Perturbation of positive semigroups

By

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Introduction. The purpose of this note is to study perturbations of generators of positive semigroups by positive operators.

Let $E$ be a complex Banach lattice and $A$ be a linear operator on $E$ with domain $D(A)$. We say that $A$ is resolvent positive if there exists $w \in \mathbb{R}$ such that $(\lambda - A): D(A) \to E$ is bijective and $(\lambda - A)^{-1}$ is a positive operator on $E$ for all $\lambda > w$. Note that the generator of a positive semigroup is resolvent positive.

Assume that $A$ generates a positive semigroup (by which we always mean a $C_0$-semigroup) and $B: D(A) \to E$ is linear and positive such that $A + B$ (with domain $D(A + B) = D(A)$) is resolvent positive.

Then it was shown by Desch [8] that $A + B$ generates a positive semigroup whenever $E$ is a space $L^1$. A simple proof is given by Voigt [20].

If $E$ is an $L^p$-space, $1 < p < \infty$, then the assertion is false, in general (see [4]). However, we show in Section 1 that in the case where the semigroup generated by $A$ is holomorphic, also $A + B$ generates a holomorphic semigroup without any restriction on the space.

Furthermore, we prove in Section 2 that $A + B$ generates a semigroup whenever $B$ is a positive rank-one perturbation of $A$. This is remarkable in view of a recent result of Desch-Schappacher [9]. If the semigroup generated by $A$ is not holomorphic, there always exists a (necessarily non positive) rank-one perturbation $B$ such that $A + B$ is not a generator.

In Section 3 we give a criterion for perturbation by multiplication operators which, in view of the Sobolev embedding theorems, is particularly useful for elliptic operators. As an illustrating example we consider Schrödinger operators.

In Section 4 the results are applied to systems of evolution equations, which obtained special attention recently (see [14]).

Concerning terminology and basic results we follow [17] and [13].

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1. Perturbation of holomorphic semigroups. Let $E$ be a complex Banach lattice (see [17]) and let $A$ be the generator of a positive semigroup $T = (T(t))_{t \geq 0}$ on $E$. We consider a positive linear operator $B: D(A) \to E$ (i.e. $Bf \geq 0$ for all $f \in D(A)_+ := D(A) \cap E_+$).

In this section we prove the following perturbation result.
Theorem 1.1. Assume that the semigroup generated by $A$ is holomorphic.
If $A + B$ is resolvent positive, then $A + C$ generates a holomorphic semigroup whenever $C: D(A) \to E$ is a linear mapping satisfying

\[ |Cu| \leq Bu \quad (u \in D(A^+)). \]

Remark. In particular, $A + B$ generates a positive holomorphic semigroup.
Using the classical perturbation result one would obtain this under the hypothesis that $\lim_{\lambda \to \infty} \|BR(\lambda, A)\| = 0$, whereas the assumption that $A + B$ be resolvent positive can be rephrased by saying $\lim_{\lambda \to \infty} r(BR(\lambda, A)) < 1$ (where $r(S)$ denotes the spectral radius of a bounded operator), see [20, Theorem 1.1].

Remark. We emphasize that it does not suffice to assume the existence of the resolvent of $Z + Y$ on a half-plane (without any norm or order condition). In order to see this it suffices to take the generator $Z$ of a holomorphic semigroup on a Banach space $G$ with empty spectrum and $Y = -2Z$. Then $Y: (D(Z), \| \cdot \|_Z) \to G$ is continuous, $Z + Y$ has empty spectrum, but $Z + Y$ does not generate a semigroup. As a concrete example one may take the generator of the Riemann-Liouville semigroup on $L^p(0, 1)$ ($1 \leq p < \infty$), see [10, Sec 23.16].

In the situation of Theorem 1.1 the semigroup generated by $A + C$ is dominated by the one generated by $A + B$. More generally, the following holds.

