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RESEARCH ARTICLE

THE SPECTRAL MAPPING THEOREM FOR ONE-PARAMETER GROUPS OF POSITIVE OPERATORS ON $C_o(X)$

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1. THE MAIN RESULT

Let A be the generator of a strongly continuous group $(T(t))_{t\in\mathbb{R}}$ on a Banach space. It is desirable to express the spectrum $\sigma(T(t))$ of the single operator $T(t)$ in terms of the spectrum $\sigma(A)$ of the generator A. Formally one would expect that

$$(1.1)\qquad \sigma(T(t)) = \overline{\exp t\sigma(A)} \qquad (t\in\mathbb{R}) .$$

For example, if H is a selfadjoint (unbounded) operator on a Hilbert space and $T(t) = \exp itH$ $(t\in\mathbb{R})$, then $A = iH$ and (1.1) holds. This follows easily from the spectral theorem. More generally, the spectral mapping theorem (that is, formula (1.1)) holds for isometric groups on arbitrary Banach spaces (see [4, 2.12, Corollary 1] and [7, 8.19]). However, it is known that (1.1) does not hold for unbounded groups even under fairly strong assumptions. In fact, there exists a strongly continuous group of positive operators on a Banach lattice (which is an intersection of weighted L^p-spaces) such that (1.1) fails (see [23]).

Our main result is the following spectral mapping theorem on $C_o(X)$ (the space of all continuous complex valued functions on a locally compact space X which vanish at infinity). Its proof will be established in several steps in sections 2 to 6.

THEOREM 1.1. *Let* $(T(t))_{t\in\mathbb{R}}$ *be a strongly continuous group of positive operators on* $C_o(X)$ *with generator* A . *Then*

$$\sigma(T(t)) = \overline{\exp t\,\sigma(A)} \qquad (t \in \mathbb{R}) .$$

There are several consequences of this theorem which remind one of the kind of conclusions the spectral theorem allows for selfadjoint operators.

COROLLARY 1.2. *Under the assumptions of the theorem the following assertions are equivalent:*

(i) A *is bounded.*

(ii) $\sigma(A)$ *is bounded.*

(iii) $\sigma(A) \subseteq \mathbb{R}$

(iv) *The group consists of multipliers, i. e.* $T(t)f = m_t f$ $(f \in C_o(X))$ *for all* $t \in \mathbb{R}$, *where* $m_t \in C^b(X)$.

(v) A *is a bounded multiplier, i. e.* $Af = mf$ *for all* $f \in C_o(X)$ *and some* $m \in C^b_{\mathbb{R}}(X)$.

Here $C^b(X)$ (*resp.* $C^b_{\mathbb{R}}(X)$) *denotes the space of all continuous complex- (resp. real-) valued bounded functions on* X .

Proof. It is evident that (i) implies (ii). Since $\sigma(A)$ is additively cyclic [8, 2.4], it follows that (ii) implies (iii). If (iii) holds, then by Theorem 1.1 $\sigma(T(t)) = \overline{\exp t\,\sigma(A)} \subseteq \mathbb{R}_+$. So (iv) follows from [21, 2.1] (or [2, 3.6]). Now assume (iv). For every $x \in X$ the function $t \to m_t(x)$ from $\mathbb{R}$ into $(0,\infty)$ is continuous and satisfies $m_{t+s}(x) = m_t(x)m_s(x)$. Thus there exists $m(x) \in \mathbb{R}$ such that $m_t(x) = \exp t\,m(x)$ $(t \in \mathbb{R})$. Since m_1 and $(m_1)^{-1} = m_{-1}$ are positive, bounded and continuous, it follows that $m = \log m_1$ is an element of $C^b_{\mathbb{R}}(X)$. It is evident that (v) holds. Finally, (v) trivially implies (i).

COROLLARY 1.3. *Assume that* A *satisfies the hypotheses of the theorem. Then*

$$\sigma(A) = \{0\} \quad \underline{\text{if and only if}} \quad A = 0.$$

Another consequence that parallels the case of a selfadjoint operator is the following: Let A be the generator of a strongly continuous group $(T(t))_{t\in\mathbb{R}}$ of positive operators on $C_0(X)$. Then every isolated point of $\sigma(A)$ is an eigen-value of algebraic multiplicity 1. For the proof of this result, though, we refer to section 6.

We want to give some introductory remarks on the method of proof and the organization of the paper.

In section 3 we show that the generator A of a positive group $(T(t))_{t\in\mathbb{R}}$ on $C_0(X)$ has the following form up to similarity: There exists an automorphism group $(T_0(t))_{t\in\mathbb{R}}$ with generator A_0 and $h \in C^b_{\mathbb{R}}(X)$ such that $Af = A_0f + hf$ for $f \in D(A) = D(A_0)$. The automorphism group $(T_0(t))_{t\in\mathbb{R}}$ is defined by a continuous flow $\varphi: \mathbb{R}\times X \to X$ via composition, i. e. $(T_0(t)f)(x) = f(\varphi(t,x))$ $(t \in \mathbb{R},\ x \in X)$ for all $f \in C_0(X)$. Moreover, $T(t)$ is given by $T(t)f = h_t\, T_0(t)f$, where $(h_t)_{t\in\mathbb{R}}$ is a cocycle given by

$$h_t(x) = \exp\int_0^t h(\varphi(s,x))ds \qquad (t \in \mathbb{R},\ x \in X) .$$

In section 2 we determine the spectrum of the single operator $T(t)$ and in sections 4 and 5 the spectrum of A by means of the properties of the multiplier h and the flow φ. The spectral mapping theorem follows then by comparing both spectra.
The complete description of $\sigma(A)$ seems to be of independent interest. Whereas the spectrum of $\sigma(A_0)$ does not give much information about the flow (in general one has $\sigma(A_0) = i\,\mathbb{R}$), the additive perturbation with the multiplier M_h (given by $M_hf = hf$) extends the spectrum, and our results show that the spectrum of $A = A_0 + M_h$ reflects a good deal of the geometric behavior of the flow.

We want to conclude this introductory section with a dual result of Theorem 1.1. As pointed out above $C_0(X)$ cannot be replaced by an arbitrary Banach lattice. However, we obtain the following result for the space $L^1(Y,\mu)$:

COROLLARY 1.4. *Let* A *be the generator of a strongly continuous group* $(T(t))_{t\in\mathbb{R}}$ *of positive operators on* $L^1(Y,\mu)$. *Then the spectral mapping theorem holds; i. e.*

$$\sigma(T(t)) = \overline{\exp t\sigma(A)} \qquad (t \in \mathbb{R}) .$$

Proof. Let $E^* = \{x' \in E' : \lim_{t\to o} \|T(t)'x' - x'\| = 0\}$. Since $(T(t))_{t\in\mathbb{R}}$ is a group of lattice isomorphisms, E^* is a closed sublattice of E'. Moreover, for every order unit u of E', $T(t)'u$ is an order unit $(t \in \mathbb{R})$. So by [6, Theorem 3.2], there exists an order unit u of E' such that $u \in D(A') \subseteq E^*$. Thus E^* is an AM-space with order unit and so E^* is a space of the type $C(X)$, X compact [20, II § 7]. Let $T(t)^*$ be the restriction of $T(t)'$ to E^* $(t \in \mathbb{R})$; i. e. $(T(t)^*)_{t\geq o}$ is the adjoint semigroup of $(T(t))_{t\geq o}$ (see [15, Chapter XIV]). Denote its generator by A^*. Then Theorem 1.1 together with [9, 1.6] implies that

$$\sigma(T(t)) = \sigma(T(t)^*) = \overline{\exp t\sigma(A^*)} = \overline{\exp t\sigma(A)} \qquad (t \in \mathbb{R}) .$$

2. THE SPECTRUM OF A SINGLE OPERATOR

If T is a linear operator on a (complex) Banach space E and N is a closed T-invariant subspace, i. e. N satisfies $T(N) \subseteq N$, then T induces in a canonical way linear operators $T_{|N}$ and $T_{/N}$ on the subspace N and the quotient $E_{/N}$ respectively. It is well known that the spectra of these operators satisfy $\sigma(T) \subseteq \sigma(T_{|N}) \cup \sigma(T_{/N})$ and simple examples show that in general this inclusion is proper. However, for lattice homomorphisms on Banach lattices one has the follwoing result:

PROPOSITION 2.1. *Suppose* E *is a Banach lattice,* $T \in \mathcal{L}(E)$ *is a lattice homomorphism (i. e.* T *satisfies* $|Tx| = T|x|$ *for all* $x \in E$*) and* I *a closed ideal such that* $T(I) = I$*, then*

$$\sigma(T) = \sigma(T_{|I}) \cup \sigma(T_{/I}) .$$

Proof. By [20, II.5.5 Corollary 1] the polar of I is a projection band in the dual space, i. e. $E' = I^o \oplus I^{o\perp}$. The assumption $T(I) = I$ implies for arbitrary $x' \in E'$:

(2.1) $\qquad T'x' \in I^o$ if and only if $x' \in I^o$.

Thus I^o is T'-invariant. Moreover for $0 \leq y' \in I^{o\perp}$, $0 \leq x' \in I^o$, $z' := x' \wedge T'y'$ one has $z' \in I^o$ and $0 \leq z' \leq T'y'$. Since T' is interval preserving (see [17, 1.2]) there exists $y_1' \in E'$ such that $T'y_1' = z'$ and $0 \leq y_1' \leq y'$, in particular $y_1' \in I^o$. By (2.1) we have $y_1' \in I^o$, thus $y_1' = 0$ and $z' = 0$. This shows that $I^{o\perp}$ is T'-invariant too, and it follows that $\sigma(T') = \sigma(T'_{|I^o}) \cup \sigma(T'_{|I^{o\perp}})$. Since the spectra of an operator and its adjoint coincide, the assertion of the proposition is a consequence of the following identifications: $(T_{/I})' \cong T'_{|I^o}$ and $(T_{|I})' \cong T'_{/I^o} \cong T'_{|I^{o\perp}}$.

In the following remark we list some consequences and extensions of the proposition. Since we don't need them in the sequel, the proofs are omitted. Moreover we show by an example that the hypotheses cannot be weakened.

REMARK 2.2. (a) If T is a lattice homomorphism with $T(I) = I$ and Z is in the center (i. e. $Z(J) \subseteq J$ for every closed ideal J), then the statement of the proposition remains true for the operators $T + Z$, ZT and TZ.

(b) The proposition remains true if the spectrum is replaced by the approximate point spectrum. However, it is false for the point spectrum.

(c) As a consequence of Proposition 2.1 one obtains the following result:

If I and J are closed ideals, T is a lattice homomorphism such that $T(I) = I$ and $T(J) = J$, then one has:

$$\text{If } I + J = E, \text{ then } \sigma(T) = \sigma(T_{|I}) \cup \sigma(T_{|J}) ;$$
$$\text{if } I \cap J = \{0\}, \text{ then } \sigma(T) = \sigma(T/_I) \cup \sigma(T/_J) .$$

(d) If we consider $E = c_o(\mathbb{Z})$ (or $\ell^p(\mathbb{Z})$), $I := \{(\xi_\nu) \, E: \xi_\nu = 0$ for $\nu \leq 0\}$, the shift operator $T \in \mathcal{L}(E)$ (i. e. $(Tx)_\nu = \xi_{\nu-1}$ for $x = (\xi_\nu) \in E$) and define $S := (2 - T)^{-1}$, then we have:

> T is a lattice isomorphism such that $T(I) \subsetneqq I$ and we have $\sigma(T) = \{z \in \mathbb{C}: |z| = 1\}$, while
> $\sigma(T_{|I}) = \sigma(T/_I) = \{z \in \mathbb{C}: |z| \leq 1\}$;
> S is a positive operator such that $S(I) = I$ and we have $\sigma(S) = \{z \in \mathbb{C}: |z - 1| = \frac{1}{2}\}$, while
> $\sigma(S_{|I}) = \sigma(S/_I) = \{z \in \mathbb{C}: |z - 1| \leq \frac{1}{2}\}$.

