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The Spectrum of Quasi-invertible Disjointness Preserving Operators

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We study (almost) invertible operators on Banach lattices which take disjoint elements to disjoint elements. We show that these operators may be decomposed into a direct sum of their strictly periodic and aperiodic parts, and then use this result to derive various properties of the spectrum.

0. INTRODUCTION

Let E be a space of functions over a set X. Two functions f, g on E are called disjoint if their supports are essentially disjoint. An operator on E is called disjointness preserving if it takes disjoint elements to disjoint elements. Every weighted composition operator on E, that is every operator of the form $Tf = h \cdot f \circ \phi$, where $h: X \to \mathbf{C}$ and $\phi: X \to X$, is disjointness preserving (for a result in the converse direction, see [1]). The notion of disjointness can be generalized to an abstract Banach lattice, which will be the context used here.

This paper studies invertible disjointness preserving operators and their spectrum on order complete Banach lattices. The structure of these operators is discussed in Section 3. The main result is that each operator can be decomposed into a direct sum of its strictly periodic and aperiodic components (Definition 3.1 and Theorem 3.10). This generalizes a well-known analogous decomposition of invertible non-singular transformations on measure spaces (see [7, 20]).

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Sections 4 and 5 discuss the spectrum of these operators. As might be expected, the spectrum possesses a good deal of rotational symmetry (cf. [21, V 4.4]). Our method is to first study the spectrum of the simple types given in the decomposition theorem. In particular, we show that if T has strict period n, then $\sigma(T)=\alpha\sigma(T)$ for every nth root of unity α, and if T is aperiodic, then $\sigma(T)$ is rotationally invariant (see Theorem 4.3 and Corollary 4.7). Our main result (Theorem 4.6) gives the connection between the spectrum of the individual components of the operator and the spectrum of the operator itself.

There are several interesting consequences of these results, which are given in Section 5. For instance, we show that isolated points of the spectrum are eigenvalues (Theorem 5.5). We also show that if the spectrum is all on one side of a line through the origin, then the operator must be a multiplication operator (see Theorem 5.3).

Our techniques actually apply to a more general class of operators which are called quasi-invertible disjointness preserving operators (Definition 2.1). Roughly speaking, on concrete function spaces this means that the operator is of the form $Tf=h\cdot f\circ\phi$, where ϕ is invertible and h is non-zero except perhaps on a negligible set. We give various characterizations of quasi-invertible disjointness preserving operators in Section 2. The result of Sections 3–5 are then presented for this slightly larger class of operators.

For the general terminology and theory of Banach lattices we refer to [21, 14, 28].

1. Preliminaries

We will throughout this paper denote by E an order complete (= Dedekind complete) complex Banach lattice (see [21, II, Sect. 11] or [28, Sect. 91]). By $E_{\mathbb{R}}$ we mean the real Banach lattice underlying E, and E_+ will denote the positive cone of $E_{\mathbb{R}}$.

Two elements $f, g\in E$ are said to be *disjoint* (denoted by $f\perp g$) if $|f|\wedge|g|=0$. If A is a subset of E, then the *disjoint completement* of E is given by $\{A\}^d=\{f\in E: f\perp g\ \forall g\in A\}$. Recall that a subspace J of E is called an *ideal* if $f\in J$, $g\in E$ and $|g|\leqslant|f|$ imply $g\in J$. A subspace B of E is called a *band* if $B^{dd}=B$. A band is always a closed ideal.

We will denote the collection of all operators (= continuous linear maps) from E into itself by $\mathscr{L}(E)$. An operator $T\in\mathscr{L}(E)$ is called *regular* if its modulus $|T|$ given by $|T|f:=\sup_{|g|\leqslant f}|Tg|$ exists. The collection of all regular operators on E will be denoted by $\mathscr{L}^r(E)$. $\mathscr{L}^r(E)$ is a subalgebra of $\mathscr{L}(E)$. It is a Banach algebra as well as a Banach lattice under the norm $\|T\|_r:=\|\,|T|\,\|$.

An operator $T \in \mathscr{L}(E)$ is called *disjointness preserving* if $f \perp g$ implies $Tf \perp Tg$ ($f, g \in E$). If $T \in \mathscr{L}(E)$ is disjointness preserving, then T is order bounded, that is T takes sets of the form $\{f \in E: g \leqslant f \leqslant h\}$ ($g, h \in E$) into sets of the same type ([3, 2.5]; for generalizations, see [1, 17]). This implies that T is regular. Moreover, $|Tf| = |T|\,|f|$ and an operator T is disjointness preserving iff $|T|$ is a lattice homomorphism (see [3] or [15]; recall that a lattice homomorphism is an operator T satisfying $|Tf| = T|f|$ for all $f \in E$).

The *center* of E, denoted by $Z(E)$, is the collection of all operators $M \in \mathscr{L}(E)$ for which there is a positive real number k such that $|Mf| \leqslant k|f|$ for all $f \in E$. $Z(E)$ is a closed commutative subalgebra and sublattice of $\mathscr{L}^r(E)$ which corresponds on concrete function spaces to the collection of all multiplication operators on E. If X denotes the maximal ideal space of $Z(E)$, then $Z(E)$ is isometricly isomorphic, as a Banach lattice as well as a Banach algebra, to $C(X)$ via the Gelfand transform. We will denote the image of $M \in Z(E)$ under the Gelfand transform by $\hat{M}$ and the inverse image of $f \in C(X)$ under the Gelfand transform by $\check{f}$.

EXAMPLE. Let (Y, Σ, μ) be a σ-finite measure space and $E = L^p(Y, \Sigma, \mu)$ ($1 \leqslant p \leqslant \infty$). Then the map $h \to M_h$ given by $M_h f = h \cdot f$ ($h \in L^\infty(Y, \Sigma, \mu)$, $f \in E$) defines an algebra and lattice isomorphism of $L^\infty(Y, \Sigma, \mu)$ onto $Z(E)$ [28, 142.11]. Thus the maximal ideal space of $Z(E)$ can be identified with the Stonean representation space of the measure algebra associated with (Y, Σ, μ).

The assumption that E is order complete implies that $Z(E)$ is also order complete, whence X is extremely disconnected. Recall that a *band projection* is an idempotent operator on E whose range is a band. For every band B, there is a unique band projection P such that $B = PE$. An idempotent operator P is a band projection iff $P \in Z(E)$. Thus, if θ is a clopen subset of X, then $\check{1}_\theta$ is a band projection, and conversely if P is a band projection, then $\hat{P}$ is the characteristic function of a clopen set in X. We will need the following characterization of the operators in $Z(E)$.

PROPOSITION 1.1. *The following are equivalent:*

(a) $T \in Z(E)$.

(b) $T \in \mathscr{L}(E)$ *and* $Tf \perp g$ *for all* $f, g \in E$ *satisfying* $f \perp g$.

(c) $T \in \mathscr{L}(E)$ *and* $TP = PT$ *for all band projections* P *on* E.

Proposition 1.1 shows in particular that every element of $Z(E)$ is disjointness preserving. We will also need the following properties of the center.

PROPOSITION 1.2. (i) *If S is any subset of E and $M_1, M_2 \in Z(E)$ satisfy $M_1 f = M_2 f$ for all $f \in S$, then $M_1 g = M_2 g$ for all $g \in \{S\}^{dd}$.*

(ii) *If $M \in Z(E)$, then* $\operatorname{Ker} M = \{ME\}^d$. *In particular,* $\operatorname{Ker} M$ *is a band.*

(iii) *If $f, g \in E$ satisfy $|g| \leqslant |f|$, then there is an operator $M \in Z(E)$ such that $g = Mf$.*

For proofs of the above results about $Z(E)$, see [12, Chap. 3] or [28, Chap. 20]. We close this section with the following lemma.