Theorem 1.2. Assume that $A + B$ generates a positive semigroup $(U(t))_{t \geq 0}$. If $C: D(A) \to E$ is linear and satisfies (1.1), then $A + C$ generates a semigroup $(V(t))_{t \geq 0}$ satisfying

\[ |V(t)f| \leq U(t)|f| \quad (f \in E) \quad \text{for all } t \geq 0. \]

We will use the following notation. For an operator $Z$ we denote by

\[ s(Z) = \sup \{ \Re(\lambda) : \lambda \in \sigma(Z) \} \]

the spectral bound of $Z$. If $Z$ is resolvent positive, then

\[ 0 \leq R(\mu, Z) \leq R(\lambda, Z) \quad \text{for } s(Z) < \lambda \leq \mu \]

(see for example [13, B-II-Lemma 1.9]).
If $Z$ generates a semigroup $(S(t))_{t \geq 0}$ we denote by $w(Z)$ the growth bound (or type) of $S$; i.e.

\[ w(Z) = \inf \{ w : \sup_{t \geq 0} \| \exp(-w t)S(t) \| < \infty \} \]

\[ = \inf \{ w > s(Z) : \sup_{\lambda > w, n \in \mathbb{N}} \| (\lambda - w)^n R(\lambda, Z)^n \| < \infty \}. \]

Now we establish some auxiliary results.
**Lemma 1.3.** If $Q, R: E \to E$ are linear such that $|Qf| \leq Rf$ for all $f \in E_+$, then $|Qf| \leq R|f|$ for all $f \in E$.

For a proof we refer to [17, p. 234].

**Lemma 1.4.** Assume that $A + B$ is resolvent positive and let $C : D(A) \to E$ be linear and satisfy (1.1). Let $\lambda \in \mathbb{C}$ such that $\lambda_0 := \text{Re} (\lambda) > \max \{w(A), s(A + B)\}$. Then

$$(1.4) \quad |[CR(\lambda, A)]^n u| \leq |BR(\lambda_0, A)]^n u| \quad (u \in E), \quad \text{for all } n \in \mathbb{N}.$$  

Moreover, $r(CR(\lambda, A)) < 1$, $\lambda \in q(A + C)$ and

$$(1.5) \quad R(\lambda_0, A + C) = R(\lambda_0, A) \sum_{n=0}^{\infty} [CR(\lambda, A)]^n.$$

**Proof.** By a result of Voigt [20] one has $r(BR(\lambda_0, A)) < 1$ and

$$(1.6) \quad R(\lambda_0, A + B) = R(\lambda_0, A) \sum_{n=0}^{\infty} [BR(\lambda_0, A)]^n.$$  

Let $\mu > s(A)$. Then by (1.1) $CR(\mu, A)$ and $BR(\mu, A) : E \to E$ are linear and satisfy

$$(1.7) \quad |CR(\mu, A) u| \leq BR(\mu, A) u \quad (u \in E).$$

We consider $D(A)$ with the graph norm $\|u\|_A = \|u\| + \|Au\|$.

Let $\mu > s(A)$. Then $BR(\mu, A) : E \to E$ is continuous as a positive linear mapping (see [17, 5.3 p. 84]). It follows from (1.7) that $CR(\mu, A) : E \to E$ is continuous. Since $\mu - A$ is an isomorphism from $(D(A), \| \cdot \|_A)$ onto $E$, it follows that $C : (D(A), \| \cdot \|_A) \to E$ is continuous as well.

For $f \in D(A)$ one has $\lim_{\mu \to \infty} \|\mu R(\mu, A)f - f\|_A = 0$.

So we conclude from (1.8)

$$(1.8) \quad |CR(\lambda, A) u| \leq CR(\mu, A) R(\lambda, A) u| \leq BR(\mu, A) R(\lambda_0, A)|u| \quad (u \in E).$$

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For $f \in D(A)$ one has $\lim_{\mu \to \infty} \|\mu R(\mu, A)f - f\|_A = 0$.

So we conclude from (1.8)

$$(1.8) \quad |CR(\lambda, A) u| = \lim_{\mu \to \infty} |\mu CR(\mu, A) R(\lambda, A) u| \leq \lim_{\mu \to \infty} \mu BR(\mu, A) R(\lambda_0, A)|u| \quad (u \in E).$$

This is (1.4) for $n = 1$. For $n \in \mathbb{N}$ the inequality follows by iteration.
As a consequence of (1.4) one has

\[ r(CR(\lambda, A)) \leq r(BR(\lambda_0, A)) < 1 \]

and so

\[ (I - CR(\lambda, A))^{-1} = \sum_{n=0}^{\infty} [CR(\lambda, A)]^n \quad \text{exists}. \]

Consequently, \((\lambda - (A + C)) = (I - CR(\lambda, A))(\lambda - A)\) is invertible and (1.5) holds.