We will apply Proposition 2.1 only to operators acting on the Banach lattice $C_o(X)$ where X is a locally compact Hausdorff space. In this case the closed ideals and the quotionts with respect to closed ideals are spaces of the same type. More precisely the following holds. Recall that an open or closed subset of a locally compact space is locally compact.

PROPOSITION 2.3. *Let X be a locally compact space, A a closed subset of X. Then $I := \{f \in C_o(X): f_{|A} = 0\}$ is a closed ideal and every closed ideal in $C_o(X)$ has this form. Moreover one has the following identifications:*

(a) The restriction map $f \to f_{|X \setminus A}$ is an isometric lattice isomorphism of I onto $C_o(X \setminus A)$.

(b) The mapping $(f + I) \to f_{|A}$ is an isometric lattice isomorphism of $C_o(X)/_I$ onto $C_o(A)$.

Observing that $C_o(X)$ is a closed ideal in $C(X \cup \{\infty\})$, the description of the closed ideals follows from [20, III 1 Example 1]. (a) is verified easily. It follows from Tietze's extension theorem that the map given in (b) is onto.

In the rest of this section we will describe the spectrum of a lattice isomorphism on $C_o(X)$. First we recall some well-known facts: Every lattice isomorphism T on $C_o(X)$ has the following form (compare with [20, III.9.1]):

(2.2) $Tf = k\, f \circ \psi \quad (f \in C_o(X))$, where $\psi: X \to X$ is a homeomorphism and $k: X \to \mathbb{R}_+$ is a continuous function such that k and k^{-1} are bounded.

One has $T^{-1}f = (1/k \circ \psi^{-1}) f \circ \psi^{-1}$, $\|T\| = \sup\{k(x):\ x \in X\}$ and $\|T^{-1}\|^{-1} = \inf\{k(x):\ x \in X\}$.

For $x \in X$ <u>the period of</u> <u>x</u> (with respect to ψ) is defined by

(2.3) $$\nu(x) := \inf\{n \in \mathbb{N}:\ \psi^n(x) = x\}, \quad \text{where} \quad \inf \emptyset = \infty .$$

The subset of all points with period less or equal than n is denoted by X_n; that is

(2.4) $$X_n := \{x \in X:\ \nu(x) \leq n\} \qquad (n \in \mathbb{N}) .$$

For $x \in X$ we define

(2.5) $$k_n(x) := [k(x)k(\psi(x))k(\psi^2(x)) \ldots k(\psi^{n-1}(x))]^{\frac{1}{n}} \qquad (n \in \mathbb{N})$$

and

(2.6) $$\hat{k}(x) := \lim_{n\to\infty} k_n(x) \quad \text{whenever this limit exists.}$$

In the following proposition we list some properties of these notions.

PROPOSITION 2.4. *Suppose* ψ *and* k *satisfy the assumptions of* (2.2), *then the following assertions hold:*

(a) $\nu(x) = \nu(\psi(x)) = \nu(\psi^{-1}(x))$ *for all* $x \in X$; *if* $\hat{k}(x)$ *exists then so do* $\hat{k}(\psi(x))$ *and* $\hat{k}(\psi^{-1}(x))$ *and we have* $\hat{k}(x) = \hat{k}(\psi(x)) = \hat{k}(\psi^{-1}(x))$.

(b) X_n *is a closed subset and* $\psi(X_n) = X_n$.

(c) *If* $\nu(x) < \infty$, *then* $\hat{k}(x)$ *exists and* $\hat{k}(x) = k_{\nu(x)}(x)$.

Proof. (a) Since ψ is one-to-one, the first assertion is obvious. We have $c := \inf\{k(x): x \in X\} > 0$ and $C := \sup\{k(x): x \in X\} < \infty$, hence for $x \in X$, $n \in \mathbb{N}$ we have

$$\left|\frac{k_n(x)}{k_n(\psi(x))} - 1\right| = \left|\left(\frac{k(x)}{k(\psi^n(x))}\right)^{\frac{1}{n}} - 1\right| \leq \left(\frac{C}{c}\right)^{\frac{1}{n}} - 1 .$$

Thus $\hat{k}(x) = \hat{k}(\psi(x))$ whenever one of these quantities exists. Applying this result to $y := \psi^{-1}(x)$, then the second assertion of (a) is proved.

(b) $\psi(X_n) = X_n$ follows from (a). That X_n is a closed subset will be proved by induction: X_1 is closed since ψ is continuous. Now assume that X_{n-1} is closed and that x is a cluster point of X_n $(n \geq 2)$. If x is a cluster point of X_{n-1} then $x \in X_{n-1} \subseteq X_n$. Otherwise there is a net (x_α) converging to x such that $\nu(x_\alpha) = n$ for every α. Since ψ is continuous we have $\psi^n(x) = \lim \psi^n(x_\alpha) = \lim x_\alpha = x$, thus $\nu(x) \leq n$ and so $x \in X_n$.

(c) Suppose $\nu := \nu(x) < \infty$, then we have $k(\psi^m(x)) = k(\psi^{m+\nu}(x))$ for every $m \in \mathbb{N}$. Let $\tilde{k} := k_\nu(x)$, then for $n = m\nu + r$ $(m, r \in \mathbb{N}$, $0 \leq r < \nu)$ we have

$$\frac{k_n(x)}{\tilde{k}} = (\tilde{k}^{-r} k(x)k(\psi(x)) \ldots k(\psi^{r-1}(x)))^{\frac{1}{n}} \to 1 .$$

Assertion (b) of the proposition implies that the mapping $\nu: X \to \mathbb{N}$, $x \to \nu(x)$ is lower semicontinuous. In general this mapping is not continuous (e. g. take $X = \mathbb{R}$, $\psi(x) = -x$) or equivalently, the set of points with a fixed period is not closed.

Before we are able to prove the main result of this section, we need two lemmas. The first deals with a special case.

LEMMA 2.5. *Suppose* T *is a lattice isomorphism on* $C_0(X)$, $Tf = k\, f \circ \psi$ $(f \in C_0(X))$, *such that* $\nu(x) = n$ *for every* $x \in X$ *and a fixed* $n \in \mathbb{N}$. *Then the spectrum of* T *is given by*

$$(2.7) \qquad \sigma(T) = \overline{\hat{k}(X)}\, \Gamma_n ,$$

where $\Gamma_n := \{z \in \mathbb{C}: z^n = 1\}$ *denotes the group of all* n-th *roots of unity.*

Proof. For $x_o \in X$, $\lambda \in \Gamma_n$ we define $x_m := \psi^m(x_o)$ $(m \in \mathbb{N})$, $\alpha_o := 1$, $\alpha_m := (\hat{k}(x_o))^{-m} \prod_{\nu=o}^{m-1} k(x_\nu)$. Since for a Dirac measure δ_x one has $T'\delta_x = k(x)\delta_{\psi(x)}$, it is easy to see that $\mu := \sum_{m=1}^{n-1} \lambda^{-m}\alpha_m\delta_{x_m}$ is an eigenvector of T' corresponding to the eigenvalue $\hat{k}(x_o)\lambda$. Since $\sigma(T)$ is closed it follows that $\overline{\hat{k}(X)\,\Gamma_n} \subseteq \sigma(T)$.
To prove the reverse inclusion we consider the operator T^n. It is given by $T^n f = \hat{k}^n f$; hence we have $\sigma(T^n) = \overline{\hat{k}^n(X)} = \overline{(\hat{k}(X))^n}$. If $\lambda \in \sigma(T)$, then $\lambda^n \in \sigma(T^n) = \overline{(\hat{k}(X))^n}$ and so $\lambda \in \overline{\hat{k}(X)}\,\Gamma_n$.

LEMMA 2.6. Let X be a locally compact space, $\psi: X \to X$ a homeomorphism, $T_o f := f \circ \psi$ $(f \in C_o(X))$. Given $x_o \in X$, $n \in \mathbb{N}$ such that $\nu(x_o) \geq 2n+1$, then for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ there is $f \in C_o(X)$ such that

$$(2.8) \qquad f(x_o) = \|f\| = 1 \quad \text{and} \quad \|T_o f - \lambda f\| \leq \frac{1}{n}.$$

Proof. By the assumption $\nu(x_o) \geq 2n+1$ there exist mutually disjoint neighborhoods V_m of $\psi^m(x_o)$ $(-n \leq m \leq n)$. Then $U := \bigcap_{m=-n}^{n} \psi^{-m}(V_m)$ is a neighborhood of x_o satisfying $\psi^m(U) \cap \psi^\ell(U) = \emptyset$ whenever $-n \leq m < \ell \leq n$. Choosing $f_o \in C_o(X)$ such that $f_o(x_o) = \|f_o\| = 1$ and $\operatorname{supp}(f_o) \subseteq U$, then $f := \sum_{m=-n}^{n} \lambda^{-m}(1 - \frac{|m|}{n})T_o^m f_o$ meets all requirements.

Before we give the description of the spectrum of an arbitrary lattice isomorphism on $C_o(X)$ we fix some notation.
For a natural number $n \in \mathbb{N}$ the group of all n-th roots of unity is denoted by Γ_n while Γ_∞ denotes the whole unit circle (i. e. $\Gamma_\infty = \Gamma = \{z \in \mathbb{C}: |z| = 1\}$). For ψ, k, T as described in (2.2) and $n \in \mathbb{N}$ we define $I_n := \{f \in C_o(X): f_{|X_n} = 0\}$, where X_n consists of all points with period less or equal than n.
By Proposition 2.4 (b) we have $T(I_n) = I_n$, hence T induces linear operators $T_{|I_n}$ on I_n and $T_{/I_n}$ on $C_o(X)/_{I_n}$. According to Proposition 2.3 we can identify I_n with $C_o(X \setminus X_n)$ and $C_o(X)/_{I_n}$ with $C_o(X_n)$. Then $T_{|I_n}$ is given by $f \to k_| f \circ \psi_|$ $(f \in C_o(X \setminus X_n))$ with $k_| := k_{|X \setminus X_n}$, $\psi_| := \psi_{|X \setminus X_n}$. Similarly $T_{/I_n}$ considered as operator on $C_o(X_n)$ is given by $f \to k_/ f \circ \psi_/$ with $k_/ := k_{|X_n}$, $\psi_/ := \psi_{|X_n}$.