LEMMA 1.3. *Let $T \in \mathscr{L}(E)$ be disjointness preserving. The following are equivalent:*

(a) *$T \in Z(E)$ and T is injective.*

(b) *If $f \in E$ and $Tf \perp f$, then $f = 0$.*

Proof. Suppose $T \in Z(E)$ and T is injective. If $Tf \perp f$, then $Tf \perp Tf$ by Proposition 1.1. Hence $Tf = 0$, which shows that $f = 0$ since T is injective.

Conversely, suppose (b) holds. Then T is clearly injective. Suppose $g \perp h$ in E and set $f := |Tg| \wedge |h|$. Then $|T| f \wedge f = |T|(|Tg| \wedge |h|) \wedge |Tg| \wedge |h| = |T^2 g| \wedge |T|(|g| \wedge |h|) \wedge |h| = 0$. By (b), $f = 0$, so $Tg \perp h$. Proposition 1.1 now implies that $T \in Z(E)$.

2. BASIC PROPERTIES

We begin by introducing the class of operators which will be investigated. Recall that an operator $T \in \mathscr{L}(E)$ is called *order continuous* if for any downward directed net f_α $(\alpha \in A)$ such that $\inf_{\alpha \in A} f_\alpha = 0$, we have $\inf_{\alpha \in A} |Tf_\alpha| = 0$. Every element of $Z(E)$ is automatically order continuous. If E has order continuous norm, (e.g., if E is an L^p space for $1 \leqslant p < \infty$), then every regular operator is order-continuous.

DEFINITION 2.1. An order continuous disjointness preserving operator $T \in \mathscr{L}(E)$ is called *quasi-invertible* if T is injective, $\{TE\}^{dd} = E$ and the adjoint T' is disjointness preserving.

EXAMPLE 2.2. Every invertible disjointness preserving operator T is quasi-invertible since $|T'| = |T|'$ is a lattice isomorphism (see [3, 2.6]). The operator $f(x) \to xf(x)$ defined on $L^p[0, 1]$ with Lebesgue measure $(1 \leqslant p \leqslant \infty)$ is an example of a non-invertible operator which is quasi-invertible. More generally, let V be a lattice isomorphism and suppose $M \in Z(E)$ is injective. Then $T := MV$ is quasi-invertible. To see this, first note that $\{TE\}^d = \{ME\}^d = 0$ by Proposition 1.2, so $\{TE\}^{dd} = E$. Furthermore, T is order continuous and disjointness preserving, since both M and V have these properties. Finally, it is easy to see that $M' \in Z(E')$ and that V' is a lattice isomorphism. Hence $T' = V'M'$ is disjointness preserving, so T is quasi-invertible.

PROPOSITION 2.3. *Let $T \in \mathscr{L}(E)$ be a disjointness preserving operator. The following are eqivalent*:

(a) *T is quasi-invertible.*

(b) *T is injective and the range of T is an order dense ideal (i.e., TE is an ideal satisfying $\{TE\}^{dd} = E$).*

Proof. (a) $\Rightarrow$ (b): This follows immediately from [6, 2.8].

(b) $\Rightarrow$ (a): Using [11, 1.2] and [3, 2.7], it is easy to see that T' is disjointness preserving. Furthermore, since the range of T is order dense, an argument similar to the proof of [14, 18.11] shows that T is order continuous.

It is well known that each disjointness preserving operator $T \in \mathscr{L}(E)$ can be naturally associated with an algebra and lattice homomorphism $\tilde{T} \in \mathscr{L}(E)$ (see [27, 3, 6]). For instance, if $E = L^p[0, 1]$ with Lebesgue measure, and $T \in \mathscr{L}(E)$ is given by $Tf(x) = h(x) f(\phi(x))$, then under the natural identification of $Z(E)$ with $L^\infty[0, 1]$, the associated operator is given by $\tilde{T}f(x) = f(\phi(x))$. Applying two theorems of Luxemburg and Schep [13], we now show that quasi-invertible disjointness preserving operators can be characterized as exactly those operators which are associated with an isomorphism $\tilde{T}$ on $Z(E)$. The existence and invertibility of $\tilde{T}$ will be crucial in the calculations involving the spectrum of T which will be given in Section 4.

THEOREM 2.4. *Let $T \in \mathscr{L}(E)$. The following are equivalent*:

(a) *T is a quasi-invertible disjointness preserving operator.*

(b) *There exists a unique algebra and lattice isomorphism $\tilde{T}$ from $Z(E)$ onto itself such that $TM = \tilde{T}(M)\, T$ for all $M \in Z(E)$.*

Moreover, if T is quasi-invertible, so is $|T|$ and these two operators have the same associated operator, i.e., $\tilde{T} = |T|^\sim$.

Proof. (a) $\Rightarrow$ (b): First suppose T is positive. Pick $M \in Z(E)$. Then $|M| \leqslant kI$ for some $k \in \mathbf{R}_+$, so applying [13, 4.2] to kT yields an operator $\tilde{T}(M) \in Z(E)$ satisfying $\tilde{T}(M)\, T = TM$. Note that by Proposition 1.2(i), $\tilde{T}(M)$ is uniquely determined since TE is order dense. An easy calculation shows that $\tilde{T}$ is an algebra and lattice homomorphism (cf. the proof of

[27, 3.1]). $\tilde{T}$ is trivially injective; that $\tilde{T}$ is surjective follows immediately from [13, 3.1] and Proposition 2.3.

If T is not positive, then since $\operatorname{Re} T \leqslant |T|$ and $\operatorname{Im} T \leqslant |T|$, it follows from [13, 4.3] that there are operators $N_1, N_2 \in Z(E)$ s.t. $\operatorname{Re} T = N_1|T|$ and $\operatorname{Im} T = N_2|T|$. Hence $T = N|T|$, where $N := N_1 + iN_2 \in Z(E)$. Thus, by the previous paragraph, for any $M \in Z(E)$ we have $TM = N|T|\, M = N|T|^{\sim}(M)|T| = |T|^{\sim}(M)\, N|T| = |T|^{\sim}(M)\, T$. It follows that $\tilde{T}(M) := |T|^{\sim}$ satisfies the required properties.

(b) $\Rightarrow$ (a): We first show that T is disjointness preserving. Suppose $f \perp g$ and denote by P and Q the band projections onto $\{f\}^{dd}$ and $\{g\}^{dd}$, respectively. Then

$$\begin{aligned} |Tf| \wedge |Tg| &= |TPf| \wedge |TQg| = |\tilde{T}(P)\, Tf| \wedge |\tilde{T}(Q)\, Tg| \\ &= \tilde{T}(P)\,|Tf| \wedge \tilde{T}(Q)\,|Tg| \leqslant (\tilde{T}(P) \wedge \tilde{T}(Q))(|Tf| + |Tg|) \\ &= \tilde{T}(P \wedge Q)(|Tf| + |Tg|) = 0. \end{aligned}$$

By [6, 2.6], T is injective. To complete the proof, we show that TE is an order dense ideal. Suppose $|g| \leqslant |Tf|$ in E. By Proposition 1.2(iii), there exists $N \in Z(E)$ such that $NTf = g$. Let $M = \tilde{T}^{-1}(N)$. Then $g = TMf \in TE$, so TE is an ideal. Now let P be the band projection onto $\{TE\}^d$ and set $Q = \tilde{T}^{-1}(P)$. Then $QT = TP = 0$. Since T is injective, it follows that $Q = 0$. Hence $P = \tilde{T}(Q) = 0$, and the proof is complete.