For the proof of Theorem 1.1 we recall that a densely defined operator \(Z\) generates a holomorphic semigroup if and only if there exist \(M \geq 0, w > w(Z)\) such that

\[ \|\lambda R(\lambda, Z)\| \leq M \quad (\text{Re}(\lambda) \geq w). \]

(This follows for example from [13, A-II-Theorem 1.14].)

**Proof of Theorem 1.1.** There exist \(M \geq 0, w > \max\{w(A), s(A + B)\}\) such that \(\|\lambda R(\lambda, A)\| \leq M \) \((\text{Re}(\lambda) \geq w)\).

It follows from Lemma 1.4 that

\[
\left\| \sum_{n=0}^{\infty} [CR(\lambda, A)]^n u \right\| \leq \sum_{n=0}^{\infty} [BR(\text{Re}(\lambda), A)]^n |u| \\
\leq \sum_{n=0}^{\infty} [BR(w, A)]^n |u| \quad (u \in E, \text{Re}(\lambda) \geq w)
\]

(where (1.3) was used for the last inequality). Hence

\[
C = \sup_{\text{Re}(\lambda) \geq w} \left\| \sum_{n=0}^{\infty} [CR(\lambda, A)]^n \right\| < \infty.
\]

Using this it follows that

\[
\|\lambda R(\lambda, A + C)\| = \left\| \lambda R(\lambda, A) \sum_{n=0}^{\infty} [CR(\lambda, A)]^n \right\| \leq Mc \quad (\text{Re}(\lambda) \geq w).
\]

So \(A + C\) generates a holomorphic semigroup.

**Proof of Theorem 1.2.** Let \(w > \max\{w(A), w(A + B)\}\).

It follows from Lemma 1.5 that

\[
|R(\lambda, A + C)u| = \left| R(\lambda, A) \sum_{n=0}^{\infty} [CR(\lambda, A)]^n u \right| \\
\leq R(\lambda, A) \sum_{n=0}^{\infty} |BR(\lambda, A)|^n |u| \\
= R(\lambda, A + B) |u| \quad (u \in E) \quad \text{for all } \lambda > w.
\]
Iterating this one obtains

\[
|R(\lambda, A + C)^n u| \leq R(\lambda, A + B)^n |u| \quad (u \in E)
\]

for all \(\lambda > w\). Since \(w > w(A + B)\) one has

\[
\sup_{\lambda > w} \|[(\lambda - w)R(\lambda, A + B)]^n\| < \infty.
\]

It follows from (1.10) that

\[
\sup_{\lambda > w} \|[(\lambda - w)R(\lambda, A + C)]^n\| < \infty.
\]

So by the Hille-Yosida theorem \(A + C\) generates a semigroup \((V(t))_{t \geq 0}\). Letting \(\lambda = 1/t\) in (1.10) one obtains

\[
|V(t)u| = \lim_{n \to \infty} |(I - (t/n)(A + C))^{-n} u|
\]

\[
\leq \lim_{n \to \infty} |(I - (t/n)(A + B))^{-n} |u| = U(t) |u|
\]

\((u \in E, t \geq 0)\). \(\Box\)

Remark. If in Theorem 1.2 the operator \(C\) is positive, then by (1.5) \(R(\lambda, A + C) > 0\) for large \(\lambda\) and so \(V(t) \geq 0\) for \(t \geq 0\). We would like to mention the following more general result of Bidard-Zerner [6]. Assume that \(Z_1, Z_2, Z_3\) are (unbounded) operators on \(E\) such that \(D(Z_1) = D(Z_2) = D(Z_3)\) and \(Z_1 f \leq Z_2 f \leq Z_3 f\) for all \(f \in D(Z_1)_+\).

Assume that \(\lambda \in \sigma(Z_1) \cap \sigma(Z_3) \cap \mathbb{R}\) such that \(R(\lambda, Z_1) \geq 0\), \(R(\lambda, Z_3) \geq 0\). Then \(\lambda \in \sigma(Z_2)\) and \(R(\lambda, Z_2) \geq 0\).