THEOREM 2.7. Let X be a locally compact space, T a lattice isomorphism on $C_o(X)$ given by $Tf = k\, f\circ\psi$. Then the following hold:

$$\sigma(T) = [\bigcup_{n\in\mathbb{N}} \overline{\hat{k}(X_n\setminus X_{n-1})}\, \Gamma_n] \cup [\bigcap_{n\in\mathbb{N}} \sigma(T_{|I_n})] . \tag{2.10}$$

The subset $R := \bigcap_{n\in\mathbb{N}} \sigma(T_{|I_n})$ is invariant under rotation and $\lambda\Gamma \subseteq \sigma(T)$ implies $\lambda\Gamma \subseteq R$ $(\lambda\in\mathbb{C})$.

If in addition $\hat{k}(x) = \lim_{n\to\infty} k_n(x)$ exists uniformly in $x\in X$, then

$$\sigma(T) \cap \mathbb{R}_+ = \overline{\hat{k}(X)} \quad \text{and} \quad \sigma(T) = \overline{\bigcup_{x\in X} \hat{k}(x)\Gamma_{\nu(x)}} . \tag{2.11}$$

Proof. We have $T(I_n) = I_n$ for every $n\in\mathbb{N}$, hence $\sigma(T) = \sigma(T_{/I_n}) \cup \sigma(T_{|I_n})$ by (2.1). It follows that $\sigma(T) = [\bigcup_{n\in\mathbb{N}} \sigma(T_{/I_n})] \cup [\bigcap_{n\in\mathbb{N}} \sigma(T_{|I_n})]$.
By induction we show that $\sigma(T_{/I_m}) = \bigcup_{n=1}^{m} \overline{\hat{k}(X_n\setminus X_{n-1})}\, \Gamma_n$.
For $m = 1$ this follows immediately from Lemma 2.5. Now assume that it is true for $m-1$. Identifying $C_o/_{I_m}$ with $C_o(X_m)$, denoting $S := T_{/I_m} \in \mathcal{L}(C_o(X_m))$, then $J_{m-1} := \{f\in C_o(X_m) : f_{|X_{m-1}} = 0\}$ is a closed ideal such that $S(J_{m-1}) = J_{m-1}$. Moreover $S_{/J_{m-1}} \cong T_{/I_{m-1}}$, and $S_{|J_{m-1}}$ satisfies the assumptions of Lemma 2.5. Thus $\sigma(S) = \sigma(S_{/J_{m-1}}) \cup \sigma(S_{|J_{m-1}}) = \sigma(T_{/I_{m-1}}) \cup \overline{\hat{k}(X_m\setminus X_{m-1})}\, \Gamma_m$. Now we show that R is invariant under rotation. Since R is cyclic [8, 2.4] we have to show that $0 < r\in R$ implies $r\Gamma \subseteq R$. The topological boundary of the spectrum is always contained in the approximative point spectrum which is cyclic [8, 2.2, 2.7]. Therefore we can assume $0 < r \in \bigcap A\sigma(T_{|I_n})$. Then there exist $f_n\in I_{2n}$ $(n\in\mathbb{N})$ such that $\|f_n\| = 1$ and $\|Tf_n - rf_n\| \le \frac{1}{2}$. Since $f_n\in I_{2n}$ there exist $x_n\in X\setminus X_{2n}$ such that $|f_n(x_n)| = \|f_n\| = 1$. Given $\lambda\in\Gamma$, by Lemma 2.6 there are functions $g_n\in C_o(X)$ such that $\|g_n\| = g_n(x_n) = 1$ and $\|g_n\circ\psi - \lambda g_n\| \le \frac{1}{2}$. Let $m\in\mathbb{N}$. For $h_n := g_n f_n$ we have $h_n\in I_m$ for $n \ge m$, $\|h_n\| = 1$ for all $n\in\mathbb{N}$ and

$$Th_n - \lambda r h_n = (Tf_n - rf_n)g_n\circ\psi + rf_n(g_n\circ\psi - \lambda g_n) .$$

Thus the sequence $(h_n)_{n\ge m}$ is an approximative eigenvector of

$T_{|I_m}$ corresponding to $r \cdot \lambda$.

The first assertion of (2.11) is proved in [2, 5.3 (b)] for compact X. The proof given there works in the locally compact case as well. To prove the second assertion of (2.11) we assume $\lambda \in \overline{\sigma(T)}$. If $\lambda\Gamma \not\subseteq \sigma(T)$, then by (2.10) $\lambda \in \overline{\bigcup \hat{k}(X_n \setminus X_{n-1})}\ \Gamma_n \subseteq \overline{\bigcup \hat{k}(x)\Gamma_{\nu(x)}}$. If $\lambda\Gamma \subseteq \sigma(T) = \sigma(T_{/I_n}) \cup \sigma(T_{|I_n})$ $(n \in \mathbb{N})$ then $\lambda\Gamma \subseteq \sigma(T_{|I_n})$ because $\lambda\Gamma \cap \sigma(T_{/I_n})$ is finite. Thus $|\lambda| \in \sigma(T_{|I_n}) \cap \mathbb{R}_+ = \overline{\hat{k}(X \setminus X_n)}$. Then there exist $(x_n) \in X$ such that $|\hat{k}(x_n) - |\lambda|| \to 0$ and $\nu(x_n) \geq n$ for every $n \in \mathbb{N}$. It follows that $\lambda\Gamma \subseteq \overline{\bigcup \hat{k}(x_n)\Gamma_{\nu(x_n)}}$ and we have $\sigma(T) \subseteq \overline{\bigcup_{x \in X} \hat{k}(x)\Gamma_{\nu(x)}}$. On the other hand given $x \in X$, then by (2.10) $\hat{k}(x)\Gamma_{\nu(x)} \subseteq \sigma(T)$ whenever $\nu(x) < \infty$. If $\nu(x) = \infty$, then $x \in X \setminus X_n$, hence $\hat{k}(x) \in \overline{\hat{k}(X \setminus X_n)} = \sigma(T_{|I_n}) \cap \mathbb{R}_+$ for every $n \in \mathbb{N}$. Thus $\hat{k}(x) \in R$ and $\hat{k}(x)\Gamma \subseteq R \subseteq \sigma(T)$ since R is invariant under rotation.

REMARK 2.8. (Aperiodic case). Using the argument given in the proof of (2.10), one can show that the whole spectrum is invariant under rotation whenever the non-periodic points are dense in X.

3. CHARACTERIZATION OF ONE-PARAMETER GROUPS OF POSITIVE OPERATORS ON $C_o(X)$

In [9] the strongly continuous semigroups of lattice homomorphisms on $C(X)$, X compact, are characterized as follows:

(3.1) Given $(T(t))_{t \in \mathbb{R}_+}$ as described above then there exist a continuous semi-flow $\varphi: \mathbb{R}_+ \times X \to X$ and functions $h, m \in C_{\mathbb{R}}(X)$ with $m >> 0$ such that

$$T(t)f = \frac{m}{m \cdot \varphi_t} \exp\left(\int_0^t h \cdot \varphi_s \, ds\right) f \cdot \varphi_t .$$

For several reasons there is no satisfactory extension of this result to spaces $C_o(X)$, X locally compact. However, in this section we will show that for strongly continuous groups of positive generators (which can be considered as special semigroups of lattice homomorphisms) there is an extension of (3.1) to the space $C_o(X)$.

First we have to explain some notation:

If X is a locally compact space, a map $\varphi: \mathbb{R}\times X \to X$ is called a <u>flow on X</u>, whenever it satisfies

$$(3.2) \qquad \begin{array}{lll} \varphi(0,x) = x & \text{for every } x\in X & \text{and} \\ \varphi(t+s,x) = \varphi(t,\varphi(s,x)) & \text{for arbitrary } s,t\in\mathbb{R}, & x\in X. \end{array}$$

Given a flow φ then each partial map $\varphi_t = \varphi(t,.)$ is a bijective transformation of X with $\varphi_t^{-1} = \varphi_{-t}$; in addition, $t \to \varphi_t$ is a group homomorphism of $(\mathbb{R},+)$ into $\mathrm{Aut}(X)$. The flow is called continuous when φ is continuous with respect to the product topology on $\mathbb{R}\times X$. Given a flow φ on X, a family $(h_t)_{t\in\mathbb{R}}$ of bounded scalar-valued functions is called a <u>cocycle of φ</u>, if the following conditions are fulfilled:

$$(3.3) \qquad \begin{array}{lll} h_0 = \mathbb{1} & (\text{i. e. } h_0(x) = 1 \text{ for every } x\in X) & \text{and} \\ h_{t+s} = h_t(h_s\circ\varphi_t) & \text{for arbitrary } s,t\in\mathbb{R}. & \end{array}$$

It follows that every h_t is non-zero and $h_t^{-1} = h_{-t}\circ\varphi_t$. The cocycle (h_t) is called continuous when the mapping $(t,x) \to h_t(x)$ is continuous with respect to the product topology on $\mathbb{R}\times X$. In this case we have $h_t \in C^b(X)$ for every $t\in\mathbb{R}$.
Given a continuous flow φ on X and an associated continuous cocycle (h_t) a group of bounded linear operators on $C_0(X)$ is given in a natural way; namely,

$$(3.4) \qquad T(t)f = h_t\, f\circ\varphi_t \quad \text{where} \quad f\in C_0(X), \quad t\in\mathbb{R}.$$

Obviously one has $\lim_{t\to 0}\|T(t)f - f\| = 0$ whenever f has compact support. Since $\|T(t)\| = \|h_t\|$ the following lemma and [19, III.4.5] then imply that $(T(t))$ is a strongly continuous group.

<u>LEMMA</u> 3.1. <u>Suppose</u> φ <u>is a flow and</u> $(h_t)_{t\in\mathbb{R}}$ <u>is a cocycle such that for every</u> $x\in X$ <u>the mapping</u> $t \to h_t(x)$ <u>is continuous</u>. <u>Then</u> $\{\|h_t\|: t\in B\}$ <u>is bounded for every bounded subset</u> $B\subseteq\mathbb{R}$.

<u>Proof.</u> If we can show that $\{h_t: 1-\varepsilon \leq t \leq 1\}$ is bounded for some $\varepsilon > 0$, then using the inequality

$$\|h_{t+s}\| = \|h_t \; h_s \circ \varphi_t\| \leq \|h_t\| \|h_s \circ \varphi_t\| = \|h_t\| \|h_s\|$$

one can easily deduce the assertion of the lemma. To prove the boundedness on some interval $[1-\varepsilon, 1]$ one can proceed as in the proof of [15, 10.2.1]. Note that the continuity assumption implies that the mapping $t \to \|h_t\| = \sup_{x \in X} |h_t(x)|$ is lower semicontinuous hence measurable.

The group defined by (3.4) consists of positive operators if and only if each function h_t is positive. The following proposition shows that all positive groups on $C_o(X)$ are of this type.