Under the identification $Z(E) \simeq C(X)$, where X is the maximal ideal space of $Z(E)$, the algebra isomorphism $\tilde{T}$ takes the form $\tilde{T}f(x) = f(\tau(x))$ $(f \in C(X))$, where τ is a homeomorphism of X onto itself. We will call τ the *associated homeomorphism* of T. If $E = C(K)$ for some extremely disconnected compact space K and T is a quasi-invertible operator defined by $Tf = h \cdot f \circ \phi$, where $h \in E$ and ϕ is a homeomorphism from K onto itself, then $\tilde{T}$ and τ may be identified with the composition operator $f \mapsto f \circ \phi$ and the homeomorphism ϕ, respectively.

3. A Decomposition Theorem

The main result of this section is that an arbitrary quasi-invertible disjointness preserving operator can be decomposed into a direct sum of operators which have one of the properties of the next definition. Recall that two bands B_1, B_2 of E are called *disjoint* if $f_1 \in B_1$ and $f_2 \in B_2$ imply $f_1 \perp f_2$. The bands B_1 and B_2 are disjoint iff $B_1 \cap B_2 = \{0\}$ iff their corresponding band projections P_1 and P_2 are disjoint (that is, $P_1 P_2 = 0$).

For the remainder of this article, X will denote the maximal ideal space of $Z(E)$, and T will denote a quasi-invertible disjointness preserving operator on E with associated isomorphism $\tilde{T}$ on $Z(E)$ and associated homeomorphism τ on X.

DEFINITION 3.1. (i) *T is said to have strict period n if $T^n \in Z(E)$ and for every band $B \neq \{0\}$, there is a band A such that $\{0\} \neq A \subset B$ and $A, \{TA\}^{dd}, ..., \{T^{n-1}A\}^{dd}$ are mutually disjoint.*

(ii) *T is said to be aperiodic if for every $n \in \mathbf{N}$ and every band $B \neq \{0\}$, there is a band A such that $\{0\} \neq A \subset B$ and $A, \{TA\}^{dd}, ..., \{T^nA\}^{dd}$ are mutually disjoint.*

EXAMPLE 3.2. Let (Y, Σ, μ) be a finite measure space and let $E = L^p(Y, \Sigma, \mu)$ $(1 \leqslant p \leqslant \infty)$. Suppose $T \in \mathscr{L}(E)$ is of the form $Tf = h \cdot f \circ \phi$ for all $f \in E$, where $h \in L^\infty(Y, \Sigma, \mu)$ is non-zero μ-a.e. and ϕ: $Y \to Y$ is a non-singular invertible bi-measurable transformation. Then T is a quasi-invertible disjointness preserving operator. Moreover, T has strict period n (resp. aperiodic) if ϕ has strict period n (aperiodic) in the measure theoretic sense (see [7]). (This may be seen by noting that the bands of E are of the form $B = \{f \in E: f(x) = 0$ for a.e. $x \in S\}$, where $S \in \Sigma$.) In particular, if $Y = [0, 1]$, Σ is the Borel sets of $[0, 1]$ and μ is a Borel measure, then T has strict period n iff $\phi^n(x) = x$ μ-a.e. and the sets $\{x \in Y: \phi^k(x) = x\}$ have measure zero for $k = 1, 2, ..., n-1$. Similarly, T is aperiodic iff the sets $\{x \in Y: \phi^k(x) = x\}$ have measure zero for each $k \in \mathbf{N}$.

LEMMA 3.3. *Let ϕ be a homeomorphism on an extremely disconnected compact space K and pick $n \in \mathbf{N}$. Then there is a clopen subset θ of K s.t. $\phi^k(\theta) \cap \theta = \varnothing$, $k = 1, 2, ..., n$, and $\bigcup_{m=-n}^{n} \phi^m(\theta) = \{x \in K: \phi^k(x) \neq x$ for $k = 1, 2, ..., n\}$.*

Proof. Let $\mathscr{M} = \{U \subset K: U$ is open and $\phi^k(U) \cap U = \varnothing$ for $k = 1, 2, ..., n\}$. It is easy to see that $\mathscr{M}$ is inductively ordered by inclusion and that $\varnothing \in \mathscr{M}$. Hence, by Zorn's lemma, there exists a maximal element θ in $\mathscr{M}$. Since K is extremely disconnected, the clopen set $\bar{\theta} \in \mathscr{M}$, whence θ is closed by maximality.

Let $V = \{x \in K: \phi^k(x) \neq x$ for $k = 1, 2, ..., n\}$. Then by construction of θ, $\theta \subset V$. Moreover, if $x \in \theta$, then $\phi^k(\phi^m(x)) = \phi^m(\phi^k(x)) \neq \phi^m(x)$ for each $k = 1, 2, ..., n$ and $m \in \mathbf{Z}$. This shows that $\phi^m(x) \in V$, so in particular $\bigcup_{m=-n}^{n} \phi^m(\theta) \subset V$.

Suppose that this last inclusion is proper. Then there exists $x \in V \backslash \bigcup_{m=-n}^{n} \phi^m(\theta)$. Since $x \neq \phi^k(x)$ for $k = 1, 2, ..., n$ and $\bigcup_{m=-n}^{n} \phi^m(\theta)$ is closed, there exists an open neighbourhood U of x such that $U \cap \phi^k(U) = \varnothing$ for $k = 1, ..., n$ and $U \cap \bigcup_{m=-n}^{n} \phi^m(\theta) = \varnothing$. This implies that $\phi^k(\theta \cup U) \cap (U \cup \theta) = \varnothing$ for $k = 1, ..., n$, contradicting the maximality of θ, and proving the lemma.

We pause now to give some topological consequences of Lemma 3.3.

COROLLARY 3.4. *Let ϕ be a homeomorphism on an extremely disconnected compact space K. Then the fixed set $K_1 = \{x \in K: \phi(x) = x\}$ is open and closed.*

Proof. Putting $n=1$ in Lemma 3.3 shows that $\{x \in K: \phi(x) \neq x\}$ is open-closed, so its complement K_1 must be so as well.

COROLLARY 3.5. *Let ϕ be a homeomorphism on an extremely disconnected compact space K. If every point $x \in K$ has finite orbit (i.e., the set $\{\phi^k(x): k \in \mathbf{N}\}$ is finite), then ϕ^n is the identity map for some $n \in \mathbf{N}$.*

Proof. By Corollary 3.4 the sets $K_n = \{x: \phi^n(x) = x\}$ are open for each $n \in \mathbf{N}$. By hypothesis we have $K = \bigcup_{n=1}^{\infty} K_n$. Since K is compact, there exists an integer N such that $K = \bigcup_{n=1}^{N} K_n$. Thus $\phi^{N!}$ is the identity map.

Remark. Although Corollary 3.5 is also valid on locally Euclidean spaces (see [16, 5.5]), it is not true for arbitrary compact spaces. For example, let $\Gamma = \{z \in \mathbf{C}: |z| = 1\}$ and $K = \bigcup_{n \in \mathbf{N}} (1/n)\, \Gamma \cup \{0\}$ with the relative topology of $\mathbf{C}$. Then K is compact. Define $\tau: K \to K$ by $\tau(z) = ze^{2\pi i/n}$ if $z \in (1/n)\, \Gamma$ and $\tau(0) = 0$. Then τ is a homeomorphism and each point of K has a finite orbit under τ, but τ^n is not the identity map for any $n \in \mathbf{N}$.

We next wish to give various characterizations of those operators which have strict period n or are aperiodic. We will use the following lemma, which follows immediately from the equivalence of (a) and (c) in Proposition 1.1.

LEMMA 3.6. *The following are equivalent:*

(a) $T \in Z(E)$.

