Perturbation of regular operators and the order essential spectrum

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ABSTRACT

For regular operators on a Banach lattice, we introduce and investigate two notions of order essential spectrum analogous to the essential spectrum and the Weyl spectrum for operators on Banach spaces. We also discuss related questions on the behaviour of the order spectrum under perturbation by r-compact operators.

§ 1. INTRODUCTION

Let E be a complex Banach lattice. A linear operator T on E is called *regular* if it is a linear combination of positive operators. The space $\mathscr{L}^r=\mathscr{L}^r(E)$ of all regular operators is a subalgebra of the algebra $\mathscr{L}=\mathscr{L}(E)$ of all bounded operators on E. Furthermore, $\mathscr{L}^r(E)$ is a Banach algebra under the norm

$$\|T\|_r := \inf\{\|P\| : P\in\mathscr{L}(E), |Tx|\le P|x| \text{ for all } x\in E\},$$

see ([14] Chap. IV, § 1) and [1]. The spectrum of an operator T in the Banach algebra $\mathscr{L}(E)$ is denoted as usual by $\sigma(T)$. Following Schaefer [15], we define the *order spectrum* (or o-spectrum) $\sigma_o(T)$ of a regular operator T, to be the spectrum of T in the Banach algebra $\mathscr{L}^r(E)$. It is obvious that $\sigma(T)\subset\sigma_o(T)$. Examples of operators T with $\sigma_o(T)\neq\sigma(T)$ are given in the Appendix.

In this article we introduce and investigate two notions of "order essential spectrum" in the algebra $\mathscr{L}^r$. Recall that the *essential spectrum* $\sigma_e(T)$ of an operator T on a Banach space X is defined to be the spectrum of the canonical

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image of T in the Banach algebra $\mathscr{L}(X)/\mathscr{K}(X)$ where $\mathscr{K}(X)$ is the ideal of all compact operators on X. We should point out that there is considerable divergence in the literature concerning the definition of the essential spectrum. What we call the essential spectrum was introduced by Wolf [20] in connection with some partial differential operators; we refer to [7] for properties of the essential spectrum for operators on Hilbert space. A related, but in general distinct, subset of the complex plane associated with an operator T is the *Weyl spectrum* $\omega(T) := \bigcap\{\sigma(T+K) : K \in \mathscr{K}(X)\}$, see [6]. This is also called the "essential spectrum" by some authors, e.g. Heuser [8, § 53] and Schechter [17, p. 180]. (A third definition of the "essential spectrum" is in Kato [10].)

To define the analogous notions for a regular operator T on the Banach lattice E, we first pick a distinguished ideal in $\mathscr{L}^r(E)$. We take this to be the ideal $\mathscr{K}^r = \mathscr{K}^r(E)$ defined as the closure (in the r-norm $\|\cdot\|_r$) of the ideal of finite rank operators in $\mathscr{L}^r(E)$, see [1]. We then define the order essential spectrum and the order Weyl spectrum in analogy with the essential spectrum and the Weyl spectrum replacing $\mathscr{K}$ and $\mathscr{L}$ by $\mathscr{K}^r$ and $\mathscr{L}^r$ respectively. Our aim is to investigate the properties of these notions of "spectra", their relations with the notions of spectrum, essential spectrum, Weyl spectrum and order spectrum; and to investigate the related question of the behaviour of the spectrum and the order spectrum under perturbation by elements of $\mathscr{K}^r$.

The choice of $\mathscr{K}^r$, and not other ideals such as $\mathscr{K} \cap \mathscr{L}^r$, for our distinguished ideal is motivated by several considerations. First, it has been shown [1] that $\sigma_o(K) = \sigma(K)$ for every $K \in \mathscr{K}^r$ and so $\sigma_o(K)$ is at most countable with 0 the only point of accumulation, but there are positive compact operators with uncountable order spectrum (see Example 2a in the Appendix). Second, in certain Banach lattices, namely c_o and l^p, $1 \leq p < \infty$, $\mathscr{K}^r$ is the only closed algebra ideal which is also a lattice ideal in $\mathscr{L}^r$ ([3] & [4]). These results, as well as results below, suggest that $\mathscr{K}^r$ plays a role in relation to $\mathscr{L}^r$ and the o-spectrum analogous to the role played by $\mathscr{K}$ in relation to $\mathscr{L}$ and the spectrum.

We end this introduction by stating some known results giving sufficient conditions for the equality of the spectrum and the order spectrum. The first result is contained in Schaefer [15; Proposition 4.1].

(1.1) THEOREM [15]. *If $\sigma_o(T)$ is totally disconnected, then $\sigma_o(T) = \sigma(T)$.*

(1.2) THEOREM [1]. *If $K \in \mathscr{K}^r$, then $\sigma(K) = \sigma_o(K)$.*

(1.3) THEOREM [2]. *If T is a lattice isomorphism on an order-complete Banach lattice, then $\sigma_o(T) = \sigma(T)$.*

§ 2. PERTURBATION BY ELEMENTS OF $\mathscr{K}^r$

It is well-known that if the difference between two operators on a Banach space is a compact operator, then they have the same spectrum except for eigenvalues. Our first result is an analogous result for the order spectrum. We denote the point spectrum (eigenvalues) of T by $\sigma_p(T)$.

(2.1) THEOREM. *If $T \in \mathscr{L}^r$ and $K \in \mathscr{K}^r$, then*

$$\sigma_o(T) \setminus \sigma_o(T+K) \subset \sigma_p(T).$$

PROOF. Let $\lambda \in \sigma_o(T) \setminus \sigma_o(T+K)$. Since

$$\lambda - T = (\lambda - T - K)\ (1 + (\lambda - T - K)^{-1}K),$$

we have that $1 + (\lambda - T - K)^{-1}K$ is not invertible in $\mathscr{L}^r$, i.e.

$$-1 \in \sigma_o((\lambda - T - K)^{-1}K).$$

But $(\lambda - T - K)^{-1}K \in \mathscr{K}^r$ and so, by Theorem 1.2, its spectrum coincides with its order spectrum, so $-1 \in \sigma((\lambda - T - K)^{-1}K)$. By the Fredholm alternative for compact operators, $1 + (\lambda - T - K)^{-1}K$ has a nonzero null space, and so $\lambda - T$ also has a nonzero null space, i.e. $\lambda \in \sigma_p(T)$. □

