}\} < \infty$.

Given a concrete operator, condition b) is frequently difficult to verify since the powers of $R(\lambda,A)$ are involved. So we take condition a) as a definition.

Definition. An operator A on an ordered Banach space E is called resolvent positive if there exists $w \in \mathbb{R}$ such that $(w,\infty) \subset \rho(A)$ and $R(\lambda,A) \geq 0$ for all $\lambda > 0$.

Thus, every generator of a strongly continuous positive semigroup is resolvent positive. The converse is not true, though. In fact, there exist many natural resolvent positive operators which are not generators. Nevertheless these operators have remarkable properties:

Let A be a resolvent positive operator on an ordered Banach space (satisfying some additonal conditions which hold in particular for a Banach lattice and the hermitian part of a C^*-algebra). If A is densely defined or E is reflexive, then the resolvent of A is the Laplace-Stieltjes transform of a strongly continuous increasing family $(S(t))_{t\geq 0}$ of operators on E ; i.e.,

$$R(\lambda,A)f = \int_0^\infty e^{-\lambda t}\, dS(t) = \int_0^\infty \lambda\, e^{-\lambda t}\, S(t)\, dt$$

for $\mathrm{Re}\lambda$ sufficiently large.

This result has interesting consequences. For instance, the "integrated semigroup" $(S(t))_{t\geq 0}$ yields a unique solution of the abstract Cauchy problem

$$u'(t) = Au(t)$$
$$u(0) = f$$

for every $f \in D(A^2)$.

These facts will be explained in section 2. For their proofs we refer to [3] which contains a detailed study of resolvent positive operators. In section 1 we state basic properties of resolvent operators. The purpose is to illustrate how "close" to generators these operators are.

In section 3 the "integrated semigroup" $(S(t))_{t\geq 0}$ is investigated. We prove a certain algebraic relation which corresponds to the semigroup property in the case of a generator. This result is a contribution to the general task in the theory of Laplace transforms to establish a relation between properties of the determining function and its Laplace (-Stieltjes) transform.
Finally, in the last section, we study perturbations of the form $A_\alpha = A + \alpha B$ where A is the generator of a positive semigroup and $B : D(A) \to E$ is positive. One always obtains a resolvent positive operator for small $\alpha > 0$. However, A_α is a generator only in special cases.
The representation of section 3 and 4 differs from that in [3] and new information is included (with proofs).

1. Elementary Properties.

Throughout this paper E denotes a real ordered Banach space with generating and normal cone (i.e. $E = E_+ - E_+$ and $E' = E'_+ - E'_+$; where $E'_+ = \{\phi \in E' : \phi(f) \geq 0 \text{ for all } f \in E_+\}$ denotes the dual cone of E_+ .) If E is a Banach lattice or the hermitian part of a C^*-algebra, then E satisfies these assumptions.

For the definition of resolvent positive operators we refer to the introduction. We first give an example.

Example 1.1. Let $\alpha \in (0,1)$. Define the operator A by

$$Af(x) = f'(x) + \frac{\alpha}{x} f(x) \qquad x \in (0,1]$$

on the space $E = C_o(0,1] := \{f \in C[0,1] : f(0) = 0\}$ with domain $D(A) = \{f \in C^1[0,1] : f'(0) = f(0) = 0\}$. Then A is a densely defined resolvent positive operator but not a generator.

Similar examples can be given on L^p-spaces (see section 4). However, there is one exceptional case, where every densely defined resolvent positive operator is a generator.

Theorem 1.2 ([2],[5]). If E_+ contains an interior point (e.g. if $E = C(K)$, K compact), then every densely defined resolvent positive operator on E is a generator of a strongly continuous positive semigroup.

Let A be a resovent positive operator. We denote by

(1.1) $$s(A) = \sup \{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$$

the spectral bound of A . One always has $s(A) < \infty$ and

(1.2) $$s(A) \in \sigma(A) \quad \text{whenever} \quad s(A) > -\infty .$$

Moreover,

(1.3) $$|\langle R(\lambda,A)f, \phi\rangle| \le \langle R(\mathrm{Re}\lambda,A)f, \phi\rangle \quad \text{for all } f \in E_+, \ \phi \in E'_+.$$

whenever $\mathrm{Re}\lambda > s(A)$. In particular, $R(\lambda,A) \geq 0$ for all $\lambda > s(A)$.

The resolvent is decreasing on $(s(A), \infty)$. If $D(A)$ is dense, then $R(\lambda,A)$ converges strongly to 0 for $\lambda \to \infty$.

Remark 1.3. It follows from the above that a positive resolvent is norm bounded in a right half plane. So it follows from [15 , Theorem 6.1] that a densely defined resolvent positive operator is a generator of a distribution semigroup of exponentiel type in the sense of Lions [15]. However, we will not make use of this theory. In fact, more specific tools (such as the Laplace-Stieltjes transform) are available in our context.

There are some automatic norm estimates for the powers of positive resolvents. Let A be a resolvent positive operator. Then for every $w > s(A)$

(1.3) $$\sup \{\|(\lambda-w)^{n-1}R(\lambda,A)^n\| : \lambda \geq w, \ n \in \mathbb{N}\} < \infty .$$

This is weaker than the Hille-Yosida norm condition. In fact, it can happen that $\lim_{\lambda\to\infty} \lambda\|R(\lambda,A)\| = \infty$. But even if $\overline{\lim}_{\lambda\to\infty} \|\lambda R(\lambda,A)\| < \infty$, A does not need to be a generator. For example, the operator A in Example 1.1 satisfies $s(A) = -\infty$ and $\|\lambda R(\lambda,A)\| \leq 1/(1-\alpha)$ for all $\lambda \geq 0$.