2. Perturbation on \(AL\)-spaces and perturbation by finite rank operators. Let \(E\) be a (real or complex) Banach lattice, \(A\) the generator of a positive semigroup \((T(t))_{t \geq 0}\) on \(E\) and \(B: D(A) \to E\) a positive linear mapping. In this section we allow \((T(t))_{t \geq 0}\) to be arbitrary but assume restrictive conditions on \(E\) or the perturbation \(B\).

Recall that \(E\) is an \(AL\)-space of \(\|u + v\| = \|u\| + \|v\|\) whenever \(u, v \in E_+\) (see [17]). Any space \(L^1(\mu)\) is an \(AL\)-space.

The following result is due to Desch [8].

**Theorem 2.1.** Assume that \(E\) is an \(AL\)-space.

If \(A + B\) is resolvent positive, then \(A + B\) generates a positive semigroup.

This result is no longer true on \(L^p(1 < p < \infty)\) or \(C_0(\Omega)\) (\(\Omega\) locally-compact but not compact); see [4] and Section 4. However, we obtain perturbation results valid in any space if we consider perturbations of finite rank.

By \(D(A)_+\) we denote the cone of all positive linear forms on \(D(A)\) (i.e. linear mappings \(\varphi: D(A) \to \mathbb{R}\) satisfying \(\varphi(u) \geq 0\) whenever \(u \in D(A)_+\)).
Theorem 2.2. Suppose that there exist $\varphi \in D(A)'_+$, $g \in E_+$ such that
$$Bf = \varphi(f)g \quad (f \in D(A)).$$

Then $A + B$ generates a positive semigroup on $E$.

Example 2.3. Let $E = L^p(0, 1)$, $1 \leq p < \infty$ and let $A$ be defined by $Af = -f'$,
$D(A) = \{ f \in L^p(0, 1) : \exists f' \in L^p(0, 1) \text{ such that } f(x) = \frac{1}{b} \int_0^y f'(y) \, dy \mid x \in (0, 1) \}.$

Then $A$ generates a positive semigroup. Let $\mu$ be a bounded positive measure on $[0, 1]$, $g \in E_+$ and define $B : D(A) \to E$ by $Bf = \int_0^1 f(x) \, d\mu(x)g$. Then $A + B$ generates a positive semigroup.

Theorem 2.2 can be extended to perturbations of the following type. The mapping $B : D(A) \to E$ is called a regular finite rank perturbation of $A$ if there exist $\varphi_i \in \text{span } D(A)'_+$, $g_i \in E$ ($i = 1 \ldots n$) such that
$$Bf = \sum_{i=1}^n \varphi_i(f)g_i \quad (f \in D(A)).$$

Corollary 2.4. If $B$ is a regular finite rank perturbation of $A$, then $A + B$ generates a semigroup.

In view of Theorem 1.2 this is an immediate consequence of Theorem 2.2. Theorem 2.2 and its corollary are remarkable in the context of results by Desch and Schappacher [9].

Let $Z$ be the generator of a semigroup on a Banach space $G$. Consider $D(Z)$ with the graph norm and denote by $D(Z)'$ its dual space. An operator $C : D(Z) \to G$ is called a rank 1 perturbation of $Z$ if $D(C) = D(Z)$ and there exist $\varphi \in D(Z)'$ and $g \in G$ such that
$$Cf = \varphi(f)g.$$

Then the following is proved in [9].

1. If $Z$ generates an analytic semigroup, then so does $Z + C$ for any rank-one perturbation $C$.
2. If $Z + C$ generates a semigroup for all rank-one perturbations $C$ of $Z$, then the semigroup generated by $Z$ is analytic.

So Corollary 2.4 shows in particular, that $D(A)'_+ - D(A)'_+ = D(A)'$ if the semigroup generated by $A$ is not analytic (in other words, the cone $D(A)_+$ is not normal in the ordered Banach space $D(A)$). On the other hand, it is easy to see that $D(A)_+$ is normal if $A$ generates a multiplication semigroup (in the sense of [13, C-II-Sec 5]).

