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Integrated solutions of Volterra integrodifferential equations and applications

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In order to treat an abstract Cauchy Problem

$$u'(t) = Au(t) \quad (t \geqq 0);\ u(0) = u_0 \tag{0.1}$$

for a linear operator A on a Banach space X, the theory of one-parameter semigroups is available and in fact highly developed. From the point of view of the theory of Laplace transforms it can be described as follows: A linear operator A is the generator of a C_0-semigroup if and only if there exists $w \geqq 0$ such that $(w,\infty) \subset \rho(A)$ (the resolvent set of A) and the function $\lambda \to (\lambda - A)^{-1}$ is a Laplace transform; i.e. there exists a strongly continuous mapping $T : [0,\infty) \to L(X)$ satisfying $\|T(t)\| \leq Me^{wt}$ $(t \geq 0)$ for some $M \geq 0$ such that

$$(\lambda - A)^{-1} = \int_0^\infty e^{-\lambda t}T(t)dt \quad (\lambda > w). \tag{0.2}$$

Of course, $(T(t))_{t\geqq 0}$ is then the C_0-semigroup generated by A and the solutions of (0.1) are given by $u(t) = T(t)x$ $(x \in D(A))$.

Starting from this observation, the theory of integrated semigroups has recently been initiated (see [1], [2], [10], [13]). Let $n \in \mathbb{N} \cup \{0\}$. A linear operator A on a Banach space X is called the generator of an n-times integrated semigroup, if there exists $w \geqq 0$ such that $(\lambda - A)$ is invertible for $\lambda > w$ and $\lambda \to (\lambda - A)^{-1}/\lambda^n$ is a Laplace transform of a strongly continuous function $S : [0,\infty) \to L(X)$ (and then $(S(t))_{t\geqq 0}$ is called the n-times integrated semigroup generated by A and the solutions of (0.1) are given by $u(t) = d^n/dt^n\, S(t)x$ (whenever this derivative exists)).

The purpose of this article is to generalize this concept to Volterra integrodifferential equations on a Banach space

$$u'(t) = \int_0^t Au(t-s)\, d\eta(s) \tag{0.3}$$

where η: $[0,\infty) \to \mathbb{R}$ is exponentially bounded and of bounded variation on each finite interval. Thus we develop a theory of n-times integrated solution families of (0.3). This concept coincides with the theory of integrated semigroups in the case when $\eta = 1_{(0,\infty)}$, and it contains the theory for the Cauchy problem of second order (i.e. the theory of integrated cosine functions) by taking $\eta(t) = t$ $(t \geq 0)$. We obtain a characterization theorem as an easy consequence of a vector-valued version of Widder's theorem proved in [2]. It coincides with the Hille-Yosida type theorem proved by Da Prato-Iannelli [5] in the case n = 0. Our approach has the advantage that also operators whose domain is not dense are admitted.

Applications to differential operators for the Cauchy problem of first and second order are given. We briefly describe the results we obtain for $A = i\Delta$. It is well known that A is the generator of a C_0-semigroup on $L^p(\mathbb{R}^N)$, $1 \leq p \leq \infty$, if and only if p = 2. However, we show that A generates, for suitable n, an n-times integrated semigroup on the spaces $L^p(\mathbb{R}^N)$ $(1 \leq p \leq \infty)$, $C_0(\mathbb{R}^N)$ and $C_b(\mathbb{R}^N)$. Since L^2-solutions of the associated Cauchy problem always exist, this can be interpreted as a regularity result for the solutions of $u'(t) = i\Delta u(t)$.

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SECTION 1. VOLTERRA INTEGRODIFFERENTIAL EQUATIONS GOVERNED BY AN n-TIMES INTEGRATED SOLUTION FAMILY

Let $A : D(A) \to X$ be a closed linear operator on a Banach space X and $\eta : [0,\infty) \to \mathbb{R}$ be a function which is of bounded variation on each compact interval $[0,T]$ $(T > 0)$.

We consider the integrodifferential problem

$$P(A,\eta) \quad \begin{cases} u'(t) = \int_0^t Au(t-s)d\eta(s) \\ \\ u(0) = x \end{cases}$$

We assume throughout that η is exponentially bounded; i.e. there exist $N \geqq 0$, $\beta \geqq 0$, such that

$$|\eta(t)| \leqq N\, e^{\beta t} \qquad (t \geqq 0). \tag{1.1}$$

Then we can define the Laplace transform $\hat{\eta}$ of η by

$$\hat{\eta}(\lambda) = \int_0^\infty e^{-\lambda t}\, d\eta(t) \qquad (\lambda > \beta).$$

We assume further that $\hat{\eta}(\lambda) \neq 0 \quad (\lambda > \beta)$.

DEFINITION 1.1. Let $n \in \mathbb{N} \cup \{0\}$. A strongly continuous family $(S(t))_{t \geqq 0}$ of bounded linear operators on X is called an n-times integrated solution family of $P(A,\eta)$ if there exist $M \geqq 0$, $w \geqq \beta$ such that

(i) $\|S(t)\| \leqq Me^{wt} \quad (t \geqq 0)$,

(ii) $S(0) = I$ if $n = 0$; $S(0) = 0$ if $n > 0$,

(iii) $(\lambda - \hat{\eta}(\lambda)A) : D(A) \to X$ is bijective $(\lambda > w)$ and

$$(\lambda - \hat{\eta}(\lambda)A)^{-1}/\lambda^n = \int_0^\infty e^{-\lambda t} S(t)dt \qquad (\lambda > w). \tag{1.2}$$

A 0-times integrated solution family is also called solution family for short.
We say $P(A,\eta)$ is governed by an n-times integrated solution family if such a family exists.

If η is the characteristic function $1_{(0,\infty)}$, then $P(A,\eta)$ is the usual abstract Cauchy problem (0.1). In that case a solution family is the same as the C_0-semigroup generated by A; an n-times integated solution family is the same as the n-times integrated semigroup generated by A (see [2]).

If $\eta(t) = t \ (t \geqq 0)$, then $P(A,\eta)$ is equivalent to the Cauchy problem of second order

$$
p_2(A) \begin{cases} u''(t) = Au(t) & (t \geq 0) \\ u(0) = x \\ u'(0) = 0, \end{cases}
$$

and the solution family of $P(A,\eta)$ is the same as the cosine family generated by A.

Throughout this section we assume now that $(S(t))_{t\geq 0}$ is an n-times integrated solution family of $P(A,\eta)$, where $n \in \mathbb{N} \cup \{0\}$. We establish existence and uniqueness results for the inhomogeneous problem

$$
P(A,\eta,f) \begin{cases} u'(t) = \int_0^t Au(t-s)d\ (s) + f(t) & (t \in [0,T]) \\ u(0) = x, \end{cases}
$$

where $f \in C([0,T],X)$ is given.

By a solution of $P(A,\eta,f)$ we understand a function $u \in C^1([0,T],X)$ satisfying

$$
\int_0^t u(t-s)d\eta(s) \in D(A),\ u'(t) = A\int_0^t u(t-s)d\eta(s) \quad (t \in [0,T]),\ u(0) = x.
$$

REMARK. Using this notion of solution and the results of this section one can easily show that, for $n = 0$, the solution families of Definition 1.1 essentially coincide with the "resolvent families" of Da Prato-Iannelli [5]). In fact, the following holds. Suppose that A is densely defined. A strongly continuous family of bounded operators $(S(t))_{t\geq 0}$ is a solution family of $P(A,\eta)$ if and only if

(a) $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$, $t \geq 0$,

(b) there exist $M \geq 0$, $w \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{wt}$ $(t \geq 0)$,

(c) for each $x \in D(A)$ the function $u(t) := S(t)x$ defines a solution of $P(A,\eta)$.

Let $x \in X$. We define $v \in C([0,T], X)$ by

$$v(t) = S(t)x + \int_0^t S(t-s)f(s)ds.$$

The following theorem shows now that solving $P(A,\eta,f)$ is reduced to verifying regularity of v.

THEOREM 1.2. Suppose $P(A,\eta)$ is governed by an n-times integrated solution family.
Then the problem $P(A,\eta,f)$ has a solution if and only if $v \in C^{n+1}([0,T],X)$. In that case the solution u is given by $u = v^{(n)}$.

For the proof we need a series of auxiliary results.

LEMMA 1.3. Let $g \in C([0,T],X)$. Then

$$u(t) = \int_0^t g(t-s)d\eta(s)$$

defines a function $u \in C([0,T],X)$. If $g \in C^1([0,T];X)$ and $g(0) = 0$, then $u \in C^1([0,T];X)$ and $u'(t) = \int_0^t g'(t-s)d\eta(s)$.

Proof. The first assertion is immediate. If $g \in C^1([0,T],X)$ such that $g(0) = 0$, then by Fubini's theorem

$$u(t) = \int_0^t \int_0^{t-s} g'(r)drd\eta(s) = \int_0^t \int_s^t g'(r-s)drd\eta(s)$$
$$= \int_0^t \int_0^r g'(r-s)d\eta(s)dr.$$ Since $v(r) = \int_0^r g'(r-s)d\eta(s)$ defines a continuous function, the claim follows. □

LEMMA 1.4. Let $x \in D(A)$. Then $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax \quad (t \geqq 0)$.