2. Positive Resolvent as Laplace-Stieltjes Transform.

Some preliminaries concerning the vector-valued Riemann-Stieltjes integral are needed. Let $S : [a,b] \to \mathcal{L}(E)$ be increasing. Then for every $f \in C[a,b]$ the Riemann-Stieltjes integral $\int_a^b f(t)\, dS(t)$ exists in the following sense.
We denote the subdivision $(t_0=a \leq t_1 \leq t_2 \leq \ldots \leq t_n=b)$ together with points $s_i \in (t_{i-1},t_i)$ by π and let $|\pi| = \max_i |t_i-t_{i-1}|$. Let

$$\sum_\pi (f) = \sum_{i=1}^n f(s_i)(S(t_{i+1})-S(t_i)) .$$

Then $\lim_{|\pi|\to 0} \sum_\pi (f)$ exists in the operator norm and this limit is denoted by the integral

$$\int_a^b f(t)\, dS(t)$$

In order to define the Laplace-Stieltjes transform let $S : [0,\infty) \to \mathcal{L}(E)$ be increasing. Let $\lambda \in \mathbb{C}$. We say that the integral $\int_0^\infty e^{-\lambda t}\, dS(t)$ <u>converges in the operator norm</u> if $\lim_{b\to\infty} \int_0^b e^{-\lambda t}\, dS(t)$ exists for the operator norm. In that case we also set $\int_0^\infty e^{-\lambda t}\, dS(t) := \lim_{b\to\infty} \int_0^b e^{-\lambda t}\, dS(t)$.

Definition 2.1. We say that E is an ideal in E'' if for $f \in E$, $g \in E''$, $0 \leq g \leq f$ implies $g \in E$.

Note: Here we identify E with a subspace of E'' (via evaluation). Then $E_+'' \cap E = E_+$ (i.e., E is an ordered subspace of E'').

Examples 2.2. a) If E is reflexive, then E is trivially an ideal in E''.

b) A Banach lattice E is an ideal in E'' if and only if the norm is order continuous (see [20,II§5]). For example, $L^p(X,\mu)$ ((X,μ) a σ-finite measure space and $1 \leq p < \infty$) and c_o have an order continuous norm, but $C[0,1]$ has not.

The following is the main result.

Theorem 2.3. Let A be a resolvent positive operator. Suppose that either $D(A)$ is dense in E or that E is an ideal in E''. Then there exists a unique strongly continuous family $(S(t))_{t \geq 0}$ of operators on E such that

$$0 = S(0) \leq S(s) \leq S(t) \qquad (0 \leq s \leq t) \quad \text{and}$$

$$(2.2) \qquad R(\lambda,A)f = \int_o^\infty e^{-\lambda t}\, dS(t) \qquad (\mathrm{Re}\lambda > s(A))$$

(where the improper integral converges in the operator norm).

Remark 2.4. The Stieltjes integral (2.2) can be transformed into a Riemann integral. In fact, the following holds.

$$(2.3) \qquad R(\lambda,A)f = \int_0^\infty \lambda e^{-\lambda t} S(t)\,dt \qquad (\mathrm{Re}\lambda > \max\{s(A),0\}) .$$

Here the improper integral converges in the operator norm and the integral $\int_0^b e^{-\lambda t} S(t)\,dt$ is the limit of the Riemann sums in the operator norm.

Remark 2.5. To the two different assumptions made in Theorem 2.3 correspond two completely different proofs. In the case where $D(A)$ is dense, using a construction due to P. Chernoff [9], one obtains that the closure of A in an enlarged space is a generator of a strongly continuous positive semigroup. In the case where E is an ideal in E'' a vector-valued version of Bernstein's theorem can be proved. We refer to [3] for details.

Theorem 2.3 is related to Bernstein's classical theorem [23, 6.7]. In fact, if A is a resolvent positive operator, then for $\lambda > s(A)$

$$(2.4) \qquad (-1)^n R^{(n)}(\lambda,A) = n!\, R(\lambda,A)^{n+1} \geq 0$$

for all $n \in \mathbb{N}$. Thus, for all $f \in E_+$, $\phi \in E'_+$, the function $\lambda \to \langle R(\lambda,A) f,\phi\rangle$ is completely monotonic on $(s(A),\infty)$.

We call the family $(S(t))_{t\geq 0}$ in Theorem 2.3 the integrated semigroup generated by A .

Example 2.6. a) Let A be the generator of a strongly continuous positive semigroup $(T(t))_{t\geq 0}$. Then $S(t)f = \int_0^t T(s)f\,ds$ for all $f \in E$, $t \geq 0$.

b) If A is the operator in Example 1.1, then

$$S(t)f(x) = \begin{cases} x^\alpha \int_0^x y^{-\alpha} f(y)\,dy & \text{if } x \leq t \\ x^\alpha \int_{x-t}^x y^{-\alpha} f(y)\,dy & \text{if } x > t \end{cases}$$

$(f \in C_o(0,1],\ x \in (0,1],\ t \geq 0)$.

The abstract Cauchy problem associated with a resolvent positive operator admits unique solutions for a large class of initial values. Note that $D(A^2)$ is dense whenever $D(A)$ is dense.

Theorem 2.7. Let A be a resolvent positive operator. Assume that either $D(A)$ is dense in E or that E is an ideal in E''. Then for every $f \in D(A^2)$ there exists a unique continuously differentiable function $u : [0,\infty) \to E$ such that $u(t) \in D(A)$ for all $t \geq 0$ and

$$(2.5) \qquad \begin{aligned} u'(t) &= Au(t) \\ u(0) &= f\,. \end{aligned}$$

If $f \geq 0$, then $u(t) \geq 0$ for all $t \geq 0$. Moreover, the solutions of (2.5) depend continuously on the initial values in the following sense: Let $f_n \in D(A^2)$ such that $\lim_{n\to\infty} f_n = f$ in the graph norm. Denote by u_n the solution of (2.5) for the initial value f_n. Then $u_n(t)$ converges to $u(t)$ in the norm uniformly on bounded intervals.

Remark. Theorem 2.7 can be proved with help of Theorem 2.3. The solution is given by $u(t) = S(t)Af + f$ where $(S(t))_{t\geq 0}$ denotes the integrated semigroup generated by A .