(2.2) REMARK. From the proof, we also get that every point in $\sigma_o(T) \setminus \sigma_o(T+K)$ is an eigenvalue of finite multiplicity (i.e., the corresponding eigenspace is finite dimensional). However, these points are not necessarily isolated. To illustrate this phenomenon, let T be defined on $l^2(\mathbb{Z})$ by $T\{x_n\} = \{y_n\}$ where $y_n = x_{n-1}$ for $n \neq 1$ and $y_1 = 0$ and let K be the rank one operator given by $K\{x_n\} = \{z_n\}$ where $z_n = 0$ for $n \neq 1$ and $z_1 = x_0$. It is easy to verify that $\sigma(T) = \sigma_o(T) = \mathbb{D}$, the unit disk $\{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ while $\sigma(T+K) = \sigma_o(T+K) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Therefore $\sigma_o(T) \setminus \sigma_o(T+K)$ consists of the open disk $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$.

(2.3) COROLLARY. *Let $T \in \mathscr{L}^r$ and $K \in \mathscr{K}^r$. Then*
(i) $\sigma_o(T+K) \setminus \sigma_o(T) \subset \sigma(T+K) \setminus \sigma(T)$;
(ii) *if $\sigma_o(T) = \sigma(T)$, then $\sigma_o(T+K) \setminus \sigma_o(T) = \sigma(T+K) \setminus \sigma(T)$.*

REMARKS. 1. The inclusion in part (i) can be proper; see Example 5.2 and Remark 5.3 below.

2. If $\sigma_o(T) = \sigma(T)$, it does not follow that $\sigma_o(T+K) = \sigma(T+K)$; see Example 5.2.

§ 3. THE ORDER ESSENTIAL SPECTRUM

(3.1) DEFINITION. Let $T \in \mathscr{L}^r$ and let $\pi : \mathscr{L}^r \to \mathscr{L}^r / \mathscr{K}^r$ be the canonical homomorphism. The *order essential spectrum* $\sigma_{oe}(T)$ of T is the spectrum of $\pi(T)$ in the algebra $\mathscr{L}^r / \mathscr{K}^r$.

We can rephrase the above definition by saying that $\lambda \notin \sigma_{oe}(T)$ if and only if $\lambda - T$ is invertible in $\mathscr{L}^r$ modulo $\mathscr{K}^r$. Of course $\lambda \notin \sigma_e(T)$ if and only if $\lambda - T$ is invertible in $\mathscr{L}$ modulo $\mathscr{K}$. A classical result of Atkinson [5], also in [8, Proposition 25.2, Aufgabe 40.1] and [17, Ch. V] states that $\lambda - T$ is invertible modulo the compacts if and only if it is invertible modulo the ideal of finite rank operators. We will give an analogous result (Theorem 3.3). First we prove a lemma which is used frequently in the sequel.

(3.2) LEMMA. *Let $T\in\mathscr{L}$ and $K\in\mathscr{K}^r$. If $T(1+K)\in\mathscr{L}^r$, then $T\in\mathscr{L}^r$.*

PROOF. By the well-known Riesz theory for compact operators, there exists a finite rank operator F such that $1+K+F$ is invertible. Since $K+F\in\mathscr{K}^r$, Theorem 1.2 implies that $\sigma_o(K+F)=\sigma(K+F)$ and so $(1+K+F)^{-1}\in\mathscr{L}^r$. If $A=T(1+K)$, then $A\in\mathscr{L}^r$ by assumption. The equation

$$T=(A+TF)(1+K+F)^{-1}$$

shows that $T\in\mathscr{L}^r$. □

(3.3) THEOREM. *Let $T\in\mathscr{L}^r$. The following statements are equivalent.*
(i) $0\notin\sigma_{oe}(T)$
(ii) *T is invertible in $\mathscr{L}^r$ modulo the ideal $\mathscr{F}$ of finite rank operators, i.e. there exist $S\in\mathscr{L}^r$ and $F_1,F_2\in\mathscr{F}$ such that $TS=1+F_1$ and $ST=1+F_2$.*

PROOF. The implication (ii)$\Rightarrow$(i) is obvious. If $0\notin\sigma_{oe}(T)$, then there exist $R\in\mathscr{L}^r$ and $K_1,K_2\in\mathscr{K}^r$ such that $TR=1+K_1$ and $RT=1+K_2$. By Atkinson's theorem, there exist $S\in\mathscr{L}$ and $F_1,F_2\in\mathscr{F}$ such that $TS=1+F_1$ and $ST=1+F_2$. We will show that S is regular. We have

$$(S-R)(1+K_1)=(S-R)TR=(1+F_2)R-(1+K_2)R=(F_2-K_2)R$$

and so $(S-R)(1+K_1)\in\mathscr{L}^r$. By Lemma 3.2, we have that $S-R\in\mathscr{L}^r$ and so $S\in\mathscr{L}^r$. This proves that (i)$\Rightarrow$(ii). □

(3.4) REMARK. There exists an operator $A\in\mathscr{L}^r$ which has in inverse in $\mathscr{L}^r$ modulo $\mathscr{K}$ and an inverse in $\mathscr{L}$ modulo $\mathscr{K}^r$ but has no inverse in $\mathscr{L}^r$ modulo $\mathscr{K}^r$. This can be shown using the fact (Example 2a in the Appendix) that there exists a positive compact operator K on $L^2(0,1)$ such that $\sigma_{oe}(K)\neq\{0\}$. Let λ be a nonzero element of $\sigma_{oe}(K)$ and let $A=\lambda 1-K$ and $B=\lambda^{-1}1$. We have that $A,B\in\mathscr{L}^r$, $AB-1$ and $BA-1$ are compact, so A is invertible in $\mathscr{L}^r$ mod $\mathscr{K}$. On the other hand, by Atkinson's theorem, there exists an operator $C\in\mathscr{L}$ such that $AC-1$ and $CA-1$ are finite rank operators and hence belong to $\mathscr{K}^r$. However $\lambda\in\sigma_{oe}(K)$ and so $0\in\sigma_{oe}(A)$, i.e. A is not invertible in $\mathscr{L}^r$ modulo $\mathscr{K}^r$.

