RESOLVENT POSITIVE OPERATORS

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ABSTRACT

Resolvent positive operators on an ordered Banach space (with generating and normal positive cone) are by definition linear (possibly unbounded) operators whose resolvent exists and is positive on a right half-line. Even though these operators are defined by a simple (purely algebraic) condition, analogues of the basic results of the theory of C_0-semigroups can be proved for them. In fact, if A is resolvent positive and has a dense domain, then the Cauchy problem associated with A has a unique solution for every initial value in the domain of A^2, and the solution is positive if the initial value is positive. Also the converse is true (if we assume that A has a non-empty resolvent set and $D(A^2) \cap E_+$ is dense in E_+). Moreover, every positive resolvent is a Laplace–Stieltjes transform of a so-called integrated semigroup; and conversely every such (increasing, non-degenerate) integrated semigroup defines a unique resolvent positive operator.

1. *Introduction*

Many problems in applied mathematics occur in the form of a *Cauchy problem*

$$\text{(CP)} \qquad \begin{aligned} u'(t) &= Au(t) \quad (t \geqslant 0), \\ u(0) &= f, \end{aligned}$$

where A is a linear operator and f (the 'initial value') an element of its domain. There is a well established notion of *well-posedness* of (CP) which is equivalent to A being the generator of a C_0-semigroup (see Goldstein [**13**, II.1.2]).

The purpose of the present paper is to show that for Cauchy problems involving an operator A on an ordered Banach space such that solutions with positive initial values remain positive, a different notion of well-posedness and of a generator seems to be more natural. In fact, we suggest considering the following class of operators:

DEFINITION. Let E be an ordered Banach space whose positive cone is generating and normal. An operator A on E is called *resolvent positive* if there exists $w \in \mathbb{R}$ such that $(w, \infty) \subset \rho(A)$ (the resolvent set of A) and $R(\lambda, A) := (\lambda - A)^{-1} \geqslant 0$ for all $\lambda > w$.

This notion is purely algebraic and no norm condition is required. By the Hille–Yosida theorem a densely defined resolvent positive operator A is the generator of a (necessarily positive) C_0-semigroup (if and) only if in addition $\sup\{\|(\lambda - a)^n R(\lambda, A)^n\| : \lambda > a, n \in \mathbb{N}\} < \infty$ for some $a \geqslant w$.

Even though resolvent positive operators do not generate a one-parameter semigroup in general, they admit a satisfactory theory which parallels the theory of semigroups to a high extent. We want to explain this in more detail.

The principal objects in the theory of one-parameter semigroups interact in the following way. To every C_0-semigroup one associates its generator whose resolvent is given by the Laplace transform of the semigroup. Moreover,

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generators of C_0-semigroups are (essentially) characterized by the fact that the associated Cauchy problem admits a unique solution for all initial values in the domain of the operator (well-posedness).

In complete analogy we find that every positive resolvent is the Laplace–Stieltjes transform of a unique so-called 'integrated semigroup', and conversely, to every (non-degenerate, increasing) integrated semigroup we associate a unique resolvent positive operator whose resolvent is the Laplace–Stieltjes transform of this integrated semigroup. Concerning the Cauchy problem, the following holds (with some reservation regarding a technical detail): a densely defined operator is resolvent positive if and only if the associated Cauchy problem admits a unique solution for every initial value f in the domain of A^2 and the solution is positive if f is positive.

The paper is organized as follows. In § 2, we discuss order conditions on the space or the domain which imply that a resolvent positive operator with a dense domain is 'automatically' the generator of a C_0-semigroup. These are exceptional phenomena. In fact, in § 3 several natural examples of resolvent positive operators which are not generators of a semigroup are constructed.

In the general theory which follows in §§ 4–8 the representation of positive resolvents as Laplace–Stieltjes transforms is essential. It is proved by two different approaches. One is based on the Hille–Yosida theorem and can be applied when A has a dense domain (§ 4). The other depends on a vector-valued version of Bernstein's theorem which we prove in § 5. Here we have to restrict the space (allowing reflexive spaces, L^1-spaces and c_0), but it is no longer necessary to assume that the domain of the operator is dense.

As we pointed out above, in our theory the semigroup is replaced by the so-called integrated semigroup. The relations between this integrated semigroup and the given resolvent positive operator are similar to those between a semigroup and its generator. They are investigated in § 6. The homogeneous Cauchy problem is considered in § 7, the inhomogeneous in § 8.

Finally, we characterize resolvent positive operators on a Banach lattice by means of Kato's inequality (§ 9). In some aspects, this result is similar to the Lumer–Phillips theorem for generators of C_0-semigroups.

General assumptions. Throughout the paper E denotes a real ordered Banach space whose positive cone E_+ is *generating and normal* (that is, $E = E_+ - E_+$ and $E' = E'_+ - E'_+$, where E'_+ denotes the dual cone). For example, E may be a Banach lattice or the hermitian part of a C*-algebra. Moreover, we assume that the norm on E is chosen in such a way that

$$\pm f \leq g \quad \text{implies} \quad \|f\| \leq \|g\| \quad (f, g \in E) \tag{1.1}$$

(which can always be done). We also note that there exists a constant $k > 0$ such that for all linear operators S, T on E one has

$$0 \leq S \leq T \quad \text{implies} \quad \|S\| \leq k\,\|T\|. \tag{1.2}$$

We refer to **[4]** and **[21]** for more details on ordered Banach spaces.

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2. *Resolvent positive operators which are automatically generators of a C_0-semigroup*

Let A be a resolvent positive operator. We introduce the following notation:

$$s(A) = \inf\{w \in \mathbb{R}: \ (w, \infty) \subset \rho(A) \text{ and } R(\lambda, A) \geq 0 \text{ for all } \lambda > w\},$$

$$D(A)_+ = E_+ \cap D(A) \quad \text{and} \quad D(A')_+ = E'_+ \cap D(A')$$

(where in the second definition we assume A to be densely defined; then A' denotes the adjoint of A). Let $s(A) < \lambda < \mu$. Then

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A) \geq 0.$$

Thus the function $R(\cdot, A)$ is decreasing on $(s(A), \infty)$.

LEMMA 2.1. *Let A be a resolvent positive operator such that $s(A) < 0$. Then*

$$R(0, A) = R(\lambda, A) + \lambda R(\lambda, A)^2 + \lambda^2 R(\lambda, A)^3 + \ldots + \lambda^{n-1} R(\lambda, A)^n + \lambda^n R(\lambda, A)^n R(0, A) \tag{2.1}$$

for all $n \in \mathbb{N}$, $\lambda \geq 0$. Consequently,

$$\sup\{\|\lambda^n R(\lambda, A)^n R(0, A)\|: \ n \in \mathbb{N}, \lambda \geq 0\} < \infty. \tag{2.2}$$

Proof. By the resolvent equation, $R(0, A) = R(\lambda, A) + \lambda R(\lambda, A)R(0, A)$ $(\lambda > 0)$. This is (2.1) for $n = 1$. Iterating this equation yields (2.1) for all $n \in \mathbb{N}$.

A subset C of E_+ is called *cofinal* in E_+ if for every $f \in E_+$ there exists $g \in C$ such that $f \leq g$. If $(T(t))_{t\geq 0}$ is a positive C_0-semigroup with generator B, then the *type* (or *growth bound*) $\omega(B)$ is defined by $\omega(B) = \inf\{w \in \mathbb{R}$: there exists $M \geq 1$ such that $\|T(t)\| \leq Me^{wt}$ for all $t \geq 0\}$. One always has $s(B) \leq \omega(B) < \infty$, but it can happen that $s(B) \neq \omega(B)$ even if B generates a positive C_0-group [**14**, **26**].

THEOREM 2.2. *Let A be a densely defined resolvent positive operator. If $D(A)_+$ is cofinal in E_+ or if $D(A')_+$ is cofinal in E'_+, then A is the generator of a positive C_0-semigroup. Moreover, $s(A) = \omega(A)$.*

Proof. (a) Assume that $s(A) < 0$. We claim that A generates a bounded C_0-semigroup, if one of the conditions in the theorem is satisfied. We first assume that $D(A)$ is cofinal. Let $f \in E_+$. Then there exists $g \in D(A)_+$ such that $f \leq g$. Let $h = -Ag$ and $k \in E_+$ such that $h \leq k$. Then

$$f \leq g = R(0, A)h \leq R(0, A)k.$$

It follows from (2.1) that

$$\lambda^n R(\lambda, A)^n f \leq \lambda^n R(\lambda, A)^n R(0, A)k \leq R(0, A)k.$$

Hence

$$\sup\{\|\lambda^n R(\lambda, A)^n f\|: \ \lambda \geq 0, n \in \mathbb{N}\} < \infty.$$

Since $E = E_+ - E_+$, it follows that

$$\{\lambda^n R(\lambda, A)^n: \ \lambda \geq 0, n \in \mathbb{N}\}$$

is strongly bounded; thus it is norm-bounded by the uniform boundedness principle. The Hille–Yosida theorem implies that A generates a bounded C_0-semigroup.

If $D(A')_+$ is cofinal in E'_+, consider $f \in E_+$, $\phi \in E'_+$. Then there exists $\psi \in E'_+$ such that $\phi \leqslant R(0, A)'\psi$. Hence by (2.1),

$$\begin{aligned}\langle \lambda^n R(\lambda, A)^n f, \phi \rangle &\leqslant \langle \lambda^n R(\lambda, A)^n f, R(0, A)'\psi \rangle \\ &= \langle \lambda^n R(\lambda, A)^n R(0, A) f, \psi \rangle \\ &\leqslant \langle R(0, A) f, \psi \rangle.\end{aligned}$$

Since E_+ and E'_+ are generating, this implies that $\{\lambda^n R(\lambda, A)^n \colon \lambda \geqslant 0,\ n \in \mathbb{N}\}$ is weakly bounded, and so norm-bounded. Again the Hille–Yosida theorem implies the claim.

(b) If $s(A)$ is arbitrary, consider $B = A - w$ for some $w > s(A)$. Then $s(B) < 0$, and so by (a), B is the generator of a bounded semigroup $(T(t))_{t \geqslant 0}$. Hence A generates the semigroup $(e^{wt}T(t))_{t \geqslant 0}$. Moreover $\omega(A) \leqslant w$.

Corollary 2.3. *Assume that* $\operatorname{int} E_+ \neq \varnothing$. *If A is a densely defined resolvent positive operator, then A is the generator of a positive C_0-semigroup and $s(A) = \omega(A)$.*

Proof. Since $\operatorname{int} E_+ \neq \varnothing$ and $D(A)$ is dense, there exists $u \in \operatorname{int} E_+ \cap D(A)$. The set $\{u\}$ is clearly cofinal in E_+.

Corollary 2.4. *Let A be a densely defined resolvent positive operator on $L^1(X, \mu)$ (where (X, μ) is a σ-finite measure space). If there exists $\phi \in D(A') \cap L^\infty(X, \mu)$ such that $\phi(x) \geqslant \varepsilon > 0$ for almost all $x \in X$, then A is the generator of a positive C_0-semigroup.*

Remark. Corollary 2.3 has been proved in **[2]** and Corollary 2.4 by Batty and Robinson **[4]** with a different approach using half-norms.

Theorem 2.5. *Let A be a densely defined resolvent positive operator. If there exist $\lambda_0 > s(A)$ and $c > 0$ such that*

$$\|R(\lambda_0, A) f\| \geqslant c\, \|f\| \quad (f \in E_+), \tag{2.4}$$

then A is the generator of a positive C_0-semigroup and $s(A) = \omega(A)$.

Proof. Let $s(A) < w \leqslant \lambda_0$. Let $B = A - w$. Then $s(B) < 0$. Since $R(0, B) = R(w, A) \geqslant R(\lambda_0, A)$, it follows from (2.4) and (1.2) that

$$\|R(0, B) f\| \geqslant k^{-1}\, \|R(\lambda_0, A) f\| \geqslant k^{-1} c\, \|f\|$$

for all $f \in E_+$. In particular,

$$\|(\lambda R(\lambda, B))^n g\| \leqslant k c^{-1}\, \|R(0, B)(\lambda R(\lambda, B))^n g\|$$

for all $g \in E_+$, $\lambda \geqslant 0$, $n \in \mathbb{N}$. Since $E = E_+ - E_+$, it follows from (2.2) that the set $\{\lambda^n R(\lambda, B)^n \colon n \in \mathbb{N}, \lambda \geqslant 0\}$ is strongly bounded and so norm-bounded. Thus by the Hille–Yosida theorem, $B = A - w$ generates a bounded positive C_0-semigroup. Hence A is the generator of a positive C_0-semigroup as well and $\omega(A) \leqslant w$.

REMARK. Theorem 2.5 (except the assertion concerning the spectral bound) is due to Batty and Robinson [4] (with a different proof).