3. The Integrated Semigroup.

In the preceding section a positive resolvent has been shown to be representable as a Laplace-Stieltjes transform of an increasing operator valued function. We now proceed in the reverse direction. Given an increasing function $S : [0,\infty) \to \mathcal{L}(E)$, we ask which conditions on S imply that the Laplace-Stieltjes transform of S is the (positive) resolvent of an operator? This question arises naturally since the Laplace-Stieltjes transform uniquely determines the determining function (up to normalization). So one expects that there exists a property of S which is equivalent to the Laplace-Stieltjes transform of S being a pseudoresolvent.

Let G be a Banach space and $S : [0,\infty) \to \mathcal{L}(G)$ a strongly continuous function satisfying $S(0) = 0$. Then for every $b > 0$ we can define the operator $\int_0^b S(t)\,dt \in \mathcal{L}(G)$ by $(\int_0^b S(t)dt)f := \int_0^b S(t)f\,dt$ (where the last expression denotes the usual Riemann integral) for all $f \in G$. Let us assume that S is of exponential growth; i.e., there exist $M \geq 0$ and $w \in \mathbb{R}$ such that

$$\|S(t)\| \leq Me^{wt} \quad \text{for all } t \geq 0 . \tag{3.1}$$

Then, for every $\lambda > w$, we can define the operator

$$R(\lambda) = \int_0^\infty \lambda e^{-\lambda t} S(t)\, dt = \lim_{b\to\infty} \int_0^b \lambda e^{-\lambda t} S(t)\,dt.$$

Theorem 3.1. The family $(R(\lambda))_{\lambda > w}$ is a pseudoresolvent if and only if

(3.2) $$S(s)S(t) = \int_0^{s+t} S(r)\,dr - \int_0^t S(r)\,dr - \int_0^s S(r)\,dr$$

for all $s,t > 0$.

Remark. a) Relation (3.2) implies that $S(s)$ and $S(t)$ commute for all $s,t > 0$.

b) Obviously, (3.2) is equivalent to

(3.3) $$S(s)S(t) = \int_0^s (S(r+t) - S(r))\,dr .$$

Proof of Theorem 3.1. We first observe that for $f \in G$, $\phi \in G'$

(3.4) $$\langle R(\lambda)f, \phi\rangle = \int_0^\infty \lambda e^{-\lambda t} \langle S(t)f,\phi\rangle\, dt \qquad (\lambda > w) .$$

This allows us to carry over the usual integration rules for numerical valued functions to our situation.

We show that for $\lambda,\mu > w$

(3.5) $$(R(\lambda)-R(\mu))/(\mu-\lambda) = \int_0^\infty \mu e^{-\mu t} \int_0^\infty \lambda e^{-\lambda s} \int_0^s (S(t+r)-S(r))\,dr\,ds\,dt.$$

$(R(\lambda)-R(\mu))/(\mu-\lambda) =$

$\mu/(\lambda(\mu-\lambda)) - 1/\lambda)\ R(\lambda) - 1/(\mu-\lambda)\ R(\mu) =$

$\mu/(\mu-\lambda) \int_0^\infty e^{-\lambda r}S(r)\,dr - \mu/(\mu-\lambda) \int_0^\infty e^{-\mu r}S(r)\,dr - 1/\lambda\ R(\lambda) =$

$\mu/(\mu-\lambda) \int_0^\infty e^{-\lambda r}S(r)(1 - e^{-(\mu-\lambda)r})dr - 1/\lambda\ R(\lambda) =$

$\mu \int_0^\infty e^{-\lambda r}S(r) \int_0^r e^{-(\mu-\lambda)t}\,dt\,dr - 1/\lambda\ R(\lambda) =$

$\int_0^\infty \mu e^{-\mu t} e^{\lambda t} \int_t^\infty e^{-\lambda r}S(r)\,dr\,dt - 1/\lambda\ R(\lambda)$ (by Fubini's theorem)

$= \int_0^\infty \mu e^{-\mu t} \int_0^\infty e^{-\lambda s}S(s+t)\,ds\,dt - 1/\lambda\ R(\lambda)$

$= \int_0^\infty \mu e^{-\mu t} (\int_0^\infty e^{-\lambda s}S(s+t)\,ds - 1/\lambda R(\lambda))\,dt$

$= \int_0^\infty \mu e^{-\mu t} \int_0^\infty e^{-\lambda s}(S(s+t)-S(s))ds\,dt$

$= \int_0^\infty \mu e^{-\mu t} \int_0^\infty \lambda e^{-\lambda s} \int_0^s (S(r+t)-S(r))drdsdt$ (by integration by parts).

On the other hand,

(3.6) $R(\mu)R(\lambda) = \int_0^\infty \mu e^{-\mu t} \int_0^\infty \lambda e^{-\lambda s}S(t)S(s)ds\,dt$.

Thus, if $(S(t))_{t\geq 0}$ satisfies (3.3), then $(R(\lambda)-R(\mu))/(\mu-\lambda) = R(\mu)R(\lambda)$ by (3.5) and (3.6).

Conversely, if $(R(\lambda))_{\lambda>w}$ is a pseudoresolvent, then the identity of the integrals in (3.5) and (3.6) for all $\lambda,\mu > w$ implies (3.3) by the uniqueness theorem for Laplace transforms [23, 7.2] . □

<u>Corollary</u> 3.2. Let $T : (0,\infty) \to \mathcal{L}(G)$ be strongly continuous such that the Riemann integral $\int_0^1 T(s)f\,ds$ exists for all $f \in G$. Assume that there exist $M \geq 0$, $w \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{wt}$ for all $t \geq 0$. Let $R(\lambda) = \int_0^\infty e^{-\lambda t}T(t)\,dt$ $(\lambda>w)$.
Then $(R(\lambda))_{\lambda>w}$ is a pseudoresolvent if and only if $(T(t))_{t\geq 0}$ is a semigroup; i.e. $T(s+t) = T(s)T(t)$ for all $s,t \geq 0$.