Proof of Theorem 2.2. There exist $\varphi \in D(A)'_+$, $g \in E_+$ such that $Bf = \varphi(f)g$ for all $f \in D(A)$.

a) We first show that $A + B$ is resolvent positive. Since $\lim_{\mu \to \infty} \|R(\mu, A)g\|_A = 0$, it follows that there exists $w \in \mathbb{R}$ such that $\varphi(R(\mu, A)g) < 1$ for all $\mu \geq w$. Define $R(\mu) \in L(E)$ by $R(\mu)f = R(\mu, A)f + \varphi(R(\mu, A)f)/(1 - \varphi(R(\mu, A)g))R(\mu, A)g (f \in E)$ for all $\mu \geq w$. 


Then $R(\mu)$ is a positive operator, and it is easy to see that $R(\mu) = (\mu - A - B)^{-1}$.

b) Now Voigt's proof of Theorem 2.1 can be adapted to the situation considered here. We merely indicate the necessary alterations in [20, Section 2].

At first, one assumes that there exists $\lambda > s(A)$ such that $\|BR(\lambda, A)\| < (1/2)$. Let $\alpha > 0$. Then for $f \in D(A)$, one has

$$i\int_0^\infty \| B \exp(-\lambda t) T(t)f \| dt = \int_0^\infty \| \exp(-\lambda t) \varphi(T(t)f) dt \| \| f \| \leq \varphi(R(\lambda, A)f) \| f \| = \| BR(\lambda, A)f \| \leq ||BR(\lambda, A)|| \| f \|.$$

Arguing as in the proof of [20, Lemma 2.1] one concludes

$$\int_0^\infty \| B \exp(-\lambda t) T(t)f \| dt \leq \gamma \| f \| (f \in D(A)) \quad \text{where} \quad \gamma := 2\|BR(\lambda, A)\| < 1.$$

So $A + B$ generates a semigroup by [21, Theorem 1]. The general case follows by replacing $B$ by $(1/n) B$ and $A$ by $A + (j/n) B$ successively ($j = 0, \ldots, n - 1$) where $n \in \mathbb{N}$ such that $\|BR(\lambda, A + B)\| < (n/2)$ for a fixed $\lambda > s(A + B)$. \hfill \Box

Remark (Ordered Banach spaces). The lattice property is not essential in the results (but convenient, in particular, for domination properties). One may consider more generally a positive semigroup on an ordered Banach space with generating and normal cone (see [5]). Then Theorem 2.2 remains valid.

Also Desch's theorem (Theorem 2.1) holds, if we suppose that the norm is additive on the positive cone (generalizing $AL$-spaces). Our proof (resp. Voigt's proof for Desch's theorem) go through without alterations.

Theorem 1.1 remains valid for $B = C$; however, some modifications (using [5], Corollary 1.7.5 for example) are necessary.

3. Perturbation by multiplication operators. Let $(\Omega, \mu)$ be a measure space, $1 \leq p < \infty$ and let $T = (T(t))_{t \geq 0}$ be a positive semigroup on $L^p(\Omega, \mu)$ with generator $A$. We assume that

$$(3.1) \quad D(A) \subset L^q(\Omega), \quad \text{where} \quad p < q \leq \infty.$$ 

Let $(1/r) + (1/q) = (1/p)$ (so that $L^rL^q \subset L^p$).

**Theorem 3.1.** Let $V \in L^r(\Omega, \mu)$ and $B: D(A) \to E$ be given by $Bf = Vf$.

a) If $T$ is holomorphic, then $A + V$ generates a holomorphic semigroup on $L^p(\Omega, \mu)$.

b) If $p = 1$, then $A + V$ generates a semigroup on $L^1(\Omega, \mu)$. 

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We will show that $B$ is relatively bounded with respect to $A$ with relative bound 0; i.e.

$$\lim_{\lambda \to \infty} \| VR(\lambda, A) \| = 0.$$  

So a) actually follows by the classical perturbation result and b) from Desch's theorem.

\textbf{Proof.} Let $F = \{ V \in L'(\Omega, \mu) : \lim_{\lambda \to \infty} \| VR(\lambda, A) \| = 0 \}$.