THEOREM 2.6. *Suppose that the norm is additive on the positive cone, that is,* $\|f+g\| = \|f\| + \|g\|$ *for all* $f, g \in E_+$ *(for example,* $E = L^1(X, \mu)$*). Let A be a densely defined operator. Then the following assertions are equivalent:*

(i) *A generates a positive* C_0*-group;*

(ii) *A and* $-A$ *are resolvent positive and there exist* $\lambda > \max\{s(A), s(-A)\}$ *and* $c > 0$ *such that*

$$\|R(\lambda, \pm A)f\| \geq c\,\|f\| \quad \text{for all } f \in E_+. \tag{2.5}$$

Proof. Assume that A generates a positive C_0-group $(T(t))_{t\in\mathbb{R}}$. Then there exist $w > 0$, $M \geq 1$ such that $\|T(-t)\| \leq Me^{wt}$ for all $t \geq 0$. This implies that $\|T(t)f\| \geq M^{-1}e^{-wt}\,\|f\|$ $(f \in E)$. Hence for $\lambda > \omega(A)$, $f \in E_+$,

$$\|R(\lambda, A)f\| = \left\|\int_0^\infty e^{-\lambda t}T(t)f\,dt\right\| = \int_0^\infty e^{-\lambda t}\,\|T(t)f\|\,dt$$
$$\geq M^{-1}\int_0^\infty e^{-\lambda t}e^{-wt}\,\|f\|\,dt = ((\lambda + w)M)^{-1}\,\|f\|.$$

Similarly for $R(\lambda, -A)$ where $\lambda > \omega(-A)$. Thus (ii) holds. The converse follows from Theorem 2.5.

REMARK. Condition (2.5) does not hold for generators of positive C_0-groups on every Banach lattice [4, Example 2.2.13].

EXAMPLE 2.7. We show by an example that condition (2.5) cannot be omitted in Theorem 2.6.

Let B be the generator of the group $(T(t))_{t\in\mathbb{R}}$ on $L^1(\mathbb{R})$ given by $T(t)f(x) = f(x+t)$. Then $D(B) = \{f \in AC(\mathbb{R})\colon f' \in L^1(\mathbb{R})\}$ and

$$R(\lambda, B)f(x) = e^{\lambda x}\int_x^\infty e^{-\lambda y}f(y)\,dy$$

and

$$-R(-\lambda, B)f(x) = R(\lambda, -B)f(x) = e^{-\lambda x}\int_{-\infty}^x e^{\lambda y}f(y)\,dy$$

for $\lambda > 0$, $f \in E$. For $n \in \mathbb{N}$ let

$$p_n(f) = (\tfrac{3}{2})^n\int_{2^{-n}}^{2^{-n+1}} |f(x)|\,dx$$

and $E_0 = \{f \in L^1(\mathbb{R})\colon \sum_{n=1}^\infty p_n(f) < \infty\}$. Then E_0 is a Banach lattice with the norm $\|f\|_0 := \|f\|_1 + \sum_{n=1}^\infty p_n(f)$. Of course, E_0 is isomorphic to $L^1(\mathbb{R}, \mu)$ for a suitable measure μ. We show that $R(\lambda, B)E_0 \subset E_0$ for all $\lambda \in \mathbb{R}\backslash\{0\}$. In fact, let $\lambda > 0$, $f \in E_0$. Then

$$p_n(R(\lambda, B)f) \leq (\tfrac{3}{2})^n\int_{2^{-n}}^{2^{-n+1}} e^{\lambda x}\int_x^\infty e^{-\lambda y}\,|f(y)|\,dy\,dx$$
$$\leq \|f\|_1(\tfrac{3}{2})^n\int_{2^{-n}}^{2^{-n+1}} dx = (\tfrac{3}{4})^n\,\|f\|_1.$$

Hence $\sum_{n=1}^{\infty} p_n(R(\lambda, B)f) < \infty$. Similarly for $\lambda < 0$. Let A be the operator on E_0 defined on $D(A) = \{f \in E_0 \cap D(B)\colon Bf \in E_0\}$ by $Af = Bf$. Then it is easy to see that $\mathbb{R}\setminus\{0\} \subset \rho(A)$ and $R(\lambda, A) = R(\lambda, B) \mid E_0$ for $\lambda \in \mathbb{R}\setminus\{0\}$. Hence $R(\lambda, A) \geqslant 0$ and $R(\lambda, -A) = -R(-\lambda, A) \geqslant 0$ for $\lambda > 0$.

We show that $D(A)$ is dense in E. Let $f \in E_0$. For $n \in \mathbb{N}$ let $f_n = f \cdot 1_{\mathbb{R}\setminus[0, 2^{-n}]}$. Then $f - f_n = f \cdot 1_{[0, 2^{-n}]}$. Hence $\|f - f_n\|_1 \to 0$ for $n \to \infty$. Moreover, $p_m(f - f_n) = 0$ for $m \leqslant n$ and $p_m(f - f_n) = p_m(f)$ for $m > n$. Hence

$$\sum_{m=1}^{\infty} p_m(f - f_n) = \sum_{m=n+1}^{\infty} p_m(f) \to 0$$

for $n \to \infty$. We have shown that $E_{00} := \{f \in E_0$: there exists $\varepsilon > 0$ such that $f|_{[0, \varepsilon]} = 0\}$ is dense in E_0. Now let $f \in E_{00}$. Then there exists $\varepsilon > 0$ such that $f|_{[0, \varepsilon]} = 0$. It is easy to see that there exists a sequence $(f_n) \subset AC(\mathbb{R})$ such that $f_n' \in L^1(\mathbb{R})$, $f_n|_{[0, \varepsilon/2]} = 0$, and $\|f - f_n\|_1 \to 0$ for $n \to \infty$. Then $f_n \in D(A)$. Since $\| \ \|_1$ and $\| \ \|_0$ are equivalent norms on $E_{\varepsilon/2} := \{f \in L^1(\mathbb{R})\colon f(x) = 0$ for $x \in [0, \varepsilon/2]$ a.e.$\} \subset E_0$, it follows that $\lim_{n\to\infty} f_n = f$ in E_0. Finally, we show that A is not a generator. In fact, assume that there exists a semigroup $(T_0(t))_{t>0}$ on E_0, which is strongly continuous for $t > 0$, such that

$$R(\lambda, A)f = \int_0^{\infty} e^{-\lambda t} T_0(t) f \, dt$$

for large $\lambda > 0$. For $f \in E_0$,

$$R(\lambda, A)f = \int_0^{\infty} e^{-\lambda t} T(t) f \, dt.$$

So it follows from the uniqueness theorem for Laplace transforms, that $T_0(t)f = T(t)f$ $(t > 0)$ for all $f \in E_{00}$. Let $t > 1$, $n \in \mathbb{N}$ and $f = 2^n 1_{[t+2^{-n}, t+2^{-n+1}]}$. Then $\|f\|_0 = \|f\|_1 = 1$. But $T_0(t)f = 2^n 1_{[2^{-n}, 2^{-n+1}]}$. Hence $\|T_0(t)f\|_0 \geqslant p_n(T_0(t)f) = (\frac{3}{2})^n$. Thus $T_0(t)$ is not continuous for $t > 1$.

3. *Perturbation and examples*

In this section we present two kinds of perturbations which demonstrate that there exist many natural resolvent positive operators which are not generators of a semigroup.

THEOREM 3.1. *Let A be a resolvent positive operator and $B\colon D(A) \to E$ a positive operator. If $r(BR(\lambda, A)) < 1$ for some $\lambda > s(A)$, then $A + B$ with domain $D(A)$ is a resolvent positive operator and $s(A + B) < \lambda$. Moreover, if* $\sup\{\|\mu R(\mu, A)\|\colon \mu \geqslant \lambda\} < \infty$ *(for example, if A is the generator of a C_0-semigroup), then also* $\sup\{\|\mu R(\mu, A + B)\|\colon \mu \geqslant \lambda\} < \infty$.

Note. By assumption, $BR(\lambda, A)$ is a positive, hence bounded operator on E; we denote by $r(BR(\lambda, A))$ its spectral radius.

Proof. Let $f \in D(A)$. Then

$$(\lambda - (A + B))f = (I - BR(\lambda, A))(\lambda - A)f.$$

Let $S_\lambda := (I - BR(\lambda, A))^{-1} = \sum_{n=0}^{\infty} (BR(\lambda, A))^n \geqslant 0$. Then

$$R(\lambda, A)S_\lambda(\lambda - (A+B))f = f \quad \text{for all } f \in D(A)$$

and

$$(\lambda - (A+B))R(\lambda, A)S_\lambda g = g \quad \text{for all } g \in E.$$

Hence $\lambda \in \rho(A+B)$ and $R(\lambda, A+B) = R(\lambda, A)S_\lambda \geqslant 0$. If $\mu > \lambda$, then $BR(\mu, A) \leqslant BR(\lambda, A)$ by (1.2), and so $r(BR(\mu, A)) \leqslant r(BR(\lambda, A)) < 1$. Hence also $\mu \in \rho(A+B)$ and $R(\mu, A+B) \geqslant 0$. Moreover, $S_\mu \leqslant S_\lambda$ and $R(\mu, A) \leqslant R(\lambda, A)$ so that

$$\mu R(\mu, A+B) = \mu R(\mu, A)S_\mu \leqslant \mu R(\mu, A)S_\lambda.$$

Hence

$$\sup\{\|\mu R(\mu, A+B)\| : \mu \geqslant \lambda\} \leqslant \sup\{\|\mu R(\mu, A)\| \, \|S_\lambda\| : \mu \geqslant \lambda\} < \infty$$

if the additional assumption is satisfied.

The following examples show that even in rather simple and natural cases perturbations as in Theorem 3.1 may yield resolvent positive operators which are not generators of semigroups.

EXAMPLE 3.2. Let $\alpha \in (0, 1)$. Define the operator A by

$$Af(x) = -f'(x) + (\alpha/x)f(x) \quad (x \in (0, 1])$$

on the space $E = C_0(0, 1] := \{f \in C[0, 1]: f(0) = 0\}$ with domain $D(A) = \{f \in C^1[0, 1]: f'(0) = f(0) = 0\}$. Then A is resolvent positive but not a generator of a semigroup. Moreover, $s(A) = -\infty$ and $\sup\{\|\mu R(\mu, A)\| : \mu \geqslant 0\} \leqslant 1/(1-\alpha)$.

Proof. Let $A_0 f = -f'$ with domain $D(A_0) = D(A)$. Then A_0 is the generator of the C_0-semigroup $(T(t))_{t\geqslant 0}$ given by

$$(T(t)f)(x) = \begin{cases} f(x-t) & x \geqslant t, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $\sigma(A_0) = \varnothing$ and

$$R(\lambda, A_0)f(x) = e^{-\lambda x} \int_0^x e^{\lambda y} f(y)\, dy \quad (\lambda \in \mathbb{C}, f \in E).$$

Let $B\colon D(A_0) \to E$ be given by $Bf(x) = \alpha f(x)/x$ $(x > 0)$, $Bf(0) = 0$. Let $f \in E$ and $g = R(0, A)f$. Then

$$|Bg(x)| = \left| \alpha/x \int_0^x f(y)\, dy \right| \leqslant \alpha \|f\|_\infty.$$

Thus $\|BR(0, A_0)\| \leqslant \alpha < 1$. So Theorem 3.1 implies that $A = A_0 + B$ is resolvent positive and $s(A) < 0$. Moreover, for $\mu \geqslant 0$ one has $\mu R(\mu, A) = \mu R(\mu, A_0)S_\mu \leqslant \mu R(\mu, A_0)S_0$, where $S_\mu = (I - BR(\mu, A_0))^{-1}$. Since $\|\mu R(\mu, A_0)\| \leqslant 1$ and $\|S_0\| \leqslant 1/(1-\alpha)$ it follows that $\sup\{\|\mu R(\mu, A)\| : \mu \geqslant 0\} \leqslant 1/(1-\alpha)$.

It remains to show that A is not a generator. One can easily check that for all

$\lambda \in \mathbb{C}$ one has $\lambda \in \rho(A)$ and

$$R(\lambda, A)f(x) = e^{-\lambda x} x^{\alpha} \int_0^x y^{-\alpha} e^{\lambda y} f(y)\, dy$$
$$= \int_0^x x^{\alpha}(x-t)^{-\alpha} f(x-t) e^{-\lambda t}\, dt \quad (f \in E).$$

Suppose that there exists a semigroup $(T(t))_{t>0}$ which is strongly continuous for $t > 0$ such that $R(\lambda, A)f = \int_0^\infty e^{-\lambda t} T(t) f\, dt$ for all $f \in E$ and all sufficiently large real λ. Then by the uniqueness theorem for Laplace transforms, for $0 < t < 1$, one would have

$$T(t)f(x) = \begin{cases} (x^{\alpha}/(x-t)^{\alpha}) f(x-t) & \text{for } x \geqslant t, \\ 0 & \text{otherwise.} \end{cases}$$

This does not define a bounded operator on $C_0(0, 1]$.