PROPOSITION 3.2. *Let $(T(t))_{t \in \mathbb{R}}$ be a strongly continuous group of positive operators on $C_o(X)$, X locally compact. Then there exist a continuous flow on X and a continuous cocycle $(h_t)_{t \in \mathbb{R}}$ of φ such that*

$$T(t)f = h_t \; f \circ \varphi_t \quad \textit{for every } f \in C_o(X), \quad t \in \mathbb{R}. \tag{3.4}$$

Moreover there are real constants $M \geq 1$, $\omega \geq 0$ such that

$$(M e^{\omega|t|})^{-1} \leq h_t(x) \leq M e^{\omega|t|} \quad \textit{for every } t \in \mathbb{R}, \; x \in X. \tag{3.5}$$

Proof. Since $T(t)$ and $T(t)^{-1} = T(-t)$ are positive operators, $T(t)$ is actually a lattice isomorphism. Then by (2.2) there exist a homeomorphism $\varphi_t \colon X \to X$ and a positive function $h_t \in C^b(X)$ such that $T(t)f = h_t f \circ \varphi_t$ for every $f \in C_o(X)$. The group property of $(T(t))$ then implies that $\varphi(t,x) := \varphi_t(x)$ defines a flow φ on X and that $(h_t)_{t \in \mathbb{R}}$ is a cocycle of φ.
It is well-known that there exist constants $M \geq 1$, $\omega > 0$ such that $\|T(t)\| \leq Me^{\alpha|t|}$ for every $t \in \mathbb{R}$, hence $0 < h_t(x) \leq \|h_t\| = \|T(t)\| \leq M e^{\omega|t|}$. Moreover $(h_t(x))^{-1} = h_{-t}(\varphi_t(x)) \leq \|h_{-t}\| = \|T(-t)\| \leq M e^{\omega|t|}$. Thus (3.5) holds and it remains to show that flow and cocycle are continuous. First we consider the flow. Since we have $\varphi_{t+s} = \varphi_t \circ \varphi_s$ and every φ_t is a homeomorphism on X, it is enough to establish continuity of φ at points $(0,x_o) \in \mathbb{R} \times X$. Given a compact neighborhood V of $x_o = \varphi(0,x_o)$, there exists a continuous function $f \colon X \to [0,1]$ with $f(x_o) = 1$ and $\operatorname{supp}(f) \subseteq V$.

We have $\lim_{t \to 0} T(t)f = f$; hence there exists $t_0 > 0$ such that $\|T(t)f - f\| < \frac{1}{2}$ for $|t| \leq t_0$. If we define $W := \{x \in X: |f(x)| > \frac{1}{2}\}$, then for $|t| \leq t_0$, $x \in W$ we have: $|h_t(x)f(\varphi(t,x)) - f(x)| < \frac{1}{2}$ and $|f(x)| > \frac{1}{2}$. Hence $f(\varphi(t,x)) > 0$ wihch implies $\varphi(t,x) \in V$. To prove the continuity of the cocycle we first remark that by strong continuity of $(T(t))$ for every fixed $f \in C_0(X)$ the mapping $(t,x) \to (T(t)f)(x)$ is continuous on $\mathbb{R} \times X$. Given compact subsets $A \subseteq \mathbb{R}$, $B \subseteq X$, the set $C := \varphi(A \times B)$ is compact, hence there exists $f \in C_0(X)$ such that $f_{|C} = 1$. For $(t,x) \in A \times B$ we have $h_t(x) = h_t(x)f(\varphi(t,x)) = (T(t)f)(x)$, thus $(t,x) \to h_t(x)$ is continuous on $A \times B$. Since continuity is a local property the cocycle (h_t) is continuous.

REMARK 3.3. Using Proposition 3.2 one can show that separate continuity of a flow φ and separate continuity of a cocycle (h_t) are sufficient for the continuity of φ and (h_t). Indeed the continuity of $x \to h_t(x)$ and $x \to \varphi(t,x)$ for fixed $t \in \mathbb{R}$ implies that $T(t)$ given by (3.4) is a bounded linear operator on $C_0(X)$. The continuity of the partial maps $t \to h_t(x)$ and $t \to \varphi(t,x)$ implies that $\lim_{t \to 0}(T(t)f)(x) = f(x)$ for every $x \in X$. By Lemma 3.1, the set $\{\|h_t\|: |t| \leq 1\}$ is bounded, hence we can apply Lebesgue's dominated convergence theorem and obtain that $(T(t))$ is weakly continuous at $t = 0$. Then $(T(t))$ is strongly continuous by [7, 1.23] and Proposition 3.2 implies that φ and (h_t) are continuous.

EXAMPLE 3.4. *Suppose that φ is a continuous flow on X.*

(a) *If m is a positive continuous function on X such that m and m^{-1} are bounded, then*

$$m_t := \frac{m}{m \circ \varphi_t} \qquad (t \in \mathbb{R}) \tag{3.6}$$

defines a continuous cocycle $(m_t)_{t \in \mathbb{R}}$ of φ and $m_t > 0$.

(b) *For $h \in C^b_{\mathbb{R}}(X)$ we define*

$$h_t(x) := \exp\left(\int_0^t h(\varphi(s,x))ds\right) \qquad (x \in X,\ t \in \mathbb{R}). \tag{3.7}$$

Then $(h_t)_{t \in \mathbb{R}}$ is a continuous cocycle of φ consisting of positive functions.

Cocycles as defined in (3.6) are always globally bounded. In general this is false for cocycles of the second type. On the other hand cocycles as described in (3.7) are differentiable with respect to t. This is not satisfied by cocycles of the first type in general. Thus neither (3.6) nor (3.7) gives a description of arbitrary cocycles. However, every positive cocycle is a product of cocycles of the form (3.6) and (3.7). More precisely we have:

LEMMA 3.5. *Suppose φ is a continuous flow on X and $(k_t)_{t\in\mathbb{R}}$ is a continuous cocycle of φ consisting of positive functions. Then there exist $h \in C^b_{\mathbb{R}}(X)$ and a positive continuous function $m: X \to \mathbb{R}$ with $0 < \inf\{m(x): x \in X\} \leq \sup\{m(x): x \in X\} < \infty$ such that*

$$(3.8) \qquad k_t(x) = \frac{m(x)}{m(\varphi(t,x))} \exp\left(\int_0^t h(\varphi(s,x))ds\right) \qquad (x \in X, \quad t \in \mathbb{R}) .$$

Proof. In view of (3.5) there exist constants $M, \omega \geq 1$ such that $(M\, e^{(\omega-1)|t|})^{-1} \leq k_t(x) \leq M\, e^{(\omega-1)|t|}$ for every $x \in X$, $t \in \mathbb{R}$. We define m and h as follows:

$$m(x) := \int_0^\infty e^{-\omega s}\, k_s(x)ds , \quad h(x) := \omega - \frac{1}{m(x)} \qquad (x \in X) .$$

Then m is a continuous function and we have

$$(M(2\omega - 1))^{-1} = \int_0^\infty e^{-\omega s}(M\, e^{(\omega-1)s})^{-1} ds \leq m(x)$$
$$\leq \int_0^\infty e^{-\omega s} M\, e^{(\omega-1)s} ds = M \quad \text{for every } x \in X .$$

In particular it follows that $h \in C^b_{\mathbb{R}}(X)$. For $x \in X$, $t \in \mathbb{R}$ we have

$$k_t(x)m(\varphi(t,x)) = \int_0^\infty e^{-\omega s}\, k_{t+s}(x)ds = e^{\omega t}\int_t^\infty e^{-\omega s}\, k_s(x)ds .$$

Now we fix $x \in X$ and define $f: \mathbb{R} \to \mathbb{R}$ as follows:

$$f(t) := \frac{k_t(x)m(\varphi(t,x))}{m(x)} = \frac{e^{\omega t}}{m(x)} \int_t^\infty e^{-\omega s}\, k_s(x)ds .$$

f is differentiable and satisfies the following differential equation: $f'(t) = \omega f(t) - \frac{k_t(x)}{m(x)} = h(\varphi(t,x))f(t)$. Moreover $f(0) = 1$, hence $f(t) = \exp(\int_0^t h(\varphi(s,x))ds$ for every $t \in \mathbb{R}$. Thus we obtain (3.8).

In the following theorem we describe the one-parameter groups of positive operators on $C_o(X)$ and characterize their generators as special perturbations of generators of automorphism groups. We use the term "automorphism" with respect to the C^*-structure of $C_o(X)$, i. e. an invertible operator $T_o \in \mathscr{L}(C_o(X))$ is called automorphism if $T_o(fg) = T_o(f)T_o(g)$ and $T_o(\bar{f}) = \overline{T_o(f)}$ for $f,g \in C_o(X)$. Since automorphisms are lattice isomorphisms (compare with [20, III. 9.1]), an automorphism group is governed by a flow and an associated cocycle. It is easy to see that the cocycle has to be trivial, i. e. $h_t(x) = 1$ for every $t \in \mathbb{R}$, $x \in X$. Moreover, the generator A_o of an automorphism group is a derivation, that is

(3.9) $D(A_o)$ is a subalgebra and A_o satisfies
$$A_o(fg) = f(A_o g) + (A_o f)g \quad \text{for all } f,g \in D(A_o).$$

THEOREM 3.6. *Let* X *be a locally compact space. For a strongly continuous group* $(T(t))_{t\in\mathbb{R}}$ *on* $C_o(X)$ *with generator* A *the following assertions are equivalent:*

(i) $(T(t))$ *is a group of positive operators (equivalently of lattice isomorphisms).*

(ii) *There exist* $h \in C^b_{\mathbb{R}}(X)$, *a continuous function* $m: X \to \mathbb{R}$ *with* $0 < \inf_{x\in X}\{m(x)\} \leq \sup_{x\in X}\{m(x)\} < \infty$ *and a continuous flow* φ *such that for every* $f \in C_o(X)$, $x \in X$
$$(T(t)f)(x) = \frac{m(x)}{m(\varphi(t,x))} \exp\left(\int_0^t h(\varphi(s,x))ds\right) f(\varphi(t,x)).$$

(iii) *There is a generator* A_o *of a strongly continuous automorphism group and functions* h, m *as in* (ii) *such that*
$$Af = m\left(A_o\left(\frac{f}{m}\right)\right) + h f \quad \textit{for} \quad f \in D(A) = \{mg: g \in D(A_o)\}.$$

Proof. (i) $\Rightarrow$ (ii) follows from Proposition 3.2 and Lemma 3.5.
(ii) $\Rightarrow$ (iii): Define A_o to be the generator of the group $(T_o(t))$ given by $T_o(t)f = f \circ \varphi_t$ $(t \in \mathbb{R},\ f \in C_o(X))$.
(iii) $\Rightarrow$ (i): The automorphism group $(T_o(t))$ generated by A_o is governed by a flow φ, i. e. $T_o(t)f = f \circ \varphi_t$. Then $T_1(t)f := \frac{m}{m \circ \varphi_t} f \circ \varphi_t$ $(f \in C_o(X))$ defines a positive group with generator A_1 given by $A_1 f = m(A_o(\frac{f}{m}))$, $D(A_1) = m\,D(A_o)$. It

follows from the Trotter product formula that the sum of a generator of a positive semigroup and a bounded positive operator generates a positive semigroup. Moreover B generates a positive semigroup if and only if $B - r\,\mathrm{Id}$ $(r \in \mathbb{R})$ does. Thus $A = A_1 + M_h = (A_1 + M_{(h+\|h\|)}) - \|h\|\,\mathrm{Id}$ generates a positive semigroup. By the same argument $-A$ generates a positive semigroup as well, i. e. A is the generator of a positive group.

The function m of assertion (ii) does not have any influence on the spectral properties of A or $T(t)$. This is a consequence of the following corollary.