We end this section by some results on the relation between $\sigma_o(T)$ and $\sigma_{oe}(T)$. The following theorem, analogous to [7, Theorem 2.3] shows that the "inessential order spectrum" $\sigma_o(T)\setminus\sigma_{oe}(T)$ is contained in the spectrum $\sigma(T)$, in other words, the "pure order spectrum" $\sigma_o(T)\setminus\sigma(T)$ is contained in $\sigma_{oe}(T)$.

(3.5) THEOREM. $\sigma_o(T)=\sigma_{oe}(T)\cup\sigma_p(T)\cup\sigma_p(T^*)$.

PROOF. For the nontrivial inclusion, assume that $\lambda\notin\sigma_{oe}(T)$, then there exist $S\in\mathscr{L}^r$ and $K\in\mathscr{K}^r$ such that $(\lambda-T)S=1+K$. If in addition $\lambda\notin\sigma_p(T)\cup\sigma_p(T^*)$ then $\lambda-T$ is injective and has dense range. Since $\lambda-T$ is also a Fredholm operator, we get that $(\lambda-T)$ is bijective. Now $(\lambda-T)^{-1}(1+K)\in\mathscr{L}^r$ and so by Lemma 3.2, we have $(\lambda-T)^{-1}\in\mathscr{L}^r$. Hence $\lambda\notin\sigma_o(T)$. □

(3.6) COROLLARY. $\sigma_o(T) \setminus \sigma_{oe}(T) \subset \sigma(T) \setminus \sigma_e(T)$.

We note that the reverse inclusion is false, see Example 5.2.

(3.7) COROLLARY. *If* $\sigma_{oe}(A) = \sigma_e(A)$, *then* $\sigma_o(A) = \sigma(A)$.

PROOF. If $\sigma_{oe}(A) = \sigma_e(A)$, then $\sigma_o(A) \subset \sigma(A)$ by Theorem 3.5. The reverse inclusion is obvious. □

The reverse implication of the above Corollary is false, see Examples 5.1 and 5.2.

It is known (see [8, Satz 51.1, Proposition 50.3] together with [7, Theorem 2.4]) that $\sigma(T) \setminus \sigma_e(T)$ is the union of isolated points and some of the "holes" in $\sigma_e(T)$, including all holes in which index $(\lambda - T) \neq 0$ [17, Ch. VII, Theorem 5.4]. (Caveat: The "essential spectrum" in [8] and [17] is different from ours.) We do not know whether a similar relation exists between $\sigma_o(T)$ and $\sigma_{oe}(T)$. We prove a partial result. It follows from the above that points in $\sigma(T)$ which lie in the unbounded component of $\Phi(T) := \mathbb{C} \setminus \sigma_e(T)$ are isolated and the corresponding spectral projections have finite rank. (See also Kato [10, pp. 242–243] but beware: Kato's "essential spectrum" is different from ours and also from [8] and [17].) We prove a similar result for the order spectra, using Corollary 3.6.

(3.8) COROLLARY. *Every point in* $\sigma_o(T)$ *which lies in the unbounded component of* $\mathbb{C} \setminus \sigma_{oe}(T)$ *is an isolated point in* $\sigma_o(T)$ *and the corresponding spectral projection has finite rank.*

PROOF. The result follows immediately from the discussion above using Corollary 3.6 and the fact that the unbounded component of $\mathbb{C} \setminus \sigma_{oe}(T)$ is contained in the unbounded component of $\mathbb{C} \setminus \sigma_e(T)$. □

§ 4. THE ORDER WEYL SPECTRUM

The Weyl spectrum of an operator T is defined by

$$\omega(T) = \bigcap \{\sigma(T+K) : K \in \mathscr{K}\},$$

see [6] and [7]. In [8] and [17] it is referred to as the "essential spectrum". In this section we study its order-theoretic analogue.

(4.1) DEFINITION. Let $T \in \mathscr{L}^r$. The *order Weyl spectrum* $\omega_o(T)$ of T is the set $\bigcap \{\sigma_o(T+K) : K \in \mathscr{K}^r\}$.

Note that, unlike the other four spectra defined above, $\omega(T)$ and $\omega_o(T)$ are not defined via the notion of invertibility in some Banach algebra. (Berberian [6, Example 3.3] showed that the spectral mapping theorem fails for the Weyl spectrum. The same example proves the failure of the same "theorem" for the order Weyl spectrum, see Example 5.4 below.) Therefore part (d) of the next theorem does not follow from Banach algebra considerations; it follows from the definition and parts (b) and (c).

(4.2) THEOREM.
(a) $\sigma_e(T) \subset \sigma_{oe}(T)$, $\omega(T) \subset \omega_o(T)$, $\sigma(T) \subset \sigma_o(T)$.
(b) $\sigma_e(T) \subset \omega(T) \subset \sigma(T)$.
(c) $\sigma_{oe}(T) \subset \omega_o(T) \subset \sigma_o(T)$.
(d) *Each of* $\omega(T)$ *and* $\omega_o(T)$ *is a nonempty compact subset of the complex plane.*

PROOF. Obvious.

Next, we show that we can replace $\mathscr{K}^r$ in the definition of $\omega_o(T)$ by the ideal $\mathscr{F}$ of finite rank operators. The corresponding result for $\omega(T)$ is in [6, Theorem 2.5].

(4.3) THEOREM. *Let* $T \in \mathscr{L}^r$. *Then*
(i) $\omega_o(T) = \bigcap \{\sigma_o(T+F) : F \text{ is of finite rank}\}$.
(ii) $\omega(T) = \bigcap \{\sigma(T+F) : F \text{ is of finite rank}\}$
$= \bigcap \{\sigma(T+K) : K \in \mathscr{K}^r\}$.