(b) *$\tilde{T}$ is the identity operator on $Z(E)$.*

(c) *τ is the identity map on X.*

LEMMA 3.7. *The following are equivalent:*

(a) *For every band $B \neq \{0\}$ there exists a band A such that $\{0\} \neq A \subset B$ and $A \cap \{TA\}^{dd} = \{0\}$.*

(b) *For every band projection $P \neq 0$ there exists a band projection Q such that $0 \neq Q \leqslant P$ and $Q\tilde{T}(Q) = 0$.*

(c) *For every $f > 0$ in E, there exists $g \in E$ such that $0 < g \leqslant f$ and $Tg \perp g$.*

(d) $I \wedge |T| = 0$.

(e) *$\tau(x) \neq x$ for all $x \in X$.*

Proof. The equivalence of (a) and (b) is clear from the correspondence of bands with band projections.

(b) $\Rightarrow$ (c): Let P be the band projection onto $\{f\}^{dd}$ and choose Q to satisfy (b). Set $g = Qf$. Then $f \geqslant g > 0$ and $QTg = QTQf = Q\tilde{T}(Q)\,Tf = 0$. Hence $g \perp Tg$.

(c) $\Rightarrow$ (d): Suppose $I \wedge |T| \neq 0$. Then since $I \wedge |T| \in Z(E)$, there is a non-zero band projection P and an $\varepsilon > 0$ such that $0 < \varepsilon P < I \wedge |T|$. Pick any $f > 0$ in the band PE, let g satisfy the properties given in (c), and let Q be the band projection onto $\{g\}^{dd}$. Note that $(I \wedge |T|)\, g \neq 0$ and that Q commutes with $I \wedge |T|$ since they are both in $Z(E)$. Consequently, $0 < (I \wedge |T|)\, g = (I \wedge |T|)\, Qg = Q(I \wedge |T|)\, g \leqslant Q|T|\, g = 0$ (since $g \perp Tg$), a contradiction. Hence $I \wedge |T| = 0$.

(d) $\Rightarrow$ (e): Let $X_1 = \{x \in X : \tau(x) = x\}$, $P = \check{1}_{X_1}$ and $E_1 = PE$. E_1 is clearly a reducing band for T and $T|_{E_1} \in Z(E_1)$ by Lemma 3.6. Hence $TP \in Z(E)$ so $|TP| \leqslant kI$ for some $k \in \mathbf{N}$. Therefore $|TP| \leqslant kI \wedge |T| \leqslant k(I \wedge |T|) = 0$. Since T is quasi-invertible, this shows that $P = 0$, whence X_1 is empty.

(e) $\Rightarrow$ (b): There is a clopen subset θ of X such that $\hat{P} = 1_\theta$. By Lemma 3.3, there is a non-empty clopen subset $U \subset \theta$ such that $U \cap \tau^{-1}(U) = \varnothing$. Let $Q = \check{1}_U$. Then it is clear from the definitions of $\tilde{T}$ and τ that $Q\tilde{T}(Q) = 0$.

The following two results follow easily by applying Lemma 3.7 successively to the powers of T.

PROPOSITION 3.8. *Let $n \in \mathbf{N}$. The following are equivalent:*

(a) *T has strict period n.*

(b) *$T^n \in Z(E)$ and for each non-zero band projection P, there is a band projection Q such that $0 \neq Q \leqslant P$ and $Q, \tilde{T}(Q), \dots, \tilde{T}^{n-1}(Q)$ are mutually disjoint.*

(c) *$T^n \in Z(E)$ and for each $f > 0$, there exists $g > 0$ such that $g \leqslant f$ and $g, Tg, \dots, T^{n-1}g$ are mutually disjoint.*

(d) *$T^n \in Z(E)$ and $I \wedge |T|^k = 0$ for each $k = 1, 2, \dots, n-1$.*

(e) *$\tau^n(x) = x$ but $\tau^k(x) \neq x$ for all $x \in X$ and each $k = 1, 2, \dots, n-1$.*

PROPOSITION 3.9. *The following are equivalent:*

(a) *T is aperiodic.*

(b) *For every band projection $P \neq 0$ and each $n \in \mathbf{N}$, there is a band projection Q_n such that $0 \neq Q_n \leqslant P$ and $Q_n, \tilde{T}(Q_n), \dots, \tilde{T}^n(Q_n)$ are mutually disjoint.*

(c) *For each* $f > 0$ *and each* $n \in \mathbf{N}$, *there exists* $g_n > 0$ *such that* $g_n < f$ *and* $g_n, Tg_n, \ldots, T^n g_n$ *are mutually disjoint.*

(d) $I \wedge |T|^n = 0$ *for all* $n \in \mathbf{N}$.

(e) $\tau^n(x) \neq x$ *for all* $x \in X$ *and all* $n \in \mathbf{N}$.

Remark. The last two propositions show that T has strict period n (or is aperiodic) iff τ has strict period n (aperiodic) in the standard terminology of topological dynamics.

We now turn to the decomposition of an arbitrary quasi-invertible disjointness preserving operator into strictly periodic parts. For each $n \in \mathbf{N}$ define

$$X_n = \{x \in X : \tau^n(x) = x \text{ but } \tau^k(x) \neq x \ (k = 1, 2, \ldots, n-1)\}$$

and set

$$X_\infty = X \setminus \overline{\bigcup_{n \in \mathbf{N}} X_n}.$$

By Corollary 3.4, the X_n are open and closed. For each $n \in \mathbf{N} \cup \{\infty\}$ define a band projection on E by $P_n = \tilde{1}_{X_n}$. Define $E_n = P_n E$ for each $n \in \mathbf{N} \cup \{\infty\}$. Since $X_i \cap X_j = \varnothing$ for $i \neq j$, the B_n are mutually disjoint bands. Since $\tau^{-1}(X_n) = X_n$ for each $n \in \mathbf{N} \cup \{\infty\}$, we have $\tilde{T}(P_n) = P_n$ by the very definition of τ and P_n. This shows that $TP_n = P_n T$, so the bands E_n are reducing bands for T. Also, by Propositions 3.8 and 3.9, $T|_{E_n}$ has a strict period n for $n \in \mathbf{N}$ and $T|_{E_\infty}$ is aperiodic. Since $X = \overline{\bigcup_{n \in \mathbf{N} \cup \{\infty\}} X_n}$, $\sup_{n \in \mathbf{N} \cup \{\infty\}} P_n = I$. Combining this with the fact that the E_n are mutually disjoint shows that $E = \bigoplus_{n \in \mathbf{N} \cup \{\infty\}} E_n$ in the sense that each $f \in E$ can be written uniquely in the form $f = f_\infty + o - \lim_{n \to \infty} \sum_{k=1}^n f_k$, where $f_\infty \in E_\infty$ and $f_k \in E_k$. We summarize these results in the following theorem.

THEOREM 3.10. *For any quasi-invertible disjointness preserving operator* T, *there exist a unique family of* T-*reducing bands* E_n $(n \in \mathbf{N} \cup \{\infty\})$ *which satisfy the following properties*:

(i) $E = \bigoplus_{n \in \mathbf{N} \cup \{\infty\}} E_n$.

(ii) $T|_{E_n}$ $(n \in \mathbf{N})$ *has strict period* n *and* $T|_{E_\infty}$ *is aperiodic.*

4. The Spectrum

We continue to assume that T is a quasi-invertible disjointness preserving operator on an order complete Banach lattice E. We will denote by $\sigma(T)$, $P\sigma(T)$, $A\sigma(T)$, $\rho(T)$ and $r(T)$ the spectrum, point spectrum, approximate point spectrum, the resolvent set and the spectral radius of T, respectively. The unit circle of $\mathbf{C}$ will be denoted by Γ.