COROLLARY 3.7. *Every strongly continuous group* $(T(t))$ *of positive operators on* $C_o(X)$ *is similar to a group of the following type:*

$$S(t)f = \exp\left(\int_o^t h \circ \varphi_s \, ds\right) f \circ \varphi_t , \tag{3.10}$$

where φ *is a continuous flow on* X *and* $h \in C^b_{\mathbb{R}}(X)$. *That is, for all* $t \in \mathbb{R}$ *and a fixed invertible operator* $M \in \mathcal{L}(C_o(X))$ *we have*

$$S(t) = M^{-1} T(t) M . \tag{3.11}$$

Proof. Choose φ, h, m according to assertion (ii) of the theorem and define $Mf := m \cdot f$ $(f \in C_o(X))$.

We conclude this section with an analog of Proposition 2.1, which can be formulated for arbitrary Banach lattices.

PROPOSITION 3.8. *Suppose* $(T(t))_{t\in\mathbb{R}}$ *is a strongly continuous group of positive operators on a Banach lattice* E, I *a closed ideal such that* $T(t)I \subseteq I$ *for every* $t \in \mathbb{R}$. *Then we have*

$$\sigma(A) = \sigma(A_{|I}) \cup \sigma(A_{/I}) . \tag{3.12}$$

Here $A_{|I}$ *(resp.,* $A_{/I}$*) denotes the generator of the positive group on* I *(resp.,* $E_{/I}$*) which is induced by* $(T(t))$ *in the natural way (compare* [8, 1.8, 1.9]).

Proof. Each $T(t)$ is a lattice isomorphism and since $T(t)I \subseteq I$, $T(-t)I \subseteq I$ we have actually $T(t)I = I$ for every $t \in \mathbb{R}$. As in the proof of Proposition 2.1 it follows that both, I^o and $I^{o\perp}$ are $T(t)'$-invariant projection bands in E'. Thus the corresponding band projections commute with every operator $T(t)'$. Considering the subspace E^* of E' on which $(T(t)')$ is strongly continuous (see [15, Chapter XIV]), this implies that $E^* = J_1 \oplus J_2$ where $J_1 = E^* \cap I^o$ and $J_2 = E^* \cap I^{o\perp}$, and both, J_1 and J_2 are invariant under $T(t)^*$. Identifying I^* with J_2, $(E/I)^*$ with J_1 and using [15, 14.3.3] we obtain $\sigma(A) = \sigma(A^*) = \sigma(A^*|_{J_2}) \cup \sigma(A^*|_{J_1}) = \sigma((A|_I)^*) \cup \sigma((A/_I)^*) = \sigma(A|_I) \cup \sigma(A/_I)$.

4. THE PERIODIC CASE

Let $\varphi\colon R \times X \to X$ be a continuous flow. The period of a point $x \in X$ is defined by

$$\tau(x) = \inf\{t > 0\colon\ \varphi(t,x) = x\}$$

where as usual the greatest lower bound of the empty set is defined to be ∞. It is easy to see that

$$(4.1) \qquad \tau(x) = 0 \quad \text{iff} \quad \varphi(t,x) = x \quad \text{for all} \quad t \in \mathbb{R},$$

and in the case $0 < \tau(x) < \infty$,

$$(4.2) \qquad \varphi(t,x) = x \quad \text{iff} \quad t = n \cdot \tau(x) \quad \text{for some} \quad n \in \mathbb{Z}.$$

In general, the mapping $x \to \tau(x)$ is not continuous (we give an example below); however, it is semicontinuous. More precisely, the set $X_\tau := \{x\ \ X\colon \tau(x) \leq \tau\}$ is closed for every $\tau \geq 0$. (This is obvious for $\tau = 0$ or $\tau = \infty$.) If $0 < \tau < \infty$, then for every $x \in X_\tau$ there exists a $\tilde{\tau}(x)$ such that $\tau/2 \leq \tilde{\tau}(x) \leq \tau$ and $\varphi(\tilde{\tau}(x),x) = x$. Now given $x_o \in X$ and an ultrafilter $\mathfrak{U}$ converging to x_o, $\tau_o := \lim_{\mathfrak{U}} \tilde{\tau}(x)$ exists and satisfies $0 < \tau/2 \leq \tau_o \leq \tau$. Since φ is jointly continuous, we have $\varphi(\tau_o,x_o) = \lim_{\mathfrak{U}} \varphi(\tau(x),x) = \lim_{\mathfrak{U}} x = x_o$. This implies that $\tau(x_o) = 0$ or $\tau_o = n\tau(x_o)$ for some $n \in \mathbb{N}$ (by

(4.2)). In both cases we have $\tau(x_o) \leq \tau$, i. e. $x_o \in (X_\tau)$.

EXAMPLE 4.1. *Let* $X = \{r \cdot e^{i\theta} : 0 \leq r \leq 1,\ \theta \in \mathbb{R}\} \subset \mathbb{C}$. *For the flow governed by the differential equation*

$$\dot{r} = r(1-r)$$
$$\dot{\theta} = 1$$

we obtain
$$\tau(0) = 0$$
$$\tau(z) = \infty \quad \text{if} \quad 0 < |z| < 1 \qquad \textit{and}$$
$$\tau(z) = 2\pi \quad \text{if} \quad |z| = 1 .$$

EXAMPLE 4.2. *Let* X *be the quotient of the rectangle* $[0,2\pi] \times [-1,1]$ *obtained by identifying the points* $(0,x_2)$ *and* $(2\pi,-x_2)$ $(x_2 \in [-1,1])$ (*that is,* X *is the Möbius strip*). *Then for the flow governed by the differential equation*

$$\dot{x}_1 = 1$$
$$\dot{x}_2 = 0$$

we obtain for $x = (x_1,x_2) \in X$:

$$\tau(x) = 2\pi \quad \text{if} \quad x_2 = 0$$
$$\tau(x) = 4\pi \quad \text{if} \quad x_2 \neq 0 .$$

For $h \in C^b_{\mathbb{R}}(X)$ we define

$$\hat{h}(x) := \lim_{t\to\infty} \frac{1}{t} \int_o^t h(\varphi(s,x))ds$$

for those $x \in X$ for which the limit exists. In general $\hat{h}$ is defined on a subset of X which is possibly empty. However, if $x \in X$ such that $\tau(x) < \infty$, then $\hat{h}(x)$ is defined and

$$\hat{h}(x) = h(x) \qquad\qquad \text{if} \quad (x) = 0$$
$$\hat{h}(x) = \frac{1}{(x)} \int_o^{\tau(x)} h(\varphi(s,x))ds \qquad \text{if} \quad (x) > 0 .$$

Thus $\hat{h}$ is defined on the subsets X_τ $(0 \leq \tau < \infty)$, and it is easy to see that $\hat{h}$ is continuous on each X_τ (in fact, the limit $\lim_{t\to\infty} \frac{1}{t} \int_o^t h(\varphi(s,x))ds$ exists uniformly in $x \in X_\tau$) .

We now consider the one-parameter group $(T_o(t))_{t\in\mathbb{R}}$ on $C_o(X)$ associated with the flow; i. e. $T_o(t)$ is defined by

$$(4.3) \qquad (T_o(t)f)(x) = f(\varphi(t,x)) \qquad (t \in \mathbb{R}, \quad x \in X)$$

for all $f \in C_o(X)$ (see section 3).
The generator of this group is denoted by A_o with domain $D(A_o)$. Let $h \in C^b_{\mathbb{R}}(X)$. By M_h we denote the multiplier on $C_o(X)$ defined by h; i. e.

$$(4.4) \qquad M_h f = h \cdot f \qquad (f \in C_o(X)) .$$

Let $A = A_o + M_h$ with domain $D(A) = D(A_o)$. Then A generates the group $(T(t))_{t\in\mathbb{R}}$ on $C_o(X)$ given by

$$(4.5) \qquad (T(t)f)(x) = h_t(x)\, f(\varphi(t,x)) \qquad (t \in \mathbb{R}, \quad x \in X)$$

(see Theorem 3.6). We will keep this notation throughout the paper.

We now describe the spectrum of A in terms of the flow and the multiplier for the periodic case.

THEOREM 4.3. *Suppose that* $\tau(x) \leq \tau$ *for all* $x \in X$ *and some* $\tau > 0$. *Then*

$$(4.6) \qquad \sigma(A) = [\, h(X_o) \cup \bigcup_{\tau(x)>o} \hat{h}(x) + i\, \frac{2\pi}{\tau(x)}\, \mathbb{Z}\,]^- .$$

Moreover, for $\lambda \in \rho(A)$ *and* $f \in C_o(X)$ *the resolvent is given by*

$$(4.7) \qquad (R(\lambda,A)f)(x) = \begin{cases} [1-\exp[\tau(x)(\hat{h}(x)-\lambda)]]^{-1} \int_o^{\tau(x)} e^{-\lambda t} T(t)f(x)\,dt \\ \text{if } \tau(x) > 0 \qquad \underline{\text{and}} \\ (\lambda - h(x))^{-1} f(x) \quad \text{if } \tau(x) = 0 \quad (x \in X) . \end{cases}$$

REMARK. For the spectral mapping theorem we will only need that the set on the right-hand side of (4.6) is included in the spectrum of A. This is the easier part of the proof and will be given first.

Proof. Let $x \in X$. Denote by δ_x die Dirac-measure at x. If $\tau(x) = 0$, then

(4.8) $$\delta_x \in D(A') \text{ and } A'\delta_x = h(x)\delta_x .$$

Hence $h(x) \in P\sigma(A') \subset \sigma(A)$.
If $\tau(x) > 0$, for $\lambda \in \mathbb{C}$ let $\mu_\lambda^x \in C_o(X)'$ be given by

$$\langle f,\mu_\lambda^x\rangle := \int_o^{\tau(x)} e^{-\lambda t}(T(t)f)(x)dt = \int_o^{\tau(x)} e^{-\lambda t} h_t(x)\ f(\varphi(t,x))dt \qquad (f \in C_o(X)).$$

Using Tietze's extension theorem, one finds $f \in C_o(X)$ such that $\langle f,\mu_\lambda^x\rangle \neq 0$, hence $\mu_\lambda^x \neq 0$. Moreover,

$$\int_o^{\tau(x)} e^{-\lambda t}\ T(t)(\lambda - A)f\ dt = f - e^{-\lambda\tau(x)}\ T(\tau(x))f$$

for all $f \in D(A)$ (see [12, 1.8]). Evaluating at x one sees that

(4.9) $$\mu_\lambda^x \in D(A') \text{ and } (\lambda - A)'\mu_\lambda^x = (1 - \exp[\tau(x)(\hat{h}(x) - \lambda)])\delta_x .$$

Hence $\lambda \in P\sigma(A') \subset \sigma(A)$ whenever $\lambda \in \hat{h}(x) + i\,\frac{2\pi}{\tau(x)}\,\mathbb{Z}$.
We have proved that the right-hand side of (4.7) is included in $\sigma(A)$.

In order to prove the reverse inclusion assume that

$$\lambda \notin (h(X_o) \cup \bigcup_{\tau(x)>o} \hat{h}(x) + i\,\frac{2\pi}{\tau(x)}\,\mathbb{Z})^- .$$

We define an operator R on $C_o(X)$. Let $f \in C_o(X)$. For $x \in X$ we set

$$(Rf)(x) = \begin{cases} (\lambda - h(x))^{-1} & \text{if } \tau(x) = 0 \\ (1 - \exp[\tau(x)(h(x) - \lambda)])^{-1} \displaystyle\int_o^{\tau(x)} e^{-\lambda t}T(t)f(x)dt & \text{if } \tau(x) > 0. \end{cases}$$

Although the mapping $x \to \tau(x)$ is not continuous, one can show that Rf is a continuous function. (For the proof let $\mathcal{U}$ be an ultrafilter on X convergent to x_o. Then the following three cases can occur:

(a) $\lim_{x\in\mathcal{U}} \tau(x) = \tau(x_o)$

(b) $\lim_{x\in\mathcal{U}} \tau(x) > \tau(x_o) = 0$

(c) $\lim_{x \in \mathcal{U}} \tau(x) = n \cdot \tau(x_o) > 0$ for some natural number $n \geq 2$.