REMARK. It follows from a result of Benyamini [5] that $C_0(0, 1]$ is isomorphic as a Banach space to a space $C(K)$ (K compact). Thus Example 3.2 yields an operator B on $C(K)$ such that $\sigma(B) = \varnothing$ and the resolvent satisfies

$$\sup\{\|\lambda^{n-1} R(\lambda, B)^n\| \colon \lambda \geqslant 0, n \in \mathbb{N}\} < \infty,$$
$$\sup\{\|\lambda R(\lambda, B)\| \colon \lambda \geqslant 0\} < \infty.$$

But B is not a generator. Of course, B is not resolvent positive by Corollary 2.3.

EXAMPLE 3.3. Let $E = L^p[0, 1]$, where $1 < p < \infty$. Choose $\alpha \in (0, (p-1)/p)$. Define the operator A by

$$Af(x) = -f'(x) + (\alpha/x) f(x)$$

with domain $D(A) = \{f \in AC[0, 1] \colon f' \in L^p[0, 1], f(0) = 0\}$. Then A is resolvent positive. Moreover, $s(A) < 0$ and $\sup\{\|\lambda R(\lambda, A)\| \colon \lambda \geqslant 0\} < \infty$. But A is not a generator.

Proof. Let $A_0 f = -f'$ with domain $D(A_0) = D(A)$. Then A_0 generates the semigroup $(T_0(t))_{t \geqslant 0}$ on E given by $T_0(t)f(x) = f(x-t)$ for $x \geqslant t$ and $T_0(t)f(x) = 0$ otherwise. Moreover, $s(A_0) = -\infty$ and $R(0, A_0)f(x) = \int_0^x f(y)\, dy$. Let $B \colon D(A) \to E$ be defined by $Bf(x) = (\alpha/x) f(x)$. Then by [7, Lemma 1], $BR(0, A_0) \in \mathscr{L}(E)$ and $\|BR(0, A_0)\| = \alpha p/(p-1)$. Hence Theorem 3.1 implies the first assertions stated above. It remains to show that A is not a generator. It is not difficult to check that the resolvent of A is given by

$$R(\lambda, A)f(x) = x^{\alpha} e^{-\lambda x} \int_0^x e^{\lambda y} y^{-\alpha} f(y)\, dy = \int_0^x x^{\alpha}(x-t)^{-\alpha} f(x-t) e^{-\lambda t}\, dt \quad (\lambda > 0).$$

Let $E_0 = \{f \in E \colon$ there exists $\delta > 0$ such that $f(x) = 0$ for almost all $x \in [0, \delta)\}$. Define $T(t) \colon E_0 \to E$ by $T(t) = 0$ if $t \geqslant 1$ and

$$T(t)f(x) = \begin{cases} x^{\alpha}(x-t)^{-\alpha} f(x-t) & \text{if } x \geqslant t, \\ 0 & \text{otherwise,} \end{cases}$$

if $0 \leqslant t < 1$. Then $T(\cdot)f$ is continuous from $[0, \infty)$ into E and

$$R(\lambda, A)f = \int_0^\infty e^{-\lambda t} T(t) f\, dt$$

for all $\lambda \geq 0$ if $f \in E_0$. Thus if there exists a semigroup $(T_1(t))_{t>0}$ which is strongly continuous for $t>0$ and such that

$$R(\lambda, A)f = \int_0^\infty e^{-\lambda t} T_1(t) f\, dt \quad (f \in E)$$

for large λ, it follows from the uniqueness theorem for Laplace transforms that $T_1(t)f = T(t)f$ for all $f \in E_0$, $t \geq 0$. But for $t \in (0, 1)$ the mapping $T(t)$ is not continuous (from E_0 with the induced norm into E). In fact, let $\beta > 0$ such that $1 - \alpha p < \beta p < 1$, and for $n \in \mathbb{N}$, let $f_n(x) = 1_{[1/n,\, 1]}(x) x^{-\beta}$. Then $f_n \in E_0$ and

$$\sup\{\|f_n\|_p \colon n \in \mathbb{N}\} < \infty.$$

But

$$\begin{aligned}
\|T(t)f_n\|_p^p &= \int_t^1 x^{\alpha p}/(x-t)^{\alpha p} f_n(x-t)^p\, dx \\
&\geq t^{\alpha p} \int_0^{1-t} f_n(y)^p / y^{\alpha p}\, dy \\
&= t^{\alpha p} \int_{1/n}^{1-t} y^{-(\alpha+\beta)p}\, dy \to \infty \quad \text{for } n \to \infty
\end{aligned}$$

since $(\alpha + \beta)p > 1$.

Next we consider multiplicative perturbations which will allow us to give examples of resolvent positive operators on L^p $(1 \leq p < \infty)$ which are not generators of semigroups.

Proposition 3.4. *Let (X, μ) be a σ-finite measure space and $E = L^p(X, \mu)$ $(1 \leq p < \infty)$ (respectively, X locally compact and $E = C_0(X)$). Let A be a resolvent positive operator. Suppose that $m\colon X \to [0, \infty)$ is measurable (respectively, continuous) such that $m(x) > 0$ a.e. (respectively, $m(x) > 0$ for all $x \in X$) and $(1/m)f \in E$ for all $f \in D(A)$. Let*

$$D(A^\#) = \{g \in E\colon\ m \cdot g \in D(A),\ (1/m)A(m \cdot g) \in E\}$$

and $A^\# g = (1/m)A(m \cdot g)$. Then $A^\#$ is a resolvent positive operator and $s(A^\#) \leq s(A)$.

Proof. For $\lambda > s(A)$ let $R^\#(\lambda)f = (1/m)R(\lambda, A)(m \cdot f)$ $(f \in E)$. Then $R^\#(\lambda)$ is a positive, and hence bounded operator. It is easy to show that $R^\#(\lambda) = (\lambda - A^\#)^{-1}$.

Example 3.5. Let $E = L^p[0, 1]$, $1 \leq p < \infty$, and A be given by $Af = f'$ with $D(A) = \{f \in AC[0, 1]\colon f' \in L^p[0, 1],\ f(1) = 0\}$. Then A is the generator of the semigroup $(T(t))_{t \geq 0}$ given by

$$T(t)f(x) = \begin{cases} f(x+t) & \text{if } x + t \leq 1, \\ 0 & \text{if } x + t > 1. \end{cases}$$

Moreover, $s(A) = -\infty$ and

$$R(\lambda, A)f(x) = e^{\lambda x} \int_x^1 e^{-\lambda y} f(y)\, dy \quad (f \in E).$$

Let $\alpha \in (0, 1/p)$ and $m(x) = x^\alpha$. Then $1/m \in L^p[0, 1]$ and since $D(A) \subset C[0, 1]$, it follows that $(1/m)f \in L^p[0, 1]$ for all $f \in D(A)$. By Proposition 3.4, the operator

$A^{\#}$ is resolvent positive, where

$$D(A^{\#}) = \{f \in E\colon\ m \cdot f \in D(A),\ (1/m)A(m \cdot f) \in E\}$$

and

$$A^{\#}f = (1/m)A(m \cdot f) = x^{-\alpha}(x^{\alpha}f)' = f' + (\alpha/x)f.$$

The domain $D(A^{\#})$ is dense in $L^p[0, 1]$. In fact,

$$D(A) \cap \{f \in L^p[0, 1]\colon\ f|_{[0,\,\varepsilon]} = 0 \text{ for some } \varepsilon > 0\} \subset D(A^{\#}).$$

But $D(A)$ is dense in $L^p[0, 1]$, and it is easy to see that every

$$f \in D(A) = \{g \in AC[0, 1]\colon\ g' \in L^p[0, 1], g(1) = 0\}$$

can be approximated by functions in $D(A)$ which vanish in a neighbourhood of 0.

We show that $A^{\#}$ is not the generator of a semigroup. In fact, assume that there exists a semigroup $(T(t))_{t>0}$ which is strongly continuous for $t > 0$ such that

$$R(\lambda, A^{\#})f = \int_0^{\infty} e^{-\lambda t}T(t)f\,dt$$

for sufficiently large λ. It is not difficult to see that

$$R(\lambda, A^{\#})f(x) = x^{-\alpha}e^{\lambda x}\int_x^1 e^{-\lambda y}f(y)y^{\alpha}\,dy = \int_0^{1-x} e^{-\lambda t}x^{-\alpha}f(x+t)(x+t)^{\alpha}\,dt.$$

As in Example 3.3 one shows that for $0 < t < 1$, $T(t)$ is given by $T(t)f(x) = x^{-\alpha}(x+t)^{\alpha}f(x+t)$ for $x + t \leq 1$. This does not define a bounded operator on $L^p[0, 1]$.

Note. For $p > 1$, $R(\lambda, A^{\#})$ is the adjoint of $R(\lambda, A)$ in Example 3.3 on $L^q(0, 1)$ (with $1/p + 1/q = 1$). Thus $\lambda R(\lambda, A^{\#})$ is norm-bounded for $\lambda \to \infty$. This is also the case for $p = 1$. In fact,

$$\|\lambda R(\lambda, A^{\#})\| = \|\lambda R(\lambda, A^{\#})'\| = \|\lambda R(\lambda, A^{\#})'1\|_{\infty}$$
$$= \sup\left\{\lambda y^{\alpha}e^{-\lambda y}\int_0^y x^{-\alpha}e^{\lambda x}\,dx\colon\ y \in [0, 1]\right\} = \|\lambda R(\lambda)\|,$$

where $R(\lambda)$ denotes the resolvent of the operator A on $C_0(0, 1]$ in Example 3.2. Hence $\|\lambda R(\lambda, A^{\#})\| \leq 1/(1-\alpha)$ $(\lambda \geq 0)$.

REMARK. In the literature, the first example of a resolvent positive operator which is not a generator was given by Batty and Davies [**3**] on $C_0(\mathbb{R})$. A similar example on $L^1(\mathbb{R})$ appears in [**4**, Example 2.2.11]. Independently, H. P. Lotz constructed an example by a renorming procedure similar to Example 2.7 (unpublished).

4. *Positive resolvent as Laplace–Stieltjes transform*

In the sequel we continuously use the notion and properties of the Laplace–Stieltjes integral as given in [**15**, Chapter III]. It is not difficult to see that any increasing function S on $[0, \infty)$ with values in $\mathcal{L}(E)$ (or E) is of bounded variation [**15**, Definition 3.2.4]. Thus $\int_0^b e^{-\lambda t}\,dS(t)$ is defined for every $\lambda \in \mathbb{C}$, $b \geq 0$. Then the improper integral $\int_0^{\infty} e^{-\lambda t}\,dS(t)$ is defined as $\lim_{b\to\infty}\int_0^b e^{-\lambda t}\,dS(t)$ in the operator norm (cf. Proposition 5.5 below).

THEOREM 4.1. *Let A be a densely defined resolvent positive operator. There exists a unique strongly continuous family $(S(t))_{t\geq 0}$ of positive operators satisfying $S(0)=0$ and $S(s)\leq S(t)$ for $0\leq s\leq t$ such that*

$$R(\lambda, A)=\int_0^\infty e^{-\lambda t}\,dS(t) \quad (\lambda > s(A)). \tag{4.1}$$

The construction of A_1 in the following proof is due to Chernoff **[10]**.

Proof. Uniqueness of the representation (4.1) follows from **[25**, 7.2]. In order to prove the existence we first assume that $s(A)<0$. Then for $\lambda>0$ one obtains, from the resolvent equation,

$$R(0, A)\lambda R(\lambda, A)=R(0, A)-R(\lambda, A)\leq R(0, A). \tag{4.2}$$

Let $\|f\|_1 := \inf\{\|R(0, A)g\|: \pm f\leq g\}$. Denote by E_1 the completion of E with respect to this norm. For $f\in E$, $\lambda>0$ one has

$$\begin{aligned}\|\lambda R(\lambda, A)f\|_1 &= \inf\{\|R(0, A)g\|: \pm\lambda R(\lambda, A)f\leq g\}\\ &\leq \inf\{\|R(0, A)\lambda R(\lambda, A)h\|: \pm f\leq h\}\\ &\leq \inf\{\|R(0, A)h\|: \pm f\leq h\} \quad \text{(by (4.2))}\\ &= \|f\|_1.\end{aligned}$$

Thus $R(\lambda, A)$ has a unique continuous extension $R_1(\lambda)$ on E_1 which satisfies

$$\|\lambda R_1(\lambda)\|\leq 1 \quad (\lambda>0). \tag{4.3}$$

It is obvious that $(R_1(\lambda))_{\lambda>0}$ is a pseudo-resolvent. Since $D(A)\subset R_1(\lambda)E_1$ $(\lambda>0)$, it has a dense image, and so it is the resolvent of a densely defined operator A_1 on E_1 **[11**, Theorem 2.6]. It follows from the Hille–Yosida theorem that A_1 is the generator of a strongly continuous contraction semigroup $(T_1(t))_{t\geq 0}$ on E_1. The operator $R(0, A)$ satisfies

$$\|R(0, A)f\|\leq \|f\|_1 \quad (f\in E). \tag{4.4}$$

(In fact, let $f\in E$ and $\pm f\leq g$. Then $\pm R(0, A)f\leq R(0, A)g$. Hence $\|R(0, A)f\|\leq \|R(0, A)g\|$. Thus $\|R(0, A)f\|\leq \inf\{\|R(0, A)g\|: \pm f\leq g\}=\|f\|_1$.) Consequently, the extension $R_1(0)$ of $R(0, A)$ onto E_1 maps E_1 into E. Moreover, $(R_1(\lambda))_{\lambda\geq 0}$ ($\lambda=0$ included) is a pseudo-resolvent too. Thus $R_1(0)=R(0, A_1)$. This implies that

$$D(A_1)=R(0, A_1)E_1\subset E. \tag{4.5}$$

The closure E_{1+} of E_+ is a cone in E_1 which is invariant under $R(\lambda, A_1)$ for $\lambda\geq 0$. This cone is proper (in fact, let $f\in E_{1+}\cap(-E_{1+})$; then $R(0, A_1)f\in E_+\cap(-E_+)$, whence $R(0, A_1)f=0$, and so $f=0$). Thus (E_1, E_{1+}) is an ordered Banach space and the semigroup $(T_1(t))_{t\geq 0}$ is positive.