Definition 3.3. A strongly continuous function $S : [0,\infty) \to \mathcal{L}(G)$ is called an integrated semigroup if $S(0) = 0$ and (3.2) holds for all $s,t \geq 0$. S is called non-degenerate if for every $f \in G$, $f \neq 0$ there exists $t > 0$ such that $S(t)f \neq 0$.

Proposition 3.4. Let $S : [0,\infty) \to \mathcal{L}(G)$ be an integrated semigroup of exponential growth. Then $R(\lambda) = \int_0^\infty \lambda e^{-\lambda t} S(t)\, dt$ (λ large) is the resolvent of an operator A if and only if S is non-degenerate. In that case A is densely defined if and only if $G_o := \bigcup_{t>0} S(t)G$ is dense.

Proof. There exists $w \in \mathbb{R}$ such that (3.1) holds. $(R(\lambda))_{\lambda>w}$ is a resolvent if and only if $R(\lambda)f = 0$ for all $\lambda > w$ implies that $f = 0$. By the uniqueness theorem for Laplace transforms, $R(\lambda)f = 0$ for all $\lambda>w$ is equivalent to $S(t)f = 0$ for all $t>0$. This proves the first assertion.

In order to show the second observe that $D(A) = R(\lambda)G$ $(\lambda>w)$ is not dense in G if and only if there exists $\phi \in G'$, $\phi \neq 0$ such that $\langle R(\lambda)f,\phi\rangle = 0$ for all $f \in G$ and $\lambda > w$. By the uniqueness theorem again, this is equivalent to $\langle S(t)f,\phi\rangle = 0$ for all $t > 0$, $f \in G$.

Thus $D(A)$ is not dense if and only if there exists $\phi \in G'$, $\phi \neq 0$ which vanishes on G_o ; i.e., if and only if G_o is not dense. □

Remark. Let $(T(t))_{t>0}$ be a strongly continuous semigroup (we do not assume anything about $t=0$). Suppose that the Riemann integral $\int_0^1 T(s)f\, ds$ exists for every $f \in G$. Then there exist

$M \geq 0$, $w \in \mathbb{R}$ such $\|T(t)\| \leq Me^{wt}$ for all $t \geq 1$ [12,(10.2.2)].
Let $R(\lambda) = \int_0^\infty e^{-\lambda t} T(t)\, dt$ $(\lambda > w)$. It follows from Corollary 3.2 that $(R(\lambda))_{\lambda > w}$ is a pseudoresolvent; and if the semigroup is non-degenerate (i.e., if $\int_0^t T(s)f\, ds = 0$ for all $t>0$ implies $f=0$), then $(R(\lambda))_{\lambda > w}$ is the resolvent of an operator A . One can consider A as the "generator" of $(T(t))_{t>0}$. Note that nearly all of the basic classes of semigroups considered by Hille and Phillips [12 ,10.6] satisfy our requirements. Thus Theorem 3.1 is also of interest for the treatment of one-parameter semigroups which are not continuous at the origin. We do not elaborate this idea.

Let $(S(t))_{t \geq 0}$ be a non-degenerate integrated semigroup on a Banach space G . Assume that $M, w \geq 0$ such that $\|S(t)\| \leq Me^{wt}$ for all $t \geq 0$.
Let A be the operator on G whose resolvent is given by

$$R(\lambda, A) = \int_0^\infty \lambda e^{-\lambda t} S(t)\, dt \qquad (\lambda > w)$$

(see Proposition 3.4). We call A <u>the generator</u> of $(S(t))_{t \geq 0}$. In the following proposition relations between the integrated semigroup and its generator are established.

<u>Proposition</u> 3.5. a) If $f \in D(A)$ then $S(t)f \in D(A)$ and $AS(t)f = S(t)Af$ for all $t \geq 0$.
b) $\int_0^t S(s)f\, ds \in D(A)$ for all $f \in E$, $t \geq 0$ and

$$(3.7) \qquad A\int_0^t S(s)f\, ds = S(t)f - tf \; .$$

c) $S(t)S(s)f \in D(A)$ for all $f \in E$, $s,t \geq 0$ and

$AS(t)S(s)f = S(t+s)f - S(t)f - S(s)f$.

Note: We do not assume that $D(A)$ is dense.

<u>Proof</u>. 1. Let $f \in E$ and $\mu \in \rho(A)$. Then for all $\lambda > w$,

$\int_0^\infty \lambda e^{-\lambda t} R(\mu,A)S(t)f\,dt = R(\mu,A)R(\lambda,A)f = R(\lambda,A)R(\mu,A)f =$ $\int_0^\infty \lambda e^{-\lambda t} S(t)R(\mu,A)f\,dt$. By the uniqueness theorem for Laplace transforms it follows that

$$(3.8)\quad S(t)R(\mu,A)f = R(\mu,A)S(t)f \quad (t \geq 0) .$$

This implies assertion a).

2. Let $f \in D(A)$. Then for all $\lambda > w$,
$\int_0^\infty \lambda^2 e^{-\lambda t} \int_0^t S(s)Af\,ds\,dt = \int_0^\infty \lambda e^{-\lambda t} S(t)Af\,dt = R(\lambda,A)Af =$ $\lambda R(\lambda,A)f - f = \int_0^\infty \lambda^2 e^{-\lambda t} S(t)f\,dt - \int_0^\infty \lambda^2 e^{-\lambda t}\,tf\,dt$. Hence $\int_0^\infty e^{-\lambda t} \int_0^t S(s)Af\,ds\,dt = \int_0^\infty e^{-\lambda t} S(t)f\,dt - \int_0^\infty e^{-\lambda t}\,tf\,dt$ for all $\lambda > w$. By the uniqueness theorem this implies

$$(3.9)\quad \int_0^t S(s)Af\,ds = S(t)f - tf \quad (t \geq 0) .$$

Now let $f \in E$ be arbitrary. Let $\mu \in \rho(A)$ and $t > 0$. Then by (3.9),