The cases (b) and (c) require lengthy computations which we omit.)
It is easy to see that R is a bounded operator on $C_o(X)$ commuting with $T(t)$ $(t \in \mathbb{R})$. This implies

$$(4.10) \qquad RD(A) \subset D(A) \quad \text{and} \quad RAf = ARf \quad (f \in D(A)) .$$

We show that

$$(4.11) \qquad R(\lambda - A)f = f \quad (f \in D(A)) .$$

Let $x \in X$. If $\tau(x) = 0$, then $R'\delta_x = (\lambda - h(x))^{-1}\delta_x$, and so $(R(\lambda - A)f)(x) = f(x)$ $(f \in D(A))$. If $\tau(x) > 0$, then $R'\delta_x = (1 - \exp[\tau(x)(h(x) - \lambda)])^{-1}\mu_\lambda^x$, and so it follows from (4.9) that $R(\lambda - A)f(x) = f(x)$ $(f \in D(A))$.
This proves (4.11). Since A is closed, it follows from (4.10) and (4.11) that $\lambda \in \rho(A)$ and $R = R(\lambda, A)$.

Now we are able to prove the spectral mapping theorem in a particular case.

<u>THEOREM</u> 4.4. <u>Assume that</u> $\tau(x) \leq \tau$ <u>for all</u> $x \in X$ <u>and some</u> $0 < \tau < \infty$. <u>Then</u>

$$(4.12) \qquad \sigma(T(t)) = [\exp(th(X_o)) \cup \bigcup_{\tau(x)>o} \exp(t\hat{h}(x)) \cdot \exp(i\tfrac{2\pi t}{\tau(x)}\mathbb{Z})]^-$$

for all $t \in \mathbb{R}$. In particular,

$$(4.13) \qquad \sigma(T(t)) = \overline{\exp t\, \sigma(A)} \qquad (t \in \mathbb{R}) .$$

<u>Proof.</u> Let $t \in \mathbb{R}$, $k = h_t$ (i. e. $k(x) = \exp \int_0^t h(\varphi(s,x))ds$), $\psi = \varphi_t$. Then $T(t)f = k\, f \circ \psi$. Moreover, for $n \in \mathbb{N}$, $k_n :=$ $(k \cdot k \circ \psi \cdot \ldots \cdot k \circ \psi^{n-1})^{1/n} = \exp[\, t \frac{1}{t \cdot n} \int_0^{t \cdot n} h(\varphi(s,x))ds\,]$. Hence (k_n) converges uniformly to $\exp t\hat{h}$. So by Theorem 2.7

$$\sigma(T(t)) = (\bigcup_{x\in X} \exp(t \cdot \hat{h}(x)) \cdot \Gamma_{\nu(x)})^- .$$

On the other hand, by Theorem 4.3,

$$\overline{\exp(t\ \sigma(A))} = (\exp(t \cdot h(X_o)) \cup \bigcup_{\tau(x)>o} \exp(t\hat{h}(x)) \cdot \exp(i\ \tfrac{2\pi t}{\tau(x)} \mathbb{Z})^- .$$

Now $\nu(x) = \infty$ iff $\frac{t}{\tau(x)}$ is irrational. In that case, $\overline{\exp(i\ \frac{2\pi t}{\tau(x)} \mathbb{Z})} = \Gamma_\infty$. And if $\nu(x) < \infty$, then it is easy to see that $\exp(i\ \frac{2\pi t}{\tau(x)} \mathbb{Z}) = \Gamma_{\nu(x)}$. This proves (4.13).

<u>REMARK</u> 4.5. In (4.13) the bar cannot be omitted. To give an example, let $X = S^1$ $(= \{z \in \mathbb{C}: |z| = 1\})$ and $\varphi(t,z) = e^{2\pi it} \cdot z$ $(t \in \mathbb{R}$, $z \in S^1)$. Let $T_o(t)f = f \circ \varphi_t$ $(f \in C(X))$. Then $\sigma(A_o) = 2\pi i\, \mathbb{Z}$. Let $t \in \mathbb{R}$ be irrational. Then $\sigma(T(t)) = \Gamma_\infty$. But $\exp(t\,\sigma(A_o)) = \exp(i2\pi t \cdot \mathbb{Z}) \neq \Gamma_\infty$ (see [12, 1.9]).

5. THE APERIODIC CASE

We keep the notation of the preceding sections.

<u>LEMMA</u> 5.1. <u>Let</u> $x_o \in X$, $n \in \mathbb{N}$ <u>and</u> $\tau(x_o) > 2n+1$. <u>Then for every</u> $\varepsilon > 0$ <u>there exists a neighborhood</u> U <u>of</u> x_o <u>such that</u>

(5.1) $\{\varphi(t,x_o):\ 0 \leq t \leq 1-\varepsilon\} \subset U$

(5.2) $\varphi_{-n}(U),\dots,\varphi_{-1}(U),U,\varphi_1(U),\dots,\varphi_n(U)$ <u>are pairwise disjoint</u>.

<u>Proof.</u> First we show the following. Let $m \in \mathbb{N}$ and $y_o \in X$ such that $\tau(y_o) > m+1$. Then there exists a neighborhood V of y_o such that

(5.3) $K := \{\varphi(t,y_o):\ 0 \leq t \leq 1-\varepsilon\} \subset V$ and

(5.4) $V, \varphi_1(V), \dots, \varphi_m(V)$ are pairwise disjoint.

If $m = 0$, one can take $V = X$. Assume that the assertion holds for $m \in \mathbb{N}$. We show that it holds for $m+1$. So by assumption

$\tau(y_o) > m+2$. By hypothesis, we find an open neighborhood V of y_o such that (5.3) and (5.4) hold. Since $\tau(y_o) > m+2$ it follows that $K \cap \varphi_{m+1}(K) = \emptyset$. So there exist open sets O_1, O_2 such that $O_1 \cap O_2 = \emptyset$, $K \subset O_1$ and $\varphi_{m+1}(K) \subset O_2$. Let $O_1' := \varphi_{-m-1}(O_2)$. Then $K \subset O_1 \cap O_1'$ and $\varphi_{m+1}(O_1 \cap O_1') \cap (O_1 \cap O_1') = \emptyset$. Let $V' := V \cap O_1 \cap O_1'$. Then V' is an open neighborhood of y_o such that $K \subset V'$ and $V', \varphi_1(V'), \ldots, \varphi_{m+1}(V')$ are pairwise disjoint. In order to prove the lemma let now $y_o = \varphi_{-n}(x_o)$. Then $\tau(y_o) > 2n+1$. So we find a neighborhood V of y_o such that (5.3) and (5.4) hold for $m = 2n$. Set $U = \varphi_n(V)$. Then U is a neighborhood of x_o satisfying (5.1) and (5.2).

LEMMA 5.2. Let $n \in \mathbb{N}$ and $x_o \in X$ such that $\tau(x_o) > 2n+1$. Let $\lambda \in \mathbb{R}$. Then given $\varepsilon > 0$ there exists $g \in D(A_o)$ such that $1 \geq \|g\| \geq |g(x_o)| \geq 1-\varepsilon$ and

$$(5.5) \qquad \|(i\lambda - A_o)g\| < 1/n .$$

Proof. Let U be an open neighborhood of x_o satisfying (5.1) and (5.2). By Proposition 2.3 (b), there exists $g_o \in C_o(X)$ such that $\|g_o\| = 1$ and $g_o(\varphi(t,x_o)) = \exp(i\lambda t)$ for $0 \leq t \leq 1$. Choose $q: X \to [0,1]$ such that $q(x) = 0$ for $x \notin U$ and $q(\varphi(t,x_o)) = 1$ for $t \in [0,1-\varepsilon]$. Let $f_o = q \cdot g_o \in C_o(X)$. Then f satisfies:

$$(5.6) \qquad \|f_o\|_\infty = 1$$

$$(5.7) \qquad f_o(\varphi(t,x_o))\exp(-i\lambda t) = 1 \qquad (0 \leq t \leq 1-\varepsilon)$$

$$(5.8) \qquad f_o(x) = 0 \quad \text{for } x \notin U .$$

Let $\alpha = \exp(i\lambda)$ and define $f \in C_o(X)$ by

$$(5.9) \qquad f(y) = \frac{1}{n}\Big(\sum_{k=o}^{n-1} \alpha^{-k}(n-k) f_o(\varphi_k(y)) + \sum_{k=1}^{n-1} \alpha^{k}(n-k) f_o(\varphi_{-k}(y))\Big) .$$

Then $\|f\| = |f_o(x_o)| = 1$ and

$$(5.10) \qquad \|T_o(1)f - \alpha f\| = \|f \circ \varphi_1 - \alpha f\| \leq \frac{1}{n} .$$

Let $g = \int_0^1 \exp(-i\lambda t)T_o(t)f\,dt$. Then $g \in D(A)$ and $\|g\| \leq 1$.

Moreover,

$$\begin{aligned} |g(x_o)| &\geq \left|\int_0^{1-\varepsilon} \exp(-i\lambda t) f(\varphi(t,x_o))dt\right| - \left|\int_{1-\varepsilon}^1 f(\varphi(t,x_o))dt\right| \\ &\geq \left|\int_0^{1-\varepsilon} \exp(-i\lambda t) f_o(\varphi(t,x_o))dt\right| - \varepsilon \\ &\qquad\qquad\qquad\qquad \text{(by (5.9) and (5.8))} \\ &= 1 - \varepsilon - \varepsilon = 1 - 2\varepsilon \qquad \text{(by (5.7))}. \end{aligned}$$

Finally, $-(i\lambda - A_o)g = e^{-i\lambda}\ T_o(1)f - f$ (by [12, 1.8]).
Hence, $\|(i\lambda - A_o)g\| = \|T_o(1)f - e^{i\lambda}f\| \leq 1/n$ (by (5.10)).

PROPOSITION 5.3. Let $\mu \in \mathbb{C}$. Assume that there exists $\delta > 0$ such that for every $\varepsilon > 0$, $n \in \mathbb{N}$ there exists $f \in D(A)$ such that $\|(\mu - A)f\| < \varepsilon$, $\|f\| = 1$ and $|f(x_o)| \geq \delta$ for some $x_o \in X$ with $\tau(x_o) > n$. Then

$$\mu + i\mathbb{R} \subset \sigma(A).$$

Proof. Let $\lambda \in \mathbb{R}$, $\varepsilon > 0$, $n > \varepsilon^{-1}$. Choose $f \in D(A)$ as in the hypothesis. By Lemma 5.2 there exists $g \in D(A_o)$ such that $\frac{1}{2} \leq |g(x_o)| \leq \|g\| \leq 1$ and $\|(i\lambda - A_o)g\| < n^{-1}$. Let $f_1 = f \cdot g$. Then $1 \geq \|f_1\| \geq \frac{\delta}{2} > 0$. Moreover, since A_o is a derivation (see (3.9)), it follows that $f_1 \in D(A_o) = D(A)$ and

$$\begin{aligned} &\|(\mu + i\lambda)f_1 - Af_1\| \\ &= \|\mu f\cdot g + i\lambda f\cdot g - (A_o f)g - (A_o g)f - h\cdot f\cdot g\| \\ &\leq \|(\mu f - Af)g\| + \|(i\lambda g - A_o g)f\| \\ &\leq \|\mu f - Af\| + \|i\lambda g - A_o g\| \leq \varepsilon + \frac{1}{n} < 2\varepsilon . \end{aligned}$$

This shows that $\mu + i\lambda \in A\sigma(A)$.