PROOF. (i) If $\lambda \notin \omega_o(T)$, then $\lambda - T = A + K$ for $K \in \mathscr{K}^r$ and A invertible in $\mathscr{L}^r$ (with $A^{-1} \in \mathscr{L}^r$). Therefore $\lambda - T = A(1 + A^{-1}K)$. Since $A^{-1}K$ is compact, there exists a finite rank operator R such that $-1 \notin \sigma(A^{-1}K + R)$, i.e. $1 + A^{-1}K + R$ has an inverse in $\mathscr{L}$. By Theorem 1.2, we have that $(1 + A^{-1}K + R)^{-1} \in \mathscr{L}^r$ and $\lambda - T + AR = A(1 + A^{-1}K + R)$ is invertible in $\mathscr{L}^r$, i.e. $\lambda \notin \sigma_o(T - AR)$. Therefore $\bigcap \{\sigma_o(T+F) : F \in \mathscr{F}\} \subset \omega_o(T)$. The reverse inclusion is obvious.

(ii) The first equation follows from the proof of (i) and the second equation follows easily since $\mathscr{F} \subset \mathscr{K}^r \subset \mathscr{K}$. □

We now prove some results describing the relation between $\sigma_o(T)$, $\omega_o(T)$ and $\sigma_{oe}(T)$. Our first result is analogous to [7, Theorem 2.2].

(4.4) THEOREM. $\sigma_o(T) = \omega_o(T) \cup \sigma_p(T)$.

PROOF. This is nothing but a restatement of Theorem 2.1. □

(4.5) COROLLARY. $\sigma_o(T) \setminus \omega_o(T) \subset \sigma(T) \setminus \omega(T)$.

The reverse inclusion is false, see Example 5.2.

We now describe the relation between $\omega_o(T)$ and $\sigma_{oe}(T)$. We show that $\omega_o(T)$ is obtained from $\sigma_{oe}(T)$ by filling in some of the "holes" in $\sigma_{oe}(T)$. The analogous result for $\omega(T)$ and $\sigma_e(T)$ is in [17, Ch. VII, Theorem 5.4], see also

[7, Theorem 2.4]. Note that the mapping $\lambda \mapsto \operatorname{ind}(\lambda - T)$ is continuous, integer valued function in the complement of $\sigma_{oe}(T)$, so it is constant in each component of $\mathbb{C} \setminus \sigma_{oe}(T)$ and vanishes at ∞. Therefore the set

$$\{\lambda \notin \sigma_{oe}(T) : \operatorname{ind}(\lambda - T) \neq 0\}$$

consists of some of the bounded components of $\mathbb{C} \setminus \sigma_{oe}(T)$.

(4.6) THEOREM. *$\omega_o(T) = \sigma_{oe}(T) \cup \{\lambda \in \mathbb{C} : \lambda - T$ is a Fredholm operator with index $\neq 0\}$.*

PROOF. It is obvious that $\omega_o(T)$ contains the set on the right side. To prove the reverse inclusion, assume that $\lambda \notin \sigma_{oe}(T)$ and that $\operatorname{ind}(\lambda - T) = 0$. Therefore, there is a finite rank operator F such that $\lambda - T - F$ is bijective (see [8, Proposition 26.2] or [17, pp. 180–181]). Furthermore $0 \notin \sigma_{oe}(\lambda - T) = \sigma_{oe}(\lambda - T - F)$, so there exists $A \in \mathscr{L}^r$ such that $1 - A(\lambda - T - F)$ is a finite rank operator (by Theorem 3.3). It follows that $(\lambda - T - F)^{-1} - A$ is of finite rank and hence $(\lambda - T - F)^{-1} \in \mathscr{L}^r$. Therefore $\lambda \notin \omega_o(T)$. □

The next corollary is similar to (3.6) and (4.5).

(4.7) COROLLARY. *$\omega_o(T) \setminus \sigma_{oe}(T) \subset \omega(T) \setminus \sigma_e(T)$.*

The reverse inclusion is false, see Example 5.2.

(4.8) COROLLARY. *If $K \in \mathscr{L}^r \cap \mathscr{K}$, then $\omega_o(K) = \sigma_{oe}(K)$.*

Our next result is a spectral inclusion theorem; cf. [6, Theorem 3.2].

(4.9) THEOREM. *Let p be a polynomial and let $T \in \mathscr{L}^r$. Then*

$$\omega_o(p(T)) \subset p(\omega_o(T)).$$

PROOF. The first part of the proof follows [6]. We may assume that p is not a constant. If $\mu \notin p(\omega_o(T))$ and if $p(\lambda) - \mu = a(\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$, then

$$p(T) - \mu = a(T - \lambda_1) \cdots (T - \lambda_n)$$

and $\lambda_j \notin \omega_o(T)$ for $1 \leq j \leq n$. Therefore $p(T) - \mu$ is a Fredholm operator with index 0.

To prove that $\mu \notin \omega_o(p(T))$, it suffices (by Theorem 4.6) to show that $\mu \notin \sigma_{oe}(p(T))$. But $\lambda_j \notin \omega_o(T)$ implies that $\lambda_j \notin \sigma_{oe}(T)$ and by the spectral mapping theorem in the algebra $\mathscr{L}^r / \mathscr{K}^r$, we get $\mu \notin \sigma_{oe}(p(T))$. □

REMARK. In Example 5.4, we will show (using Berberian's example) that the inclusion in Theorem 4.9 may be proper.

Next we determine the relation between the conditions that various spectra are equal to their "order analogues".

(4.10) THEOREM. *Consider the following assertions for $T \in \mathscr{L}^r$.*
(a) $\sigma_e(T) = \sigma_{oe}(T)$.
(b) $\omega(T) = \omega_o(T)$.
(c) $\sigma(T+K) = \sigma_o(T+K)$ *for every* $K \in \mathscr{K}^r$.
(d) $\sigma(T) = \sigma_o(T)$.
The following implications are valid:
(a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d).

PROOF. The implication (a) $\Rightarrow$ (c) follows from Corollary 3.7 and the obvious fact that σ_e and σ_{oe} are stable under perturbation by elements of $\mathscr{K}^r$. The implication (c) $\Rightarrow$ (b) follows from Theorem 4.3. To prove that (b) implies (c), assume the $\omega(T) = \omega_o(T)$. By Theorem 4.4, we have $\sigma_o(T) = \sigma_p(T) \cup \omega_o(T)$ and hence $\sigma_o(T) = \sigma_p(T) \cup \omega(T) \subset \sigma(T)$. Therefore $\sigma_o(T) = \sigma(T)$. Applying the above argument to $T+K$ in place of T, we get (c). □

The implications (b) $\Rightarrow$ (a) and (d) $\Rightarrow$ (c) are false, see Examples 5.1 and 5.2.