LEMMA 4.1. *Suppose for some $n \in \mathbf{N}$ that $I \wedge |T|^k = 0$ for each $k \in \{1, 2,..., n\}$. Then there is a band projection P on E such that P, $\tilde{T}(P)$,..., $\tilde{T}^n(P)$ are mutually disjoint and $\bigvee_{k=0}^{2n} \tilde{T}^k(P) = I$.*

Proof. By Lemma 3.7, $\tau^k(x) \neq x$ for all $x \in X$ and $k = 1, 2,..., n$. Lemma 3.3 now shows that there exists a clopen subset θ of X such that $\tau^k(\theta) \cap \theta = \varnothing$ $(k = 1, 2,..., n)$ and $\bigcup_{k=-n}^{n} \tau^k(\theta) = X$. Thus, $P = \check{1}_\theta$ satisfies the conditions of the lemma.

We first consider the periodic case.

LEMMA 4.2. *Suppose $T \in Z(E)$. Then*

(i) $(\lambda I - T)^{-1} \in Z(E)$ *for all* $\lambda \in \rho(T)$.

(ii) $\sigma(T) = A\sigma(T) = \sigma(\hat{T}) = A\sigma(\hat{T})$.

Proof. Statement (i) follows immediately from [3, 3.3]. To prove (ii), note that (i) implies that $\sigma(T)$ in $\mathscr{L}(E)$ coincides with the spectrum of $\hat{T}$ in the Banach algebra $C(X)$. Since $\sigma(\hat{T}) = A\sigma(\hat{T})$ for functions in $C(X)$, (ii) now follows easily.

THEOREM 4.3. *Suppose T has strict period n for some $n \in \mathbf{N}$. Then for any nth root of unity α there is an invertible operator $M \in Z(E)$ such that $M^{-1}TM = \alpha T$. Moreover, $\sigma(T) = A\sigma(T) = \alpha\sigma(T)$ and $P\sigma(T) = \alpha P\sigma(T)$.*

Proof. By Lemma 4.1, there is a projection P such that P, $\tilde{T}(P)$,..., $\tilde{T}^{n-1}(P)$ are disjoint and $\bigvee_{k=0}^{2n-2} \tilde{T}^k(P) = I$. By Lemma 3.6, $\tilde{T}^n(P) = P$ so in fact $\bigvee_{k=0}^{n-1} \tilde{T}^k(P) = \sum_{k=0}^{n-1} \tilde{T}^k(P) = I$. Define $M = \sum_{k=0}^{n-1} \alpha^{-k} \tilde{T}^k(P)$. Since $|M| = I$, M is invertible. Moreover, $\tilde{T}(M) = \sum_{k=0}^{n-1} \alpha^{-k} \tilde{T}^{k+1}(P) = \alpha \sum_{k=1}^{n} \alpha^{-k} \tilde{T}^k(P) = \alpha \sum_{k=0}^{n-1} \alpha^{-k} \tilde{T}^k(P) = \alpha M$, because $\tilde{T}^n(P) = P$. Therefore, $M^{-1}TM = M^{-1}\tilde{T}(M)\, T = \alpha T$.

To complete the proof, we show that $\sigma(T) = A\sigma(T)$. By Lemma 4.2, $\sigma(T^n) = A\sigma(T^n)$ and hence $(\sigma(T))^n = (A\sigma(T))^n$. Since $\sigma(T) = \alpha\sigma(T)$ and $A\sigma(T) = \alpha A\sigma(T)$ for every nth root of unity, $\sigma(T) = A\sigma(T)$ as asserted.

COROLLARY 4.4. *Suppose T is a quasi-invertible lattice homomorphism with strict period n. Then $\lambda \in \sigma(T)$ iff $|\lambda| \in \sigma(T)$ and $\lambda = |\lambda|\, \alpha$, for some nth root of unity α.*

Proof. Since $T^n \in Z(E)_+$, $\sigma(T^n) \subset \mathbf{R}_+$. The corollary now follows from the spectral mapping theorem and Theorem 4.3.

We now turn to the general case.

LEMMA 4.5. *Suppose that for some* $m \in \mathbf{N}$, $I \wedge |T|^k = 0$ *for each* $k \in \{1, 2, \ldots, 2m\}$. *Then for any* $f \in E$ *there exists a band projection* Q *such that* $\tilde{T}^{-m}(Q), \tilde{T}^{1-m}(Q), \ldots, \tilde{T}^{-1}(Q), Q, \tilde{T}(Q), \ldots, \tilde{T}^m(Q)$ *are mutually disjoint and* $\|\sum_{k=0}^m \tilde{T}^k(Q) f\| \geqslant \frac{1}{4}\|f\|$.

Proof. By Lemma 4.1, there is a band projection P such that $P, \tilde{T}(P), \ldots, \tilde{T}^{2m}(P)$ are mutually disjoint and $\bigvee_{k=0}^{4m} \tilde{T}^k(P) = I$. Thus,

$$\|f\| \leqslant \left\| \sum_{k=0}^{4m} \tilde{T}^k(P)|f| \right\| = \left\| \sum_{j=0}^{3} \sum_{k=0}^{m} \tilde{T}^{mj+k}(P)|f| \right\|$$

$$\leqslant \sum_{j=0}^{3} \left\| \sum_{k=0}^{m} \tilde{T}^{mj+k}(P)|f| \right\| = \sum_{j=0}^{3} \left\| \sum_{k=0}^{m} \tilde{T}^{mj+k}(P) f \right\|.$$

This implies that for some $j_0 \in \{0, 1, 2, 3\}$, $\|\sum_{k=0}^m \tilde{T}^{mj_0+k}(P) f\| \geqslant \frac{1}{4}\|f\|$. Set $Q = \tilde{T}^{mj_0}(P)$. Since $\tilde{T}$ is a lattice isomorphism, it is clear that Q satisfies the required properties.

Remark. Lemma 4.5 is a weakened functional analytic version of the well-known Rohlin–Halmos lemma of ergodic theory. A more usual formulation would be that if $I \wedge |T^k| = 0$ for $k = 1, 2, \ldots, n$ ($n \in \mathbf{N}$) then for any $f \in E$ there is a band projection P such that $P, \tilde{T}(P), \ldots, \tilde{T}^n(P)$ are mutually disjoint and $\|\sum_{k=0}^n \tilde{T}^k(P) f\| \geqslant \frac{1}{2}\|f\|$. If E is an L^p space ($1 \leqslant p \leqslant \infty$), it follows from [4] that the lower bound of $\frac{1}{2}\|f\|$ may be improved to the classical $(1-\varepsilon)\|f\|$ bound. We do not know if this can be done in general.

We shall say that a set $S \subset \mathbf{C}$ is *rotationally invariant* if $\lambda \in S$ implies $\lambda e^{i\theta} \in S$ for all $\theta \in [0, 2\pi)$. The following theorem is our main result.

THEOREM 4.6. *Let* E_n ($n \in \mathbf{N}$) *be the T-reducing bands given in the decomposition Theorem* 3.10, *so* $T|_{E_n}$ *is the component of* T *which has strict period n. Set* $F_n = \{E_1 + \cdots + E_n\}^d$ *and define* $R = \bigcap_{n=1}^{\infty} (\sigma(T|_{F_n}))$. *Then* R *is rotationally invariant and* $\sigma(T) = \bigcup_{n=1}^{\infty} \sigma(T|_{E_n}) \cup R$.

Note. Some of the bands E_n and F_n may consist of zero only. We then interpret $\sigma(T|_{\{0\}})$ to be the empty set.