Proof. Let $\mu > w$. Then for all $y \in X$, $\lambda > w$ one has by definition of $(S(t))_{t \geqq 0}$

$$\int_0^\infty e^{-\lambda t} S(t)(\mu-\hat{\eta}(\mu)A)^{-1} y\,dt = (\lambda-\hat{\eta}(\lambda)A)^{-1}/\lambda^n\ (\mu-\hat{\eta}(\mu)A^{-1} y$$

$$= (\mu-\hat{\eta}(\mu)A)^{-1}(\lambda-\hat{\eta}(\lambda)A)^{-1} y/\lambda^n = (\mu-\hat{\eta}(\mu)A)^{-1}\int_0^\infty e^{-\lambda t} S(t) y\,dt$$

$$= \int_0^\infty e^{-\lambda t}(\mu-\hat{\eta}(\mu)A)^{-1} S(t) y\,dt.$$

By the uniqueness theorem for Laplace transforms it follows that

$$S(t)\ (\mu-\hat{\eta}(\mu)A)^{-1} = (\mu-\hat{\eta}(\mu)A)^{-1} S(t). \tag{1.3}$$

Now let $x \in D(A)$ and $y = (\mu-\hat{\eta}(\mu)A)x$. Then by (1.3), $S(t)x = (\mu-\hat{\eta}(\mu)A)^{-1}S(t)y \in D(A)$ and $(\mu-\hat{\eta}(\mu)A)S(t)x = S(t)y = S(t)(\mu-\hat{\eta}(\mu)A)x$. This implies that $AS(t)x = S(t)Ax$ (since $\hat{\eta}(\mu) \neq 0$). □

<u>LEMMA 1.5</u>. For all $x \in X$, $t \geq 0$ we have

$$\int_0^t \int_0^s S(s-r)x\,d\eta(r)\,ds \in D(A) \qquad \text{and}$$

$$A\int_0^t \int_0^s S(s-r)x\,d\eta(r)\,ds = S(t)x - (t^n/n!)x. \tag{1.4}$$

<u>Proof</u>. For $y \in D(A)$ and $\lambda > w$ we have

$$\int_0^\infty \lambda^{n+1}\, e^{-\lambda t}\, t^n/n!\ y\,dt = y$$

$$= \lambda \int_0^\infty \lambda^n\, e^{-\lambda t}\, S(t)y\,dt - \hat{\eta}(\lambda)\int_0^\infty \lambda^n e^{-\lambda t} S(t)Ay\,dt$$

$$= \int_0^\infty \lambda^{n+1}\, e^{-\lambda t}\, S(t)y\,dt - \int_0^\infty \lambda^n e^{-\lambda t}\int_0^t S(t-s)Ay\,d\eta(s)\,dt$$

$$= \int_0^\infty \lambda^{n+1}\, e^{-\lambda t}\, S(t)y\,dt - \int_0^\infty \lambda^{n+1} e^{-\lambda t}\int_0^t \int_0^s S(s-r)Ay\,d\eta(r)\,ds\,dt.$$

Cancelling by λ^{n+1} one obtains from the uniqueness theorem that

$$\int_0^t \int_0^s S(s-r)Ay\,d\eta(r)ds = S(t)y - t^n/n!\; y \quad (y \in D(A),\ t \geq 0). \tag{1.5}$$

Now let $x \in X$. Define $y = (\lambda-\hat{\eta}(\lambda)A)^{-1}x$ where $\lambda > w$ is fixed. Let $z = \int_0^t \int_0^s S(s-r)x\,d\eta(r)ds$, where $t > 0$. We have to show that $z \in D(A)$ and

$$Az = S(t)x - t^n/n!\; x. \tag{1.6}$$

Using Lemma 1.4 and (1.5) we obtain

$$z = (\lambda-\hat{\eta}(\lambda)A) \int_0^t \int_0^s S(s-r)y\,d\eta(r)ds$$

$$= \lambda \int_0^t \int_0^s S(s-r)y\,d\eta(r)ds - \hat{\eta}(\lambda)[S(t)y - t^n/n!\; y] \in D(A) \text{ and}$$

$$(\lambda-\hat{\eta}(\lambda)A)z = \lambda z - \hat{\eta}(\lambda)[S(t)x-t^n/n!\; x], \text{ which gives (1.6).} \quad \square$$

<u>COROLLARY 1.6</u>. If $y \in D(A)$, then $S(\cdot)y \in C^1([0,T],X)$ and

$$d/dt\; S(t)\; y = \begin{cases} t^{n-1}/(n-1)!\; y + \int_0^t S(t-r)Ay\,d\eta(r) & (n > 0) \\ \int_0^t S(t-r)Ay\,d\eta(r) & (n = 0). \end{cases} \tag{1.7}$$

The following lemma gives us the key to prove Theorem 1.2. We show that the function v (given by $v(t) = S(t)x + \int_0^t S(t-s)f(s)ds$) solves some integrated version of the problem $P(A,\eta,f)$.

<u>LEMMA 1.7</u>. For all $t \in [0,T]$ one has

$$\int_0^t \int_0^{t-s} v(r)dr\,d\eta(s) \in D(A) \quad \text{and}$$

$$v(t) = t^n/n!\; x + A \int_0^t \int_0^{t-s} v(r)dr\,d\eta(s) + \int_0^t (t-s)^n/n!\; f(s)ds. \tag{1.8}$$

<u>Proof</u>. Introducing the expression for $S(t)$ given by (1.4) into the following function v_0 yields:

$$v_0(t) := \int_0^t S(t-s)f(s)ds$$

$$= A\int_0^t \int_0^{t-s} \int_0^w S(w-r)f(s)d\eta(r)dwds + \int_0^t (t-s)^n/n!f(s)ds$$

$$= A\int_0^t \int_0^{t-s} \int_r^{t-s} S(w-r)f(s)dwd\eta(r)ds + \int_0^t (t-s)^n/n!f(s)ds$$

$$= A\int_0^t \int_0^{t-s} \int_0^{t-s-r} S(w)f(s)dwd\eta(r)ds + \int_0^t (t-s)^n/n!f(s)ds$$

$$= A\int_0^t \int_0^{t-r} \int_0^{t-s-r} S(w)f(s)dwdsd\eta(r) + \int_0^t (t-s)^n/n!f(s)ds$$

$$= A\int_0^t \int_0^{t-r} \int_s^{t-r} S(w-s)f(s)dwdsd\eta(r) + \int_0^t (t-s)^n/n!f(s)ds$$

$$= A\int_0^t \int_0^{t-r} \int_0^w S(w-s)f(s)dsdwd\eta(r) + \int_0^t (t-s)^n/n!f(s)ds$$

$$= A\int_0^t \int_0^{t-r} v_0(w)dwd\eta(r) + \int_0^t (t-s)^n/n!f(s)ds.$$

Moreover by (1.4),

$$\int_0^t \int_0^{t-r} S(w)xdwd\eta(r) = \int_0^t \int_r^t S(w-r)xdwd\eta(r) = \int_0^t \int_0^w S(w-r)xd\eta(r)dw \in D(A)$$

and

$$A\int_0^t \int_0^{t-r} S(w)xdwd\eta(r) = S(t)x - t^n/n!x.$$

Since $v(t) = v_0(t) + S(t)x$ we obtain (1.8) as a consequence of this and the above identity for v_0. □

<u>PROOF OF THEOREM 1.2</u>. Let u be a solution of $P(A,\eta,f)$. Then by (1.7)

$$\int_0^t d/ds[S(t-s)u(s)]ds = -\int_0^t S'(t-s)u(s)ds + \int_0^t S(t-s)u'(s)ds$$

$$= -\int_0^t (t-s)^{n-1}/(n-1)!\ u(s)ds - \int_0^t \int_0^{t-s} S(t-s-r)Au(s)d\eta(r)ds$$

$$+\int_0^t S(t-s)\int_0^s Au(s-r)d\eta(r)ds + \int_0^t S(t-s)f(s)ds$$

in the case when $n > 0$, whereas the first term has to be replaced by 0 if $n = 0$. Now by Fubini's theorem the second term equals

$$-\int_0^t\int_0^{t-r} S(t-s-r)Au(s)dsd\eta(r) = -\int_0^t\int_r^t S(t-s)Au(s-r)dsd\eta(r)$$

$$= -\int_0^t\int_0^s S(t-s)Au(s-r)d\eta(r)ds,$$ and so it cancels with the third term.

Now

$$\int_0^t d/ds[S(t-s)u(s)]ds =$$

$$= S(t-s)u(s)|_0^t = \begin{cases} -S(t)x & \text{if } n > 0 \\ u(t)-S(t)x & \text{if } n = 0. \end{cases}$$

Hence $v(t) = S(t)x + \int_0^t S(t-s)f(s)ds$

$$= \begin{cases} \int_0^t (t-s)^{n-1}/(n-1)!\ u(s)ds & \text{if } n > 0 \\ u(t) & \text{if } n = 0. \end{cases}$$

This shows one implication to hold.

In order to prove the other assume that $v \in C^{n+1}([0,T];X)$. Using Lemma 1.3 and the fact that A is closed one obtains by differentiating (1.8) that $\int_0^t v(t-s)d\eta(s) \in D(A)$ and

$$v'(t) = t^{n-1}/(n-1)!\ x + A\int_0^t v(t-s)d\eta(s) + \int_0^t (t-s)^{n-1}/(n-1)!\ f(s)ds \quad \text{if } n > 0 \tag{1.9}$$

and $v'(t) = A\int_0^t v(t-s)d\eta(s) + f(t)$ if $n = 0$.