§ 5. SOME COUNTEREXAMPLES

In this section, we give some counterexamples which have been referred to above, see remarks after (2.3), (3.6), (3.7), (4.5), (4.7), (4.9) and (4.10).

(5.1) EXAMPLE. We construct an operator T such that $\omega(T) = \omega_o(T)$ and hence $\sigma(T) = \sigma_o(T)$ but $\sigma_e(T) \neq \sigma_{oe}(T)$. This gives counterexamples to the implication (b) $\Rightarrow$ (a) in Theorem 4.10, to the converse of Corollary 3.7 and to the reverse inclusion of Corollary 4.7.

There exists a positive compact operator K on $L^2(0,1)$ such that $\sigma(K)$ is real, the spectral radius $r(K) = 1$ and $\sigma_o(K)$ contains the unit circle $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$ (see Example A2(a) in the Appendix). Let $T = \frac{1}{2}K \oplus S$ on $L^2(0,1) \oplus l^2(\mathbb{Z}^+)$ where S is the forward shift on l^2, i.e. $S\{x_n\} = \{y_n\}$ where $y_1 = 0$ and $y_n = x_{n-1}$ for $n \geq 2$. It is easy to verify that $\sigma(T) = \sigma_o(T) = \sigma(S) = \sigma_o(S) = \mathbb{D}$ where $\mathbb{D}$ is the unit disk $\{z \in C : |z| \leq 1\}$. Now $SS^* = 1$ and $SS^* - 1$ has rank one, so S is essentially unitary and so $\sigma_e(S)$ is included in the unit circle Γ. On the other hand λS is unitarily equivalent to S for every unimodular complex number λ, so $\sigma_e(S) = \Gamma$. Therefere $\sigma_e(T) = \Gamma \cup \{0\}$. Since ind $(\lambda - T) = -1$ for $0 < |\lambda| < 1$, the well known relation between σ_e and ω [17, Ch. VII, Theorem 5.4] implies that $\omega(T) = \mathbb{D}$. On the other hand $\omega_o(T) \subset \sigma_o(T) \subset \mathbb{D}$ and $\omega_o(T) \supset \omega(T) = \mathbb{D}$, thus $\omega_o(T) = \omega(T) = \mathbb{D}$.

By Theorem 3.5, $\sigma_o(K) \setminus \sigma(K) \subset \sigma_{oe}(K)$, so $\sigma_{oe}(K)$ contains the unit circle Γ and $\sigma_{oe}(T)$ contains $\frac{1}{2}\Gamma$. This shows that $\sigma_e(T) \neq \sigma_{oe}(T)$.

(5.2) EXAMPLE. We give an example of operators $T \in \mathscr{L}^r$ and $K \in \mathscr{K}^r$ such that $\sigma(T) = \sigma_o(T)$ but $\sigma(T+K) \neq \sigma_o(T+K)$ and so (by Theorem 4.10) $\omega(T) \neq \omega_o(T)$ and $\sigma_e(T) \neq \sigma_{oe}(T)$. This gives a counterexample to several assertions (see remarks following (2.3), (3.6), (3.7), (4.5) and (4.10)).

Ando constructed a unitary operator U on $l^2(\mathbb{Z}^+)$ (see Example A1 in the

Appendix) such that the order spectral radius $r_o(U)=r>1$. Let V be the operator on $l^2(\mathbb{Z})$ defined by $V\{x_n\}=\{y_n\}$ where $y_1=0$ and $y_n=x_{n-1}$ for $n\neq 1$ and let $T=(1/2r)U\oplus V$ on $l^2(\mathbb{Z}^+)\oplus l^2(\mathbb{Z})$. Since V can be written as $S^*\oplus S$ where S is the forward shift of $l^2(\mathbb{Z}^+)$, we have $\sigma(T)=\sigma_o(T)=\mathbb{D}$, the unit disk.

Now let R be the rank one operator on $l^2(\mathbb{Z})$ defined by $R\{x_n\}=\{y_n\}$ where $y_1=x_0$ and $y_n=0$ for $n\neq 1$ and let $K=0\oplus R$. We have $T+K=(1/2r)U\oplus W$ where W is the bilateral shift on $l^2(\mathbb{Z})$ which is a unitary operator. Therefore $\sigma(T+K)\subset\{z\in\mathbb{C}:|z|=1 \text{ or } 1/2r\}$, but $\sigma_o(T+K)$ includes $\sigma_o((1/2r)U)$ and so contains a complex number z with $|z|=\frac{1}{2}$. We have $\sigma(T+K)\neq\sigma_o(T+K)$.

(5.3) REMARK. In the above example, we have $\sigma_o(T)\setminus\sigma_o(T+K)\neq$ $\neq\sigma(T)\setminus\sigma(T+K)$. This shows that the inclusion in Corollary 2.3(i) can be proper.

(5.4) EXAMPLE. We show that the spectral mapping theorem fails for ω_o. We use Berberian's example [6, Example 3.3] where he showed that the spectral mapping theorem fails for ω.

Let $T=S\oplus(S^*+2)$, where S is the forward shift on $l^2(\mathbb{Z}^+)$ and so S^* is the backward shift. Let $p(z)=z(z-2)$. Since ind $(T)=-1$, $0\in\omega(T)\subset\omega_o(T)$ and so $0=p(0)\in p(\omega_o(T))$. We will show that $0\notin\omega_o(p(T))$.

We have $p(T)=S(S-2)\oplus S^*(S^*+2)$. It is easy to verify that $\sigma_{oe}(S)=$ $=\sigma_{oe}(S^*)=\{\lambda\in C:|\lambda|=1\}$. It follows from the spectral mapping theorem in the algebra $\mathscr{L}^r/\mathscr{K}^r$ that $0\notin\sigma_{oe}(S(S-2))$ and therefore $0\notin\sigma_{oe}(p(T))$. Since $\operatorname{ind}(S)=-1$, $\operatorname{ind}(S^*)=1$ and $S-2$ and S^*+2 are invertible, we have $\operatorname{ind}(p(T))=0$. By Theorem 4.6, we have $0\notin\omega(p(T))$. □

§ 6. THE ESSENTIAL SPECTRUM MEETS EVERY COMPONENT OF THE ORDER ESSENTIAL SPECTRUM

In [15], Schaefer proved that every non-void disconnecting subset of $\sigma_o(T)$ intersects $\sigma(T)$. We give below (Theorem 6.2) an analogous result for $\sigma_{oe}(T)$ and $\sigma_e(T)$. We need the following lemma in the proof of Theorem 6.2.