We now determine the spectrum of A and prove the spectral mapping theorem in the aperiodic case. Note, in contrast to the periodic case (cf. Remark 4.5), here the set $\exp(t\sigma(A))$ is automatically closed (see (5.12)).

THEOREM 5.4. If the flow φ is aperiodic (that is, if for every open subset O of X and every $n \in \mathbb{N}$ there exists $x \in O$ such that $\tau(x) > n$), then

(5.11) $\sigma(A) = \sigma(A) + i\,\mathbb{R}$

Moreover,

(5.12) $\sigma(T(t)) = \exp(t\,\sigma(A)) \quad (t \in \mathbb{R})$.

Proof. Let $\lambda \in \sigma(A)$. If $\lambda + i\,\mathbb{R} \not\subset \sigma(A)$, we find $\beta \in \mathbb{R}$ such that $\lambda + i\beta$ is in the boundary of $\sigma(A)$, hence in the approximative point spectrum $A\sigma(A)$. It follows from Proposition 5.3 that $\lambda + i\mathbb{R} \subset \sigma(A)$, a contradiction. (5.12) then follows from $\sigma(T(t)) \supset \exp(t\,\sigma(A)$ (which holds in general [7, 2.16]) and the real spectral mapping theorem $\sigma(T(t)) \cap \mathbb{R}_+ = \exp(t\,\sigma(A) \cap \mathbb{R})$ (see [3, 5.7]).

6. THE GENERAL CASE AND FURTHER CONSEQUENCES

We consider now the case where φ is an arbitrary flow. Using the notation of section 4 we denote by

$$J_\tau = \{f \in C_o(X)\colon f(x) = 0 \text{ for all } x \in X_\tau\} \text{ for } \tau > 0 .$$

Recall that $C_o(X)/J_\tau$ is isomorphic to $C_o(X_\tau)$ (by Proposition 2.3). Moreover, since X_τ is invariant under the flow, the ideal J_τ is invariant under the group $(T(t))$.

THEOREM 6.1. *The spectrum of* A *is given by*

(6.1) $$\sigma(A) = \Big(h(X_o) \cup \bigcup_{o<\tau(x)<\infty} \hat{h}(x) + i\,\frac{2\pi}{\tau(x)}\mathbb{Z}\Big)^- \cup R$$

where $R = \bigcap_{\tau > o} \sigma(A_{|J_\tau})$. *Moreover,*

(6.2) $R + i\,\mathbb{R} = R$ *and*

(6.3) $\lambda + i\,\mathbb{R} \subset \sigma(A)$ *implies* $\lambda \in R$.

REMARK. (6.2) and (6.3) say that R is exactly that part of $\sigma(A)$ which is invariant under imaginary translations.

Proof. It follows from Theorem 4.3 that for $\tau > 0$,

$$\sigma(A/J_\tau) = (h(X_o) \cup \bigcup_{o<\tau(x)\leq\tau} \hat{h}(x) + i\frac{2\pi}{\tau(x)}\mathbb{Z})^- .$$

So (6.1) follows from $\sigma(A) = \sigma(A_{/J_\tau}) \cup \sigma(A_{|J_\tau})$ for all $\tau > 0$ (see (3.12)). In order to prove (6.2) recall that for $\tau > 0$, $\sigma(A_{|J_\tau})$ is additively cyclic [8, 2.4].

Moreover, $\sigma(A_{|J_\tau})_{\tau>o}$ is decreasing. Thus, if (6.2) is false, there exist $\rho \in R$ and $\tau_o > 0$ such that $\rho + i\,\mathbb{R} \not\subset \sigma(A_{|J_{\tau_o}})$ and ρ is a boundary point of $\sigma(A_{|J_\tau})$ for all $\tau \geqslant \tau_o$. Consequently $\rho \in A\sigma(A_{|J_\tau})$ for all $\tau \geqslant \tau_o$. Let $n \in \mathbb{N}$, $\varepsilon > 0$. There exists $f \in J_\tau$ with $\tau > n$ such that $\|(\rho - A)f\| < \varepsilon$ and $\|f\| = 1$. Since $f \in J_\tau$, there exists $x_o \notin X_\tau$ such that $|f(x_o)| \geqslant \frac{1}{2}$. Applying Proposition 5.3 to $A_{|J_{\tau_o}}$ we conclude that $\rho + i\,\mathbb{R} \subset \sigma(A_{|J_{\tau_o}})$. This is a contradiction.

Finally we prove (6.3). By Theorem 4.3 we have $\sigma(A_{/J_\tau}) \cap \{\lambda \in \mathbb{C}: 0 < |\mathrm{Im}\,\lambda| < 2\pi/\tau\} = \emptyset$ and $\sigma(A_{|J_\tau})$ is additively cyclic by [8, 2.4]. Therefore, from $\lambda + i\,\mathbb{R} \subseteq \sigma(A) = \sigma(A_{/J_\tau}) \cup \sigma(A_{|J_\tau})$ it follows that $\lambda + i\,\mathbb{R} \subseteq \sigma(A_{|J_\tau})$.
This finishes the proof.

We now prove the spectral mapping theorem.

Proof of Theorem 1.1. By Corollary 3.7 we can assume that $(T(t))$ is given by (3.10). Let $t \in \mathbb{R}\setminus\{0\}$, $\lambda \in \sigma(T(t))$. We have to show that $\lambda \in \overline{\exp(t\,\sigma(A))}$. For $\tau > 0$ we have by Proposition 2.1, $\sigma(T(t)) = \sigma(T(t)_{|J_\tau}) \cup \sigma(T(t)_{/J_\tau})$. If $\lambda \in \sigma(T(t)_{/J_\tau})$ for some $\tau > 0$, we are finished by Theorem 4.4. So we can assume that $\lambda \notin \sigma(T(t)_{/J_\tau})$ for all $\tau > 0$. This implies $\lambda \in \bigcap_{\tau>o} \sigma(T(t)_{|J_\tau})$. Consequently $|\lambda| \in \bigcap_{\tau>o} \sigma(T(t)_{|J_\tau})$ (by [20, V. 4.4]).
By the real spectral mapping theorem [3, 5.7], this implies that $\rho := t^{-1}\log|\lambda| \in \sigma(A_{|J_\tau})$ for all $\tau > 0$. So by Theorem 6.1, $\rho + i\,\mathbb{R} \subset \sigma(A)$. In particular, $\lambda \in \{\exp t(\rho + is) : s \in \mathbb{R}\} \subset \exp(t\sigma(A))$.

The following consequence of Theorem 6.1 was mentioned in Section 1.

THEOREM 6.2. *Let* A *be the generator of a positive group on* $C_o(X)$. *If* λ_o *is an isolated point in* $\sigma(A)$, *then* λ_o *is a pole of order* 1 *of the resolvent*. In particular, every isolated point in $\sigma(A)$ is *an eigenvalue*.

Proof. Again, by Corollary 3.7 we can assume that A generates a group of the form (3.10). Let λ_o be an isolated point in $\sigma(A)$. It follows from Theorem 6.1 that $\lambda_o \notin \sigma(A_{|J_\tau})$ for some $\tau > 0$. Thus $R(\lambda,A)_{|J_\tau}$ is holomorphic in λ_o, and by [12, 1.2 (c)] it is enough to show that λ_o is a pole of order 1 of $R(\lambda,A)_{/J_\tau}$. Let $A_{-2} := \frac{1}{2\pi i}\int_c (\lambda-\lambda_o)\, R(\lambda,A)\, d\lambda$, where c is a circumference of sufficiently small radius. We have to show that $A_{-2/J_\tau} = 0$ (cf. [24, VIII. 8]). For that, it is enough to show that $(A_{-2}f)(x) = 0$ for all $f \in C_o(X)$ and all $x \in X_\tau$. Let $f \in C_o(X)$, $x \in X_\tau$. By [12, 1.8],

$$\int_o^{\tau(x)} \exp(-\lambda t)\, T(t)f\, dt = R(\lambda,A)f - \exp(-\lambda\tau(x))\, T(\tau(x))R(\lambda,A)f$$

$(\lambda \in \rho(A))$. Evaluating at x we obtain:

$$R(\lambda,A)f(x) = (1-\exp[\tau(x)(\hat{h}(x)-\lambda)])^{-1}\int_o^{\tau(x)} e^{-\lambda t}\, T(t)f(x)\, dt$$

$(\lambda \in \rho(A))$. The function $\lambda \to \int_o^{\tau(x)} \exp(-\lambda t)\, T(t)f(x)\, dt$ is holomorphic at λ_o and the function $\lambda \to (1-\exp[\tau(x)(\hat{h}(x)-\lambda)])^{-1}$ has a pole of order at most 1 at λ_o. This implies that

$$\frac{1}{2\pi i}\int_c (\lambda-\lambda_o)R(\lambda,A)f(x)\, d\lambda = 0. \quad \text{Hence} \quad (A_{-2}f)(x) = 0.$$

As in the discrete case (see (2.11)), the real part of $\sigma(A)$ can be described more precisely if the Cesàro-means of h converge uniformly on X. We use the same notation as in the preceding sections, i. e. $\varphi\colon \mathbb{R}\times X \to X$ is a continuous flow, the group $(T_o(t))_{t\in\mathbb{R}}$ is given by $T_o(t)f = f\circ\varphi_t$ $(f \in C_o(x)$ and its generator is denoted by A_o. We let $h \in C^b_{\mathbb{R}}(X)$, and the operator A on $C_o(X)$ with domain $D(A) = D(A_o)$ is given by $Af = A_of + h\cdot f$ $(f \in D(A))$. Moreover, we denote the Cesàro-means of h by

$$(6.4) \qquad C_t(h)(x) := \frac{1}{t}\int_o^t h(\varphi(s,x))\, ds \qquad (t > 0).$$

We first determine the real spectrum of A by means of the range of $C_t(h)$ $(t > 0)$.

PROPOSITION 6.3. *The real spectrum of* A *is given by the following expressions.*

(6.5) $\sigma(A) \cap \mathbb{R} = \{r \in \mathbb{R} : \forall \varepsilon > 0 \ \exists (t_n) \subset (0,\infty), \ \lim_{n\to\infty} t_n = \infty$ *such that* $(r-\varepsilon, r+\varepsilon) \cap C_{t_n}(h)(X) \neq \emptyset \quad \forall n \in \mathbb{N}\}$.

(6.6) $\sigma(A) \cap \mathbb{R} = \bigcap_{t>t_o} \overline{C_t(h)(X)}$ *for all* $t_o > 0$.