Now let $S(t)f=\int_0^t T_1(s)f\,ds\in D(A_1)\subset E$ for $f\in E$. Then $S(t)$ is a positive operator on E $(t\geq 0)$. It is clear from the definition that $0=S(0)\leq S(s)\leq S(t)$ for $0<s<t$. Moreover, let $t>0$. Then for $f\in E_{1+}$,

$$A_1\int_0^t T_1(s)f\,ds=T_1(t)f-f.$$

Hence

$$\int_0^t T_1(s)f\,ds = R(0, A_1)f - R(0, A_1)T_1(t)f \leqslant R(0, A_1)f.$$

Thus,

$$S(t) \leqslant R(0, A) \quad (t \geqslant 0). \tag{4.6}$$

In particular, $\sup\{\|S(t)\|: t \geqslant 0\} < \infty$. We now show that $S(\cdot): [0, \infty) \to \mathscr{L}(E)$ is strongly continuous. Let $f \in D(A)$ and $g = Af$. Then

$$\begin{aligned}\|S(t+h)f - S(t)f\| &= \|R(0, A)(S(t+h)g - S(t)g)\| \\ &\leqslant \|S(t+h)g - S(t)g\|_1 \to 0\end{aligned}$$

for $h \to 0$. Here we made use of (4.4). Thus $S(\cdot)$ is strongly continuous on a dense subspace. Since $S(\cdot)$ is bounded, this implies the strong continuity on the whole space. Since $S(\cdot)$ is bounded, the integral in (4.1) converges in the operator norm for $\lambda > 0$. Let $f \in E$. Then

$$\int_0^\infty e^{-\lambda t}\,dS(t)f = \int_0^\infty e^{-\lambda t}T_1(t)f\,dt = R(\lambda, A_1)f = R(\lambda, A)f.$$

Thus (4.1) holds and the proof is finished in the case when $s(A) < 0$.

Now let $s(A)$ be arbitrary. For $w > s(A)$ consider the operator $B = A - w$. Then $s(B) < 0$, so by what we have proved above, there exists a strongly continuous increasing function $S_w(\cdot): [0, \infty) \to \mathscr{L}(E)_+$ satisfying $S_w(0) = 0$, such that

$$R(\mu, B) = \int_0^\infty e^{-\mu t}\,dS_w(t)$$

for $\mu > 0$. Hence

$$R(\lambda, A) = R(\lambda - w, B) = \int_0^\infty e^{-\lambda t}e^{wt}\,dS_w(t) = \int_0^\infty e^{-\lambda t}\,dS(t)$$

for all $\lambda > w$, where $S(t) = \int_0^t e^{ws}\,dS_w(s)$. Clearly, $S(\cdot)$ is strongly continuous, increasing and satisfies $S(0) = 0$. Because of the uniqueness theorem [**25**, 7.2], it does not depend on w and so (4.1) holds for all $\lambda > s(A)$. This proves the theorem in the general case.

EXAMPLE 4.5. (a) In order to illustrate the construction in the proof of Theorem 4.1, consider the operator $A^\#$ on $L^1[0, 1]$ given in Example 3.5. Then

$$R(0, A^\#)f(x) = x^{-\alpha}\int_x^1 f(y)y^\alpha\,dy.$$

Thus

$$\begin{aligned}\|f\|_1 = \|R(0, A^\#)\,|f|\,\| &= \int_0^1 x^{-\alpha}\int_x^1 |f(y)|\,y^\alpha\,dy\,dx \\ &= \int_0^1 |f(y)|\,y^\alpha\int_0^y x^{-\alpha}\,dx\,dy \\ &= 1/(1-\alpha)\int_0^1 |f(y)|\,y\,dy.\end{aligned}$$

Hence $E_1 = L^1([0, 1], (1-\alpha)^{-1} y\, dy)$ and

$$T_1(t)f(x) = \begin{cases} x^{-\alpha}(x+t)^{\alpha} f(x+t) & \text{if } x \leq 1-t, \\ 0 & \text{otherwise,} \end{cases}$$

for all $f \in E_1$, $t \geq 0$.

(b) Let A be the generator of a strongly continuous positive semigroup $(T(t))_{t \geq 0}$. Then $S(t)f = \int_0^t T(s)f\, ds$ for all $f \in E$, $t \geq 0$.

(c) If A is the operator in Example 3.2, then

$$S(t)f(x) = \begin{cases} x^{\alpha} \displaystyle\int_0^x y^{-\alpha} f(y)\, dy & \text{if } x \leq t, \\ x^{\alpha} \displaystyle\int_{x-t}^x y^{-\alpha} f(y)\, dy & \text{if } x > t \end{cases}$$

($f \in C_0(0, 1]$, $x \in (0, 1]$, $t \geq 0$).

5. *Approach via Bernstein's theorem*

A numerical-valued function is the Laplace–Stieltjes transform of an increasing function if and only if it is completely monotonic (by Bernstein's theorem [**25**, 6.7]). Now let A be a resolvent positive operator. Then

$$(-1)^n R^{(n)}(\lambda, A) = n!\, R(\lambda, A)^{n+1} \geq 0$$

for all $n \in \mathbb{N}$, $\lambda > s(A)$. Thus the function $R(\cdot, A)$ is automatically completely monotonic whenever it is positive. And in fact we can use Bernstein's theorem to prove the representation theorem (Theorem 4.1) if additional assumptions on the space are made. On the other hand, it is not necessary to assume that A has a dense domain.

DEFINITION 5.1. We say that E is an *ideal* in E'' if for $f \in E$, $g \in E''$, $0 \leq g \leq f$ implies $g \in E$.

Note. Here we identify E with a subspace of E'' (via evaluation). Then $E''_+ \cap E = E_+$ (that is, E is an ordered subspace of E'').

LEMMA 5.2. *Suppose that E is an ideal in E''. Then the norm is order continuous in the following sense. If $(f_n)_{n \in \mathbb{N}}$ is a decreasing sequence in E_+, then $(f_n)_{n \in \mathbb{N}}$ converges strongly* (*and* $\lim_{n \to \infty} f_n = \inf_{n \in \mathbb{N}} f_n$).

Proof. Define $F \in E''_+$ by $F(\phi) = \inf_{n \in \mathbb{N}} \langle f_n, \phi \rangle$ ($\phi \in E'_+$). Then $F \in E''$ and $0 \leq F \leq f_n$ for all $n \in \mathbb{N}$. Hence $F \in E$ by assumption. It follows from Dini's theorem that $\lim_{n \to \infty} \langle f_n, \phi \rangle = \langle F, \phi \rangle$ uniformly on $U^0_+ := \{\phi \in E'_+ : \|\phi\| \leq 1\}$. Let $N(f) := \sup\{\langle f, \phi \rangle : \phi \in U^0_+\}$. Since E_+ is normal, $\|f\|_N := N(f) + N(-f)$ defines an equivalent norm on E (see [**4**]). But $\lim_{n \to \infty} \|f_n - F\|_N = 0$.

EXAMPLE 5.3. A Banach lattice E is an ideal in E'' if and only if the norm is order continuous, in the sense of Lemma 5.2 (see [**22**, II, §5]). For example, $L^p(X, \mu)$ ((X, μ) a σ-finite measure space and $1 \leq p < \infty$) and c_0 have order continuous norm, but $C[0, 1]$ has not.

DEFINITION 5.4. A function $f\colon (a, \infty) \to E$ is called *completely monotonic* if f is infinitely differentiable and $(-1)^n f^{(n)}(\lambda) \geq 0$ for all $\lambda > a$, $n = 0, 1, 2, \ldots$.

Let $\alpha\colon [0, \infty) \to E_+$ be an increasing function with values in E or $\mathscr{L}(E)$ satisfying $\alpha(0) = 0$. We show that the Laplace–Stieltjes transform of α is a completely monotonic function. Moreover, the abscissa of convergence does not depend on which of the natural topologies is considered. In order to formulate this more precisely, we say that the integral $\int_0^\infty e^{-\lambda t}\, d\alpha(t)$ converges weakly (strongly, in the weak operator topology, or in the operator norm, respectively) if $\lim_{b\to\infty} \int_0^b e^{-\lambda t}\, d\alpha(t)$ exists in the corresponding topology.

PROPOSITION 5.5. *Let $w \in \mathbb{R}$. Consider the following assertions:*

(i) *$\int_0^\infty e^{-wt}\, d\alpha(t)$ converges weakly;*

(ii) *for all $\lambda > w$ there exist $M \geq 0$ and an element $\alpha(\infty)$ of E (of $\mathscr{L}(E)$, respectively) such that $\|\alpha(t) - \alpha(\infty)\| \leq M e^{\lambda t}$ for all $t \geq 0$;*

(iii) *$\int_0^\infty e^{-\lambda t}\, d\alpha(t)$ converges in the norm whenever* $\operatorname{Re}\lambda > w$.

Then (i) *implies* (ii) *and* (ii) *implies* (iii). *If* (i) *holds, then the function $\lambda \to \int_0^\infty e^{-\lambda t}\, d\alpha(t)$ is completely monotonic on (w, ∞). Moreover,*

$$\int_0^\infty e^{-\lambda t}\, d\alpha(t) = \int_0^\infty \lambda e^{-\lambda t} \alpha(t)\, dt \quad (\operatorname{Re}\lambda > \max\{0, w\}). \tag{5.1}$$

Proof. It follows by standard arguments that (ii) implies (iii) (cf. [**15**, Theorem 6.2.1]). We show that (i) implies (ii). It suffices to consider the case when α takes values in E (the other case is treated analogously). At first we consider the case when $w \geq 0$. Let $\lambda > w$. Then for $\phi \in E'_+$, $t \geq 0$ one has

$$\begin{aligned} 0 \leq e^{-\lambda t}\langle \alpha(t), \phi\rangle &\leq e^{-\lambda t}\langle \alpha(t), \phi\rangle + \lambda \int_0^t e^{-\lambda s}\langle \alpha(s), \phi\rangle\, ds \\ &= \int_0^t e^{-\lambda s}\, d\langle \alpha(s), \phi\rangle \\ &\leq \int_0^\infty e^{-s\lambda}\, d\langle \alpha(s), \phi\rangle. \end{aligned}$$

Hence $\sup_{t\geq 0} e^{-\lambda t}\langle \alpha(t), \phi\rangle < \infty$. Since $E'_+ - E'_+ = E'$, it follows that $(e^{-\lambda t}\alpha(t))_{t\geq 0}$ is weakly bounded, and hence norm bounded. This proves (ii) in the case when $w \geq 0$. Now assume that $w < 0$. Let $\beta(t) = \int_0^t e^{-ws}\alpha(s)\, ds$. Then $\int_0^\infty d\beta(t) = \int_0^\infty e^{-ws}\, d\alpha(s)$ exists weakly. Let $0 < \delta < -w$. Then by the first case $\|\beta(t)\| \leq M e^{\delta t}$ $(t \geq 0)$ for some $M \geq 0$. Moreover, $\alpha(\infty) := \int_0^\infty d\alpha(t) = \int_0^\infty e^{ws}\, d\beta(s)$ exists (since (iii) holds for β and $\lambda > 0$). Hence

$$\begin{aligned} \|\alpha(\infty) - \alpha(t)\| = \left\| \int_t^\infty d\alpha(t) \right\| &= \left\| \int_t^\infty e^{ws}\, d\beta(s) \right\| \\ &= \left\| -e^{wt}\beta(t) + (-w)\int_t^\infty e^{ws}\beta(s)\, ds \right\| \\ &\leq (-w) M \int_t^\infty e^{ws} e^{\delta s}\, ds \\ &= Mw/(w+\delta) e^{(w+\delta)t} \quad (t \geq 0). \end{aligned}$$

Since $\delta \in (0, -w)$ was arbitrary, this shows (ii) to hold. Furthermore, if (iii) holds, then $r(\lambda) = \int_0^\infty e^{-\lambda t}\, d\alpha(t)$ is C^∞ on (w, ∞) and

$$(-1)^n r(\lambda)^{(n)} = \int_0^\infty e^{-\lambda t} t^n\, d\alpha(t) \geqslant 0 \quad (n = 0, 1, 2, \ldots).$$

Finally (5.1) is proved by integrating by parts.