This shows that v is a solution if $n = 0$. If $n > 0$, since A is closed, one can differentiate (1.9) n-times and obtains that

$$\int_0^t v^{(n)}(t-s)d\eta(s) \in D(A) \quad \text{and}$$

$$v^{(n+1)}(t) = A\int_0^t v^{(n)}(t-s)d\eta(s) + f(t).$$

Since $v^{(n)}(0) = x$, this shows that $v^{(n)}$ is a solution of $P(A,\eta,f)$. □

Next we give sufficient conditions on the initial value x and the inhomogeneity f for $P(A,\eta,f)$ to have a solution.

THEOREM 1.8. Let $x \in D(A^{n+1})$ and $f \in C^{n+1}([0,T],X)$ such that $f^{(k)}(0) \in D(A^{n-k})$ $(k = (0,1,\ldots,n-1)$. *Then* $P(A,\eta,f)$ *has a unique solution with initial value* x.

Proof. We let $v(t) = S(t)x + v_o(t)$ where $v_o(t) = \int_0^t S(s)f(t-s)ds$. First we show that $S(.)x \in C^{n+1}[0,\infty),X)$. In fact, we claim that $S(.)y \in C^k([0,\infty),X)$ for all $y \in D(A^k)$ and

$$d^k/dt^k\, S(t)y = \begin{cases} t^{n-k}/(n-k)!\, y + \int_0^t S(t-s)^{(k-1)}Ayd\eta(s) \\ \quad \text{if } k = 1,2,\ldots,n \qquad \text{and} \\ \int_0^t S(t-s)^{(k-1)}Ayd\eta(s) \quad \text{if } k > n. \end{cases} \tag{1.10}$$

For k = 1 this follows from Corollary 1.6. Assume now that the assertion is true for k. Let $y \in D(A^{k+1})$. Then $S(.)^{(k-1)}Ay \in C^1([0,\infty),X)$ by the inductive assumption. So by Lemma 1.3,

$$d/dt \int_0^t S(t-s)^{(k-1)}Ayd\eta(s) = \int_0^t S(t-s)^{(k)}Ayd\eta(s).$$

This shows (1.10) to hold for k+1.
Next we show that $v_o \in C^{n+1}([o,T],X)$. In fact, we show by induction that $v_o \in C^k([0,T],X)$ and

$$v_o^{(k)}(t) = S(t)f^{(k-1)}(0) + d/dtS(t)f^{(k-2)}(0) + \dots + d^{k-1}/dt^{k-1}\, S(t)f(0) + \int_0^t S(s)f^{(k)}(t-s)ds \qquad (1.11)$$

$$(k = 1,\dots,n+1).$$

For k = 1 this is proved by differentiating v_o by the usual rule. Assume that (1.11) holds for $k \leq n+1$. Since by (1.10) $S(.)y \in C^k$ if $y \in D(A^k)$, it follows that $v_o^{(k)} \in C^1$ and we obtain (1.11) for k+1 by differentiating.

We have shown that $v \in C^{n+1}([0,T];X)$ which implies the claim by Theorem 1.2. □

COROLLARY 1.9. *If $x \in D(A^{n+1})$, then there exists a unique function $u \in C^1[0,\infty),X)$ such that $u(t) \in D(A)$ $(t \geq 0)$ and*

$$u'(t) = \int_0^t An(t-s)d\eta(s) \qquad (t \geq 0)$$

$$u(0) = x.$$

Proof. One deduces from (1.10) that $u(t) = v^{(n)}(t) \in D(A)$ $(t \geq 0)$. □

SECTION 2. REAL CHARACTERIZATION

Throughout this section we assume that A is a closed linear operator and $\eta:[0,\infty) \to \mathbb{R}$ a function satisfying the assumptions of Section 1.

THEOREM 2.1. *Let $n \in \mathbb{N} \cup \{0\}$, $w \geq \beta$, $M \geq 0$. The problem $P(A,\eta)$ is governed by an (n+1)-times integrated solution family $(S(t))_{t\geq 0}$ satisfying*

$$\lim\sup_{h\downarrow 0} \|S(t+h) - S(t)\| \leq Me^{wt} \qquad (t \geq 0) \qquad (2.1)$$

if and only if $(\lambda-\hat{\eta}(\lambda)A)$ is bijective whenever $\lambda > w$ and

$$\|(\lambda-w)^{k+1}\, d^k/d\lambda^k((\lambda-\hat{\eta}(\lambda)A)^{-1}/\lambda^n)/k!\| \leq M \;(\lambda > w,\ k = 0,1,\dots). \qquad (2.2)$$

Note that (2.2) makes sense since the function $\lambda \to (\lambda - \hat{\eta}(\lambda)A)^{-1}$ is C^∞ on (w,∞).

Theorem 2.1 is an immediate consequence of the vector-valued version of Widder's theorem given in [2, Theorem 4.1].

THEOREM 2.2. Suppose that A is densely defined. Let $n \in \mathbb{N} \cup \{0\}$. Then the following assertions are equivalent.

(i) Problem $P(A,\eta)$ is governed by an n-times integrated solution family.

(ii) There exist $w \geqq \beta$, $M \geqq 0$ such that $(\lambda-\hat{\eta}(\lambda)A)$ is bijective whenever $\lambda > w$ and

$$\|(\lambda-w)^{k+1}\, d^k/dt^k((\lambda-\hat{\eta}(\lambda)A)^{-1}/\lambda^n)k!\| \leqq M \quad (\lambda > w,\ k = 0,1,2,\ldots). \tag{2.3}$$

Proof. (i) $\to$ (ii). Let $(T(t))_{t\geqq 0}$ be an n-times integrated solution family satisfying $\|T(t)\| \leqq Me^{wt}$ $(t \geqq 0)$, where $w \geqq \beta$. Then for $k = 0,1,2,\ldots,\lambda > w$,

$$\|(\lambda-w)^{k+1}\, d^k/d\lambda^k[(\lambda-\hat{\eta}(\lambda)A)^{-1}/\lambda^n]/k!\|$$
$$= \|(\lambda-w)^{k+1}\, d^k/d\lambda^k \int_0^\infty e^{-\lambda t}\, T(t)dt\|/k!$$
$$= \|(\lambda-w)^{k+1} \int_0^\infty (-t)^k e^{-\lambda t}\, T(t)dt\|/k!$$
$$\leqq (\lambda-w)^{k+1}\, M \int_0^\infty t^k e^{-\lambda t}\, e^{wt} dt/k! = M.$$

(ii) $\to$ (i). By Theorem 2.1 it follows from condition (ii) that $P(A,\eta)$ is governed by an $(n+1)$-times integrated solution family $((S(t))_{t\geqq 0}$ satisfying (2.1). It follows from Corollary 1.6 that the function $S(.)y$ is continuously differentiable for all $y \in D(A)$. Because of (2.1) it follows from this that $S(.)x$ is continuously differentiable for all $x \in D(A)^- = X$. Define $T:[0,\infty) \to L(X)$ by $T(t)x = d/dt\, S(t)x$ $(t \geqq 0,\ x \in X)$. Then T is strongly continuous and satisfies $\|T(t)\| \leqq Me^{wt}$ $(t \geqq 0)$. Hence for $\lambda > w$ one obtains by integrating by parts,

$$(\lambda-\hat{\eta}(\lambda)A)^{-1}x/\lambda^n = \lambda \int_0^\infty e^{-\lambda t} S(t)xdt = \int_0^\infty e^{-\lambda t} T(t)xdt \quad (x \in X).$$

Thus $(T(t))_{t\geqq 0}$ is an n-times integrated solution family of $P(A,\eta)$. □

REMARK. In the case when $n = 0$, Theorem 2.2 is known. In fact, if η is the characteristic function of $(0,\infty)$, then it is just the Hille-Yosida theorem; if $\eta(t) = t$, then it is the characterization theorem for generators of cosine functions due to Sova; for arbitrary η finally, it is essentially the generation theorem of Da Prato-Ianelli [5].

SECTION 3. A COMPLEX CONDITION

The real condition discussed in the preceding section has the advantage of involving merely real values λ. Moreover, at least when the operator is densely defined, it gives the best value of the integration order n.

However, the disadvantage of the condition is that all derivatives of $(\lambda-\hat{\eta}(\lambda)A)^{-1}$ have to be estimated, which might be impossible in concrete cases.

Now we discuss the condition that $(\lambda-\hat{\eta}(\lambda)A)^{-1}$ is polynomially bounded in a right half plane. In general, this condition is much easier to verify. However, usually not the best integration order n is obtained (cf. Remark 3.3).

We start with the underlying result for vector-valued Laplace transforms.

PROPOSITION 3.1. Let Y be a Banach space, $M \geqq 0$, $w \geqq 0$, $\alpha \in \mathbb{R}$ and let $r : [\mathrm{Re}\,\lambda > w] \to Y$ be an analytic function satisfying

$$\|r(\lambda)\| \leqq M|\lambda|^{\alpha} \qquad (\mathrm{Re}\,\lambda > w). \tag{3.1}$$

Then there exists a locally Hölder continuous function $f : [0,\infty) \to Y$ such that

(a) for all $w' > w$ there exists $M' \geqq 0$ such that

$$\|f(t)\| \leqq M'e^{w't} \qquad (t \geqq 0),$$

(b) $f(0) = 0$,

(c) $r(\lambda) = \lambda^{[\alpha]+2} \int_0^\infty e^{-\lambda t} f(t)\,dt \quad (\mathrm{Re}\,\lambda > w)$.