$\mu\int_0^t S(s)R(\mu,A)f\,ds - \int_0^t S(s)f\,ds = \int_0^t S(s)AR(\mu,A)f\,ds =$ $S(t)R(\mu,A)f - tR(\mu,A)f$. Hence,

$\int_0^t S(s)f ds = R(\mu,A)\mu\int_0^t S(s)f ds - R(\mu,A)S(t)f + tR(\mu,A)f \in D(A)$.
Now it follows from (3.9) that
$R(\mu,A)\ A\int_0^t S(s)f ds = \int_0^t S(s)AR(\mu,A)f ds = S(t)R(\mu,A)f - tR(\mu,A)f$
$= R(\mu,A)(S(t)f - tf)$. This implies (3.7) since $R(\mu,A)$ is injective. Thus b) is proved.
Finally, c) follows from b) and (3.2). □

We return now to positive resolvents. First we determine the abscissa of convergence of the Laplace-Stieltjes transform.

Proposition 3.6. Let $S : [0,\infty) \to \mathcal{L}(E)$ be increasing and satisfy $S(0) = 0$. Let $w \in \mathbb{R}$. Consider the following assertions.

(i) $\int_0^\infty e^{-wt}\, d\langle S(t)f,\phi\rangle$ converges for every $f \in E_+$, $\phi \in E'_+$ and $\lambda > w$.

(ii) For every $\lambda > w$ there exists $M \geq 0$ such that $\|S(t)\| \leq Me^{\lambda t}$ for all $t \geq 0$.

(iii) $\int_0^\infty e^{-\lambda t}\, dS(t)$ converges in the operator norm for $\mathrm{Re}\lambda > w$.

Then (i) implies (iii). Moreover, if $w \geq 0$, then (i) implies (ii), and (ii) implies (iii). Finally, if (i) holds, then

$$\int_0^\infty e^{-\lambda t} dS(t) = \int_0^\infty \lambda e^{-\lambda t} dS(t) \qquad (\mathrm{Re}\lambda > \max\{0,w\}) .$$

Consequence. Let $s := \inf \{w\in\mathbb{R} : \int_0^\infty e^{-wt}\, dS(t)$ converges in the operator norm$\}$. Then the integral $\int_0^\infty e^{-\lambda t}\, dS(t)$ converges in the operator norm whenever $\mathrm{Re}\lambda > w$, but it does not converge for any $\lambda < w$. Thus s is the abscissa of convergence for the weak, strong and uniform topology. If $(S(t))_{t\geq 0}$ is the integrated semigroup generated by a resolvent positive operator A according to Theorem 2.3, then $s = s(A)$. Moreover, Proposition 3.6 shows that $(S(t))_{t\geq 0}$ is of exponential growth.

<u>Theorem</u> 3.7. Let $(S(t))_{t\geq 0}$ be a non-degenerate integrated semigroup of exponential growth. If $S(t) \geq 0$ for all $t\geq 0$, then S is increasing and there exists a unique resolvent positive operator A such that

$$R(\lambda,A) = \int_0^\infty e^{-\lambda t}\, dS(t) \qquad (\mathrm{Re}\lambda > s(A)) .$$

<u>Proof</u>. Let $R(\lambda) = \int_0^\infty \lambda e^{-\lambda t} S(t)dt$ $(\lambda > w)$, where $w\in\mathbb{R}$ is such that $\|S(t)\| \leq Me^{wt}$ $(t\geq 0)$ for some $M\geq 0$. It follows from Theorem 3.1 and Proposition 3.4 that $(R(\lambda))_{\lambda > w}$ is the resolvent of a unique operator A . Let $f \in E_+$, $\phi \in E'_+$. Since $R(\lambda,A) \geq 0$ for $\lambda > w$, it follows that $\langle R(\lambda,A)f, \phi\rangle$ is a completely monotonic function in λ on $[w,\infty)$ (see section 2). Since $\langle R(\lambda,A)f, \phi\rangle = \int_0^\infty e^{-\lambda t} d\langle S(t)f,\phi\rangle$ $(\mathrm{Re}\lambda > s(A))$, it follows from Bernstein's theorem [23,6.7] and the uniqueness theorem that $\langle S(t)f,\phi\rangle$ is increasing in t . □

It is well known that a semigroup is automatically of exponential growth. We show that the same is true for integrated semigroups under some additional assumption.

Proposition 3.8. Assume that E is a Banach lattice or the hermitian part of a C^*-algebra.
Let $(S(t))_{t\geq 0}$ be an integrated semigroup. If $(S(t))_{t\geq 0}$ is increasing and there exists $r > 0$ such that $\|S(r)\| < 1$, then $(S(t))_{t\geq 0}$ is of exponential growth.

Remark 3.9. Concerning the additional assumption in Proposition 3.8 the following is to say. There exists a densely defined resolvent positive operator A such that for the integrated semigroup $(S(t))_{t\geq 0}$ generated by A one has $\|S(t)\| \geq 1$ for all $t>0$.
For the resolvent this implies that $\inf_{\lambda>s(A)} \|R(\lambda,A)\| > 0$ (Note however that $R(\lambda,A)$ tends to 0 strongly for $\lambda\to\infty$).
Nevertheless the assumption in Proposition 3.8 is not a too severe restriction. In fact, for an integrated semigroup $(S(t))_{t\geq 0}$ generated by a resolvent positive operator A one has $\overline{\lim}_{t\to 0} 1/t\|S(t)\| < \infty$ if and only if $\overline{\lim}_{\lambda\to\infty} \|\lambda R(\lambda,A)\| < \infty$. We will see in section 4 (Remark 4.2) that there are many resolvent positive operators which satisfy this (even more restrictive) condition.
b) The assumption on the space is made in order to be able to conclude that $0 \leq S \leq T$ implies that $\|S\| \leq \|T\|$ for all $S,T \in \mathscr{L}(E)$. The assumption on E can be omitted if $\lim_{t\to 0} \|S(t)\| = 0$. This will be seen in the proof.