Since $L^\infty \cap L' \subset F$, it suffices to show that $F$ is closed. Let $V_n \in F$, $V \in L'$ such that $\| V_n - V \|_{L'} \to 0 (n \to \infty)$. By the closed graph theorem the embedding (3.1) is continuous if $D(A)$ carries the graph norm. Moreover,

$$\| VR(\lambda, A) \| \leq \| V - V_n \|_{L'} \| R(\lambda, A) \|_{L'(L^p, L^q)} + \| V_n R(\lambda, A) \|$$

$$\leq \text{const} \| V - V_n \|_{L'} \| R(\lambda, A) \|_{L'(L^p, D(A))} + \| V_n R(\lambda, A) \|$$

for all $\lambda > s(A)$, $n \in \mathbb{N}$.

Hence

$$\lim_{\lambda \to \infty} \| VR(\lambda, A) \| = 0$$

(and $(\lambda - A - B)^{-1} = (\lambda - A)^{-1} \sum_{n=0}^{\infty} [VR(\lambda, A)]^n$ exists and is positive for $\lambda$ sufficiently large). \(\square\)

Remark. One cannot omit condition (3.1). To see this it suffices to take $0 \leq m \in L^p \setminus L^\infty$, $Af = -mf$ with $D(A) = \{ f \in L^p : mf \in L^p \}$ and $V = 2m$.

We consider concrete examples.

1. Schrödinger semigroups. a) Let $1 < p < \infty$ and define $A$ on $L^p(\mathbb{R}^N)$ by $D(A) = W^{2,p}(\mathbb{R}^N)$, $Af = Af$. Then $A$ generates the Gaussian semigroup which is holomorphic and positive. We conclude from Theorem 3.1 that $A + V$ generates a holomorphic semigroup on $L^p(\mathbb{R}^N)$ whenever $0 \leq V \in L'(\mathbb{R}^N)$ for some $r$ satisfying $r \geq \max \{ p, (N/2) \}$ if $p \neq (N/2)$ and $r > (N/2)$ if $p = (N/2)$. In fact, by the Sobolev imbedding theorem one has

$$W^{2,p} \subset \begin{cases} L^\infty & \text{if } (N/2) < p \\ \bigcup_{p \leq q < \infty} L^q & \text{if } (N/2) = p \\ L^q & \text{for } (1/q) = ((1/p) - (2/N)) & \text{if } (N/2) > p. \end{cases}$$

So the claim follows from Theorem 3.1.
b) Kato [11] defines the operator $A + V$ on $L^1(\mathbb{R}^N)$ considering a well-known class of potentials $K_N$, where

$$K_N = \{ V \in L^1_{\text{loc}}(\mathbb{R}^N) : VD(A_1) \subset L^1 \quad \text{and} \quad \lim_{\lambda \to \infty} \| VR(\lambda, A_1) \| = 0 \}$$

and $A_1$ is the Laplacian on $L^1(\mathbb{R}^N)$ (i.e. $D(A_1) = \{ f \in L^1 : Af \in L^1 \}$, $A_1 f = Af$).

So by Theorem 1.1 $A_1 + V$ generates a positive holomorphic semigroup on $L^1$ whenever $0 < V \in K_N$.

It is shown by Kato [11] that this semigroup interpolates on $L^p(\mathbb{R}^N)$ ($1 \leq p < \infty$). This gives another proof of the result of a) in the case $p > (N/2)$ since then $L^p(\mathbb{R}^N) \subset K_N$ (see [2, Proposition 4.3]).

Remark. A perturbation theory for a larger class $K_N$ has been developed by Voigt [19].

2. Bounded domain. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with boundary of class $C^\infty$.

Then the operator $A$ defined by $D(A) = W^{1,p}_0 \cap W^{2,p}$, $Af = Af$, generates a positive holomorphic semigroup on $L^p(\Omega)$ ($1 < p < \infty$) (see [1]).

By the same argument as in 1.a) one sees that $A + V$ generates a positive holomorphic semigroup on $L^p(\Omega)$ whenever $0 \leq V \in L^r(\Omega)$ where $r \geq \max \{ p, (N/2) \}$ if $p \neq (N/2)$ and $r > (N/2)$ if $p = (N/2)$.

3. Elliptic operators. In Example 1 and 2 one can replace the Laplacian by any strictly elliptic real differential operator of second order with (sufficiently) regular coefficients. Indeed, those operators, with domain $W^{2,p}(\mathbb{R}^N)$ (resp. $W^{1,p}_0(\Omega) \cap W^{2,p}(\Omega)$), generate a positive holomorphic semigroup on $L^p(\mathbb{R}^N)$ (resp. $L^p(\Omega)$), $1 < p < \infty$.