Here a function $f : [0,\infty) \to Y$ is said to be locally Hölder continuous if there exist $\varepsilon > 0$ and $c : [0,\infty) \to [0,\infty)$ such that

$$\|f(s) - f(t)\| \leq c(T)\, |s - t|^{\varepsilon} \quad (s,t \in [0,T])$$

for all $T > 0$.

REMARK. We are indebted to Jan Prüss for the argument in the following proof which leads to the Hölder continuity of f.

Proof. Let $w' > w$ and $\gamma := w' + i\mathbb{R}$. Define

$$f(t) = 1/2\pi i \int_{\gamma} e^{\mu t} (1/\mu)^{[\alpha]+2}\, r(\mu)\, d\mu \qquad (t \geq 0).$$

The integral converges since

$$\|e^{\mu t} (1/\mu)^{[\alpha]+2} r(\mu)\| \leq M e^{w't} (1/|\mu|)^{2+[\alpha]-\alpha} \quad (\mu \in \gamma) \tag{3.2}$$

This implies (a).

We show that f is Hölder continuous. Let $\varepsilon > 0$ such that $[\alpha]-\alpha+2-\varepsilon > 1$. For $\lambda \in \gamma$ we have

$$|e^{\lambda s} - e^{\lambda t}| \leq |\lambda|\, e^{w'T}\, |t-s| \qquad \text{and}$$

$$|e^{\lambda s} - e^{\lambda t}| \leq 2e^{w'T} \quad \text{for all } s,\ t \in [0,T].$$

Consequently, $|e^{\lambda s} - e^{\lambda t}| \leq |e^{\lambda s} - e^{\lambda t}|^{\varepsilon} \cdot |e^{\lambda s} - e^{\lambda t}|^{1-\varepsilon}$

$$\leq |\lambda|^{\varepsilon} e^{\varepsilon w'T} |t - s|^{\varepsilon}\ 2^{1-\varepsilon}\ e^{w'(1-\varepsilon)T}$$

$$\leq 2\ e^{w'T}\ |\lambda|^{\varepsilon} |t-s|^{\varepsilon} \quad (s,\ t \in [0,T],\ \lambda \in \gamma).$$

Now $f(t) - f(s) = 1/2\pi i \int_{\gamma} (e^{\mu t} - e^{\mu s})\, (1/\mu)^{[\alpha]+2}\, r(\mu)\, d\mu.$

The integrand is bounded by

$$|e^{\mu t} - e^{\mu s}| \cdot (1/|\mu|)^{[\alpha]+2}\, \|r(\mu)\|$$

$$\leq\ 2e^{w'T} |\mu|^{\varepsilon} \cdot |t-s|^{\varepsilon} \cdot (1/|\mu|)^{[\alpha]+2} \cdot M \cdot |\mu|^{\alpha}$$

$$\leq 2e^{w'T}.M.(1/|\mu|)^{[\alpha]-\alpha+2-\varepsilon}.|t-s|^{\varepsilon}$$

which is integrable over γ.

Hence $\|f(s) - f(t)\| \leq ce^{w'T}.|t-s|^{\varepsilon}$ $(s,t \in [0,T])$. Next we show (c) to hold. Let $\lambda > w'$. Then

$$\lambda^{[\alpha]+2} \int_0^{\infty} e^{-\lambda t} f(t)dt$$

$$= \lambda^{[\alpha]+2} \int_0^{\infty} e^{-\lambda t}\ 1/2\pi i \int_{\gamma} e^{\mu t}\ (1/\mu)^{[\alpha]+2}\ r(\mu)d\mu dt$$

$$= \lambda^{[\alpha]+2}\ 1/2\pi i \int_{\gamma} \int_0^{\infty} e^{-(\lambda-\mu)t}dt\ (1/\mu)^{[\alpha]+2}\ r(\mu)d\mu$$

$$= \lambda^{[\alpha]+2}\ 1/2\pi i \int_{\gamma} 1/(\lambda-\mu)\cdot(1/\mu)^{[\alpha]+2}\ r(\mu)d\mu$$

$= r(\lambda)$ by residue calculus.

Finally, to see (b), we observe that $\mu \to (1/\mu)^{[\alpha]+2}\cdot r(\mu)$ is analytic in $[\mathrm{Re} > w]$ and $\|(1/\mu)^{[\alpha]+2}\cdot r(\mu)\| \leq M(1/|\mu|)^{2+[\alpha]-\alpha}$. So

$$f(0) = \lim_{R\to\infty}\ 1/2\pi i \int_{\gamma(R)} (1/\mu)^{[\alpha]+2}\cdot r(\mu)d\mu = 0,$$

where $\gamma(R)$ denotes the closed path consisting of the line $\{is : -R \leq s \leq R\}$ and the semicircle $\{Re^{i\xi} : -\pi/2 \leq \xi \leq \pi/2\}$. □

<u>THEOREM 3.2.</u> Assume that there exist $w > \beta$, $M \geq 0$, and $\alpha \geq -1$ such that

(a) $(\lambda-\hat{\eta}(\lambda)A)$ is bijective for $\mathrm{Re}\ \lambda > w$ and

(b) $\|(\lambda-\hat{\eta}(\lambda)A)^{-1}\| \leq M|\lambda|^{\alpha}$ $(\mathrm{Re}\ \lambda > w)$.

Then $P(A,\eta)$ is governed by an $[\alpha]+2$ - times integrated solution family $(S(t))_{t\geq 0}$.

Moreover, S is Hölder continuous in the following sense: There exist $\varepsilon > 0$ and $c : [0,\infty) \to [0,\infty)$ such that

$$\|S(t) - S(s)\| \leq c(T)\cdot|s-t|^{\varepsilon}\ (s,t \in [0,T]) \text{ for all } T > 0.$$

This is an immediate consequence of Proposition 3.1.

REMARK 3.3. a) In the case when $\alpha = -1$ a much stronger result can be obtained (see Da Prato-Iannelli [5]). In particular, if A is densely defined and η is the characteristic function of $(0,\infty)$, then A is the generator of a holomorphic semigroup.

b) Assume that $(S(t))_{t\geq 0}$ is the n-times integrated solution family of $P(A,\eta)$. Then

$$\|(\lambda-\hat{\eta}(\lambda)A)^{-1}\| = \|\int_0^\infty \lambda^n e^{-\lambda t} S(t)\, dt\| \leq M|\lambda|^n/(\mathrm{Re}\lambda - w) \text{ for } \mathrm{Re}\,\lambda > w$$

seems to be the best possible estimate on a right half plane, and so Theorem 3.2 would merely yield an n+2 -times integrated solution family. So Theorem 3.2 gives a convenient criterion; however, in general one looses an integration order of 2.

SECTION 4. RELATION TO DISTRIBUTION SEMIGROUPS

In 1960 J.L. Lions developed the theory of distribution semigroups as an approach to abstract Cauchy problems $u' = Au$, $u(0) = x$ which are well posed in a weaker sense than the classical one that leads to C_0-semigroups ([11]). As a particular class he defined the "distribution semigroups of exponential growth" and for their generators he gave a very simple characterization.

THEOREM 4.1 ([11, Theorem 6.1]). Let $A:D(A) \to X$ be a densely defined linear operator. The following are equivalent.

(i) A is the generator of a distribution semigroup of exponential growth.

(ii) For some positive constants and c and some $n \in \mathbb{N}$ $\{\lambda:\mathrm{Re}\,\lambda > \omega\} \subset \rho(A)$ and $\|(\lambda-A)^{-1}\| \leq c|\lambda|^n$ for $\mathrm{Re}\,\lambda > \omega$.

By Theorem 3.2 we immediately obtain the following.

THEOREM 4.2. Let $A:D(A) \to X$ be linear and densely defined. The following are equivalent.

(i) A generates a distribution semigroup of exponential growth.

(ii) There exists some $n \in \mathbb{N}$ such that A generates an n-times integrated semigroup.

A more detailed analysis is possible by the theory of "smooth distribution semigroups" developed by M. Balabane and H. Emami-Rad ([3], [4]).

DEFINITION 4.3 ([3], [4]). Let $\mathcal{D}(\mathbb{R}_+)$ denote the test functions with support in $(0,\infty)$ and let $k \in \mathbb{N}$. By $p_k(\phi) := \Sigma_{i=1}^{k} \| t^i \phi^{(i)} \|_{L^1}$ a norm is defined on $\mathcal{D}(\mathbb{R}_+)$. Let T_k denote the completion of $\mathcal{D}(\mathbb{R}_+)$ with respect to p_k.

A smooth distribution semigroup of order k is a continuous linear mapping G from T_k to $L(X)$ such that

(i) $G(\phi * \Psi) = G(\phi)G(\Psi)$ for $\phi, \Psi \in T_k$.

(ii) There exists a dense subspace D in X such that for $x \in D$ there is a continuous function $g_x : \mathbb{R}_+ \to X$ with $g_x(0) = x$ and $G(\phi)x = \int_0^\infty \phi(t) g_x(t)dt$ for all $\phi \in T_k$.

In the following we study how smoothness of order k translates into terms of integrated semigroups.

THEOREM 4.4. Let A be linear and densely defined. The following are equivalent.