Proof. Recall that, if $(S(t))_{t\in\mathbb{R}}$ is a positive strongly continuous group on $C_o(X)$ with generator B, then

(6.7) $-\omega(-B) \leq \lambda \leq \omega(B)$ for all $\lambda \in \sigma(B) \cap \mathbb{R}$.

Here $\omega(B)$ denotes the *type* of $((S(t))_{t\geq o}$, that is, $\omega(B) = \lim_{t\to\infty} \frac{1}{t} \log\|S(t)\|$.

For $f \in C^b_{\mathbb{R}}(X)$ we let $s(f) := \sup_{x\in X} f(x)$ and $i(f) := \inf_{x\in X} f(x)$.

Now A generates the group $(T(t))_{t\in\mathbb{R}}$ biven by

$$T(t)f(x) = \exp(t\cdot C_t(h)(x)) \cdot f(\varphi(t,x)) \quad \text{(see (3.11))}.$$

Hence,

(6.8) $\omega(A) = \lim_{t\to\infty} s(C_t(h))$

Indeed,
$$\begin{aligned}\omega(A) &= \lim_{t\to\infty} \frac{1}{t} \log\|T(t)\| \\ &= \lim_{t\to\infty} \frac{1}{t} \log(\sup_{x\in X} \exp(t\cdot C_t(h)(x))) \\ &= \lim_{t\to\infty} s(C_t(h)).\end{aligned}$$

Similarly,

(6.9) $-\omega(-A) = \lim_{t\to\infty} i(C_t(h))$.

Now let $r \in \mathbb{R}$, $t_o > 0$ such that $r \notin \overline{C_{t_o}(h)(X)}$. Then there exists $\varepsilon > 0$ such that $(r-\varepsilon, r+\varepsilon) \cap C_{t_o}(h)(X) = \emptyset$. Let $X_1 = \{x \in X : C_{t_o}(h)(x) < r\}$ and $X_2 = \{x \in X : C_{t_o}(h)(x) > r\}$. Then $X_1 \cap X_2 = \emptyset$, $X_1 \cup X_2 = X$ and X_1, X_2 are both open and closed; in particular, X_1, X_2 are invariant under the flow. Let $(T_j(t))_{t\in\mathbb{R}}$

be the restriction of the group to $C_o(X_j)$ $(j = 1, 2)$. Then $\frac{1}{t_o} \log\|T_1(t_o)\| = s(C_{t_o}(h)_{|X_1}) < r$. Hence $\exp(t_o r) > r(T_1(t_o))$. Similarly, $\exp(t_o r) < r(T_2(-t_o))^{-1}$. Hence $\exp(t_o r) \notin \sigma(T_1(t_o)) \cup \sigma(T_2(t_o)) = \sigma(T(t_o))$. Consequently, $r \notin \sigma(A)$. We have thus proved that $\sigma(A) \cap \mathbb{R} \subset \bigcap_{t>o} \overline{C_t(h)(X)}$. Moreover, it is trivial that $\bigcap_{t>t_o} \overline{C_t(h)(X)}$ is included in the set on the right-hand side of (6.4). So it remains to show that this set is included in $\sigma(A) \cap \mathbb{R}$. Let $r \in \mathbb{R}$ and assume that $r \notin \sigma(A)$. By [3, 5.2] and [3, 6.4], there exists an open and closed subset X_1 of X, such that $\omega(A_{|J_1}) < r < -\omega(-A_{|J_2})$, where J_j denotes the band $J_j = \{f \in C_o(X): f(x) = 0$ for all $x \notin X_j\}$ $(j = 1, 2)$ where $X_2 = X \setminus X_1$. (Note: every projection band B in $C_o(X)$ can be identified with $C_o(U)$ for some open and closed subset U of X). So it follows from (6.8) and (6.9) that

$$\lim_{t\to\infty} s(C_t(h)_{|X_1}) < r < \lim_{t\to\infty} i(C_t(h)_{|X_2}) . \tag{6.10}$$

Thus for a suitable $\varepsilon > 0$ there exists $t_o > 0$ such that $C_t(h)X \cap (r-\varepsilon, r+\varepsilon) = \emptyset$ for all $t > t_o$; that is, r is not an element of the set on the right-hand side of (6.4).

REMARK. The analog of Proposition 6.3 for a single operator appears in [2, 5.1].

COROLLARY 6.4. *If* $C_t(h)$ *converges uniformly to* $\hat{h}$ $(t \to \infty)$, *then* $\sigma(A) \cap \mathbb{R} = \overline{(\hat{h}(X))}$.

COROLLARY 6.5. *Let* $r \in \mathbb{R}$. *The following are equivalent.*

(i) $\sigma(A) \cap \mathbb{R} = \{r\}$

(ii) $\lim_{t\to\infty} C_t(h) = r \cdot \mathbb{1}$ *uniformly* (*where* $\mathbb{1}(x) = 1$ *for all* $x \in X$).

The flow φ is called *uniquely ergodic*, if for every $g \in C_o(X)$ the net $(C_t(g))_{t>o}$ converges uniformly to a constant $(t \to \infty)$. (In the case that X is compact, one has the following characterization: φ is uniquely ergodic if and only if there exists exactly one probability measure which is invariant under φ). We obtain the following characterization from Corollary 6.5.

COROLLARY 6.6. *The following are equivalent.*

(i) φ *is uniquely ergodic.*

(ii) $\sigma(A_o + M_g) \cap \mathbb{R}$ *contains a single point for every* $g \in C_o(X)$.

THEOREM 6.7. *Assume that* $(C_t(h))_{t>o}$ *converges uniformly to* $\hat{h}$ $(t \to \infty)$. *Then*

$$\sigma(A) = [h(X_o) \cup \bigcup_{o<\tau(x)<\infty} (\hat{h}(x) + i\,\frac{2\pi}{\tau(x)}\mathbb{Z}) \cup \bigcup_{\tau(x)=\infty}(\hat{h}(x) + i\,\mathbb{R})]^- \tag{6.11}$$

Proof. Let $R = \bigcap_{\tau>o}\sigma(A_{|J_\tau})$. It follows from Corollary 6.4 applied to $A_{|J_\tau}$ that $\sigma(A_{|J_\tau}) \cap \mathbb{R} = \overline{\hat{h}(X \setminus X_\tau)}$ $(\tau > 0)$. So we conclude from Theorem 6.1 that the set

$$M := (h(X_o) \cup \bigcup_{o<\tau(x)<\infty} (\hat{h}(x) + i\,\frac{2\pi}{\tau(x)}\mathbb{Z}) \cup \bigcup_{\tau(x)=\infty}(\hat{h}(x) + i\,\mathbb{R}))^-$$

is included in $\sigma(A)$. In order to show the reverse inclusion let $r \in R \cap \mathbb{R}$. We have to show that $r + i\,\mathbb{R} \subset M$. If $r \in \overline{\hat{h}(X_\infty)}$ (where $X_\infty = X \setminus \bigcup_{\tau>o} X_\tau$) this is evident. So assume that $r \notin \overline{\hat{h}(X_\infty)}$. Since $r \in R \cap \mathbb{R} = \bigcap_{\tau>o} \overline{\hat{h}(X \setminus X_\tau)}$, there exists a sequence (x_n) in X such that $n < \tau(x_n) < \infty$ and $\lim_{n\to\infty} \hat{h}(x_n) = r$. This implies

$$r + i\,\mathbb{R} \subset \overline{\{\hat{h}(x_n) + i\,\frac{2\pi m}{\tau(x_n)} : \quad n,m \in \mathbb{Z}\}} \subset M .$$

In order to apply Theorem 6.7, it is useful to know conditions under which $C_t(h)$ converges uniformly. The following is a consequence of Proposition 6.3.

PROPOSITION 6.8. *If* $\sigma(A) \cap \mathbb{R}$ *is totally disconnected, then* $C_t(h)$ *converges uniformly* $(t \to \infty)$.

Proof. Let $\varepsilon > 0$. Since $\sigma(A) \cap \mathbb{R}$ is completely disconnected, there exist real numbers $r_o < r_1 < \dots < r_n$ such that $r_i - r_{i-1} < \varepsilon$

$(i = 1,\dots,n)$ and $\sigma(A) \cap \mathbb{R} \subset \bigcup_{i=1}^{n} (r_{i-1}, r_i)$. Using the argument which implies (6.10), we obtain a partition $X_1,\dots,X_n$ of X consisting of open and closed subsets and $t_o > 0$ such that $C_t(h)(X_i) \subset (r_{i-1}, r_i)$ for $i = 1,\dots,n$ and all $t > t_o$. This implies that $\|C_{t_1}(h) - C_{t_2}(h)\| < \varepsilon$ for all $t_1, t_2 > t_o$. We have proved that $C_t(h)$ is a Cauchy-net for the uniform norm.

REMARK 6.9. The preceding result can be applied if A_o has compact resolvent. In fact, in that case A has compact resolvent as well (see [10, VIII, 3.17]). So $\sigma(A) \cap \mathbb{R}$ is finite. Hence Proposition 6.8 implies that $C_t(h)$ converges uniformly $(t \to \infty)$.

On the other hand A_o rarely has compact resolvent. The following is essentially the only example:

Let $X = S^1$ $(= \{z \in \mathbb{C}: |z| = 1\})$ and $\tau > 0$. Let $T^\tau(t)f(z) = f(z \cdot e^{2\pi i t/\tau})$ $(z \in S^1,\ t \in \mathbb{R},\ f \in C(X))$ and A^τ be the generator of the strongly continuous group $(T^\tau(t))_{t\in\mathbb{R}}$. Then A^τ has compact resolvent. (This is easy to show.)

Conversely, assume that A_o has compact resolvent. We show that X is the disjoint union of compact sets $X_o, X_1, \dots, X_n$ such that X_o is finite, and $T_o(t)_{|C(X_o)} = I$ $(t \in \mathbb{R})$ and for $i = 1,\dots,n$, the space X_i is isomorphic to S^1 and $T_o(t)_{|C(X_i)} = T^{\tau_i}(t)$ for all $t \in \mathbb{R}$ and one $\tau_i > 0$.
Indeed, since A_o has compact resolvent, $\sigma(A_o)$ consists only of isolated points. So it follows from (6.1) that $\tau(x) < \infty$ for all $x \in X$. We denote by $o(x) := \{\varphi(t,x) : 0 \leq t \leq \tau(x)\}$ the orbit of $x \in X$. We claim that there exist only finitely many orbits. In fact, let $F_x := R(1, A_o)'\delta_x$ where δ_x is the Dirac measure in $x \in X$. Then $F_x = \delta_x$ if $\tau(x) = 0$ and

$$F_x(f) = [1 - \exp(-\tau(x))]^{-1} \int_o^{\tau(x)} f(\varphi(s,x))e^{-s}\, ds$$

if $\tau(x) > 0$. Thus, if $o(x) \neq o(y)$, then $\|F_x - F_y\| \geq 1$. Since $R(1, A_o)'$ is compact, it follows that $\{F_x: x \in X\}$ is finite. So there exists only a finite number of distinct orbits. Define $X_o := \{x \in X: \tau(x) = 0\}$ and let $X_1,\dots,X_n$ be the non-trivial orbits.

Generators of positive groups with compact resolvent on arbitrary Banach lattices have been characterized by Uhlig [22] (see also [11]).

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