For the proof of Proposition 3.8 we need the following Lemma.

Lemma 3.10. Let $(S(t))_{t\geq 0}$ be an integrated semigroup and $w \in \mathbb{R}$. Let

$$S_w(t) = \int_0^t e^{-sw}\, dS(s) \qquad (t\geq 0) \quad . \tag{3.10}$$

Then $(S(t))_{t\geq 0}$ is also an integrated semigroup.

We omit the proof of the Lemma.

Remark. Let A be a densely defined resolvent positive operator which generates the integrated semigroup $(S(t))_{t\geq 0}$. Let $w\in\mathbb{R}$. Then $A-w$ is resolvent positive and generates the integrated semigroup $(S_w(t))_{t\geq 0}$ given by (3.10).

Proof of Proposition 3.8. a) We assume that $\|S(1)\| < 1$. Let $t > 0$. We show that

$$S(t) \leq S(t)S(1) + S(1) \quad . \tag{3.11}$$

$$\begin{aligned} S(t) &\leq \int_t^{t+1} S(s)\,ds \\ &= \int_0^{t+1} S(s)\,ds - \int_0^t S(s)\,ds - \int_0^1 S(s)\,ds + \int_0^1 S(s)\,ds \\ &= S(t)S(1) + \int_0^1 S(s)\,ds \\ &\leq S(t)S(1) + S(1) \quad . \end{aligned}$$

Iterating (3.11) yields

$$S(t) \leq S(t)S(1)^n + \sum_{k=1}^n S(1)^k \qquad (n\in\mathbb{N}) \quad . \tag{3.12}$$

Since $\|S(1)\| < 1$ this implies that $S(t) \leq \sum_{k=1}^{n} S(1)^k$.
Consequently $(S(t))_{t\geq 0}$ is norm bounded.

b) Let $\|S(r)\| < 1$, where $0 < r < 1$. Choose $w > 0$ such that $e^{-wr} \|S(1)\| + (1 - e^{-wr}) \|S(r)\| < 1$.
Let $(S_w(t))_{t\geq 0}$ be the integrated semigroup given by (3.10).
Then integrating (3.10) by parts yields

$$\begin{aligned} S_w(1) &= e^{-w} S(1) + w \int_0^1 e^{-ws} S(s)\,ds \\ &\leq e^{-w} S(1) + w \int_r^1 e^{-ws} ds\; S(1) + w \int_0^r e^{-ws} ds\; S(r) \\ &= e^{-wr}\, S(1) + (1 - e^{-wr}) S(r) \ . \end{aligned}$$

Hence $\|S_w(1)\| < 1$.
By a) we get that $\sup_{t\geq 0} \|S_w(t)\| < \infty$. But
$S(t) = \int_0^t e^{sw}\, dS_w(s) = e^{wt} S_w(t) - w \int_0^t e^{sw}\, S_w(s)\, ds \leq e^{wt} S_w(t)$
for all $t \geq 0$. Consequently, $\|S(t)\| \leq Me^{wt}$ for all $t \geq 0$ and some $M > 0$. □

4. Perturbation.

The following result allows one to perturbate generators of positive semigroups so that resolvent positive operators are obtained. This method yields natural examples of densely defined operators which are resolvent positive but not generators of a semigroup, in general.

Theorem 4.1. Let A be a resolvent positive operator and $B : D(A) \to E$ a positive operator (i.e., $Bf \geq 0$ for all $f \in D(A) \cap E_+$).
For $\alpha \geq 0$, consider the operator $A_\alpha = A + \alpha B$ with domain $D(A_\alpha) = D(A)$.
Then there exists $\gamma \in (0,\infty]$ such that for $0 \leq \alpha \in \mathbb{R}$, A_α is resolvent positive if and only if $\alpha < \gamma$.
Moreover, $s(A_\alpha)$ is an increasing function of $\alpha \in [0,\gamma)$.

Proof. 1. Since E_+ is generating and normal, there exists a constant $c > 0$ such that for all $S,T \in \mathscr{L}(E)$, $0 \leq S \leq T$ implies $\|S\| \leq c\|T\|$. Thus by the spectral radius formula we obtain that

$$(4.1) \qquad 0 \leq S \leq T \text{ implies } r(S) \leq r(T) .$$

2. Let $\lambda > s(A)$. Then $(\lambda - A_\alpha) = [I - \alpha BR(\lambda,A)](\lambda-A)$. Thus for $0 < \alpha < r(BR(\lambda,A)))^{-1}$ one has $\lambda \in \sigma(A_\alpha)$ and

$$(4.2) \qquad R(\lambda,A_\alpha) = R(\lambda,A) \textstyle\sum_{n=0}^{\infty} (\alpha BR(\lambda,A)))^n \geq 0 .$$

Since $BR(\mu,A)$ is a decreasing function of $\mu \in (s(A),\infty)$, it follows by (4.1) that A_α is resolvent positive and $s(A_\alpha) \leq \lambda$.
We have proved that

$$(4.3) \qquad 0 < \gamma := \sup\{\alpha \geq 0 : A_\beta \text{ is resolvent positive for all } \beta \in [0,\alpha)\} .$$

3. Let $0 \leq \alpha \leq \beta$. Suppose that $\lambda \in \rho(A_\alpha) \cap \rho(A_\beta)$ such that $R(\lambda, A_\alpha) \geq 0$ and $R(\lambda, A_\beta) \geq 0$. We claim that

(4.4) $$R(\lambda, A_\alpha) \leq R(\lambda, A_\beta) .$$

For $f \in E_+$ we have

$$R(\lambda, A_\beta)f - R(\lambda, A_\alpha)f = R(\lambda, A_\beta)[(A_\beta - \lambda) + (\lambda - A_\alpha)]R(\lambda, A_\alpha)f =$$
$$R(\lambda, A_\beta)(\beta - \alpha)BR(\lambda, A_\alpha)f \geq 0 .$$