(i) A generates a smooth distribution semigroup of order k.

(ii) A generates a k-times integrated semigroup $(S(t))_{t \geq 0}$ with the additional property $\|S(t)\| \leq ct^k$ for $t \geq 0$ (and a suitable constant c).

To prove this we need a few lemmas.

LEMMA 4.5. Let $T \in \mathcal{D}(\mathbb{R}_+)'$ and suppose $T(\phi^{(k)}) = 0$ for all $\phi \in \mathcal{D}(\mathbb{R}_+)$ and some fixed $k \in \mathbb{N}$.

Then T is a polynomial $\Sigma_{i=0}^{k-1} a_i t^i$ for suitable scalar constants $a_0, \ldots, a_{k-1}$.

For the proof we refer to books about the theory of distributions.

LEMMA 4.6. (i) The norms $p_k(\phi) := \Sigma_{i=1}^{k} \| t^i \phi^{(i)} \|_{L^1}$ and $\phi \to \| t^k \phi^{(k)} \|_{L^1}$ are equivalent norms on $\mathcal{D}(\mathbb{R}_+)$.

(ii) The set $\{\phi^{(k)} : \phi \in \mathcal{D}(\mathbb{R}_+)\}$ is dense in $L^1(\mathbb{R}_+, t^k dt)$.

PROOF. (i): For all $\phi \in \mathcal{D}(\mathbb{R}_+)$ the function $t \to |\phi(t)|$ is absolutely continuous and $|\phi(t)|' = (\operatorname{sign} \phi(t))\phi(t)'$.

This implies for $i < k$ and $\phi \in \mathcal{D}(\mathbb{R}_+)$

$$\int_0^\infty t^i |\phi^{(i)}(t)| dt = - \frac{1}{i+1} \int_0^\infty t^{i+1} (\operatorname{sign} \phi^{(i)}(t)) \phi^{(i+1)}(t) dt$$

$$\leq \frac{1}{i+1} \int_0^\infty t^{i+1} |\phi^{(i+1)}(t)| dt.$$

Therefore all terms in the sum $p_k(\phi)$ can be estimated by $c \, \|t^k \phi^{(k)}\|_{L^1}$ for some constant c.

(ii): Let $h \in L^\infty(\mathbb{R}_+)$ be such that $\int_0^\infty h(t)\phi^{(k)}(t) t^k \, dt = 0$ for $\phi \in \mathcal{D}(\mathbb{R}_+)$. By Lemma 4.5 we have $h(t)t^k = \Sigma_{i=0}^{k-1} \, a_i t^i$ a.e. in $(0,\infty)$ or $h(t) = \Sigma_{i=0}^{k-1} a_i \, t^{i-k}$. The right hand side is bounded only if all a_i's vanish. Therefore $h = 0$.

PROOF OF THEOREM 4.4. (i) $\Rightarrow$ (ii): Let G be the smooth distribution semi-group of order k generated by A. For all x in some dense subspace D we have a continuous function $g_x : \mathbb{R}_+ \to X$ such that

$$G(\phi)x = \int_0^\infty \phi(t) g_x(t) \, dt \text{ for } \phi \in \mathcal{D}(\mathbb{R}_+).$$

Integration by parts yields

$$G(\phi)x = \int_0^\infty (-1)^k \phi^{(k)}(t) H_x(t) t^k \, dt$$

where $H_x(t) := (1/t^k) \int_0^t \frac{1}{(k-1)!} (t-s)^{k-1} g_x(s) ds$. Since the function g_x is unique $x \to H_x(t)$ defines a linear operator from D to X. We show that this operator is bounded. By Lemma 4.6(i) the boundedness of G means $\|G(\phi)\| \leq c \| t^k \phi^{(k)}\|_{L^1}$ for all $\phi \in \mathcal{D}(\mathbb{R}_+)$. Fix some $x \in D$ and some $x' \in X'$. Then

$$\left|\int_0^\infty (-1)^k \phi^{(k)}(t) \, \langle H_x(t), x' \rangle \, t^k \, dt\right| = |\langle G(\phi)x, x' \rangle|$$

$$\leq c \, \|t^k \phi^{(k)}\|_{L^1} \|x\| \|x'\| = c \, \|\phi^{(k)}\|_{L^1(t^k dt)} \|x\| \, \|x'\| \, .$$

Since by Lemma 4.6(ii) the functions $\phi^{(k)}$ are dense in $L^1(t^k dt)$ the functional $\phi^{(k)} \to \int_0 (-1)^k \phi^{(k)}(t) \langle H_x(t),x'\rangle\, t^k\, dt$ extends to the whole of $L^1(t^k dt)$. Therefore $\langle H_x(\cdot),x'\rangle \in L^\infty$ and $\|\langle H_x(.),x'\rangle\|_\infty \leq c\, \|x\|\, \|x'\|$. Hence $\|H_x(t)\| \leq c\, \|x\|$ for $t > 0$. This shows that $x \to H_x(t)$ is bounded. Let $S(t)$ denote the bounded extension of $x \to t^k H_x(t) = \int_0^t \frac{1}{(k-1)!}\, t^k\, g_x(s)ds$ to the whole space X. Then $\|S(t)\| \leq ct^k$. Since $t \to S(t)$ is strongly continuous on D we obtain a strongly continuous family $(S(t))_{t\geq 0}$ of bounded operators $S(t)$ that fulfills $G(\phi) = \int_0^\infty (-1)^k \phi^{(k)}(t)\, S(t)dt$ for $\phi \in T_k$. Plugging in $\phi(t) = e^{-\lambda t}$ (which is a function in T_k for $\lambda > 0$) we get $(\lambda - A)^{-1} = G(e^{-\lambda t}) = \lambda^k \int_0^\infty e^{-\lambda t}\, S(t)dt$ for $\lambda > 0$. By definition this implies that $(S(t))_{t\geq 0}$ is a k-times integrated semigroup with generator A.

(ii) => (i): Define $G(\phi) := \int_0^\infty (-1)^k \phi^{(k)}(t)\, S(t)dt$. It is not difficult to prove that G is a smooth distribution semigroup of order k (For the functional equation use [2, Th. 3.1]). To see that Def. 4.3 (ii) holds note that for $x \in D(A^k)$ the function $t \to S(t)x$ is k-times continuously differentiable with $S^{(k)}(0)x = x$ by Theorem 1.2 and Theorem 1.8 for $f = 0$ and $\eta = 1_{(0,\infty)}$. Integration by parts yields

$$G(\phi) = \int_0^\infty \phi(t)\, S^{(k)}(t)x\, dt \text{ for } \phi \in \mathcal{D}(R_+).$$

It remains to show that A coincides with the generator B of G. By definition ([3, Definition 4]) the set $\{G(\phi)x : x \in X,\ \phi \in \mathcal{D}(R_+)\}$ is a core of B and $BG(\phi)x = G(-\phi')x$. By Lemma 1.5, $\int_0^t S(s)x\, ds \in D(A)$ and

$$A \int_0^t S(s)x\, ds = S(t)x - (t^k/k!)x \text{ for all } x \in X. \text{ Hence}$$

$$G(\phi)x = \int_0^\infty (-1)^{k+1} \phi^{(k+1)}(t) \int_0^t S(s)x\, dsdt \in D(A) \text{ and } AG(\phi)x = G(-\phi')x$$

for all $x \in X$, $\phi \in \mathcal{D}(R_+)$. This implies $A = B$. □

SECTION 5. INTEGRATED COSINE FUNCTIONS

We assume $\eta(t) = t$ $(t \geq 0)$ throughout this section so that $P(A,\eta)$ is equivalent to

$$P_2(A)\begin{cases} u''(t) = Au(t), \ t \geq 0 \\ u(0) = x \\ u'(0) = 0 \end{cases}$$

in the sense that u is a classical solution of $P(A,\eta)$ if and only if $u \in C^2([0,\infty),X) \cap C([0,\infty), D(A))$ and u satisfies $P_2(A)$.

Now a solution family of $P_2(A)$ is the same as a cosine function (in fact, one can show, A is the genrator of a cosine function (in the classical sense) if and only if there exist $\omega \geq 0$ and $M \geq 0$ and a strongly continuous function $C:[0,\infty) \to L(X)$ satisfying $\|C(t)\| \leq Me^{\omega t}$ $(t \geq 0)$ such that $(\lambda^2-A)^{-1}$ exists for $\lambda > \omega$ and $\lambda(\lambda^2-A)^{-1} = \int_0^\infty e^{-\lambda t}C(t)dt$. Of course, in that case $(C(t))_{t\geq 0}$ is the cosine function generated by A.). So in the case considered here we use the term n-times integrated cosine function ($n = 0, 1,2,\ldots$) for n-times integrated solution families of $P_2(A)$. We also use the term sine function for once-integrated cosine function. For convenience we recollect the precise meaning from Definition 1.1: An operator A is called the generator of an n-times integrated cosine function if $(\omega,\infty) \subset \rho(A)$ (the resolvent set of A) for some $\omega \geq 0$ and the function $\lambda \to (\lambda^2-A)^{-1}/\lambda^{n-1}$ $(\lambda > \omega)$ is a Laplace transform (i.e. there exists a strongly continuous family $(S(t))_{t\geq 0}$ of bounded operators in $L(X)$ such that $\|S(t)\| \leq Me^{\omega t}$ $(t \geq 0)$ for some $M \geq 0$ and $(\lambda^2-A)^{-1}/\lambda^{n-1} = \int_0^\infty e^{-\lambda t}S(t)dt$). Of course, in that case, $(S(t))_{t\geq 0}$ is the n-times integrated cosine function generated by A.