THEOREM 5.6. *Assume that E is an ideal in E''. Let $f\colon (a, \infty) \to E$ be a completely monotonic function. Then there exists a uniquely determined normalized increasing function $\alpha\colon (0, \infty) \to E$ such that*

$$f(\lambda) = \int_0^\infty e^{-\lambda t}\, d\alpha(t) \quad (\lambda > a). \tag{5.2}$$

REMARK. Here we call α *normalized* if for all $\phi \in E'_+$ the function $h\colon t \to \langle \alpha(t), \phi \rangle$ is normalized (that is, $h(0) = 0$, $h(t) = \frac{1}{2}(h(t+0) + h(t-0))$ for $t > 0$).

Proof. Let $\phi \in E'_+$. Then $\lambda \to \langle f(\lambda), \phi \rangle$ is completely monotonic. So by Bernstein's theorem there exists a unique normalized increasing function $\alpha_\phi\colon (0, \infty) \to \mathbb{R}$ such that

$$\langle f(\lambda), \phi \rangle = \int_0^\infty e^{-\lambda t}\, d\alpha_\phi(t) \quad (\lambda > a). \tag{5.3}$$

From the uniqueness theorem [25, 7.2] it follows that $\alpha_\phi(t)$ is additive and positive homogeneous in ϕ for every $t \geqslant 0$. Thus for every $t \geqslant 0$ there exists a unique $\alpha_\phi(t) \in (E')'_+$ such that $\langle \phi, \alpha(t) \rangle = \alpha_\phi(t)$ for all $\phi \in E'_+$. Let $\lambda > \max\{a, 0\}$. Then for every $\phi \in E'_+$,

$$\langle f(\lambda), \phi \rangle \geqslant \int_0^t e^{-\lambda s}\, d\alpha_\phi(s) = e^{-\lambda t} \alpha_\phi(t) + \lambda \int_0^t e^{-\lambda s} \alpha_\phi(s)\, ds \geqslant e^{-\lambda t} \alpha_\phi(t).$$

Consequently, $\alpha(t) \leqslant e^{\lambda t} f(\lambda)$, and our assumption on E implies that $\alpha(t) \in E_+$. Since the integral in (5.3) converges for every $\lambda > a$ and $\phi \in E'_+$, we conclude from Proposition 5.5 that the integral $\int_0^\infty e^{-\lambda t}\, d\alpha(t)$ converges in the norm for every $\lambda > a$. Finally, (5.3) implies that

$$\langle f(\lambda), \phi \rangle = \int_0^\infty e^{-\lambda t}\, d\langle \alpha(t), \phi \rangle = \left\langle \int_0^\infty e^{-\lambda t}\, d\alpha(t), \phi \right\rangle$$

for all $\phi \in E'_+$. Hence (5.2) holds.

REMARK. There are other results related to Theorem 5.6. Schaefer [**20**] obtained a characterization of completely monotonic sequences with values in an ordered locally convex space as moments of an increasing function on $[0, 1]$ (Hausdorff moment problem). Another vector-valued version of Bernstein's theorem has been obtained by Bochner [**6**].

THEOREM 5.7. *Suppose that E is an ideal in E''. Let A be a resolvent positive operator. Then there exists a unique strongly continuous family $(S(t))_{t \geqslant 0}$ of* *operators on E such that*

$$0 = S(0) \leq S(s) \leq S(t) \quad (0 \leq s \leq t)$$

and

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t}\, dS(t) \quad (\lambda > s(A)). \tag{5.4}$$

Proof. Uniqueness follows from [**25**, 7.2]. We show the existence of the representation (5.4). Let $f \in E_+$. Then $R(\cdot, A)f$ is a completely monotonic function from $(s(A), \infty)$ into E. By Theorem 5.6 there exists a unique normalized increasing function $S(\cdot, f)$: $(0, \infty) \to E$ such that

$$R(\lambda, A)f = \int_0^\infty e^{-\lambda t}\, dS(t, f) \quad (\lambda > s(A)). \tag{5.5}$$

From the uniqueness theorem it follows that for every $t \geq 0$ the mapping $f \to S(t, f)$ from E_+ into E_+ is additive and positive homogeneous. Since $E = E_+ - E_+$, there exists a unique linear operator $S(t)$ on E such that $S(t)f = S(t, f)$ for all $f \in E_+$. Since $S(\cdot)f$ is increasing for all $f \in E_+$, $S(\cdot)$ is increasing. Let $\mu > s(A)$. Then it follows immediately from the definition that $R(\mu, A)S(\cdot)f$ and $S(\cdot)R(\mu, A)f$ are also normalized for all $f \in E_+$. Moreover, for all $\lambda > s(A)$,

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\, d(R(\mu, A)S(t)) &= R(\mu, A)R(\lambda, A)\\ &= R(\lambda, A)R(\mu, A) = \int_0^\infty e^{-\lambda t}\, d(S(t)R(\lambda, A)).\end{aligned}$$

Hence it follows from the uniqueness theorem that

$$S(t)R(\mu, A) = R(\mu, A)S(t) \quad (t \geq 0). \tag{5.6}$$

Now let $f \in D(A)$. Then for all $\lambda > \max\{s(A), 0\}$,

$$\begin{aligned}\int_0^\infty \lambda^2 e^{-\lambda t} tf\, dt = f &= R(\lambda, A)(\lambda f - Af)\\ &= \int_0^\infty \lambda^2 e^{-\lambda t} S(t)f\, dt - \int_0^\infty \lambda e^{-\lambda t} S(t)Af\, dt\\ &= \int_0^\infty \lambda^2 e^{-\lambda t} S(t)f\, dt - \int_0^\infty \lambda^2 e^{-\lambda t} \int_0^t S(s)Af\, ds\, dt.\end{aligned}$$

Thus

$$\int_0^\infty e^{-\lambda t}\left(tf - S(t)f + \int_0^t S(s)Af\, ds\right) dt = 0$$

for all $\lambda > \max\{0, s(A)\}$. Consequently, by the uniqueness theorem,

$$S(t)f = tf + \int_0^t S(s)Af\, ds \quad (t \geq 0). \tag{5.7}$$

This implies that $S(\cdot)f$ is continuous for all $f \in D(A)$. Now let $g \in E_+$, $t > 0$. Then $\lim_{s \downarrow t} S(s)g =: h_+$ and $\lim_{s \uparrow t} S(s)g =: h_-$ exist by Lemma 5.2. We have to show that $h_+ = h_-$. Let $\lambda > s(A)$. Then by (5.6),

$$\begin{aligned} R(\lambda, A)h_+ &= \lim_{s\downarrow t} R(\lambda, A)S(s)g = \lim_{s\downarrow t} S(s)R(\lambda, A)g \\ &= S(t)R(\lambda, A)g \quad (\text{since } R(\lambda, A)g \in D(A)) \\ &= \lim_{s\uparrow t} S(s)R(\lambda, A)g = R(\lambda, A)\Big(\lim_{s\uparrow t} S(s)g\Big) \\ &= R(\lambda, A)h_-. \end{aligned}$$

Since $R(\lambda, A)$ is injective, it follows that $h_+ = h_-$.

6. *The integrated semigroup*

Let A be a resolvent positive operator. We assume that there exists a strongly continuous increasing function S: $[0, \infty) \to \mathscr{L}(E)$ satisfying $S(0) = 0$ such that

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t}\, dS(t) \quad (\lambda > \lambda_0) \tag{6.1}$$

for some $\lambda_0 \geq s(A)$. By the results of the two preceding sections such a representation of $R(\lambda, A)$ exists when either A is densely defined or E is an ideal in E''. Note that $(S(t))_{t\geq 0}$ is uniquely determined by A, and we call $(S(t))_{t\geq 0}$ the *integrated semigroup generated by* A. (Of course, this terminology is motivated by the case when A is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ because then $S(t) = \int_0^t T(s)\, ds$.)

The following proposition shows that $s(A)$ is determined by the asymptotic behaviour of $S(t)$ for $t \to \infty$.

PROPOSITION 6.1. (a) *If* $\lambda \in \mathbb{C}$ *satisfies* $\operatorname{Re}\lambda > s(A)$, *then* $\lambda \in \rho(A)$ *and*

$$\begin{aligned} R(\lambda, A) &= \int_0^\infty e^{-\lambda t}\, dS(t) \\ &= \int_0^\infty \lambda e^{-\lambda t} S(t)\, dt \quad \textit{if in addition } \operatorname{Re}\lambda > 0. \end{aligned} \tag{6.2}$$

Moreover, $s(A) \in \sigma(A)$ *(the spectrum of* A*) whenever* $s(A) > \infty$.

(b) *If* $s(A) \geq 0$, *then* $s(A) = \inf\{w > 0$: *there exists* $M \geq 0$ *such that* $\|S(t)\| \leq Me^{wt}$ *for all* $t \geq 0\}$.

(c) *If* $s(A) < 0$, *then* $\lim_{t\to\infty} S(t) = R(0, A)$ *and* $s(A) = \inf\{w < 0$: *there exists* $M \geq 0$ *such that* $\|R(0, A) - S(t)\| \leq Me^{wt}$ *for all* $t \geq 0\}$.

Proof. For $f \in E_+$, $\phi \in E'_+$ denote by $s_{f\phi}$ the abscissa of convergence of the integral $\int_0^\infty e^{-\lambda t}\, d\langle S(t)f, \phi\rangle$. Then by [25, Chapter 5, Theorem 10.1], $s_{f\phi}$ is a singular point of the analytic function $\lambda \to \int_0^\infty e^{-\lambda t}\, d\langle S(t)f, \phi\rangle$ $(\operatorname{Re}\lambda > s)$. Consequently, $s := \sup\{s_{f\phi}: f \in E_+, \phi \in E'_+\} \leq s(A)$. Since $\|R(\lambda, A)\| \geq \operatorname{dist}(\lambda, \sigma(A))^{-1}$ for all $\lambda \in \sigma(A)$, it follows from the uniform boundedness principle that $\lambda \in \rho(A)$ whenever $\operatorname{Re}\lambda > s(A)$. So (6.2) holds by Proposition 5.5.

Suppose that $-\infty < s(A) \notin \sigma(A)$. Then $(s(A) - \delta, \infty) \subset \rho(A)$ for some $\delta > 0$ and so $s \leq s(A) - \delta$. Hence $\langle R(\lambda, A)f, \phi\rangle \geq 0$ for all $f \in E_+$, $\phi \in E'_+$, $\lambda > s(A) - \delta$. This implies that $R(\lambda, A) \geq 0$ for $\lambda > s(A) - \delta$, which contradicts the definition of

$s(A)$. This finishes the proof of (a). Assertions (b) and (c) follow directly from Proposition 5.5.

REMARK 6.2. (a) It follows from Proposition 6.1 that $s(A)$ is the *abscissa of convergence* of the Laplace–Stieltjes transform (6.2) (no matter whether the weak or strong operator topology or the operator norm is considered, see Proposition 5.5).

(b) If A is the generator of a positive C_0-semigroup, then Proposition 6.1 implies that $R(\lambda, A) = \int_0^\infty e^{-\lambda t}T(t)\,dt$, where the integral converges in the *operator norm*, for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > s(A)$. (Here $\int_0^b e^{-\lambda t}T(t)\,dt$ is defined strongly.) However, it may happen that $s(A) < \omega(A)$ (see [**14**]). Thus $s(A)$ is determined by the asymptotic behaviour of the *integrated* semigroup but not of the semigroup.

Now we establish the relations between A and the integrated semigroup. The operators A and $S(t)$ commute. In fact,

$$S(t)R(\lambda, A) = R(\lambda, A)S(t) \quad (\lambda > s(A),\ t > 0). \tag{6.4}$$

(This is proved as (5.6).) As a consequence,

$$f \in D(A) \text{ implies } S(t)f \in D(A) \text{ and } AS(t)f = S(t)Af \quad (t \geq 0). \tag{6.5}$$

Given $f \in E$, by a *solution* of the inhomogeneous Cauchy problem

$$\begin{aligned} v'(t) &= Av(t) + f \quad (t \geq 0), \\ v(0) &= 0, \end{aligned} \tag{6.6}$$

we understand a function $v \in C^1([0, \infty), E)$ satisfying $v(t) \in D(A)$ for all $t \geq 0$ such that (6.6) holds.

PROPOSITION 6.3. *Let $f \in D(A)$. Then $v(t) = S(t)f$ $(t \geq 0)$ defines a solution v of* (6.6). *Conversely, if $f \in E$ and v is a solution of* (6.6), *then $v(t) = S(t)f$ for all $t \geq 0$.*

Proof. The proof of Theorem 5.7 shows that (5.7) holds, and consequently $S(\cdot)f$ is a solution for every $f \in D(A)$.