4. Let $\beta > 0$ such that A_β is resolvent positive. We have to show that $\beta < \gamma$. Assume that, on the contrary, $\beta \geq \gamma$. Then by 3. we have $0 \leq R(\lambda, A_\alpha) \leq R(\lambda, A_\beta)$ for all $\lambda > \max\{s(A_\alpha), s(A_\beta)\}$. Since $\lim_{\lambda \to s(A)} \|R(\lambda, A_\alpha)\| = \infty$ (by (1.2)), it follows that $s(A_\alpha) \leq s(A_\beta)$.
Let $w > s(A_\beta)$. Then for $\lambda \geq w$ and $0 \leq \alpha < \gamma$ one has $0 \leq BR(\lambda, A_\alpha) \leq BR(\lambda, A_\beta) \leq BR(w, A_\beta)$. Consequently, $M := \sup\{\|BR(\lambda, A_\alpha)\| : 0 \leq \alpha < \gamma , \lambda \geq w\} < \infty$. Let $\lambda \geq w$ and $\alpha \in [0, \gamma)$ such that $\gamma - \alpha < 1/M$. Then $(\lambda - A_\gamma) = (\lambda - (A + \alpha B)) - (\gamma - \alpha)B = [1 - (\gamma - \alpha)BR(\lambda, A_\alpha)](\lambda - A_\alpha)$. Thus $\lambda \in \rho(A_\gamma)$ and $R(\lambda, A_\gamma) = R(\lambda, A_\alpha) \sum_{n=0}^{\infty} [(\gamma - \alpha)BR(\lambda, A_\alpha)]^n \geq 0$. This implies that A_γ is resolvent positive and (4.3) applied to A_γ instead of A yields a contradiction.

<u>Remark</u> 4.2. It can be seen from (4.2) in the proof that for small $\alpha > 0$ one has
$\overline{\lim}_{\lambda \to \infty} \|\lambda R(\lambda, A_\alpha)\| < \infty$ if $\overline{\lim}_{\lambda \to \infty} \|\lambda R(\lambda, A)\| < \infty$ (which is satisfied if A is a generator of a strongly continuous semigroup).

Even in rather simple and natural cases perturbations as in Theorem 4.1 may yield resolvent positive operators which are not generators of a semigroup. One is given in Example 1.1. Another is the following.

Example 4.3. Let $E = L^p[0,1]$, where $1 < p < \infty$. Choose $\alpha \in (0, (p-1)/p)$. Define the operator A by

$$Af(x) = -f'(x) + (\alpha/x)f(x)$$

with domain $D(A) = \{f \in AC[0,1] : f' \in L^p[0,1], f(0) = 0\}$. Then A is densely defined and resolvent positive. Moreover, $s(A) < 0$ and $\sup\{\|\lambda R(\lambda,A)\| : \lambda \geq 0\} < \infty$. But A is not a generator of a semigroup.
This can be proved via Theorem 4.1 by perturbating the generator A_o given by $A_o f = -f'$ with domain $D(A_o) := D(A)$ by the operator $B : D(A_o) \to E$ given by $Bf(x) = \alpha f(x)/x \quad (x \in (0,1])$.

Starting from a generator (by which we always mean the generator of a strongly continuous semigroup in the following), the perturbation in Theorem 4.1 yields a generator again in two special cases. The first is based on a result by Desch and Schappacher [10,Theorem 1] and the automatic continuity of positive mappings which we prove in the appendix. The second concerns generators of holomorphic semigroups. We start with a preliminary result.

Proposition 4.4. Let A be the generator of a strongly continuous positive semigroup and $B : D(A) \to E$ a positive operator. Let

(4.5) $\eta = \sup \{\alpha \geq 0 : A_\beta$ is a generator for all $\beta \in [0,\alpha)\}$.

Let γ be as in Theorem 4.1. Then $\eta \leq \gamma$; i.e., the semigroup generated by A_β is positive for all $\beta < \eta$.
Moreover, if $\beta \geq 0$ is such that A_β generates a positive semigroup, then $\beta \leq \eta$.

Proof. Assume that, in contrast to the first assertion of the theorem, $\eta > \gamma$. Then A_γ is a generator. We show that A_γ is resolvent positive, which contradicts Theorem 4.1.
Since A_γ is a generator, there exists $w \in \mathbb{R}$ such that $[w,\infty) \subset \rho(A_\gamma)$. If S is a bounded operator from G into H we denote by $\|S\|_H^G$ its norm (G , H being Banach spaces). Consider $D(A)$ as a Banach space with the graph norm. Since $\sup_{\lambda\geq w} \{\|\lambda R(\lambda,A_\gamma)\| < \infty$, we have $M := \sup_{\lambda\geq w} \|R(\lambda,A_\gamma)\|_{D(A)}^E < \infty$.
The operator B is positive and so continuous from $D(A)$ into E (see Appendix). Consequently, $N := \sup_{\lambda\geq w} \|BR(\lambda,A_\gamma)\| \leq \|B\|_E^{D(A)} M < \infty$.
Let $\alpha \in [0,\gamma)$ such that $(\gamma-\alpha)N < 1$. Let $\lambda \geq w$. Then $(\lambda - A_\alpha) = (\lambda-A_\gamma) - (\alpha-\gamma)B = (I - (\alpha-\gamma)BR(\lambda,A_\gamma))(\lambda - A_\gamma)$.
So it follows that $\lambda \in \rho(A_\alpha)$.
Consequently, $s(A_\alpha) \leq w$ for all $\alpha > \gamma - N^{-1}$.
Let $\lambda \geq w$. Then $(\lambda - A_\alpha)$ is a bounded invertible operator in $\mathcal{L}(D(A),E)$ for all $\alpha > \gamma - N^{-1}$ and $(\lambda - A_\gamma)$ is an invertible operator in $\mathcal{L}(D(A),E)$. Moreover, $\lim_{\alpha\to\gamma} (\lambda-A_\alpha) =$

$(\lambda - A_\gamma)$ in $\mathcal{L}(E,D(A))$. This implies that $\lim_{\alpha\to\gamma} (\lambda-A_\alpha)^{-1} = (\lambda - A_\gamma)^{-1}$ in $\mathcal{L}(E,D(A))$. Since $(\lambda-A_\alpha)^{-1} \geq 0$, it follows that $(\lambda - A_\gamma)^{-1} \geq 0$. Thus A_γ is resolvent positive.