(6.1) LEMMA. *If $C\in\mathscr{K}\cap\mathscr{L}^r$ and $C^2-C\in\mathscr{K}^r$, then $C\in\mathscr{K}^r$.*

PROOF. There exists a finite rank operator F such that $1\notin\sigma(C+F)$. Let $S=C+F$, then $S^2-S\in\mathscr{K}^r$. It follows from [1, Corollary 2.9] that $\sigma(S)=\sigma_o(S)$ and so $(1-S)^{-1}\in\mathscr{L}^r$. We have $S=(1-S)^{-1}(S-S^2)\in\mathscr{K}^r$ and so $C\in\mathscr{K}^r$. □

Following [15], an open subset Ω of the complex plane $\mathbb{C}$ is said to *disconnect* a subset Δ of $\mathbb{C}$ if $\Omega\cap\Delta$ is nonempty, is not all of Δ and is relatively open and relatively closed in Δ.

(6.2) THEOREM. *Let Ω be a nonempty open subset of C and assume that Ω disconnects $\sigma_{oe}(T)$. Then $\sigma_e(T)\cap\Omega$ is nonempty.*

PROOF. The mapping $\phi:\mathscr{L}^r/\mathscr{K}^r\to\mathscr{L}/\mathscr{K}$ given by $\phi(A+\mathscr{K}^r)=A+\mathscr{K}$ is a well-defined algebra-homomorphism and $\phi(1)=1$. Moreover ϕ is continuous since

$$\|A+\mathscr{K}\|=\inf\{\|A+K\|:K\in\mathscr{K}\}\le\inf\{\|A+K\|_r:K\in\mathscr{K}^r\}$$
$$=\|A+\mathscr{K}^r\|.$$

Since Ω disconnects $\sigma_{oe}(T)$, there exists a rectifiable simple closed curve $\gamma\subset\Omega$ such that $\Omega\cap\sigma_{oe}(T)$ is in the interior of γ. Let

$$p=\frac{1}{2\pi i}\int_\gamma R(\lambda,T+\mathscr{K}^r)d\lambda$$

where $R(\lambda,x)$ is the resolvent $(\lambda 1-x)^{-1}$. Therefore p is a projection in the algebra $\mathscr{L}^r/\mathscr{K}^r$, $p\neq 0$, $p\neq 1$. We can write p in the form $P+\mathscr{K}^r$ where $P\in\mathscr{L}^r$. The fact that p is a projection means that $P^2-P\in\mathscr{K}^r$.

Since ϕ is a continuous homomorphism, we have

$$\phi(p)=\frac{1}{2\pi i}\int_\gamma R(\lambda,T+\mathscr{K})d\lambda.$$

If $\Omega\cap\sigma_e(T)=\emptyset$, then $\phi(p)=0$, i.e. $P\in\mathscr{K}$. By Lemma 6.1, we have $P\in\mathscr{K}^r$ and so $p=0$, a contradiction. □

(6.3) COROLLARY. *If $\sigma_{oe}(T)$ is totally disconnected, then $\sigma_e(T)=\sigma_{oe}(T)$ and consequently $\sigma(T)=\sigma_o(T)$.*

This Corollary contains Schaefer's result (Theorem 1.1) and also Theorem 1.2 [1, Theorem 2.6] as special cases. The case when $\sigma_{oe}(T)=\{0\}$ has been proved by Raubenheimer [12].

We point out that in view of Corollary 3.8, the condition that $\sigma_{oe}(T)$ is totally disconnected is equivalent to the condition that $\sigma_o(T)$ is totally disconnected and so Corollary 6.3 does not give a larger class of operators for which $\sigma(T)=\sigma_o(T)$ than the class given by Schaefer's result.

(6.4) REMARK. In contrast with Corollary 6.3, the condition that $\sigma(T)$ is totally disconnected does not imply that $\sigma(T)=\sigma_o(T)$. The compact operator in Example A2 in the Appendix provides a counterexample.

§ 7. MORE ON THE EQUATION $\sigma_o(T)=\sigma(T)$

We have seen (Example 5.2) that $\sigma_o(T)=\sigma(T)$ does not imply that $\sigma_{oe}(T)=\sigma_e(T)$. However, we will now show that if $\sigma_o(T)=\sigma(T)$, then, except for "holes", $\sigma_{oe}(T)$ agrees with $\sigma_e(T)$.

(7.1) THEOREM. *If $\sigma_o(T)=\sigma(T)$, then the complement of $\sigma_{oe}(T)$ and the complement of $\sigma_e(T)$ in $\mathbb{C}$ have the same unbounded component.*

PROOF. As usual we denote $\mathbb{C}\setminus\sigma_e(T)$ by $\Phi(T)$. Likewise we denote $\mathbb{C}\setminus\sigma_{oe}(T)$ by $\Phi_o(T)$. The unbounded components of $\Phi(T)$ and $\Phi_o(T)$ are denoted by

Φ^∞ and Φ_o^∞ respectively. Let $\lambda \in \Phi^\infty$. If $\lambda \in \sigma_{oe}(T)$, then $\lambda \in \sigma_o(T)$ and so $\lambda \in \sigma(T) \cap \Phi^\infty$. Such points are known to be isolated points of $\sigma(T)$, see the discussion preceding Corollary 3.8. Since $\sigma_{oe}(T) \subset \sigma_o(T) = \sigma(T)$, λ is also an isolated point in $\sigma_{oe}(T)$ and by Theorem 6.2, $\lambda \in \sigma_e(T)$, a contradiction. Thus $\lambda \in \Phi_o(T)$. The above shows that $\Phi^\infty \subset \Phi_o^\infty$. The reverse implication is obvious. □

(7.2) COROLLARY. *Let $K \in \mathscr{K} \cap \mathscr{L}^r$. Then $\sigma_o(K) = \sigma(K)$ if and only if $\sigma_{oe}(K) = \sigma_e(K) = \{0\}$.*

It follows from Corollary 6.3 that $\sigma_o(T) = \sigma(T)$ if $\sigma_{oe}(T)$ is countable and in particular if $\sigma_o(T)$ is countable. The compact operator in Example A2 in the Appendix shows that the countability of $\sigma(A)$ is not sufficient. In this connection we point out the following result.