Proof. The only point of the theorem which is not obvious is that R is rotationally invariant. Suppose this is not the case, so there exists a complex number $\lambda \in \partial R$ (where ∂R denotes the topological boundary of R) such that there exists $\mu_m \in |\lambda| \Gamma \setminus R$ satisfying $\lim_{m \to \infty} \mu_m = \lambda$. Fix $\alpha \in \Gamma$ such that $\lambda\alpha \notin R$. Then there exists $n_0 \in \mathbf{N}$ such that $\lambda\alpha \notin \sigma(T|_{F_n})$ for each $n \geqslant n_0$, since $\sigma(T|_{F_n}) \supset \sigma(T|_{F_{n+1}})$ for each $n \in \mathbf{N}$. Given $\varepsilon > 0$, there exists an even natural number $n \geqslant n_0$ and a complex number $\mu \in |\lambda| \Gamma$ such that $|\lambda|/n < \varepsilon/3$, $|\mu - \lambda| < \varepsilon/3$ and $\mu \in \partial\sigma(T|_{F_{2n}}) \subset A\sigma(T|_{F_{2n}})$. Pick $f \in F_{2n}$ such that $\|Tf - \mu f\| < \varepsilon/3$ and $\|f\| = 1$. By Lemma 4.5 (applied to $T|_{F_{2n}}$) there exists a band projection P such that $\tilde{T}^{-n}(P), \tilde{T}^{1-n}(P), \ldots, \tilde{T}^{-1}(P), P, \tilde{T}(P), \ldots, \tilde{T}^{n}(P)$ are mutually disjoint and $\|\sum_{k=-n/2}^{n/2} \tilde{T}^k(P)|f|\| \geqslant \frac{1}{4}$.

Define $M \in Z(E)$ by $M = (1/n) \sum_{k=1-n}^{n-1} \alpha^{-k}(n-|k|)\, \tilde{T}^k(P)$.

We claim that

$$\|Mf\| \geqslant \tfrac{1}{8} \tag{1}$$

and

$$\|TMf - \lambda\alpha Mf\| < \varepsilon. \tag{2}$$

Since ε is arbitrary, these two statements imply that $\lambda\alpha \in \sigma(T|_{F_{n_0}})$, a contradiction.

It remains to prove (1) and (2). First note that since

$$|M| \geqslant \frac{1}{n} \sum_{k=-n/2}^{n/2} (n-|k|)\, \tilde{T}^k(P) \geqslant \frac{1}{2} \sum_{k=-n/2}^{n/2} \tilde{T}^k(P),$$

we have

$$\|Mf\| = \|\,|M|\,|f|\,\| \geqslant \frac{1}{2} \left\| \sum_{k=-n/2}^{n/2} \tilde{T}^k(P)\,|f| \right\| \geqslant \frac{1}{8},$$

which proves (1). Furthermore,

$$|\tilde{T}(M) - \alpha M| = \frac{1}{n} \sum_{k=1-n}^{n} \tilde{T}^k(P) \leqslant \frac{1}{n} I.$$

Note also that $|\tilde{T}(M)| \leqslant I$. Therefore,

$$\begin{aligned}
\|TMf - \lambda\alpha Mf\| &= \|\tilde{T}(M)\, Tf - \lambda\alpha Mf\| \\
&\leqslant \|\tilde{T}(M)\, Tf - \tilde{T}(M)\, \mu f\| \\
&\quad + \|\tilde{T}(M)\, \mu f - \tilde{T}(M)\, \lambda f\| + \|\tilde{T}(M)\, \lambda f - \lambda\alpha Mf\| \\
&\leqslant \|\tilde{T}(M)\|\, \|Tf - \mu f\| \\
&\quad + |\mu - \lambda|\, \|\tilde{T}(M)\|\, \|f\| + |\lambda|\, \|\tilde{T}(M) - \alpha M\|\, \|f\| \\
&\leqslant \varepsilon/3 + \varepsilon/3 + |\lambda| \cdot 1/n < \varepsilon,
\end{aligned}$$

which proves (2).

Remarks 1. It can be seen from the proof of the last theorem that it remains valid if the full spectrum is replaced throughout by the approximate point spectrum.

2. It might be thought that $\sigma(T) = \overline{\bigcup_{n \in \mathbf{N} \cup \{\infty\}} \sigma(T|_{E_k})}$. The following

example shows that this is not true in general, even if T has no aperiodic component.

Let $I=\{(2n,l): n\in\mathbf{N},\ 1\leqslant l\leqslant 2n\}\subset\mathbf{N}\times\mathbf{N}$ and $E=l^p(I)$ $(1\leqslant p\leqslant\infty)$. Let $\phi: I\to I$ be given by

$$\phi(2n,l)=\begin{cases}(2n,l+1) & \text{if } 1\leqslant l<2n\\ (2n,1) & \text{if } l=2n\end{cases}$$

and define $h\in l^\infty(I)$ by

$$h(2n,l)=\begin{cases}1 & \text{if } 1\leqslant l\leqslant n\\ \frac{1}{2} & \text{if } n<l\leqslant 2n.\end{cases}$$

Then $Tf=h\cdot f\circ\phi$ $(f\in E)$ defines a lattice isomorphism on E. Using the notation of Theorem 4.6 we have $E_\infty=0$; $E_n=0$ if $n\in\mathbf{N}$ is odd and $E_{2n}=\{f\in E: f(2m,l)=0 \text{ for } m\neq n,\ 1\leqslant l\leqslant 2m\}$ $(n\in\mathbf{N})$. Let $n\in\mathbf{N}$. Since $T^{2n}|_{E_{2n}}\in Z(E_{2n})$, we have

$$r(T|_{E_{2n}})=\|T^{2n}|_{E_{2n}}\|^{1/2n}=\|h\cdot h\circ\phi\cdot\ \cdots\ \cdot h\circ\phi^{2n-1}|_{E_{2n}}\|_\infty^{1/2n}=1/\sqrt{2}.$$

Similarly, $r(T^{-1}|_{E_{2n}})=\sqrt{2}$. Consequently, $\sigma(T|_{E_{2n}})\subset\{\lambda\in\mathbf{C}: |\lambda|=1/\sqrt{2}\}$. On the other hand, for $m\in\mathbf{N}$,

$$\begin{aligned}\|T^m\|&=\sup\{|(h\cdot h\circ\phi\cdot\ \cdots\ \cdot h\circ\phi^{m-1})(2n,l)|: n\in\mathbf{N},\ 1\leqslant l\leqslant 2n\}\\ &\geqslant|(h\cdot h\circ\phi\cdot\ \cdots\ \cdot h\circ\phi^{m-1})(2m,1)|=1.\end{aligned}$$

Thus $r(T)\geqslant 1$. We have shown that $\overline{\bigcup_{n\in\mathbf{N}\cup\{\infty\}}\sigma(T|_{E_n})}\subset\{\lambda\in\mathbf{C}: |\lambda|=1/\sqrt{2}\}\not\supset\sigma(T)$.

We single out the most important special case of the previous theorem. In the invertible case it was stated without proof in [9]. Many special cases on concrete function spaces are known; see [18, 19, 8, 24].

COROLLARY 4.7. *If T is aperiodic, then $\sigma(T)$ and $A\sigma(T)$ are rotationally invariant.*

5. Some Consequences

In this section we derive various consequences of the previous results. We will continue to use the notation established in the previous sections and in Theorem 4.6.

We shall call T *band indecomposable* if T has no non-trivial reducing bands. Clearly T is band indecomposable iff T commutes with no non-trivial band projections.

EXAMPLE 5.1. Let (Y, μ) be a finite measure space, let $\phi: Y \to Y$ be an invertible measure preserving transformation, and suppose $h \in L^\infty(Y, \mu)$ is non-zero a.e. Define $T: L^p(Y, \mu) \to L^p(Y, \mu)$ $(1 \leqslant p \leqslant \infty)$ by $Tf = h \cdot f \circ \phi$. Then T is band indecomposable iff ϕ is ergodic.