Conversely, let $f \in E$ and assume that v is a solution of (6.6). Let $t > 0$. Define $w(s) = S(t-s)v(s)$ $(s \in [0, t])$. Since $v(s) \in D(A)$ by hypothesis, it follows that

$$w'(s) = -AS(t-s)v(s) - v(s) + S(t-s)v'(s) = -v(s) + S(t-s)f.$$

Consequently,

$$0 = w(t) - w(0) = \int_0^t (S(t-s)f - v(s))\,ds.$$

Hence $\int_0^t v(s)\,ds = \int_0^t S(s)f\,ds$ $(t \geq 0)$. This implies that $v = S(\cdot)f$.

REMARK. The inhomogeneous Cauchy problem (6.6) has a unique solution for all $f \in E$ if and only if A is the generator of a C_0-semigroup (see [**9**, Theorem 3.1]).

PROPOSITION 6.4. *For all $f \in E$, $t \geq 0$ one has*

$$\int_0^t S(s)f\,ds \in D(A) \quad \textit{and} \quad A\int_0^t S(s)f\,ds = S(t)f - tf.$$

Proof. Let $\lambda \in \rho(A)$ and

$$v(t)=\lambda\int_0^t S(s)R(\lambda,A)f\,ds+tR(\lambda,A)f-S(t)R(\lambda,A)f \quad (t\geq 0).$$

Then $v(t)\in D(A)$ (by (6.4)) for all $t\geq 0$. Moreover, $v(0)=0$ and it follows from Proposition 6.3 that $v'(t)=S(t)f$ $(t\geq 0)$. Consequently, $\int_0^t S(s)f\,ds=v(t)\in D(A)$ and

$$(\lambda-A)\int_0^t S(s)f\,ds=(\lambda-A)v(t)=\lambda\int_0^t S(s)f\,ds+tf-S(t)f.$$

Hence $A\int_0^t S(s)f\,ds=S(t)f-tf$.

Proposition 6.5. *Let $s,t>0$. Then*

$$S(s)S(t)=\int_0^{s+t}S(r)\,dr-\int_0^s S(r)\,dr-\int_0^t S(r)\,dr. \tag{6.7}$$

In particular, $S(s)S(t)f\in D(A)$ for all $f\in E$ and

$$AS(s)S(t)f=S(s+t)f-S(s)f-S(t)f.$$

Moreover, $S(s)S(t)=S(t)S(s)$.

Proof. Let $s>0$, $f\in E$. For $t\geq 0$ let

$$v(t)=\int_0^{s+t}S(r)f\,dr-\int_0^s S(r)f\,dr-\int_0^t S(r)f\,dr.$$

Then by Proposition 6.4, $v(t)\in D(A)$ and

$$Av(t)=S(s+t)f-S(s)f-S(t)f=v'(t)-S(s)f.$$

Since $v(0)=0$, it follows from Proposition 6.3 that $v(t)=S(t)S(s)f$ $(t\geq 0)$.

The functional equation (6.7) corresponds to the semigroup property. In fact, assume that $(T(t))_{t>0}$ is a strongly continuous family of operators such that $S(t):=\int_0^t T(r)\,dr$ exists strongly. Then $(S(t))_{t\geq 0}$ satisfies (6.7) if and only if $T(s+t)=T(s)T(t)$ for all $s,t>0$. This leads us to the following definition.

Definition 6.6. A strongly continuous family $(S(t))_{t\geq 0}$ in $\mathscr{L}(E)$ is called *integrated semigroup* if $S(0)=0$ and (6.7) holds for all $s,t>0$. Moreover, $(S(t))_{t\geq 0}$ is called *non-degenerate* if for all $0\neq f\in E$ there exists $t>0$ such that $S(t)f\neq 0$.

Note that the integrated semigroup generated by a resolvent positive operator is non-degenerate and increasing. Next we show that conversely, every non-degenerate, increasing integrated semigroup is generated by a resolvent positive operator.

The following result is due to H. Engler (oral communication) after a special case had been proved by the author.

Proposition 6.7. *Let $(S(t))_{t\geq 0}$ be an increasing integrated semigroup. Then there exist $M\geq 0$ and $w\in\mathbb{R}$ such that $\|S(t)\|\leq Me^{wt}$ for all $t\geq 0$.*

Proof. For all $t \geq 0$ we have

$$\int_0^{t+2} S(r)\,dr = \int_0^t S(r)\,dr + \int_0^2 S(r)\,dr + S(2)S(t).$$

Hence

$$S(t+1) \leq \int_t^{t+2} S(r)\,dr = \int_0^2 S(r)\,dr + S(2)S(t).$$

Consequently, there exists $q > 1$ such that $\|S(t+1)\| \leq q(1 + \|S(t)\|)$ $(t \geq 0)$. Hence, $\|S(n)\| \leq q + q^2 + \ldots + q^n \leq Nq^n$ $(n \in \mathbb{N})$ for $N := q/(q-1)$. Let $w := \log q$ and $M := kNe^w$ (where k is given by (1.2)). Then for $t \geq 0$ there exist $n \in \mathbb{N}$ and $s \in [0, 1)$ such that $t = n + s$. Thus

$$\|S(t)\| \leq k\,\|S(n+1)\| \leq kNq^{n+1} = kNe^{(1-s)w}e^{tw} \leq Me^{tw}.$$

THEOREM 6.8. *Let $(S(t))_{t \geq 0}$ be a non-degenerate, increasing integrated semigroup. Then there exists a unique resolvent positive operator A such that $R(\lambda, A) = \int_0^\infty e^{-\lambda t}\,dS(t)$ $(\lambda > s(A))$. Moreover, A is given by $D(A) = \{f \in E$: there exists a (necessarily unique) $g \in E$ such that $S(t)f = tf + \int_0^t S(r)g\,dr$ for all $t \geq 0\}$ and $Af = g$.*

Proof. By Proposition 6.7 there exist $M \geq 0$, $w \geq 0$ such that $\|S(t)\| \leq Me^{wt}$ $(t \geq 0)$. Let

$$R(\lambda) = \int_0^\infty e^{-\lambda t}\,dS(t) = \int_0^\infty \lambda e^{-\lambda t} S(t)\,dt \quad (\lambda > w).$$

Define A as above. We show that $(w, \infty) \subset \rho(A)$ and $R(\lambda, A) = R(\lambda)$ for all $\lambda > w$. Let $f \in D(A)$. Then $(d/dt)S(t)f = f + S(t)Af$ $(t \geq 0)$; hence

$$\begin{aligned} R(\lambda)(\lambda - A)f &= \int_0^\infty \lambda^2 e^{-\lambda t} S(t)f\,dt - \int_0^\infty \lambda e^{-\lambda t} S(t)Af\,dt \\ &= \int_0^\infty \lambda e^{-\lambda t} \frac{d}{dt} S(t)f\,dt - \int_0^\infty \lambda e^{-\lambda t} S(t)Af\,dt \\ &= \int_0^\infty \lambda e^{-\lambda t}(S(t)Af + f)\,dt - \int_0^\infty \lambda e^{-\lambda t} S(t)Af\,dt = f \quad (\lambda > w). \end{aligned}$$

Conversely, let $\lambda > w$, $h \in E$. We claim that $R(\lambda)h \in D(A)$ and $(\lambda - A)R(\lambda)h = h$. For $t \geq 0$ we have

$$\begin{aligned} \frac{d}{dt} S(t)R(\lambda)h &= \int_0^\infty \lambda e^{-\lambda s} \frac{d}{dt}(S(t)S(s)h)\,ds \\ &= \int_0^\infty \lambda e^{-\lambda s}(S(t+s) - S(t))h\,ds \quad \text{(by (6.7))} \\ &= \int_0^\infty \lambda e^{-\lambda s} S(t+s)h\,ds - S(t)h \\ &= \int_0^\infty \lambda e^{-\lambda s}(S(t+s) - S(s))h\,ds + R(\lambda)h - S(t)h \end{aligned}$$

$$= \int_0^\infty \lambda^2 e^{-\lambda s} \int_0^s (S(t+r) - S(r))h \, dr \, ds + R(\lambda)h - S(t)h$$

$$= \int_0^\infty \lambda^2 e^{-\lambda s} S(t)S(s)h \, ds + R(\lambda)h - S(t)h \quad \text{(by (6.7))}$$

$$= S(t)(\lambda R(\lambda)h - h) + R(\lambda)h.$$

By the definition of A this implies that $R(\lambda)h \in D(A)$ and $AR(\lambda)h = \lambda R(\lambda)h - h$.

We conclude this section by studying differentiability of the integrated semigroup in 0. Let $(S(t))_{t\geq 0}$ be a non-degenerate increasing integrated semigroup with generator A. It follows from Proposition 6.3 that $(d/dt)|_{t=0} S(t)f = f$ for all $f \in D(A)$; that is,

$$\lim_{t\downarrow 0}(1/t)S(t)f = f \quad (f \in D(A)). \tag{6.8}$$

PROPOSITION 6.9. *The following assertions are equivalent:*
(i) $\sup_{0<t\leq 1} t^{-1} \|S(t)\| < \infty$;
(ii) $\limsup_{\lambda\to\infty} \|\lambda R(\lambda, A)\| < \infty$.
Moreover, (i) *holds and* $D(A)$ *is dense if and only if*

$$\lim_{t\downarrow 0} \frac{1}{t} S(t)f = f \quad \textit{for all } f \in E. \tag{6.9}$$

Proof. Let $w \in \mathbb{R}$. Then Condition (i) as well as Condition (ii) holds for A if and only if it holds for $A - w$ (observe that the integrated semigroup $(S_w(t))_{t\geq 0}$ generated by $A - w$ is given by

$$S_w(t) = \int_0^t e^{-ws} \, dS(s) = e^{-wt}S(t) + w\int_0^t e^{-sw}S(s) \, ds).$$

Thus we can assume that $s(A) < 0$.

Assume that (i) holds. Then $M := \sup_{0<t<\infty}(1/t) \|S(t)\| < \infty$. Hence

$$\|\lambda R(\lambda, A)\| = \left\| \int_0^\infty \lambda^2 e^{-\lambda t} S(t) \, dt \right\| \leq \int_0^\infty \lambda^2 t e^{-\lambda t}(1/t) \|S(t)\| \, dt \leq M \int_0^\infty \lambda^2 t e^{-\lambda t} \, dt = M$$

for all $\lambda > 0$. Conversely, assume that $\sup_{\lambda\geq 0} \|\lambda R(\lambda, A)\| < \infty$. Let $t > 0$. Choose $\lambda = t^{-1}$. Then

$$0 \leq \frac{1}{t} S(t) = \frac{1}{t} \int_0^t dS(s) = e\lambda \int_0^t e^{-1} \, dS(s) \leq e\lambda \int_0^t e^{-\lambda s} \, dS(s) \leq e\lambda R(\lambda, A).$$

This implies that $\sup_{t\geq 0} t^{-1} \|S(t)\| < \infty$. Moreover, by Proposition 6.5, $S(t)S(s)f \in D(A)$ for all $f \in E$ and $\lim_{t\downarrow 0} t^{-1}S(t)S(s)f = S(s)f$. Hence (6.9) implies that $D(A)$ is dense. The converse follows from (6.8).

REMARK. The argument in the proof is due to G. Greiner (in the context of positive semigroups, cf. [**17**, C-III, Definition 2.8]).

EXAMPLE 6.10. (a) The operators given in Examples 3.2, 3.3, and 3.5 satisfy the equivalent conditions of Proposition 6.9. Moreover, they all are densely defined.

(b) Consider the operator $-A$, where A is defined as in Example 2.7 on $E_0 \subset L^1(\mathbb{R})$. Let $(S(t))_{t\geq 0}$ be the integrated semigroup generated by $-A$. Then

$$S(t)f(x) = \int_0^t f(x-s)\,ds = \int_{x-t}^x f(y)\,dy.$$

We show that $\underline{\lim}_{t\to 0} t^{-1}\|S(t)\| = \infty$. Let $t_n = 2^{-n}$ and $f_n = 2^n 1_{[-2^{-n},0]}$. Then $\|f_n\|_0 = \|f_n\|_1 = 1$. Moreover,

$$(S(t_n)f_n)(x) = \begin{cases} 0 & \text{for } x \leq -2^{-n}, \\ 2^n x + 1 & \text{for } -2^{-n} < x \leq 0, \\ 1 - 2^n x & \text{for } 0 < x \leq 2^{-n}, \\ 0 & \text{for } 2^{-n} < x. \end{cases}$$

Hence

$$p_{n+1}(S(t_n)f_n) = (\tfrac{3}{2})^{n+1} \int_{2^{-n-1}}^{2^{-n}} S(t_n)f_n(x)\,dx = (\tfrac{3}{2})^{n+1}(\tfrac{1}{8})2^{-n}.$$

Hence,

$$\|(1/t_n)S(t_n)\| \geq t_n^{-1} p_{n+1}(S(t_n)f_n) = \tfrac{1}{8}(\tfrac{3}{2})^{n+1} \to \infty \quad \text{for } n \to \infty.$$

7. *The homogeneous Cauchy problem*

Let A be an operator on E and $f \in D(A)$. By a solution of the homogeneous Cauchy problem

$$\text{(7.1)} \qquad \begin{aligned} u'(t) &= Au(t) \quad (t \geq 0), \\ u(0) &= f, \end{aligned}$$

we understand a function $u \in C^1([0,\infty), E)$ satisfying $u(t) \in D(A)$ for all $t \geq 0$ such that (7.1) holds.