(7.3) THEOREM. *Suppose that $T \in \mathscr{L}^r(E)$ and that $\sigma(T)$ is countable. Let f be a non-constant analytic function defined on a neighbourhood of $\sigma_o(T)$. If $\sigma_o(f(T)) = \sigma(f(T))$, then $\sigma_o(T) = \sigma(T)$.*

PROOF. By the spectral mapping theorem, we have $\sigma_o(T) \subset f^{-1}f(\sigma(T)) = f^{-1}(\Lambda)$ for a countable set Λ. Therefore $\sigma_o(T)$ is countable since f is analytic. By Corollary 6.3, or [15, p. 84], $\sigma_o(T) = \sigma(T)$. □

(7.4) REMARK. In Theorem 7.3, the condition that $\sigma(T)$ is countable cannot be omitted. To see this consider the operator $T = A \oplus K$, where A is a diagonal operator on l^2 whose diagonal entries form a dense subset of $\{z \in \mathbb{D} : \operatorname{Im} z \leq 0\}$ and where K is a compact operator on l^2 with $\sigma(K) \subset [-1, 1]$ and with $\sigma_o(K)$ containing the unit circle Γ (see Example A2 in the Appendix). Therefore $\sigma(T^2) = \sigma_o(T^2) = \mathbb{D}$. On the other hand, $i \in \sigma_o(K) \subset \sigma_o(T)$ while $\sigma(T) = \sigma(A) \cup \sigma(K)$ does not contain i.

§ 8. OPERATORS IN THE CENTRE OF E

The centre $\mathscr{Z}(E)$ of E consists of all operators $T \in \mathscr{L}(E)$ for which there exists $c > 0$ such that $|Tx| \leq c|x|$ for all $x \in E$; see [21, Ch. 20]. For example if $E = L^p(\Omega, \mu)$ for a σ-finite measure space (Ω, μ), $1 \leq p \leq \infty$, then $\mathscr{Z}(E)$ consists of all multiplication operators M_f, $f \in L^\infty$ where $M_f g = fg$, see [21, Theorem 142.11]. Similarly if $E = C_0(X)$ for a locally compact space X, then $\mathscr{Z}(E)$ consists of all multiplication operators where the multiplier is a bounded continuous function [21, Example 142.2].

(8.1) LEMMA. *If $T \in \mathscr{Z}(E)$ is a Fredholm operator, then* ind $(T) = 0$.

PROOF. Let $\mathscr{N}(T)$ denote the null space of T and $\mathscr{R}(T)$ denotes the range of T. From [21, Theorem 140.5(ii) and Corollary 144.3(ii)] we have $\mathscr{N}(T) = \mathscr{R}(T)^\perp$ and so dim $(\mathscr{N}(T)) \leq$ dim $(E/\mathscr{R}(T))$. In other words ind $(T) \leq 0$.

However, the adjoint T^* of T belongs to $\mathscr{Z}(E^*)$ and is a Fredholm operator, so ind $(T^*) \leq 0$ and ind $(T) = -\text{ind } (T^*) \geq 0$. Therefore ind $(T) = 0$. □

Schep [18, Theorem 1.11] showed that if $T \in \mathscr{Z}(E)$ and if E has no atoms, then $\sigma(T) = \sigma_e(T)$. Our next theorem contains Schep's result (with a different proof) and other related results (cf. [18, Theorem 3.3]).

(8.2) THEOREM. *Let $T \in \mathscr{Z}(E)$.*

(a) *Every point in $\sigma(T) \setminus \sigma_e(T)$ is an isolated point in $\sigma(T)$ and the corresponding spectral projection is a finite rank band projection.*

(b) *$\sigma_e(T) = \sigma_{oe}(T) = \omega(T) = \omega_o(T)$ and $\sigma_o(T) = \sigma(T)$.*

(c) *If E has no atoms, then the six spectra in (b) coincide.*

PROOF. If $\lambda \notin \sigma_e(T)$, then $\lambda - T$ is a Fredholm operator, so $\mathscr{N} = \mathscr{N}(\lambda - T)$ is finite dimensional. Also $\mathscr{N}(\lambda - T)$ is a band [21, Theorem 140.5], so there exists a band projection $P \in \mathscr{Z}(E)$ with range $\mathscr{N}$. We have $(\lambda - T)|\mathscr{R}(P) = 0$ and $(\lambda - T)|\mathscr{R}(1-P)$ is an injective Fredholm operator in the centre of $\mathscr{R}(1-P)$ and so by Lemma 8.1, $(\lambda - T)|\mathscr{R}(1-P)$ is invertible. Therefore, if $\lambda \in \sigma(T)$, it is an isolated point in $\sigma(T)$ and P is the corresponding spectral projection. This proves part (a). Furthermore $\lambda - T - P$ is injective and by Lemma 8.1, we have that $\lambda - T - P$ is also surjective. The algebra $\mathscr{Z}(E)$ is a full subalgebra of $\mathscr{L}(E)$ (see, e.g., [18, Theorem 1.8]) so $(\lambda 1 - T - P)^{-1} \in \mathscr{Z}(E) \subset \mathscr{L}^r(E)$ and so $\lambda \notin \sigma_{oe}(T)$. We conclude that $\sigma_{oe}(T) \subset \sigma_e(T)$. The reverse inclusion is obvious and so $\sigma_{oe}(T) = \sigma_e(T)$. By Theorem 4.10, we have that $\sigma(T) = \sigma_o(T)$ and $\omega(T) = \omega_o(T)$. Further $\omega(T) \setminus \sigma_e(T) = \{\lambda : \lambda 1 - T \text{ is Fredholm with index } \neq 0\}$, so, by Lemma 8.1, we have $\omega(T) = \sigma_e(T)$. This ends the proof of (b).