THEOREM 5.2. *Suppose T is band indecomposable and E is infinite dimensional. Then $\sigma(T)$ is either a disk or an annulus. In other words, $\sigma(T) = \{z \in \mathbf{C}: r_0 \leqslant |z| \leqslant r_1\}$ for some $r_0, r_1 \in \mathbf{R}_+$.*

Proof. By [3, 4.6], $|\sigma(T)| = \{|z|: z \in \sigma(T)\}$ is an interval. Thus we only need to show that $\sigma(T)$ is rotationally invariant. We claim that T is in fact aperiodic. If this is not the case, then T has strict period n for some $n \in \mathbf{N}$ by Theorem 3.10. As in the proof of Theorem 4.3, there is a band projection P such that $P, \tilde{T}(P), ..., \tilde{T}^{n-1}(P)$ are mutually disjoint and $\sum_{k=0}^{n-1} \tilde{T}^k(P) = I$. Suppose $f, g \in PE$ and $f \perp g$. Let Q be the projection onto $\{f\}^{dd}$, and let $\bar{Q} = \sum_{k=0}^{n-1} \tilde{T}^k(Q)$. Since the $\tilde{T}^k(Q)$ are mutually disjoint, $\bar{Q}$ is a band projection. Moreover, since $\tilde{T}^n(Q) = Q$ we have $\tilde{T}(\bar{Q}) = \bar{Q}$, so $T\bar{Q} = \bar{Q}T$. By assumption this says that $\bar{Q} = 0$ or $\bar{Q} = I$. The first case implies $f = 0$ and the second forces $g = 0$. By [14, 26.4], PE is one dimensional and hence E is n-dimensional, contrary to assumption. Therefore T is aperiodic and hence $\sigma(T)$ is rotationally invariant by Corollary 4.7.

Remark. The previous theorem is stated without proof by Kitover [9] under the additional assumption that T is invertible. In the special case given in Example 5.1, it was proven by Parrott [18, 2.9].

Our next result generalizes theorems of Schaefer, Wolff and Arendt [23], Arendt [3], and in part, those of Wickstead [26, 27].

THEOREM 5.3. *Suppose that for each $r > 0$, $\sigma(T) \cap r\Gamma$ lies in a half plane of the form $H_r := \{z \in \mathbf{C}: \theta_r \leqslant \arg z < \theta_r + \pi\}$ for some $\theta_r \in (-\pi, \pi]$ (which may depend on r). Then there exists a T-reducing band B such that $T|_B \in Z(B)$ and $T|_{B^d}$ is quasi-nilpotent.*

Proof. Let $B := E_1$, so $T|_B$ has strict period 1, i.e., $T|_B \in Z(B)$. Suppose $0 \neq \lambda \in \sigma(T|_{B^d})$. Then by Theorem 4.6, $\lambda \in \bigcup_{n=2}^\infty \sigma(T|_{E_n}) \cup R$. Since λ cannot be in the rotationally invariant set R, $\lambda \in \sigma(T|_{E_n})$ for some integer $n \geqslant 2$. Choose an nth root of unity $e^{i\theta}(\theta \in (-\pi, \pi])$ such that $|\theta| \geqslant \pi/2$. Then by Theorem 4.3, $\lambda, \lambda e^{i\theta}, \lambda e^{-i\theta} \in \sigma(T|_{E_n}) \subset \sigma(T)$. But at least one of these points lies outside the open half plane H_r, contrary to assumption. Thus, $\sigma(T|_{B^d}) \subset \{0\}$ and the proof is complete.

Remark. There do exist quasi-nilpotent quasi-invertible disjointness preserving operators. For example, let T be the weighted shift on $l^1(\mathbf{Z})$ given by $(Tf)_n = (1/n) f_{n+1}$. Then T is aperiodic, so $\sigma(T)$ is rotationally invariant. But T is also compact, which forces $\sigma(T) = \{0\}$ (cf. [26, 4.1]).

An example on $L^1[0, 1]$ can be found in [21, V problem 9]. On the other hand, it is not hard to see that such a quasi-nilpotent operator must be aperiodic.

One other related result is the following:

PROPOSITION 5.4. *Suppose T is invertible and $\sigma(T)$ does not contain a circle $r\Gamma$ about the origin for any $r > 0$. Then $T^n \in Z(E)$ for some $n \in \mathbf{N}$.*

Proof. The rotationally invariant set $R = \bigcap_{k=1}^{\infty} \sigma(T|_{F_k})$ is empty by assumption. Since the sets $\sigma(T|_{F_k})$ are decreasing and compact, there exists an integer m such that $\sigma(T|_{F_m}) = \varnothing$, i.e., $F_m = \{E_1 + \cdots + E_m\}^d = \{0\}$. This shows that $T^{m!} \in Z(E)$.

We next show, as a partial analogue to the situation for normal operators on a Hilbert space, that non-zero isolated points of $\sigma(T)$ are eigenvalues. As was pointed out in [26], the exclusion of zero is necessary (cf. the remark after Theorem 5.3). This result is also true for another class of disjointness preserving operators; see [26].

THEOREM 5.5. *If $\lambda \neq 0$ is an isolated point of $\sigma(T)$, then λ is a pole of order 1 of the resolvent $R(\mu) = (\mu I - T)^{-1}$. In particular, it is an eigenvalue.*

Proof. By Theorem 4.6, it may be assumed that T has strict period n for some $n \in \mathbf{N}$. Without loss of generality (since $\lambda \neq 0$), it may be assumed that $\lambda = 1$. So we must show that $f(\mu) := (\mu - 1)\, R(\mu)$ $(\mu \in P(T))$ is norm bounded in a neighborhood of 1. Since $(\mu - T) \sum_{k=0}^{n-1} (T^k/\mu^{k+1}) = I - T^n/\mu^n$ the resolvent is given by $R(\mu) = (I - T^n/\mu^n)^{-1} \cdot \sum_{k=0}^{n-1} (T^k/\mu^k)$ for μ in some neighborhood of 1 $(\mu \neq 1)$. Hence, it is enough to show that $g(\mu) := (1 - \mu)$ $(I - T^n/\mu^n)^{-1}$ $(\mu \neq 1)$ is bounded in some neighborhood of 1. Let $h = (\widehat{T^n})$. Then 1 must also be an isolated point of $\sigma(T^n)$ and hence also of $\sigma(h)$ in $C(X)$. Thus the set $\theta = \{x \in X: h(x) = 1\}$ must be open and closed.

Set $P = \hat{1}_\theta$, so $T^n P = P$ and $1 \notin \sigma(T^n|_{(T-P)E})$. Thus the function $\mu \to g(\mu)$ $(I - P)$ is bounded in some neighborhood of 1. Moreover, $g(\mu)\, P = \mu^n(1 - \mu)/(\mu^n - 1)$, which shows that the function $\mu \to g(\mu)\, P$ and hence $g(\mu) = g(\mu)[(I - P) + P]$ is bounded in a neighborhood of 1, and the theorem now follows.

Recall that $\mathscr{L}^r(E)$, the set of regular operators on E form a Banach algebra under the r-norm $\|S\|_r = \|\,|S|\,\|$. The spectrum of a regular operator S with respect to $\mathscr{L}^r(E)$ is called the order spectrum of S and is denoted by $\sigma_0(S)$. It is clear that $\sigma(S) \subset \sigma_0(S)$; this inclusion may be strict, see [2, 22] for examples and further discussion. In the case discussed here, however, we now show that this cannot occur.