THEOREM 5.1. Let $n \in \mathbb{N} \cup \{0\}$. Suppose that A and -A are generators of an n-times integrated semigroup. Then A^2 is the generator of an n-times integrated cosine function.

Proof. There exists $\omega \geq 0$ such that $(\omega,\infty) \subset \rho(A) \cap \rho(-A)$ and $(\lambda-A)^{-1}/\lambda^n$ and $(\lambda+A)^{-1}/\lambda^n$ are Laplace transforms (in the sense made precise above). So $(\lambda^2-A^2)^{-1} = (\lambda-A)^{-1}(\lambda+A)^{-1}$ exists for $\lambda > \omega$ and by the resolvent equation we have

$$(\lambda^2-A^2)^{-1} = -(\lambda-A)^{-1}(-\lambda-A)^{-1} = \frac{1}{2\lambda}\left((\lambda+A)^{-1} + (\lambda-A)^{-1}\right).$$

So $(\lambda^2-A^2)^{-1}/\lambda^{n-1}$ is a Laplace transform. □

In the following theorem we consider semigroups other than C_o-semigroups. We say, that a linear operator A is the generator of a semigroup, if there exists a strongly continuous function $T:(0,\infty) \to L(X)$ and $\omega \geq 0$ such that $(\omega,\infty) \subset \rho(A)$, $\int_0^\infty e^{-\lambda t} \|T(t)x\| \, dt < \infty$ for every $x \in X$, $\lambda > \omega$ and $(\lambda-A)^{-1}x = \int_0^\infty e^{-\lambda t} T(t)x \, dt$ $(x \in X, \lambda > \omega)$. Clearly such T is unique and one shows as in [2, Theorem 1.3] that T has the semigroup property $T(s + t) = T(s)T(t)$ $(s,t > 0)$. Thus we call $(T(t))_{t>0}$ the semigroup generated by A. (Of course, then $\int_0^t T(s)ds = S(t)$ defines an integrated semigroup $(S(t))_{t\geq 0}$ whose generator is A).

<u>THEOREM 5.2</u>. Let A be the generator of a sine function $(S(t))_{t\geq 0}$. Then A generates a semigroup $(T(t))_{t>0}$ which is given by

$$T(t)x = 1/(2\sqrt{\pi}\, t^{3/2}) \int_0^\infty s e^{-s^2/4t} S(s)x \, ds \quad (t > 0, x \in X) \tag{5.1}$$

where the integral exists in the sense of Bochner).

In particular $(T(t))_{t>0}$ has an analytic extension to $[\mathrm{Re}\, t > 0]$. If in addition D(A) is dense and $\overline{\lim}_{t\to 0} \|S(t)\| /t < \infty$ then A generates a C_o-semigroup.

<u>Proof</u>. Let $\Psi(s,t) := 1/(2\sqrt{\pi}\, t^{3/2})\, s e^{-s^2/4t}$ $(s \geq 0,\ t > 0)$. Then $\int_0^\infty e^{-\lambda t} \Psi(s,t)dt = e^{-s\sqrt{\lambda}}$ $(s > 0, \lambda > 0)$ (see [7, p. 50-51]). There exists $\omega \geq 0$, $M \geq 0$ such that $\|S(t)\| \leq Me^{\omega t}$ $(t \geq 0)$. The argument given in [5, p. 139] shows that $\int_0^\infty s e^{-s^2/4t} \|S(s)x\| \, ds < \infty$ for all $t > 0$, $x \in X$. Moreover by Fubini's theorem

$$\int_0^\infty e^{-\lambda t} \|T(t)x\| \, dt \leq \int_0^\infty e^{-\lambda t} \int_0^\infty 1/(2\sqrt{\pi}\, t^{3/2}) s e^{-s^2/4t} \|S(s)x\| \, ds \, dt$$

$$= \int_0^\infty e^{-\lambda t} \|S(s)x\| \; 1/(2\sqrt{\pi}\, t^{3/2})\, s e^{-s^2/4t} \, dt ds$$

$$= \int_0^\infty \|S(s)x\| \; e^{-s\sqrt{\lambda}} \, ds < \infty \quad (\lambda > \omega^2).$$

Hence $\int_0^\infty e^{-\lambda t}\, T(t)x\, dt$ exists for $\lambda > \omega^2$ and (by Fubini's theorem again)

$$\int_0^\infty e^{-\lambda t} T(t)x\, dt = \int_0^\infty e^{-s\sqrt{\lambda}} S(s)x\, ds = (\lambda - A)^{-1}x.$$

To see the additional assertion note that $t \to T(t)$ is strongly continuous at $t = 0$ on $D(A)$ (Observe that for large enough λ we have
$T(t)(\lambda-A)^{-1} = \int_0^\infty e^{-\lambda s}\, T(s+t)ds = e^{\lambda t}\int_t^\infty e^{-\lambda s}\, T(s)ds$).
Further $\overline{\lim}_{t\to 0} \|S(t)\| /t < \infty$ implies $\|S(t)\| \leq cte^{\omega t}$ $(t \geq 0)$. Since

$$\begin{aligned}
\|T(t)\| &= \|1/(2\sqrt{\pi}\, t^{3/2}) \int_0^\infty se^{-s^2/4t}\, S(s)x\, ds\| \\
&= \|1/(2\sqrt{\pi}\, t^{3/2}) \int_0^\infty 4tve^{-v^2}\, S(2v\sqrt{t})\, dv\| \\
&\leq 4/(\sqrt{\pi}t) \int_0^\infty c\sqrt{t}ve^{-v^2}\, e^{2\omega v\sqrt{t}}\, dv \\
&\leq 4/(\sqrt{\pi}) \int_0 cve^{-v^2}\, e^{2\omega v}\, dv \text{ for } 0 < t \leq 1
\end{aligned}$$

the strong continuity of $(T(t))_{t>0}$ (at $t = 0$) extends to the whole space. □

The preceding theorem shows that generators of sine functions are also of interest in the context of Cauchy problems of first order. In addition to the regularity property described above they have nice properties with respect to perturbations (which is not true for all generators of integrated semigroups, see the example in [9 , Section 5]). The following perturbation theorem will illustrate this. First we prepare some notation. We denote by $(S(t))_{t\geq 0}$ the sine function generated by the (unperturbed) operator A and by $(S^B(t))_{t\geq 0}$ the sine function generated by $A + B$.

Since we shall also make a statement about the spaces where the sine functions are differentiable we use the abbreviations

$$C^n := \{x \in X : t \to S(t)x \text{ is a } C^n\text{-function}\} \quad \text{and}$$

$$C_B^n := \{x \in X : t \to S^B(t)x \text{ is a } C^n\text{-function}\}.$$

THEOREM 5.3. Let A generate a sine function and $B \in L(\overline{D(A)}, X)$. Then

$A + B$ generates a sine function, too. Furthermore $C_B^n = C^n$ for $n = 1,2$.

Proof. Let $(S(t))_{t\geq 0}$ be the sine function generated by A. For some positive constants M and ω we have $\|S(t)\| \leq Me^{\omega t}$ for $t \geq 0$. Consider now the integral equation on $\mathbb{R}_+$

$$S^B(t) = S(t) + \int_0^t S^B(s)BS(t-s)\,ds. \tag{5.2}$$

We show first that the term $BS(t-s)$ in (5.2) is well defined. By Lemma 1.5 we have $\int_0^t \int_0^s S(r)x\,drds \in D(A)$ for all $x \in X$ and $t \geq 0$. Differentiating this twice we obtain $S(t)x \in \overline{D(A)}$.

Multiplying (5.2) by $e^{-(\omega+\alpha)t}$ we obtain the equivalent integral equation

$$U(t) = e^{-(\omega+\alpha)t}S(t) + \int_0^t U(s)Be^{-(\omega+\alpha)(t-s)}S(t-s)ds. \tag{5.3}$$

Consider now the Banach space

$$E := \{V:\mathbb{R}_+ \to L(X) : V \text{ is strongly continuous and bounded}$$

(equipped with the norm $|V| := \sup_{t\geq 0} \|V(t)\|$) and fix some $\alpha > M\,\|B\|$. It is not difficult to verify that $(JV)(t) := \int_0^t V(s)Be^{-(\omega+\alpha)(t-s)}S(t-s)ds$ defines a bounded linear operator J on E with $\|J\|_{L(E)} \leq \frac{M\,\|B\|}{\alpha} < 1$. Write now the integral equation (5.3) in the form

$$(I - J)U = (e^{-(\omega+\alpha)t}S(t))_{t\geq 0}. \tag{5.4}$$

Obviously the operator function on the right hand side is in E and the operator $I - J$ is invertible. This shows that we have a unique strongly continuous and bounded solution $(U(t))_{t\geq 0}$ of (5.3). Therefore $(S^B(t))_{t\geq 0} := (e^{(\omega+\alpha)t}U(t))_{t\geq 0}$ solves (5.2) and fulfills $\|S^B(t)\| \leq M'e^{(\omega+\alpha)t}$ for $t \geq 0$.