4. Systems of evolution equations. Let $A$ and $D$ be generators of positive semigroups on a Banach lattice $E$ (resp. $F$), and let $B: D(D) \to E$ and $C: D(A) \to F$ be linear and positive. We consider the operator

$$\mathcal{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

on $E \times F$ with domain $D(\mathcal{A}) = D(A) \times D(D)$.

Proposition 4.1. The operator $\mathcal{A}$ is resolvent positive if and only if there exists $\lambda > \max \{ s(A), s(D) \}$ such that $r(BR(\lambda, D) CR(\lambda, A)) < 1$.

Remark. One has $r(BR(\lambda, D) CR(\lambda, A)) = r(CR(\lambda, A) BR(\lambda, D))$.

Proof. a) Assume that $\mathcal{A}$ is resolvent positive. Let $\lambda > s(\mathcal{A})$. We write $\mathcal{A} = \mathcal{A}_1 + \mathcal{B}$ where

$$\mathcal{A}_1 = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix},$$
both with domain $D(A) \times D(D)$. It follows from [20, Theorem 1.1] that
\[
\mathcal{B} R(\lambda, A) = \begin{bmatrix}
0 & BR(\lambda, D) \\
CR(\lambda, A) & 0
\end{bmatrix}
\]
has spectral radius $< 1$.

In particular,
\[
Q(\lambda) := \begin{bmatrix}
I & -BR(\lambda, D) \\
- CR(\lambda, A) & I
\end{bmatrix}
\]
is invertible and has positive inverse. It follows from [15, Lemma 2.1 and (2.1)] that
\[
Q(\lambda) := CR(\lambda, A) I
\]
is invertible and has positive inverse. It follows from [15, Lemma 2.1 and (2.1)] that
\[
A(\lambda) := (I - BR(\lambda, D)) CR(\lambda, A)
\]
is invertible and $|A(\lambda)| > 0$. Since
\[
BR(\lambda, D) CR(\lambda, A)
\]
is a positive operator one concludes $r(BR(\lambda, D) CR(\lambda, A)) < 1$ (by [16, App. 2.3]).

b) Assume that $\lambda_0 > \max \{s(A), s(D)\}$ such that
\[
r(BR(\lambda_0, D) CR(\lambda_0, A)) < 1.
\]
Since $R(\lambda, D)$ and $R(\lambda, A)$ are decreasing in $\lambda$ (by (1.3)), it follows that
\[
r(BR(\lambda, D) CR(\lambda, A)) < 1
\]
for all $\lambda \geq \lambda_0$. Consequently, by [15, Lemma 2.1] the operator $Q(\lambda)$ is invertible for $\lambda \geq \lambda_0$
and $Q(\lambda)^{-1} \geq 0$ (this can be seen from [15, (2.1)]). Thus
\[
(\lambda - A) = Q(\lambda) \begin{bmatrix}
\lambda - A & 0 \\
0 & \lambda - D
\end{bmatrix}
\]
has positive inverse for $\lambda \geq \lambda_0$. □

**Corollary 4.2.** $\mathcal{A}$ is resolvent positive whenever $B$ or $C$ is bounded.

Using the results of Section 1 and 2 one obtains the following conclusion.

**Theorem 4.3.** Assume that $r(BR(\lambda, D) CR(\lambda, A)) < 1$ for some $\lambda > \max \{s(A), s(D)\}$.

a) If the semigroups generated by $A$ and $D$ are holomorphic, then $\mathcal{A}$ generates a holomorphic positive semigroup.

b) If $E$ and $F$ are AL-spaces, then $\mathcal{A}$ generates a positive semigroup.

**Remark.** A systematic investigation of matrices of unbounded operators is given by Nagel [14], [15]. The problem under which condition a matrix of unbounded operators generates a positive semigroup is treated by [7].

We conclude by an example where $\mathcal{A} + \mathcal{B}$ does not generate a semigroup.

Let $F = C_0[0, 1] = \{ f \in C[0, 1]: f(0) = 0 \}$, $Af = -f'$,
\[
D(A) = \{ f \in C^1[0, 1]: f'(0) = f(0) = 0 \}.
\]
Then $A$ generates a positive contraction semigroup on $F$.