If E has no atom, then there is no nonzero finite rank band projection in $\mathscr{L}^r(E)$ and so part (a) implies that $\sigma(T) = \sigma_e(T)$. This together with (b) establishes (c). □

(8.3) COROLLARY. *If $T \in \mathscr{Z}(E)$ and $K \in \mathscr{K}^r(E)$, then $\sigma(T+K) = \sigma_o(T+K)$.*

APPENDIX

We describe some (mostly known) examples of operators with different spectrum and o-spectrum. These examples have been referred to extensively in the main body of the paper and have been used as building blocks of other examples.

(A1) EXAMPLE. (Ando), see [15]. Let

$$C_n = \begin{pmatrix} \cos \pi/4n & -\sin \pi/4n \\ \sin \pi/4n & \cos \pi/4n \end{pmatrix}$$

and let $U_n = C_n \otimes \cdots \otimes C_n$ (n factors) and $U = \sum_{n=1}^{\infty} \oplus U_n$. Then U is a unitary operator on l^2 and is regular with order-spectral radius $r_o(U) \geq \sqrt{2}$. See [15] for details.

(A2) EXAMPLE. This is a class of convolution operators. Let G be locally compact abelian group. Consider the operator $C_{\mu,p}$ on $L^p(G)$ $(1 \leq p \leq \infty)$ given by $C_\mu f = \mu * f$ for a bounded Borel measure μ on G. Then $C_{\mu,p}$ is a bounded regular operator and [1, Theorem 3.4]

$$\sigma_o(C_{\mu,p}) = \sigma(\mu)$$

where $\sigma(\mu)$ is the spectrum of μ in the measure algebra $M(G)$. On the other hand

$$\sigma(C_{\mu,2}) = \overline{\text{range } \hat{\mu}}$$

since $C_{\mu,2}$ is unitarily equivalent to the operator of multiplication by $\hat{\mu}$ on $L^2(\hat{G})$.

2a) Let G be the circle group $\Gamma := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. There exists a positive measure μ on G such that $\|\mu\| = 1$, $\hat{\mu}(n)$ is real for every n, $\lim_{|n| \to \infty} \hat{\mu}(n) = 0$ and $\sigma(\mu)$ contains the unit circle Γ (see [1, Example 3.7]). It follows that $C_{\mu,2}$ is compact, positive (i.e., order-preserving), hermitian operator. We have $\sigma(C_{\mu,2}) = \{0\} \cup \{\hat{\mu}(n) : n \in \mathbb{Z}\}$ is a countable subset of $[-1, 1]$. On the other hand, $\sigma_o(C_{\mu,2}) = \sigma(\mu)$ contains the unit circle Γ.

For $1 < p < \infty$, an interpolation theorem of Krasnosel'skii [11] implies that $C_{\mu,p}$ is compact, and by a theorem of Schaefer [16], we have $\sigma(C_{\mu,p}) = \sigma(C_{\mu,2}) = \{0\} \cup \{\hat{\mu}(n) : n \in \mathbb{Z}\}$ but $\sigma_o(C_{\mu,p}) = \sigma(\mu) \supset \Gamma$.

We also point out that $\sigma_{oe}(C_{\mu,p}) \neq \{0\}$. This follows from Theorem 3.5 or Theorem 4.10.

2b) There exists a compact abelian group G and a positive measure $\mu \in M(G)$ with norm 1 such that $\hat{\mu}(\hat{G})$ is real but the spectrum of μ is the whole closed unit disk $\mathbb{D}$ (see 13, § 5.4.4]). Again, we have $C_{\mu,2}$ hermitian and so has real spectrum but $\sigma_o(C_{\mu,2}) = \mathbb{D}$.

2c) In Examples 2a and 2b we have regular hermition operator T on $L^2(G)$, for a compact abelian group G, such that $\sigma_o(T)$ is not real. We now give another example on l^2. First, we note that the convolution operators $C_{\mu,p}$ are also defined for any locally compact amenable group G and that $\sigma_o(C_{\mu,p}) = \sigma(\mu)$ [1, Theorem 3.4]. Jenkins [9] constructed a discrete countable amenable group G such that there exists a measure $\mu \in M(G) = l^1(G)$ satisfying $\mu = \mu^*$ but $\sigma(\mu)$ is not real. Therefore the convolution operator $C_{\mu,2}$ on $l^2(G)$ is the desired example.

2d) If T is any of the hermitian operators $C_{\mu,2}$ in Example 2a, 2b or 2c above and if $U = \exp(iT)$, then U is a regular unitary operator with $\sigma_o(U) \neq \sigma(U)$; cf. Ando's example (Example 1).

We end by a remark on regular resolvents.

(A3) REMARK. The equality $\sigma(T) = \sigma_o(T)$ for a regular operator T can be reformulated by saying that all resolvents $(\lambda 1 - T)^{-1}$ are regular. In this connection, we point out the curious fact that there exists an operator T (necessarily non-regular) such that $(\lambda 1 - T)^{-1}$ is not regular for any λ in the resolvent set of T. If P is a non-regular projection, then $\sigma(P)=\{0,1\}$ and for $\lambda\notin\{0,1\}$, we have $(\lambda 1-P)^{-1}=\lambda^{-1}1-\lambda^{-1}(\lambda-1)^{-1}P$ which is never regular. One example of a non-regular projection on l^2 is in [19]. Another example is the orthogonal projection of $L^2(\mathbf{T})$ onto $H^2(\mathbf{T})$ where $\mathbf{T}$ is the unit circle $\{z:|z|=1\}$ and $H^2(\mathbf{T})$ is the Hardy space; we omit the details. More generally, if T is any algebraic non-regular operator, e.g., the Fourier Transform on $L^2(\mathbb{R})$, then no resolvent of T is regular. (An operator is called algebraic if it satisfies a polynomial equation.) To see this, observe that every resolvent of T is algebraic. If any resolvent $R=(\lambda-T)^{-1}$ is regular and if p is the minimal polynomial of R, then $\sigma(R)=\sigma_o(R)=\{\lambda\in\mathbb{C}:p(\lambda)=0\}$. This implies that $R^{-1}\in\mathscr{L}^r$ and so $T\in\mathscr{L}^r$.

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