THEOREM 7.1. *Assume that A is resolvent positive and either $D(A)$ is dense or E is an ideal in E'' (see § 5). For every $f \in D(A^2)$ there exists a unique solution of the Cauchy problem* (7.1). *Furthermore, denote by $(S(t))_{t\geq 0}$ the integrated semigroup generated by A. Then $u(t) = S(t)Af + f$ for all $t \geq 0$. Moreover, if $f \geq 0$, then $u(t) \geq 0$ for all $t \geq 0$.*

The solutions of (7.1) *depend continuously on the initial values in the following sense: let $f_n \in D(A^2)$ such that* $\lim_{n\to\infty} f_n = f$ *in the graph norm. Denote by u_n the solution of* (7.1) *for the initial value f_n. Then $u_n(t)$ converges to $u(t)$ in the norm uniformly on bounded intervals.*

Proof. Uniqueness follows from Proposition 6.3. In order to prove existence we assume that $s(A) < 0$ (otherwise one considers $A - w$ instead of A for some $w > s(A)$). Denote by $(S(t))_{t\geq 0}$ the integrated semigroup generated by A. Let $f \in D(A^2)$ and define $u(t) = S(t)Af + f$ $(t \geq 0)$. Then by Proposition 6.3, $u'(t) = AS(t)Af + Af = Au(t)$ $(t \geq 0)$. Thus u is the solution of (7.1). Now let $f_n \in D(A^2)$ such that $\lim_{n\to\infty} f_n = f$ in the graph norm. Let $u_n(t) = S(t)Af_n + f_n$. Since $(S(t))_{t\geq 0}$ is strongly continuous, it follows that $u_n(t)$ converges in the norm to $u(t)$ uniformly on bounded intervals. Finally, assume that $0 \leq f \in D(A)$. Then using

(6.5) and Proposition 6.3 one obtains

$$u(t) = S(t)Af + f = AS(t)f + f = \frac{d}{dt}(S(t)f).$$

Hence $u(t) \geq 0$, since $S(\cdot)$ is increasing.

REMARKS 7.2. (a) If $D(A)$ is dense, then $D(A^2)$ is also dense. In fact, let $\lambda \in \rho(A)$; then

$$\begin{aligned} E = D(A)^- = (R(\lambda, A)E)^- &= (R(\lambda, A)D(A)^-)^- \subset (R(\lambda, A)D(A))^- \\ &= ((R(\lambda, A))^2E)^- = D(A^2)^-. \end{aligned}$$

(b) In general, there does not exist a continuously differentiable solution of (7.1) for every initial value in $D(A)$. In fact, if $D(A)$ is dense, this would imply that A is the generator of a C_0-semigroup (see [**13**, II, Theorem 1.2]).

(c) The continuous dependence of the solutions on the initial values is no longer guaranteed if in Theorem 7.1 one replaces the graph norm by the norm. In fact, if $D(A)$ is dense, this implies that for every $t \geq 0$, the operator $T_0(t)$ given by $T_0(t)f = S(t)Af + f$ (from $D(A^2)$ into E) has a continuous extension $T(t)$ on E. It is not difficult to see that then $(T(t))_{t \geq 0}$ is a C_0-semigroup whose generator is A.

(d) Under the hypothesis of Theorem 7.1 the number $s(A)$ determines the asymptotic behaviour of the solutions of (7.1). In fact, suppose that $s(A) < 0$. Then by Proposition 6.1, $\|S(t) - R(0, A)\| \leq Me^{-\delta t}$ $(t \geq 0)$ for some $\delta > 0$. Hence

$$\|u(t)\| = \|S(t)Af + f\| = \|S(t)Af - R(0, A)Af\| \leq Me^{-\delta t}\|Af\| \quad (t \geq 0).$$

Thus the solutions tend to 0 for $t \to \infty$. (See [**17**, C–IV, §1] for this and related results in the context of positive C_0-semigroups.)

Next we prove a converse of Theorem 7.1.

THEOREM 7.3. *Let A be densely defined such that $\rho(A) \neq \varnothing$. Assume that the following conditions hold:*

(a) *for every $f \in D(A^2)$ the Cauchy problem* (7.1) *has a unique solution u, and $u(t) \geq 0$ for all $t \geq 0$ whenever $f \geq 0$;*

(b) *$D(A^2) \cap E_+$ is dense in E_+.*

Then A is resolvent positive.

REMARKS. 1. By Theorem 7.1 Condition (a) is also necessary. We do not know whether every densely defined resolvent positive operator A satisfies (b). It is certainly the case if in addition $\limsup_{\lambda \to \infty} \|\lambda R(\lambda, A)\| < \infty$.

2. If A is a resolvent operator such that $D(A)_+$ is dense in E_+, then (b) holds as well since $\lim_{\lambda \to \infty} \lambda R(\lambda, A)f = f$ for all $f \in D(A)$.

We first prove a lemma.

LEMMA 7.4. *Let A be an operator satisfying $\rho(A) \neq \varnothing$. The following assertions are equivalent:*

(i) *for every $f \in D(A^2)$ there exists a unique solution of the homogeneous problem* (7.1)*;*

(ii) *for every* $g \in D(A)$ *there exists a unique* $v \in C^1([0, \infty), E)$ *satisfying* $v(0) = 0$ *and* $v(t) \in D(A)$ *for all* $t \geq 0$ *such that* $v'(t) = Av(t) + g$ $(t \geq 0)$.

Proof. Let $\lambda \in \mathbb{C}$. If u is as in (i), then $u_1(t) = e^{-\lambda t}u(t)$ defines a function satisfying $u_1'(t) = (A - \lambda)u_1(t)$, $u_1(0) = f$. If v is a function as in (ii), then

$$v_1(t) = e^{-\lambda t}v(t) + \lambda \int_0^t e^{-\lambda s}v(s)\, ds$$

defines a function satisfying $v_1'(t) = (A - \lambda)v_1(t) + g$, $v_1(0) = g$. Thus A satisfies (i) or (ii), respectively, if and only if $A - \lambda$ satisfies (i) or (ii), respectively. Since $\rho(A) \neq \emptyset$, we can assume that $0 \in \rho(A)$ (considering $A - \lambda$ if necessary). Let $g \in D(A)$ and $u \in C^1([0, \infty), E)$ such that $u(t) \in D(A)$ $(t \geq 0)$. Let $v(t) = u(t) - A^{-1}g$. Then $u'(t) = Au(t)$ $(t \geq 0)$ and $u(0) = A^{-1}g$ if and only if $v'(t) = Av(t) + g$ $(t \geq 0)$ and $v(0) = 0$. This proves the claim.

Proof of Theorem 7.3. By Lemma 7.4 the assumptions of the theorem imply that for every $f \in D(A)$ there exists a unique solution $v(\cdot, f)$ of the inhomogeneous problem (6.6). Let E_1 denote the Banach space $D(A)$ endowed with the graph norm $\| \ \|_A$. For $f \in E_1$ let $S_1(t)f = v(t, f)$; then the uniqueness of the solutions of (6.6) implies that $S_1(t)$: $E_1 \to E_1$ is linear for every $t \geq 0$. Moreover the mapping $S_1(\cdot)f$ from $[0, \infty)$ into E_1 is continuous for every $f \in E_1$. We show that $S_1(t)$ is a bounded operator for every $t \geq 0$.

Consider the Fréchet space $C([0, \infty), E_1)$ of all continuous E_1-valued functions on $[0, \infty)$ topologized by the family $(p_n)_{n \in \mathbb{N}}$ given by $p_n(f) = \sup_{0 \leq t \leq n} \|f(t)\|_A$. Let W: $E_1 \to C([0, \infty), E_1)$ be defined by $Wf = S_1(\cdot)f$. Then W is linear. We show that W is closed. Then it follows that W and so also $S_1(t)$ is continuous $(t \geq 0)$.

Let $\lim_{n \to \infty} f_n = f$ in E_1 and suppose that $\lim_{n \to \infty} Wf_n = v$ in $C([0, \infty), E_1)$. Then $\lim_{n \to \infty} S_1(t)f_n = v(t)$ and $\lim_{n \to \infty} AS_1(t)f_n = Av(t)$ in the norm of E uniformly on compact intervals. Since

$$S_1(t)f_n = tf_n + \int_0^t AS_1(s)f_n\, ds,$$

it follows that $v(t) = tf + \int_0^t Av(s)\, ds$ for all $t \geq 0$. From the uniqueness of the solutions of (6.6) (by Lemma 7.4) it follows that $v(t) = v(t, f) = S_1(t)f$. Hence $v = Wf$.

Next we show that $(S_1(t))_{t \geq 0}$ satisfies the functional equation (6.7). Let $f \in E_1$, $s > 0$. For $t > 0$ let

$$w(t) = \int_0^{t+s} S_1(r)f\, dr - \int_0^t S_1(r)f\, dr - \int_0^s S_1(r)f\, dr.$$

Then $w(0) = 0$ and

$$w'(t) = S_1(t+s)f - S_1(t)f = Aw(t) + S_1(s)f.$$

Hence $w(t) = S_1(t)S_1(s)f$ $(t \geq 0)$ by the uniqueness of the solutions of (6.6). We have shown that $(S_1(t))_{t \geq 0}$ is an integrated semigroup on E_1. It follows from the definition that it is non-degenerate. Now we define the integrated semigroup $(S(t))_{t \geq 0}$ on E by the similarity transformation $S(t) = (\mu - A)S_1(t)R(\mu, A)$ where $\mu \in \rho(A)$ is fixed (observe that $(\mu - A)$ is an isomorphism from E_1 onto E).

We claim that $S(t)f = S_1(t)f$ for $f \in D(A)$, $t \geq 0$. Let $f \in D(A)$ and $w(t) = R(\mu, A)v(t, f)$. Then

$$w(0) = 0, \quad w'(t) = R(\mu, A)Av(t, f) + R(\mu, A)f = Aw(t) + R(\mu, A)f.$$

Hence $w(t) = v(t, R(\mu, A)f)$; that is,

$$S(t)f = (\mu - A)v(t, R(\mu, A)f) = v(t, f) \quad (t \geq 0).$$

This proves the claim. As a consequence, $R(\mu, A)S(t)f = S(t)R(\mu, A)f$ whenever $f \in E$. This in turn implies that $S(t)f \in D(A)$ and $AS(t)f = S(t)Af$ for all $f \in D(A)$.

We show that $(S(t))_{t \geq 0}$ is increasing. Let $f \in D(A^2) \cap E_+$. By hypothesis there exists a positive solution u of (7.1). Let $v(t) = \int_0^t u(s)\, ds$. Then $v(0) = 0$ and

$$v'(t) = u(t) = \int_0^t Au(s)\, ds + f = Av(t) + f.$$

Thus $v(t) = S(t)f$ $(t \geq 0)$. Consequently $S(s)f = v(s) \leq v(t) = S(t)f$ whenever $0 \leq s \leq t$, $f \in D(A^2) \cap E_+$. Since $D(A^2) \cap E_+$ is dense in E_+, this implies that $(S(t))_{t \geq 0}$ is increasing. Denote by B the generator of $(S(t))_{t \geq 0}$. It remains to show that $B = A$. If $f \in D(A)$, then

$$S(t)f = \int_0^t S(r)Af\, dr + tf \quad (t \geq 0).$$

Hence $f \in D(B)$ and $Bf = Af$ by the definition of B. Conversely, let $f \in D(B)$. Then

$$S(t)f = \int_0^t S(r)Bf\, dr + tf \quad (t \geq 0).$$

Hence

$$\begin{aligned} S(t)R(\mu, A)Bf + R(\mu, A)f &= (d/dt)(S(t)R(\mu, A)f) \\ &= AS(t)R(\mu, A)f + R(\mu, A)f \\ &= S(t)AR(\mu, A)f + R(\mu, A)f \quad (t \geq 0). \end{aligned}$$

Since $(S(t))_{t \geq 0}$ is non-degenerate, this implies that $R(\mu, A)Bf = AR(\mu, A)f = \mu R(\mu, A)f - f$. Hence $f \in D(A)$ and $Bf = Af$.

8. *The inhomogeneous Cauchy problem*

Let A be a resolvent positive operator. We assume that A has a dense domain or that E is an ideal in E'' (see §5). Let F: $[0, \infty) \to E$ be a continuous function. Given $f \in E$, we consider the inhomogeneous Cauchy problem

$$\begin{aligned} &u'(t) = Au(t) + F(t) \quad (t \geq 0), \\ &u(0) = f. \end{aligned} \tag{8.1}$$

By a solution of (8.1) we understand a function $u \in C^1([0, \infty), E)$ satisfying $u(t) \in D(A)$ for all $t \geq 0$ such that (8.1) holds.