We prove the last assertion. Assume that $\beta \geq 0$ such that A_β generates a positive semigroup. Then $\beta < \gamma$ by Theorem 4.1 . Let $\alpha \in [0,\beta]$. Then $s(A_\alpha) \leq s(A_\beta)$ by Theorem 4.1. There exists $w > s(A_\beta)$ such that $\sup \{\|(\lambda-w)^n R(\lambda,A_\beta)^n\| : \lambda \geq w, n \in \mathbb{N}\} < \infty$. Since by (4.4) $0 \leq R(\lambda,A_\alpha) \leq R(\lambda,A_\beta)$, it follows that $R(\lambda,A_\alpha)^n \leq R(\lambda,A_\beta)^n$ for all $n \in \mathbb{N}$, $\lambda \geq w$. Hence $\sup \{\|(\lambda-w)^n R(\lambda,A_\alpha)^n\| : \lambda \geq w, n \in \mathbb{N}\} < \infty$. By the Hille-Yosida theorem A_α is a generator. Consequently, $\beta \leq \gamma$.□

<u>Theorem</u> 4.5. Let A be the generator of a strongly continuous semigroup and B a positive operator from $D(A)$ into $D(A)$. Then $A + B$ with domain $D(A+B) = D(A)$ is a generator of a strongly continuous semigroup.

<u>Proof</u>. Considering the graph norm on $D(A)$ and the cone $D(A)_+ := D(A) \cap E_+$, $D(A)$ is an ordered Banach space with generating cone (which is in general not normal, though).

Since $B : D(A) \to D(A)$ is positive, it follows from the theorem in the appendix, that B is continuous. Thus by [10 , Theorem 1], $A + B$ with domain $D(A)$ is the generator of a strongly continuous semigroup. It follows from Proposition 4.4 that the semigroup is positive. □

In Theorem 4.5 we made a special assumption on B (namely that B maps into $D(A)$). In the next result an additional assumption on A implies that $A + \alpha B$ is a generator for small $\alpha > 0$.

<u>Theorem</u> 4.6. Let A be the generator of a positive holomorphic semigroup and $B : D(A) \to E$ a positive operator. Let

$$\iota := \{\beta \geq 0 : A+\alpha B \text{ generates a holomorphic semigroup for all } \alpha \in [0,\beta)\} .$$

Then $\iota > 0$ and the semigroup generated by $A + \alpha B$ is positive for all $\alpha \in [0,\iota)$.

<u>Proof</u>. Since B is positive , B is continous when $D(A)$ is considered with the graph norm; i.e., B is relatively A-bounded.
It follows from [13 , IX Theorem 2.4] that $\iota > 0$. Proposition 4.4 yields the last assertion. □

<u>Question</u>. In the situation of Theorem 4.6 we have three constants; namely, ι, η (given by (4.4)) and γ (according to Thorem 4.1), which satisfy $0 < \iota \leq \eta \leq \gamma$. Can it happen that these constants are different?

Appendix (Automatic Coninuity of Positive Linear Mappings).

Theorem. Let E, F be Banach spaces and E_+ (resp., F_+) a closed cone in E (resp., F). Assume that E_+ is generating (i.e., $E = E_+ - E_+$) and F_+ is proper (i.e., $F_+ \cap (-F_+) = \{0\}$). If $T : E \to F$ is a linear positive mapping (i.e., $TE_+ \subset F_+$), then T is continuous.

Proof. a) If $g \in F$ such that $\langle g,\phi\rangle = 0$ for all $\phi \in F'_+$, then $g = 0$. In fact, since $F_+ \cap (-F_+) = \{0\}$ it follows from [19, Chapter IV, 1.5 Corollary] that $\overline{(F'_+ - F'_+)}^{\sigma(F',F)}$ $= (F_+ \cap (-F_+))^{\circ} = F'$. So the assumption implies that $\langle g,\phi\rangle = 0$ for all $\phi \in F'$. Hence $g = 0$.

b) We show that T has a closed graph (which implies continuity). Let $f_n \to f$ in E and $Tf_n \to g$ in F. We have to show that $Tf = g$. Let $\phi \in F'_+$. Then $\psi := \phi \circ T$ is a positive linear form on E. Thus ψ is continuous [19, Ch.V, 5.5], [8, A2]. Consequently, $\langle g,\phi\rangle = \lim_{n\to\infty} \langle Tf_n,\phi\rangle = \lim_{n\to\infty} \langle f_n,\psi\rangle = \langle f,\psi\rangle = \langle Tf,\phi\rangle$. It follows from a) that $g = Tf$. □

Remark. The theorem is false if E_+ is merely total (i.e., $\overline{(E_+ - E_+)} = E$) but not generating and $F \neq \{0\}$.

In fact, then there exists a subspace M of E such that $\dim E/M = 1$ and $E_+ - E_+ \subset M$. There exixts a linear form ϕ on E such that $M = \ker \phi$. Hence ϕ is positive. But ϕ is not continuous (since otherwise $E = \overline{E_+ - E_+} \subset \ker\phi$). Let $T = \phi \times f$ for some $f \in F$, $f \neq 0$. Then $Tg = 0$ for all $g \in E_+ \subset M$. Hence T is positive but not continuous. □

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