Let $\lambda > \omega + \alpha$. With the abbreviations $Q(\lambda) := \int_0^\infty e^{-\lambda t}S^B(t)dt$ and $R(\lambda^2) := (\lambda^2 - A)^{-1}$ we have

$$\begin{aligned} Q(\lambda) - R(\lambda^2) &= \int_0^\infty e^{-\lambda t} (S^B(t) - S(t))\, dt \\ &= \int_0^\infty e^{-\lambda t} \int_0^t S^B(s)BS(t-s)\, dsdt \\ &= \int_0^\infty \int_0^t e^{-\lambda s}S^B(s)B\, e^{-\lambda(t-s)}S(t-s)dsdt \\ &= \int_0^\infty \int_s^\infty e^{-\lambda s}S^B(s)B\, e^{-\lambda(t-s)}S(t-s)dtds \\ &= \int_0^\infty e^{-\lambda s}S^B(s)B \int_0^\infty e^{-\lambda u}S(u)duds \\ &= Q(\lambda)BR(\lambda^2). \end{aligned}$$

This shows $Q(\lambda)(I - BR(\lambda^2)) = R(\lambda^2)$. The operator $I - BR(\lambda^2)$ is invertible since $\|BR(\lambda^2)\| \leq \|B\| \, \|\int_0^\infty e^{-\lambda t}S_t dt\| \leq \frac{\|B\|M}{\lambda - \omega} < 1$. A simple calculation shows now that $Q(\lambda) = R(\lambda^2)(I - BR(\lambda^2))^{-1}$ is the inverse of λ^2-A-B. By definition this implies that $(S^B(t))_{t\geq 0}$ is a sine function with generator $A + B$.

It remains to show the additional assertion. The inclusions $C^n \subset C^n_B$ for $n = 1,2$ can be verified by simply differentiating (5.2). Since $-B \in L(\overline{D(A + B)},X)$ we also have $S(t) = S^B(t) - \int_0^t S(s)BS^B(t-s)ds$ which shows that the converse inclusions hold, too. □

SECTION 6. APPLICATIONS

I. Differential Operators on $\mathbb{R}^n$ as Generators of Integrated Semigroups

We consider elliptic differential operators with constant coefficients $A = \Sigma_{|\alpha|\leq m}\, a_\alpha D^\alpha$ on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, $C_b(\mathbb{R}^n)$, $UC_b(\mathbb{R}^n)$ (= the space of bounded, uniformly continuous functions) and $C_o(\mathbb{R}^n)$.

Such an operator A is called elliptic if the polynomial $\Sigma_{|\alpha|\leq m}\, a_\alpha\, i^{|\alpha|}x^\alpha$ is ellitpic (i.e. $\Sigma_{|\alpha|=m}\, a_\alpha x_1^{\alpha_1} \dots x_n^{\alpha_n} = 0 \Rightarrow x_1 = \dots = x_n = 0$).

It is known that A (with some suitable domain) generates an analytic C_o-semigroup on $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, if m is even and $\mathrm{Re}\, \Sigma_{|\alpha|=m}\, a_\alpha\, i^{|\alpha|}x^\alpha < 0$ for $x \neq 0$ (see e.g. [8, Chapter 2, Theorem 12.7]).

On the other hand if $\mathrm{Re}\,\Sigma_{|\alpha|\leq m}\, a_\alpha\, i^{|\alpha|}x^\alpha = 0$ for $x \in \mathbb{R}^n$ and $m \geq 2$ then it follows from [10, Theorem 1.14] that A does not generate a semigroup on $L^p(\mathbb{R}^n)$ unless $p = 2$.

However M. Balabane and H. Emami-Rad showed that in this case A and $-A$ (with suitable domain) generate smooth distribution semigroups of order k where $k > \frac{n}{2}$ on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. By Theorem 4.4 this is equivalent to saying that A and $-A$ generate a k-times integrated semigroup $(S(t))_{t\geq 0}$ on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, with $\|S(t)\| \leq ct^k$ for $t \geq 0$ and with $k > \frac{n}{2}$.

The following theorem deals with a more general situation. We will give a direct proof which remains completely within the framework of integrated semigroups. Our proof (like the proof of the Balabane-Emami-Rad result) is based on a lemma on Fourier multipliers due to Mikhlin. We are indebted to Mikhael Balabane who demonstrated to us how one can - by sophisticated enough techniques - eliminate problems, so to say, with a twinkle of the eye.

THEOREM 6.1. Let A be elliptic on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and suppose

$$\sup_{x\in\mathbb{R}^n} \mathrm{Re}\,\Sigma_{|\alpha|\leq m}\, a_\alpha i^{|\alpha|}x^\alpha < \infty\,. \tag{6.1}$$

Then $A := \overline{A_{|S}}$ (where S denotes the space of rapidly decreasing functions) generates a k-times integrated semigroup for $k > \frac{n}{2}$.

We state the lemma mentioned above in the following form.

LEMMA 6.2 ([16, p. 96]). Let $k > \frac{n}{2}$ and define

$$E := \{u:\mathbb{R}^n \to \mathbb{C} : u \text{ is a } C^k\text{-function and } |x|^{|\alpha|}|D^\alpha u(x)| \leq c$$
$$\text{for } x \in \mathbb{R}^n, |\alpha| \leq k \text{ and some constant } c \geq 0\}.$$

If E is equipped with the norm $\|u\|_E := \Sigma_{|\alpha|\leq k} \|\, |x|^{|\alpha|}D^\alpha u\|_\infty$ then $u \to T_u$, where $T_u f := F^{-1}(uFf)$, defines a bounded operator from E to M_p for $1 < p < \infty$.

Here F means the Fourier transform and $M_p \subset L(L^p(\mathbb{R}^n))$ is the set of bounded L^p-multipliers. For the theory of multipliers we refer to [10] and [16].

PROOF OF THEOREM 6.1. Let us abbreviate $p(x) := \Sigma_{|\alpha|\leq m} a_\alpha i^{|\alpha|} x^\alpha$. We show that the functions

$$u_t := \frac{1}{(k-1)!} \int_0^t (t - s)^{k-1} e^{sp}\, ds \qquad (6.2)$$

give, by $S(t) := T_{u_t}$, an exponentially bounded k-times integrated semigroup $(S(t))_{t\geq 0}$.

Let w be a positive constant such that $\mathrm{Re}\, p(x) \leq w$ for $x \in \mathbb{R}^n$. We show now that $u_t \in E$ for $t \geq 0$ and $\|u_t\|_E \leq Me^{\omega t}$ for some M and $\omega \geq 0$. Ellipticity of p implies that for some constants c, $L \geq 0$

$$|p(x)| \geq c|x|^m \quad \text{for } |x| > L. \qquad (6.3)$$

We divide now $\mathbb{R}^n$ in two parts, the closed ball with radius L centered at the origin and the rest.

For $|x| \leq L$ we obviously have

$$|x|^{|\alpha|}|D^\alpha u_t(x)| \leq M'e^{\omega' t} \text{ for } |\alpha| \leq k \text{ and } t \geq 0 \text{ (}M'\text{ and } \omega' \text{ suitable).}$$

Let, therefore $|x| > L$. Carrying out the integration in (6.2) one obtains

$$u_t = e^{tp}/p^k - \Sigma_{j=1}^k \frac{1}{(k-j)!} t^{k-j}/p^j. \qquad (6.4)$$

By induction one shows the following.

$D^\alpha(e^{tp}/p^k) = e^{tp} q/p^r$ where $r \in \mathbb{N}$ and q is some polynomial of $\mathrm{ord}(q) \leq rm - (k - |\alpha|)m - |\alpha|$. (6.5)

For any $j \in \mathbb{N}$ we have $D^\alpha(1/p^j) = q/p^r$ where $r \in \mathbb{N}$ and q is some polynomial of $\mathrm{ord}(q) \leq rm - jm - |\alpha|$. (6.6)

Using this in combination with (6.3) and $|q(x)| \leq c|x|^{\mathrm{ord}(q)}$ one easily shows that $|x|^{|\alpha|}|D^\alpha u_t(x)| \leq M''e^{\omega'' t}$ for $|\alpha| \leq k$, $|x| \geq L$ and $t \geq 0$ (M'', ω'' suitable).

Lemma 6.2 implies now that the multipliers T_u are bounded and fulfill

$$\|T_{u_t}\| \leq Ce^{\omega t} \text{ for } t \geqq 0 \text{ (and constants } C, \ \omega \geqq w). \tag{6.7}$$

Next we prove that for $\lambda > \omega$ the operator $\lambda - A$ is invertible with inverse $T_{r(\lambda)}$, where $r(\lambda) := \frac{1}{\lambda - p}$. The fact that $T_{r(\lambda)}$ is a bounded multiplier can be seen by means of (6.6), just as before. On S we further have $(\lambda-A)T_{r(\lambda)} = T_{r(\lambda)}(\lambda-A) = I$. Since $T_{r(\lambda)}$ is bounded and A is the closure of $A|_S$ this implies $T_{r(\lambda)} = (\lambda-A)^{-1}$.