THEOREM 8.1. *Assume that $F \in C^2([0, \infty), E)$. If $f \in D(A)$ such that $Af + F(0) \in D(A)$, then the inhomogeneous problem* (8.1) *has a unique solution.*

Proof. Denote by $(S(t))_{t\geq 0}$ the integrated semigroup generated by A. Asssume that u is a solution of (8.1). For $s \in [0, t]$ let $v(s) = S(t-s)u(s)$. Then by Proposition 6.3,

$$v'(s) = -AS(t-s)u(s) - u(s) + S(t-s)u'(s) = -u(s) + S(t-s)F(s).$$

It follows that

$$-S(t)f = v(t) - v(0) = -\int_0^t u(s)\,ds + \int_0^t S(t-s)F(s)\,ds.$$

Hence

$$\int_0^t u(s)\,ds = S(t)f + \int_0^t S(s)F(t-s)\,ds.$$

Differentiating and using Proposition 6.3 one obtains

$$u(t) = S(t)(Af + F(0)) + f + \int_0^t S(s)F'(t-s)\,ds \quad (t \geq 0). \tag{8.2}$$

This shows uniqueness. In order to prove existence define u by (8.2). Then $u \in C^1([0, \infty), E)$ and

$$u'(t) = AS(t)(Af + F(0)) + Af + F(0) + S(t)F'(0) + \int_0^t S(s)F''(t-s)\,ds.$$

Let us show that $u(t) \in D(A)$. We have $S(t)(Af + F(0)) + f \in D(A)$ by (6.5). Moreover,

$$\begin{aligned}\int_0^t S(s)F'(t-s)\,ds &= \int_0^t S(s)F'(0)\,ds + \int_0^t S(s)\int_0^{t-s} F''(r)\,dr\,ds \\ &= \int_0^t S(s)F'(0)\,ds + \int_0^t \int_0^{t-r} S(s)F''(r)\,ds\,dr.\end{aligned}$$

It follows from Proposition 6.4 that this term is in $D(A)$ and

$$\begin{aligned}A\int_0^t S(s)F'(t-s)\,ds &= S(t)F'(0) - tF'(0) + \int_0^t (S(t-r)F''(r) - (t-r)F''(r))\,dr \\ &= S(t)F'(0) - tF'(0) + \int_0^t S(r)F''(t-r)\,dr + tF'(0) - \int_0^t F'(r)\,dr \\ &= S(t)F'(0) + \int_0^t S(r)F''(t-r)\,dr - F(t) + F(0).\end{aligned}$$

Hence $u(t) \in D(A)$ and

$$\begin{aligned}Au(t) &= AS(t)(Af + F(0)) + Af + S(t)F'(0) + \int_0^t S(r)F''(t-r)\,dr - F(t) + F(0) \\ &= u'(t) - F(t).\end{aligned}$$

Hence (8.1) holds.

9. *A characterization of resolvent positive operators by Kato's inequality*

Up to this point we assumed that a resolvent positive operator was given. Now we find conditions on A which imply that A is resolvent positive.

Throughout this section we assume that E is a *Banach lattice with order continuous norm and that there exists a strictly positive linear form ϕ on E.* Then $\|f\|_\phi := \langle |f|, \phi \rangle$ defines a norm on E. We denote by (E, ϕ) the completion of E with respect to this norm. Then (E, ϕ) is an AL-space (and is isomorphic to a space of type L^1 [**22**, II.8.5]). Moreover, E is an *ideal* in (E, ϕ); that is, if $f, g \in (E, \phi)$, $|g| \leq f$, and $f \in E$, then also $g \in E$ (see [**22**, IV.9.3]). For example, let $E = L^p(X, \mu)$ $(1 \leq p < \infty)$, where (X, μ) is a σ-finite measure space. Let $\phi \in L^q(X, \mu)$ (where $(1/p) + (1/q) = 1$) and $\phi(x) > 0$ μ-a.e. Then $(E, \phi) = L^1(X, \phi\mu)$.

For $f \in E$ we denote by $\operatorname{sign} f$ the unique operator on E satisfying $(\operatorname{sign} f)f = |f|$, $|(\operatorname{sign} f)g| \leq |g|$ $(g \in E)$, $(\operatorname{sign} f)g = 0$ if $|f| \wedge |g| = 0$ (see [**17**, C-I, §8]).

THEOREM 9.1. *A densely defined operator A on E is resolvent positive if and only if the following assertions hold:*

(K) *there exist a strictly positive $\phi \in D(A')$ and $\lambda_0 \in \mathbb{R}$ such that $A'\phi \leq \lambda_0 \phi$ and*

$$\langle (\operatorname{sign} f)Af, \phi \rangle \leq \langle |f|, A'\phi \rangle \quad (f \in D(A)) \quad (\textit{Kato's inequality});$$

(R) *$(\mu_0 - A)D(A) = E$ for some $\mu_0 > \lambda_0$ (range condition). Moreover, in that case, A is closable in (E, ϕ) and its closure is the generator of a positive C_0-semigroup on (E, ϕ).*

REMARKS. (a) Condition (K) involves an abstract version of Kato's inequality. See [**17**, Chapter A-II] and [**1**] for further information and the relation to the classical inequality.

(b) In some aspects the theorem is similar to the Lumer–Phillips theorem [**11**, Theorem 2.24]. The condition that A is dissipative is replaced by Kato's inequality and the existence of a strictly positive subeigenvector of A'. In contrast to dissipativity, this condition is satisfied by A if and only if it holds for $A + w$ $(w \in \mathbb{R})$.

Proof. Assume that the conditions of the theorem hold. Considering $(A - \lambda_0)$ instead of A we can assume that $\lambda_0 = 0$. Let $N(f) = \langle f^+, \phi \rangle$ for all $f \in E$. The function N is the restriction to E of the canonical half-norm (see [**2**]) on (E, ϕ). By [**1**, Proposition 2.4] it follows from Kato's inequality that A is N-dissipative [**2**]. Since $D(A)$ is dense in E, it is also dense in (E, ϕ). Thus it follows from [**2**, Theorem 2.4] that A is closable in (E, ϕ) and the closure A_1 of A is N-dissipative. Since $E = (\mu_0 - A)D(A) \subset (\mu_0 - A_1)D(A_1)$, $\mu_0 - A_1$ also has dense range. So it follows from [**2**, Remark 4.2] (see also [**19**]) that A_1 generates an N-contraction semigroup, that is, a positive contraction semigroup on (E, ϕ). In particular, A_1 is resolvent positive and $s(A_1) \leq 0$. It follows from (R) that $\mu_0 \in \rho(A)$ and $R(\mu_0, A) = R(\mu_0, A_1)|_E$. Moreover,

$$Af = A_1 f \quad \text{and} \quad D(A) = \{f \in D(A_1) \cap E \colon A_1 f \in E\}. \tag{9.1}$$

Let $\mu \geq \mu_0$. Then for $f \in E_+$, $R(\mu, A_1)f \leq R(\mu_0, A_1)f \in E$. Since E is an ideal in (E, ϕ), it follows that $R(\mu, A_1)E \subset E$ for all $\mu \geq \mu_0$. This together with (9.1) implies that $\mu \in \rho(A)$ and $R(\mu, A) = R(\mu, A_1)|_E$ for all $\mu \geq \mu_0$. Thus A is resolvent positive and $s(A) \leq \mu_0$.

Conversely, let A be resolvent positive. We may assume that $s(A) < 0$. Then it can be seen from the proof of [**1**, Proposition 1.5] that there exists a strictly positive $\phi \in D(A')$ such that $A'\phi \leq 0$. Consider the canonical half-norm N on

(E, ϕ). Then

$$\begin{aligned} N(\lambda R(\lambda, A)f) = \langle (\lambda R(\lambda, A)f)^+, \phi\rangle &\leq \langle \lambda R(\lambda, A)f^+, \phi\rangle \\ &= \langle AR(\lambda, A)f^+ + f^+, \phi\rangle \\ &= \langle R(\lambda, A)f^+, A'\phi\rangle + \langle f^+, \phi\rangle \\ &\leq \langle f^+, \phi\rangle = N(f) \end{aligned}$$

for all $f \in E$, $\lambda > 0$. Hence A is N-dissipative. Since $(1-A)D(A) = E$ is dense in (E, ϕ), it follows from [**2**, Remark 4.2] that A is closable in (E, ϕ) and its closure generates a positive semigroup on (E, ϕ). Hence (K) follows from [**1**, Proposition 1.1] (observe that for $f, g \in E$, by the uniqueness of the signum operator the vector $(\operatorname{sign} f)g$ is the same no matter whether $(\operatorname{sign} f)$ is considered as an operator on E or (E, ϕ)).

REMARK 9.2. If A is a densely defined resolvent positive operator such that $\limsup_{\lambda\to\infty} \|\lambda R(\lambda, A)\| < \infty$, then

$$\langle (\operatorname{sign} f)Af, \psi\rangle \leq \langle |f|, A'\psi\rangle$$

holds for all $f \in D(A)$, $\psi \in D(A')_+$. This can be proved in the same way as [**1**, Proposition 1.1] if $T(t)$ is replaced by $(1-tA)^{-1}$ ($t > 0$ small), because

$$Af = \frac{d}{dt}\bigg|_{t=0} (1-tA)^{-1}f \quad \text{for all } f \in D(A).$$

References

1. W. ARENDT, 'Kato's inequality. A characterization of generators of positive semigroups', *Proc. Roy. Irish Acad.* 84 (1984) 155–174.
2. W. ARENDT, P. CHERNOFF, and T. KATO, 'A generalization of dissipativity and positive semigroups', *J. Operator Theory* 8 (1982) 167–180.
3. C. J. K. BATTY and E. B. DAVIES, 'Positive semigroups and resolvents', *J. Operator Theory* 10 (1983) 357–364.
4. C. J. K. BATTY and D. W. ROBINSON, 'Positive one-parameter semigroups on ordered Banach space', *Acta Appl. Math.* 2 (1984) 221–296.
5. Y. BENYAMINI, 'Separable G spaces are isomorphic to $C(K)$ spaces', *Israel J. Math.* 14 (1973) 287–296.
6. S. BOCHNER, 'Completely monotone functions in partially ordered spaces', *Duke Math. J.* 9 (1942) 519–526.
7. D. W. BOYD, 'The spectrum of the Cesàro operator', *Acta Sci. Math.* 29 (1968) 31–34.
8. O. BRATTELI, T. DIGERNES, and D. W. ROBINSON, 'Positive semigroups on ordered Banach spaces', *J. Operator Theory* 9 (1983) 371–400.
9. J. A. VAN CASTEREN, *Generators of strongly continuous semigroups* (Pitman, Boston, 1985).
10. P. R. CHERNOFF, unpublished note, Berkeley, 1981.
11. E. B. DAVIES, *One-parameter semigroups* (Academic Press, London, 1980).
12. W. DESCH, 'Perturbations of positive semigroups in AL-spaces', preprint, Graz, 1985.
13. J. A. GOLDSTEIN, *Semigroups of linear operators and applications* (Oxford University Press, New York, 1985).
14. G. GREINER, J. VOIGT, and M. WOLFF, 'On the spectral bound of the generator of semigroups of positive operators', *J. Operator Theory* 5 (1981) 245–256.
15. E. HILLE and R. S. PHILLIPS, *Functional analysis and semi-groups* (American Mathematical Society, Providence, Rhode Island, 1957).
16. T. KATO, 'Superconvexity of the spectral radius, and convexity of the spectral bound and type', *Math Z.* 180 (1982) 265–273.
17. R. NAGEL (ed.), *One parameter semigroups of positive operators,* Lecture Notes in Mathematics 1184 (Springer, Berlin, 1986).
18. R. D. NUSSBAUM, 'Positive operators and elliptic eigenvalue problems', *Math. Z.* 186 (1984) 247–264.

19. D. W. ROBINSON, 'Continuous semigroups on ordered Banach spaces', *J. Funct. Anal.* 51 (1983) 268–284.
20. H. H. SCHAEFER, 'A generalized moment problem', *Math. Ann.* 146 (1962) 326–330.
21. H. H. SCHAEFER, *Topological vector spaces* (Springer, Berlin, 1971).
22. H. H. SCHAEFER, *Banach lattices and positive operators* (Springer, Berlin, 1974).
23. J. VOIGT, 'On the abscissa of convergence for Laplace transform of vector valued measures', *Arch. Math.* 39 (1982) 455–462.
24. D. V. WIDDER, *The Laplace transform* (Princeton University Press, 1946).
25. D. V. WIDDER, *An introduction to transform theory* (Academic Press, New York, 1971).
26. M. WOLFF, 'A remark on the spectral bound of the generator of a semigroup of positive operators with application to stability theory', *Functional analysis and approximation,* proceedings of a conference at Oberwolfach, 1980, International Series of Numerical Mathematics 60 (Birkhäuser, Basel, 1981).

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