THEOREM 5.6. *For every $\lambda \in \rho(T)$, the resolvent $(\lambda I - T)^{-1}$ is order bounded, that is $\sigma(T) = \sigma_0(T)$.*

Proof. Since $|Tf| = |T|\,|f|$ for all $f \in E$, $\|T^n\| = \||T^n|\|$ for all $n \in \mathbf{N}$. Consequently the spectral radius of T in $\mathscr{L}^r(E)$ is the same as it is in $\mathscr{L}(E)$.

Now suppose $T \in Z(E)$. Then by Lemma 4.2, $(\lambda I - T)^{-1} \in Z(E) \subset \mathscr{L}^r(E)$ so the theorem holds here.

Next suppose T has strict period n for some $n \in \mathbf{N}$. Then by the spectral mapping theorem and the last paragraph, $(\sigma(T))^n = \sigma(T^n) = \sigma_0(T^n) = (\sigma_0(T))^n$. Furthermore, by Theorem 4.3, $\sigma(T) = \alpha\sigma(T)$ for all nth roots of unity α. Since $\sigma(T) \subset \sigma_0(T)$, this implies that $\sigma(T) = \sigma_0(T)$ in this case as well.

Now consider the general case. The preceding paragraph shows that $(\lambda I - T)^{-1}$ is order bounded on $E_1 + E_2 + \cdots + E_n$ for each $n \in \mathbf{N}$. Hence it suffices to show that $(\lambda I - T)^{-1}$ is order bounded on $F_n = \{E_1 + \cdots + E_n\}^d$ for some $n \in \mathbf{N}$. We may choose n so that $\sigma(T|_{F_n}) \cap |\lambda|\,\Gamma = \varnothing$ for if there was no integer with this property, $\bigcap_{n=1}^{\infty} \sigma(T|_{F_n}) \cap |\lambda|\,\Gamma \neq \varnothing$, so $\lambda \in \sigma(T)$ by Theorem 4.6, a contradiction. By [3, 4.2], there is a band projection P commuting with $T|_{F_n}$ such that $T|_{F_{n_2}}$ is invertible and $r(T|_{F_{n_1}}) < |\lambda| < (r(T|_{F_{n_2}})^{-1})^{-1}$, where $F_{n_1} = PF_n$ and $F_{n_2} = (I-P)\,F_n$. The first paragraph now shows that $|\lambda|\,\Gamma \cap \sigma_0(T) = \varnothing$, so the theorem follows.

Remark. If $E = L^p(Y, \Sigma, \mu)$ $(1 \leqslant p \leqslant \infty,\ p \neq 2)$, where (Y, Σ, μ) is a σ-finite measure space, then every isometry of E onto itself is disjointness preserving [10]. Hence the last theorem shows that $\sigma(T) = \sigma_0(T)$ for every isometry of E. When $p = 2$, however, this is no longer true; see [22] for an example.

We shall now discuss the special case $T = MU$, where $M \in Z(E)$ is injective and U is an isometric lattice isomorphism (cf. Example 2.2). It is clear that the associated operators and hence the associated homeomorphisms of T and U are the same. Now define $\bar{T} \in \mathscr{L}(C(X))$ by $\bar{T}f = \hat{M} \cdot f \circ \tau$. Analogously, for any $N \in Z(E)$ we define $\bar{N} \in Z(C(X))$ to be multiplication by $\hat{N}$. Note that $\bar{T} = \bar{M}\bar{U}$. Moreover, if P is a band projection on E then $\bar{P}$ is a band projection on $C(X)$.

THEOREM 5.7. *Using the above notation, we have* $\sigma(T) = \sigma(\bar{T})$.

Proof. It is easy to see that T has strict period n iff $\bar{T}$ has this property. If $\bar{E}_n$ denotes the bands in $C(X)$ given in Theorem 3.10 applied to $\bar{T}$, so $\bar{T}|_{\bar{E}_n}$ has strict period n, it follows that $\bar{E}_n = \bar{P}_n\bar{E} = \{\check{P}_n f : f \in C(X)\}$. It can be easily verified that the powers of T and $\bar{T}$ are given by the formulas

$$T^n = M_n U^n \qquad \text{and} \qquad \bar{T}^n = \bar{M}_n \bar{U}^n \tag{$*$}$$

where $M_n = \prod_{k=0}^{n-1} \tilde{T}^k(M) = \prod_{k=0}^{n-1} (U^k M U^{-k})$.

(a) We claim that $r(T) = r(\bar{T})$. Indeed, since U and $\bar{U}$ are isometries,

$\|T^n\| = \|M_n\| = \|\bar{M}_n\| = \|\bar{T}^n\|$. The claim now follows from the spectral radius formula.

(b) Assume that T has strict period n. Then $\overline{U^n} \in Z(E)_+$ and is an isometry, whence $U^n = I$. It follows that $(\bar{U})^n = \overline{U^n}$ and hence by (*), $\bar{T}^n = (\bar{T})^n$. By Lemma 4.2, $\sigma(N) = \hat{N}(X) = \sigma(\bar{N})$ for any $N \in Z(E)$. Therefore, by Theorem 4.3 (applied to T and $\bar{T}$), $\lambda \in \sigma(T)$ iff $\lambda^n \in \sigma(T^n)$ if $\lambda^n \in \sigma(\overline{T^n})$ iff $\lambda \in \sigma(\bar{T})$ so the theorem is proved in this case.

Now consider the general case. By the last paragraph and Theorem 4.6, we only need to show that $\sigma(T|_{F_n}) = \sigma(\bar{T}|_{\bar{F}_n})$ (where $\bar{F}_n = \{\bar{E}_1 + \cdots + \bar{E}_n\}^d$) for some $n \in \mathbf{N}$. If $\lambda \in \mathbf{C} \backslash \sigma(\bar{T}|_{\bar{F}_{n_0}})$ for some $n_0 \in \mathbf{N}$, then $\lambda\Gamma \cap \sigma(T|_{F_n}) = \varnothing$ for some integer $n \geqslant n_0$. By [3, 4.2] there is a band projection P such that $r(T|_{PF_n}) < |\lambda| < (r(T|_{(I-P)F_n})^{-1})^{-1}$. Now apply (a) to $T|_{PF_n}$ and $T|_{(T-P)F_n}$ to see that $r(\bar{T}|_{\bar{P}\bar{F}_n}) < |\lambda| \leqslant r(T|_{(I-P)F_n})^{-1})^{-1}$ so $\lambda \notin \sigma(\bar{T})$. The inclusion $\rho(\bar{T}|_{\bar{F}_n}) \subset \rho(T|_{F_n})$ is proven analogously.

Theorem 5.7 can be applied in the following situation. Let (Y, Σ, μ) be a σ-finite measure space, $\phi\colon Y \to Y$ an invertible measure preserving transformation and supose $h \in L^\infty(Y, \Sigma, \mu)$ is non-zero μ-almost everywhere. Then for each $p \in [1, \infty]$, the operator defined by $T_p f = h \cdot f \circ \phi$ ($f \in L^p(Y, \Sigma, \mu)$) defines a quasi-invertible disjointness preserving operator on $L^p(Y, \Sigma, \mu)$.

COROLLARY 5.8. *$\sigma(T_p)$ is independent of $p \in [1, \infty]$.*

Proof. Choose $p \in [1, \infty]$ and let $E = L^p(Y, \Sigma, \mu)$. Then $Z(E)$ is isometricly isomorphic, both as an algebra and lattice, to $L^\infty(Y, \Sigma, \mu)$ via the map $\psi(m) f = mf$ ($m \in L^\infty(Y, \Sigma, \mu)$, $f \in E$). (See [12, p. 162] or [28, 142.11].) By Theorem 5.7 $\sigma(T_p) = \sigma(\bar{T}) = \sigma(\psi^{-1}\bar{T}\psi) = \sigma(T_\infty)$.

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