To see strong continuity of the family $(S(t))_{t\geqq 0}$ observe first that we have

$$u_t = p \int_0^t u_s \, ds + t^k/k!. \tag{6.8}$$

We fix now some $\mu > \omega$ and obtain from (6.8)

$$u_t r(\mu) = (\mu r(\mu) - 1) \int_0^t u_s \, ds + (t^k/k!) r(\mu). \tag{6.9}$$

Recalling that $\|u_t\|_E \leq Me^{\omega t}$ for $t \geqq 0$ we see that the right hand side of (6.9) is continuous with respect to $\| \ \|_E$. By Lemma 6.2 we obtain that $T_{u_t r(\mu)} = T_{u_t} T_{r(\mu)} = T_{u_t}(\mu - A)^{-1}$ is continuous with respect to the M_p-norm. This implies in particular that $t \to T_{u_t} = S(t)$ is strongly continuous on the dense subspace D(A). By the exponential boundedness of the family $(S(t))_{t\geqq 0}$ we obtain strong continuity on the whole space.

The only thing that remains to show is $(\lambda-A)^{-1} = \lambda^k \int_0^\infty e^{-\lambda t} S(t) dt$ for $\lambda > \omega$. Obviously $r(\lambda) = \lambda^k \int_0^\infty e^{-\lambda t} u_t \, dt$ for $\lambda > \omega$. Fixing again some $\mu > \omega$ we obtain $r(\lambda) r(\mu) = \lambda^k \int_0^\infty e^{-\lambda t} u_t r(\mu) dt$. As we just showed $t \to u_t r(\mu)$ is $\| \ \|_E$-continuous. Therefore by Lemma 6.2 we have

$$T_{r(\lambda)} T_{r(\mu)} = \int_0^\infty e^{-\lambda t} T_{u_t r(\mu)} \, dt = \int_0^\infty e^{-\lambda t} T_{u_t} \, dt \, T_{r(\mu)}$$

or $(\lambda-A)^{-1} = \int_0^\infty e^{-\lambda t} S_t dt$ on D(A). Boundedness of both sides implies the same on the whole space. □

The situation looks different on L^1 and L^∞, since Lemma 6.2 does not apply here anymore. But the following technique can be used. One shows that (for some large enough $k \in \mathbb{N}$) the functions

$$\mathcal{F}^{-1}u_t = \mathcal{F}^{-1} \frac{1}{(k-1)!} \int_0^t (t-s)^{k-1} e^{sp}\, ds$$

are in L^1 and that the family of convolution operators $(S(t))_{t \geq 0}$, given by $S(t)f := (\frac{1}{2\pi})^{n/2}\mathcal{F}^{-1}u_t * f$ is a k-times integrated semigroup with generator A. One way to show that the functions $\mathcal{F}^{-1}u_t$ are in L^1 is to show that the functions m_t are in $H^{k,2}$ (= the k-th Sobolev space in L^2) for $k > \frac{n}{2}$ and then to use the fact that $\mathcal{F}^{-1}$ maps $H^{k,2}$ boundedly into L^1. This procedure demands a severe restriction on the polynomial p, namely $\mathrm{ord}(p) > \frac{n}{2}$. The result is then

<u>THEOREM 6.3 ([10, Part II, Theorem 1.9]).</u> Let $p(x) := \Sigma_{|\alpha| \leq m}\, a_\alpha\, i^{|\alpha|} x^\alpha$ be an elliptic polynomial of $\mathrm{ord}(p) > \frac{n}{2}$ such that $\sup_{x \in \mathbb{R}^n} \mathrm{Re}\, p(x) < \infty$. Then for $k > \frac{n}{2} + 1$ the operator $A = \Sigma_{|\alpha| \leq m}\, a_\alpha\, D^\alpha$ (with distributional domain) generates a norm continuous k-times integrated semigroup on the spaces $L^1(\mathbb{R}^n)$, $L^\infty(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$, $UC_b(\mathbb{R}^n)$ and $C_0(\mathbb{R}^n)$.

A better result is possible for $n = 1$.

<u>THEOREM 6.4 ([10, Part II, Theorem 1.1]).</u> Suppose that $\sup_{x \in \mathbb{R}} \mathrm{Re}\, \Sigma_{j=0}^m a_j (ix)^j < \infty$. Then $A = \Sigma_{j=0}^m a_j \left(\frac{d}{dx}\right)^j$ (with distributional domain) generates a once integrated semigroup on the spaces $L^1(\mathbb{R})$, $L^\infty(\mathbb{R})$, $C_b(\mathbb{R})$, $UC_b(\mathbb{R})$ and $C_0(\mathbb{R})$.

II. <u>Differential Operators on $\mathbb{R}^n$ as Generators of Integrated Cosine Functions</u>

As in 6.I we consider elliptic differential operators $A = \Sigma_{|\alpha| \leq m}\, a_\alpha D^\alpha$ on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, $C_b(\mathbb{R}^n)$, $UC_b(\mathbb{R}^n)$ and $C_0(\mathbb{R}^n)$. We study the question under what conditions they generate k-times integrated cosine functions. Since the techniques are essentially the same as in I. We do not go so much into detail, but only show the various points where the proofs have to be changed to yield results for this setting here. Again we treat the reflexive L^p-spaces separately and we obtain

THEOREM 6.5. Let A be elliptic on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and suppose that $\Sigma_{|\alpha|\leq m} a_\alpha i^{|\alpha|} x^\alpha$ is real valued and bounded above.
Then $A := \overline{A|_S}$ generates a k-times integrated cosine function for $k > \frac{n}{2}$.

Proof. Define $p(x) := \Sigma_{|\alpha|\leq m} a_\alpha i^{|\alpha|} x^\alpha$. The proof consists in showing that the functions

$$u_t := \frac{1}{(k-1)!} \int_0^t (t-s)^{k-1} \cosh(s\sqrt{p})ds \tag{6.10}$$

give, by $S(t) := T_{u_t}$, an exponentially bounded family of L^p-multipliers $(S(t))_{t\geq 0}$. The rest is more or less a copy of the proof of Theorem 6.1. As in the proof of Theorem 6.1 we divide $\mathbb{R}^n$ in the ball $\{x \in \mathbb{R}^n : |x| \leq L\}$ and the rest. Since p is a polynomial with $p(x) \leq c^2$ $(x \in \mathbb{R}^n)$ for some $c \geq 0$, we have

$$\mathrm{Re}\, \sqrt{p(x)} \leq c \text{ for } x \in \mathbb{R}^n. \tag{6.11}$$

By (6.11) we have $|x|^{|\alpha|}|D^\alpha u_t(x)| \leq Me^{\omega t}$ for $|x| \leq L$, $|\alpha| \leq k$ and $t \geq 0$.
For $|x| > L$ we write (6.10) in the form

$$u_t = \begin{cases} \sinh(t\sqrt{p})/\sqrt{p} & \text{for } k = 1, \\ \sinh(t\sqrt{p})/p^{k/2} - \Sigma_{i=1}^{(k-1)/2} \frac{1}{(k-2i)!} t^{k-2i}/p^i, & k \geq 3, \text{ odd} \\ \cosh(t\sqrt{p})/p^{k/2} - \Sigma_{i=1}^{(k-1)/2} \frac{1}{(k-2i)!} t^{k-2i}/p^i, & k \text{ even.} \end{cases} \tag{6.12}$$

We ignore now the difference between the functions cosh and sinh since we are only interested in estimating the right hand side of (6.12) and we denote both functions by the same symbol U.
By induction one shows then

$D^\alpha(U(t\sqrt{p})/p^{k/2}) = U(t\sqrt{p})q/p^{r/2}$ where $r \in \mathbb{N}$ and q is some polynomial of order $\mathrm{ord}(q) \leq \frac{rm}{2} - (k - |\alpha|)\frac{m}{2} - |\alpha|$.

Together with (6.11) and (6.6) this implies

$|x|^{|\alpha|}|D^\alpha u_t(x)| \leq M'e^{\omega' t}$ for $|\alpha| \leq k$, $|x| \geq L$ and $t \geq 0$ (M' and ω' suitable). Lemma 6.2 implies now that the multipliers T_{u_t} are bounded and fulfill $\|T_{u_t}\| \leq Ce^{\omega'' t}$ for $t \geq 0$ (and suitable constants C and ω''). If one replaces now (6.8) by $u_t = p\int_0^t (t-s)u_s\, ds + t^k/k!$ the rest consists in copying the proof of Theorem 6.1 with minor changes. □

We turn now to the cases L^1 and L^∞ applying the same technique as in 6.I. Again the restriction $\mathrm{ord}(p) > \frac{n}{2}$ comes up and we obtain

<u>THEOREM 6.6</u> Let $p(x) := \Sigma_{|\alpha|\leq m}\, a_\alpha\, i^{|\alpha|} x^\alpha$ be a real valued polynomial of order $> \frac{n}{2}$ which is elliptic and bounded above.

Then for $k > \frac{n}{2} + 2$ the operator $A = \Sigma_{|\alpha|\leq m}\, a_\alpha\, D^\alpha$ (with distributional domain) generates a norm continuous k-times integrated cosine function on the spaces $L^1(\mathbb{R}^n)$, $L^\infty(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$, $UC_b(\mathbb{R}^n)$ and $C_o(\mathbb{R}^n)$.

For the case $N = 1$ we can again improve our result.

<u>THEOREM 6.7.</u> Suppose that the polynomial $\Sigma_{j=0}^m\, a_j (ix)^j$ is real valued and bounded above.
Then $A = \Sigma_{j=0}^m\, a_j (\frac{d}{dx})^j$ (with distributional domain) generates a norm continuous sine function on the spaces $L^1(\mathbb{R})$, $L^\infty(\mathbb{R})$, $C_b(\mathbb{R})$, $UC_b(\mathbb{R})$ and $C_o(\mathbb{R})$.

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