Let $B: D(A) \to F$ be given by
\[
Bf(x) = (1/x) f(x) \quad \text{if } x \neq 0 \quad \text{and } Bf(0) = 0.
\]
Then $\mathcal{A} + \mathcal{B} = \begin{bmatrix} A & 0 \\ B & A \end{bmatrix}$ on $E = F \times F$ is resolvent positive but does not generate a
semigroup (where $\mathcal{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ and $\mathcal{B} = \begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}$).
Proof. We have

\[(\lambda - \mathcal{A} - \mathcal{B})^{-1} = \begin{bmatrix} R(\lambda, A) & 0 \\ R(\lambda, A) B R(\lambda, A) & R(\lambda, A) \end{bmatrix} \]

for \( \lambda > s(A) = -\infty \), where \((R(\lambda, A)f)(x) = \exp(-\lambda x) \int_0^x \exp(\lambda y)f(y)\,dy\).

For \( f \in F \) we compute

\[
(R(\lambda, A)BR(\lambda, A)f)(x) = \exp(-\lambda x) \int_0^x \exp(\lambda y)BR(\lambda, A)f(y)\,dy
\]

\[
= \exp(-\lambda x) \int_0^x \exp(\lambda y)B\exp(\lambda z)f(z)\,dz\,dy
\]

\[
= \exp(-\lambda x) \int_0^x (1/y) \exp(\lambda z)f(z)\,dz\,dy
\]

\[
= \int_0^x \exp(-\lambda(x-z))f(z)\log(x/z)\,dz
\]

\[
= \int_0^x \exp(-\lambda t)\log(x/(x-t))f(x-t)\,dt
\]

where

(4.1) \((W(t)f)(x) = \log(x/(x-t))f(x-t)\) if \( x > t \) and

(4.2) \((W(t)f)(x) = 0\) if \( x \leq t \).

If \( \mathcal{A} + \mathcal{B} \) were the generator of a semigroup \( U \) on \( E = F \times F \), then by the uniqueness of Laplace transforms it would be of the form

\[
U(t)(f, g) = \begin{bmatrix} T(t)f & 0 \\ W(t)f & T(t)g \end{bmatrix}.
\]

But (4.1) does not define a bounded operator on \( C_0(0, 1) \). \( \square \)

The example demonstrates the sharpness of the Hille-Yosida theorem. In fact, let

\( \mathcal{B}(\lambda) = (\lambda - \mathcal{A} - \mathcal{B})^{-1} \) and \( R(\lambda) = R(\lambda, A) \).

Then

(4.2) \( \sup_{\lambda > 0} \| (\lambda \mathcal{B}(\lambda))^n \| \leq \text{const} \cdot n \)

for all \( n \in \mathbb{N} \).
However, \( \mathcal{A} + \mathcal{B} \) does not generate a semigroup, i.e.
\[
\sup_{\lambda > w} \sup_{n \in \mathbb{N}} \| [\lambda - w] \mathcal{R}(\lambda)]^n \| = \infty \quad \text{for all } w \geq 0.
\]

**Proof of (4.2).** It is easy to show by induction that
\[
\mathcal{R}(\lambda)^n = \left[ \begin{array}{cc} R(\lambda)^n & 0 \\ \sum_{k=1}^{n} R(\lambda)^{n+1-k} BR(\lambda)^k & R(\lambda)^n \end{array} \right].
\]

But \( R(\lambda)^{n+1-k} BR(\lambda)^k \leq R(\lambda)^{n+1-k} BR(0) R(\lambda)^{k-1} \) for all \( \lambda \geq 0 \).

Hence
\[
\lambda^n \| R(\lambda)^{n+1-k} BR(\lambda)^k \| \leq \lambda^n \| R(\lambda)^{n+1-k} \| \| BR(0)\| \| R(\lambda)^{k-1} \| \leq \| BR(0)\|
\]

since \( \| R(\lambda)\| \leq 1 \). Thus
\[
\| \lambda^n \mathcal{R}(\lambda)^n \| \leq \text{const} \cdot n \quad \text{for all } \lambda > 0.
\]

**References**


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